Robust Kalman Filtering for Discrete Time-Varying Uncertain Systems With Multiplicative Noises

Fuwen Yang, Zidong Wang, and Y. S. Hung

***Abstract*—In this note, a robust finite-horizon Kalman filter is designed for discrete time-varying uncertain systems with both additive and multiplicative noises. The system under consideration is subject to both deterministic and stochastic uncertainties. Sufficient conditions for the filter to guarantee an optimized upper bound on the state estimation error variance for admissible uncertainties are established in terms of two discrete Riccati difference equations. A numerical example is given to show the applicability of the presented method.**

***Index Terms*—Additive noise, multiplicative noise, norm-bounded uncertainty, robust Kalman filtering, time-varying system.**

I. Introduction

The control and filtering problems for the systems with multiplicative noises have recently received much attention, since the signals contaminated by multiplicative noises are common in many practical systems, such as image processing systems [9], [10], [18], communication systems [20], and aerospace systems (see, e.g., [11] and the references therein). Different from the case of the additive noise, the second order statistics of the multiplicative noise is usually unknown, as it depends on the real state of the system. This gives rise to more difficulties in the research. So far, there have been several approaches to dealing with the control and filtering problems for systems with multiplicative noises, including the game-theoretic method [2], [16], the linear matrix inequality (LMI) approach [1], [3], [4], [21], and the Riccati equation approach [6], to name just a few.

On the other hand, due to unmodeled dynamics, parameter variations, model reduction and linearization, the systems inherently contain the modeling parameter uncertainties. Different kinds of descriptions have been introduced in the literature to account for the uncertainties, such as norm-bounded uncertainty, convex uncertainty, integral quadratic constraint (IQC) uncertainty, linear fractional transformation (LFT) uncertainty. Accordingly, different methods for studying robust control and filtering problems of these uncertain systems have been proposed (see, e.g., [7], [8], [15], [23]–[25], [27], [28], and the references therein). Very recently, Petersen *et al.* investigated the control problem in [12] and [13] for a class of systems with both stochastic modeling uncertainty and deterministic modeling uncertainty, where the stochastic uncertainty has been expressed as a multiplicative noise. It should be pointed out that, compared to the control case, the corresponding robust filtering problem for systems with stochastic and deterministic uncertainties has gained much less attention. This situation motivates our present investigation.

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In this note, the robust Kalman filtering problem is considered for discrete time-varying systems with two kinds of parameter uncertainties. One is the norm-bounded deterministic uncertainty and the other is the so-called stochastic uncertainty, namely, the multiplicative noise. It is worth emphasizing that the robust Kalman filtering problems have been intensively investigated in the past decade. However, most of the papers have been concerned with the uncertain systems *without* multiplicative noise (see, e.g., [5], [14], [17], [19], and [26]). In [21] and [22], Wang and Balakrishnan have proposed an LMI method to cope with the *stationary* robust filtering problem for the uncertain systems with multiplicative noises over an *infinite horizon*, and a practical example on communication channel filtering problem has been given in [22].

In this note, a robust *finite-horizon* Kalman filter is designed for the uncertain systems with multiplicative noises where the signals are *nonstationary*. The problem addressed is the design of a linear filter that yields an estimation error variance with an optimized guaranteed upper bound for all admissible uncertainties. Sufficient conditions for designing such an optimized filter are derived in terms of two discrete Riccati difference equations, which might be suitable for recursive computation in online applications.

The remainder of this note is organized as follows. In Section II, the robust Kalman filtering problem for discrete time-varying systems subject to norm-bounded parameter uncertainty and multiplicative noises is formulated. An algorithm for the filter design is developed in Section III, which guarantees the upper bound on the state estimation error variance and simultaneously minimizes this upper bound. An example is given in Section IV, and some concluding remarks are drawn in Section V.

II. Problem Formulation and Preliminaries

Consider the following class of discrete time-varying uncertain systems with multiplicative noises defined on $k \in [0, N]$:

$$\begin{aligned} x_{k+1} &= (A_k + H_{1,k}F_kE_k + A_{s,k}\eta_k)\,x_k + B_kw_k \\ y_k &= (C_k + H_{2,k}F_kE_k + C_{s,k}\varsigma_k)\,x_k + v_k \end{aligned} \tag{1}$$

where $x_k \in R^n$ is the state, $y_k \in R^p$ is the measured output, $w_k \in R^{q1}$ is the process noise, $v_k \in R^{q2}$ is the measurement noise, $\eta_k \in R$ and $\varsigma_k \in R$ are the multiplicative noises, A_k, $A_{s,k}$, B_k, C_k, $C_{s,k}$, $H_{1,k}$, $H_{2,k}$ and E_k are known real time-varying matrices with appropriate dimensions, whereas $F_k \in R^{i\times j}$ is the norm-bounded time-varying uncertainty, i.e.,

$$F_kF_k^T \le I, \quad \forall k. \tag{2}$$

The parameter uncertainty F_k is said to be admissible if it satisfies (2).

The noise signals w_k, v_k, η_k and ς_k are all Gaussian white noise sequences. They, together with the initial state x_0, have the following statistical properties:

$$\begin{aligned} &E[w_k] = 0 \quad E[v_k] = 0 \quad E[\eta_k] = 0 \\ &E[\varsigma_k] = 0 \quad E[x_0] = \bar{x}_0 \end{aligned} \tag{3}$$

$$E\left\{\begin{bmatrix} w_k \\ v_k \\ \eta_k \\ \varsigma_k \\ x_0 \end{bmatrix}\begin{bmatrix} w_j \\ v_j \\ \eta_j \\ \varsigma_j \\ x_0 \end{bmatrix}^T\right\} = \begin{bmatrix} Q_k\delta_{kj} & 0 & 0 & 0 & 0 \\ 0 & R_k\delta_{kj} & 0 & 0 & 0 \\ 0 & 0 & \delta_{kj} & 0 & 0 \\ 0 & 0 & 0 & \delta_{kj} & 0 \\ 0 & 0 & 0 & 0 & S_0 \end{bmatrix} \tag{4}$$

where E stands for the mathematical expectation operator, δ_{kj} denotes the Kronecker delta function, which is equal to unity for $k = j$ and zero elsewhere. The known matrices Q_k, R_k, and S_0 represent the second-order statistics of the noises and the initial state.

Remark 1: The deterministic uncertainty in F_k and the stochastic uncertainties in η_k, ς_k can be scaled and absorbed in the matrices E_k and $A_{s,k}$, $C_{s,k}$, respectively. Therefore, it is reasonable to assume that the deterministic uncertainty in F_k satisfies (2) and the stochastic uncertainties in η_k, ς_k satisfy (4). It is worth mentioning that all the noise signals in (1) are modeled as the zero-mean Gaussian white noise sequences. This, however, does not cause any loss of generality, since color noise can be whitened *a priori* to the zero-mean Gaussian white noise. ■

Note that the system matrix and output matrix in (1) contain both deterministic parametric uncertainties $H_{1,k}F_kE_k$, $H_{2,k}F_kE_k$ and the stochastic parametric uncertainties $A_{s,k}\eta_k$, $C_{s,k}\varsigma_k$, respectively. Due to the complexity in the uncertainties, it is not easy to predict the system states in the form of (1). In the following, we transform (1) into an uncertain system with state-dependent noises:

$$\begin{aligned} x_{k+1} &= (A_k + H_{1,k}F_kE_k)\,x_k + \bar{w}_k \\ y_k &= (C_k + H_{2,k}F_kE_k)\,x_k + \bar{v}_k \end{aligned} \tag{5}$$

where

$$\bar{w}_k = A_{s,k}\eta_k x_k + B_k w_k \quad \bar{v}_k = C_{s,k}\varsigma_k x_k + v_k. \tag{6}$$

It can be seen that the noise signals $\bar{w}_k$ and $\bar{v}_k$ depend on the system state x_k and their second-order statistics are unknown.

The statistical properties of the noise signals (6) can be described as follows:

$$E[\bar{w}_k] = 0, \quad E[\bar{v}_k] = 0 \tag{7}$$

$$\begin{aligned} &E\left\{\begin{bmatrix}\bar{w}_k\\ \bar{v}_k\end{bmatrix}\begin{bmatrix}\bar{w}_k^T & \bar{v}_k^T\end{bmatrix}\right\} \\ &= \begin{bmatrix} A_{s,k}\tilde{P}_kA_{s,k}^T + B_kQ_kB_k^T & 0 \\ 0 & C_{s,k}\tilde{P}_kC_{s,k}^T + R_k\end{bmatrix} \\ &:= \tilde{\Phi}_k \end{aligned} \tag{8}$$

where

$$\tilde{P}_k := E\left[x_k x_k^T\right]. \tag{9}$$

Now, consider the following filter for the uncertain system (5):

$$\hat{x}_{k+1} = \hat{A}_k\hat{x}_k + \hat{K}_k(y_k - C_k\hat{x}_k) \tag{10}$$

where $\hat{x}_k \in R^n$ is the state estimate, $\hat{A}_k$ and $\hat{K}_k$ $(0 \le k \le N)$ are the filter parameters to be determined.

The objective of this note is twofold. First, we intend to design a finite-horizon filter of the structure (10), such that for all admissible uncertainties in F_k (F_k meets $F_kF_k^T \le I$), there exist a sequence of positive–definite matrices Θ_k $(0 < k \le N)$ satisfying

$$E\left[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\right] \le \Theta_k, \quad \forall k. \tag{11}$$

That is, the finite upper bound on the state estimation error variance is guaranteed. Second, we shall minimize the bound Θ_k and obtain an optimized filter eventually. This problem will be referred to as a finite-horizon robust Kalman filtering problem. Note that the bound Θ_k is independent of the second order statistics of the system state, i.e., $\tilde{P}_k$ defined in (9).

III. Finite-Horizon Robust Kalman Filter Design

In this section, we shall discuss the robust Kalman filter design problem over finite horizon. Define a new state vector

$$\tilde{x}_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} \tag{12}$$

then an augmented system follows from the system (5) and the filter (10) that

$$\tilde{x}_{k+1} = \left(\tilde{A}_k + \tilde{H}_kF_k\tilde{E}_k\right)\tilde{x}_k + \tilde{B}_kd_k \tag{13}$$

where

$$\begin{aligned} d_k &= \begin{bmatrix}\bar{w}_k\\ \bar{v}_k\end{bmatrix} \quad \tilde{A}_k = \begin{bmatrix} A_k & 0 \\ \hat{K}_kC_k & \hat{A}_k - \hat{K}_kC_k\end{bmatrix} \\ \tilde{H}_k &= \begin{bmatrix} H_{1,k} \\ \hat{K}_kH_{2,k}\end{bmatrix} \quad \tilde{E}_k = \begin{bmatrix} E_k & 0\end{bmatrix} \\ \tilde{B}_k &= \begin{bmatrix} I & 0 \\ 0 & \hat{K}_k\end{bmatrix}. \end{aligned} \tag{14}$$

Denote the state covariance matrix of the augmented system (13) by

$$\tilde{\Sigma}_k := E\left[\tilde{x}_k\tilde{x}_k^T\right] = E\left\{\begin{bmatrix}x_k\\ \hat{x}_k\end{bmatrix}\begin{bmatrix}x_k\\ \hat{x}_k\end{bmatrix}^T\right\}. \tag{15}$$

According to (13) and (15), the Lyapunov equation that governs the evolution of the covariance matrix $\tilde{\Sigma}_k$ can be written as

$$\tilde{\Sigma}_{k+1} = \left(\tilde{A}_k + \tilde{H}_kF_k\tilde{E}_k\right)\tilde{\Sigma}_k\left(\tilde{A}_k + \tilde{H}_kF_k\tilde{E}_k\right)^T + \tilde{B}_k\tilde{\Phi}_k\tilde{B}_k^T \tag{16}$$

where $\tilde{\Phi}_k$ is defined in (8). The initial value is $\tilde{\Sigma}_0$ and, for later use, we set $[I \quad 0]\tilde{\Sigma}_0[I \quad 0]^T = S_0$, $[I \quad -I]\tilde{\Sigma}_0[I \quad -I]^T = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] := S_1$.

It is noted that the uncertainty F_k appears in (16). Therefore, it is impossible to give the exact value of the covariance matrix $\tilde{\Sigma}_k$. An alternative way is to find a set of upper bounds for $\tilde{\Sigma}_k$ and then obtain the minimum with respect to the filter parameters $\hat{A}_k$ and $\hat{K}_k$.

We now recall some useful lemmas.

Lemma 1 [1]: Let Y be a symmetric matrix, A, H and E be real matrices, X be a symmetric positive–definite matrix and a matrix F satisfy $FF^T \le I$. Then

$$(A + HFE)X(A + HFE)^T - Y \le 0 \tag{17}$$

if and only if there exists a constant $\alpha > 0$ such that

$$\alpha^{-1}I - EXE^T > 0, \tag{18}$$

$$A\left(X^{-1} - \alpha E^TE\right)^{-1}A^T + \alpha^{-1}HH^T - Y \le 0. \tag{19}$$

■

Lemma 2 [26]: Given matrices A, H, E, and F with compatible dimensions such that $FF^T \le I$. Let X be a symmetric positive–definite matrix and $\alpha > 0$ be an arbitrary positive constant such that $\alpha^{-1}I - EXE^T > 0$, then the following inequality holds

$$(A + HFE)X(A + HFE)^T \le A\left(X^{-1} - \alpha E^TE\right)^{-1}A^T + \alpha^{-1}HH^T. \tag{20}$$

■

Lemma 3 [19]: For $0 \le k \le N$, suppose that $X = X^T > 0$, $f_k(X) = f_k^T(X) \in R^{n\times n}$ and $g_k(X) = g_k^T(X) \in R^{n\times n}$. If there exists $Y = Y^T \ge X$ such that

$$f_k(Y) \ge f_k(X) \tag{21}$$

and

$$g_k(Y) \ge f_k(Y) \tag{22}$$

then the solutions A_k and B_k to the following difference equations:

$$A_{k+1} = f_k(A_k) \quad B_{k+1} = g_k(B_k) \quad A_0 = B_0 > 0 \tag{23}$$

satisfy $A_k \le B_k$. ■

Remark 2: Lemma 1 is known as the S-procedure technique, which is often utilized to convert the inequality involving norm-bounded uncertainty like (17) into an equivalent matrix inequality with an extra scalar parameter α. Lemma 2 is a direct result. Lemma 3 will be used to give the upper bound for the covariance matrix $\tilde{\Sigma}_k$. ■

Now, we are in the position to introduce the notions of *quadratic filter* and *identity quadratic filter* for the uncertain system (13).

Definition 1 [14], [26]: The filter (10) is said to be *a quadratic filter* associated with Σ_k and P_k if there exist symmetric positive–definite matrices Σ_k and $P_k (0 \le k \le N)$ such that for all admissible uncertainty F_k satisfying (2), the following inequality holds:

$$(\tilde{A}_k + \tilde{H}_k F_k \tilde{E}_k)\Sigma_k(\tilde{A}_k + \tilde{H}_k F_k \tilde{E}_k)^T - \Sigma_{k+1} + \tilde{B}_k \Phi_k \tilde{B}_k^T \le 0 \tag{24}$$

where

$$\Phi_k = \begin{bmatrix} A_{s,k}P_kA_{s,k}^T + B_kQ_kB_k^T & 0 \\ 0 & C_{s,k}P_kC_{s,k}^T + R_k \end{bmatrix}. \tag{25}$$

■

According to Lemma 1, (24) holds for all admissible uncertainty F_k satisfying (2), if and only if there exist a sequence of positive scalars $\alpha_k > 0$ such that

$$\tilde{A}_k(\Sigma_k^{-1} - \alpha_k \tilde{E}_k^T \tilde{E}_k)^{-1}\tilde{A}_k^T + \alpha_k^{-1}\tilde{H}_k\tilde{H}_k^T - \Sigma_{k+1} + \tilde{B}_k\Phi_k\tilde{B}_k^T \le 0 \tag{26}$$

and

$$\alpha_k^{-1} I - \tilde{E}_k \Sigma_k \tilde{E}_k^T > 0. \tag{27}$$

Definition 2: The filter (10) is said to be an *identity quadratic filter* associated with Σ_k and P_k if there exist symmetric positive–definite matrices Σ_k and $P_k (0 \le k \le N)$ and positive scalars $\alpha_k (0 \le k \le N)$, such that for all admissible uncertainty F_k satisfying (2), both the following equation:

$$\Sigma_{k+1} = \tilde{A}_k(\Sigma_k^{-1} - \alpha_k \tilde{E}_k^T \tilde{E}_k)^{-1}\tilde{A}_k^T + \alpha_k^{-1}\tilde{H}_k\tilde{H}_k^T + \tilde{B}_k\Phi_k\tilde{B}_k^T \tag{28}$$

and (27) are satisfied, where Φ_k is given in (25). ■

Based on Definition 2, we have the following conclusion.

Theorem 1: If $\Sigma_0 = \tilde{\Sigma}_0$ and $\tilde{P}_k \le P_k$, then

$$\tilde{\Sigma}_k \le \Sigma_k \tag{29}$$

where $\tilde{\Sigma}_k$ and Σ_k satisfy (16) and (28), respectively.

Proof: $P_k \ge \tilde{P}_k$ implies that $\Phi_k \ge \tilde{\Phi}_k$ and $\tilde{B}_k\Phi_k\tilde{B}_k^T \ge \tilde{B}_k\tilde{\Phi}_k\tilde{B}_k^T$. Thus, (29) follows directly from Lemma 2 and Lemma 3. ■

Theorem 1 gives an upper bound for the covariance matrix $\tilde{\Sigma}_k$ provided that $\tilde{P}_k \le P_k$. Next, we shall first find the upper bound P_k for $\tilde{P}_k$, then construct an upper bound Σ_k for $\tilde{\Sigma}_k$ and select the filter parameters $\hat{A}_k$ and $\hat{K}_k$ that minimize

$$\begin{aligned}\Theta_k &:= [I \quad -I]\,\Sigma_k\,[I \quad -I]^T \\ &\ge [I \quad -I]\,\tilde{\Sigma}_k\,[I \quad -I]^T \\ &= E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T].\end{aligned} \tag{30}$$

Our main results are summarized in the following theorem that provides a constructive approach to designing the identity quadratic filter with optimized upper bound Θ_k. For the purpose of clarity, we only give the sketch of the proof.

Theorem 2: Consider (1). Let $\alpha_k > 0$ be a sequence of positive scalars. If the following two discrete-time Riccati difference equations:

$$\begin{aligned}\Theta_{k+1} = &-\left[\alpha_k^{-1}H_{1,k}H_{2,k}^T + A_k(\Theta_k^{-1} - \alpha_k E_k^T E_k)^{-1}C_k^T\right] \\ &\cdot R_{1,k}^{-1}\left[\alpha_k^{-1}H_{2,k}H_{1,k}^T + C_k(\Theta_k^{-1} - \alpha_k E_k^T E_k)^{-1}A_k^T\right] \\ &+ A_k\left(\Theta_k^{-1} - \alpha_k E_k^T E_k\right)^{-1}A_k^T + \alpha_k^{-1}H_{1,k}H_{1,k}^T \\ &+ A_{s,k}P_kA_{s,k}^T + B_kQ_kB_k^T, \quad \Theta_0 = S_1\end{aligned} \tag{31}$$

and

$$\begin{aligned}P_{k+1} = &\alpha_k^{-1}H_{1,k}H_{1,k}^T + A_k\left(P_k^{-1} - \alpha_k E_k^T E_k\right)^{-1}A_k^T \\ &+ B_kQ_kB_k^T + A_{s,k}P_kA_{s,k}^T, \quad P_0 = S_0\end{aligned} \tag{32}$$

have positive–definite solutions Θ_k and P_k such that $\alpha_k^{-1}I - E_kP_kE_k^T > 0$ and $P_k > \Theta_k$, then an identity quadratic filter (10) with parameters

$$\begin{aligned}\hat{A}_k = &A_k + \left(A_k - \hat{K}_kC_k\right)\Theta_kE_k^T \\ &\cdot\left(\alpha_k^{-1}I - E_k\Theta_kE_k^T\right)^{-1}E_k\end{aligned} \tag{33}$$

$$\begin{aligned}\hat{K}_k = &\Big[\alpha_k^{-1}H_{1,k}H_{2,k}^T \\ &+ A_k\left(\Theta_k^{-1} - \alpha_kE_k^TE_k\right)^{-1}C_k^T\Big]R_{1,k}^{-1}\end{aligned} \tag{34}$$

where

$$\begin{aligned}R_{1,k} = &\alpha_k^{-1}H_{2,k}H_{2,k}^T + C_{s,k}P_kC_{s,k}^T + R_k \\ &+ C_k(\Theta_k^{-1} - \alpha_kE_k^TE_k)^{-1}C_k^T\end{aligned} \tag{35}$$

will be such that the state estimation error variance satisfies boundedness condition

$$E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] \le \Theta_k, \ \forall k. \tag{36}$$

Moreover, the filter (10) with parameters (33) and (34) minimizes the bound Θ_k.

Proof: First, substituting (14) into (16) and according to (9), we have

$$\begin{aligned}\tilde{P}_{k+1} = &(A_k + H_{1,k}F_kE_k)\tilde{P}_k(A_k + H_{1,k}F_kE_k)^T \\ &+ B_kQ_kB_k^T + A_{s,k}\tilde{P}_kA_{s,k}^T, \\ \tilde{P}_0 = &S_0.\end{aligned} \tag{37}$$

Comparing (37) to (32), and according to Lemmas 2 and 3, we conclude that $\tilde{P}_k \le P_k$.

Next, suppose that Σ_k is of the following form [5], [19]:

$$\Sigma_k = \begin{bmatrix} P_k & P_k - \Theta_k \\ P_k - \Theta_k & P_k - \Theta_k \end{bmatrix}. \tag{38}$$

Substitute (14), (33), (34), and (38) into the right-hand side of (28), and consider the relationships (31) and (32). Then, direct algebraic manipulations show the right-hand side of (28) is given by

$$\tilde{A}_k(\Sigma_k^{-1} - \alpha_k\tilde{E}_k^T\tilde{E}_k)^{-1}\tilde{A}_k^T + \alpha_k^{-1}\tilde{H}_k\tilde{H}_k^T + \tilde{B}_k\Phi_k\tilde{B} = \begin{bmatrix} P_{k+1} & P_{k+1} - \Theta_{k+1} \\ P_{k+1} - \Theta_{k+1} & P_{k+1} - \Theta_{k+1} \end{bmatrix}. \tag{39}$$

This means (38) is a solution to (28). Now, from the initial conditions in (31) and (32), we know that (16) and (28) have the same initial conditions. It then follows from Theorem 1 that $\tilde{\Sigma}_k \leq \Sigma_k$. Therefore, we have

$$\begin{aligned} E[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] &= [\,I \quad -I\,]\,\tilde{\Sigma}_k\,[\,I \quad -I\,]^T \\ &\leq [\,I \quad -I\,]\,\Sigma_k\,[\,I \quad -I\,]^T \\ &= \Theta_k, \quad \forall k \end{aligned} \tag{40}$$

which implies (36).

Substituting (14) and (38) into (28), we have

$$\begin{aligned} \Theta_{k+1} =& [\,I \quad -I\,]\,\Sigma_{k+1}\,[\,0 \quad I\,]^T \\ =& \alpha_k^{-1}\left(H_{1,k} - \hat{K}_k H_{2,k}\right)\left(H_{1,k} - \hat{K}_k H_{2,k}\right)^T \\ &+ A_{s,k} P_k A_{s,k}^T + B_k Q_k B_k^T \\ &+ \hat{K}_k\left(C_{s,k} P_k C_{s,k}^T + R_k\right)\hat{K}_k^T \\ &+ [\,A_k - \hat{K}_k C_k \quad \hat{K}_k C_k - \hat{A}_k\,]\left(\Sigma_k^{-1} - \alpha_k \tilde{E}_k^T \tilde{E}_k\right)^{-1} \\ &\cdot [\,A_k - \hat{K}_k C_k \quad \hat{K}_k C_k - \hat{A}_k\,]^T. \end{aligned} \tag{41}$$

We take the partial derivatives of Θ_{k+1} with respect to $\hat{A}_k$ and $\hat{K}_k$ as follows:

$$\begin{aligned} \frac{\partial \Theta_{k+1}}{\partial \hat{A}_k} =& [\,A_k - \hat{K}_k C_k \quad \hat{K}_k C_k - \hat{A}_k\,] \\ &\cdot (\Sigma_k^{-1} - \alpha_k \tilde{E}_k^T \tilde{E}_k)^{-1}\,[\,0 \quad -I\,]^T = 0 \end{aligned} \tag{42}$$

$$\begin{aligned} \frac{\partial \Theta_{k+1}}{\partial \hat{K}_k} =& -2\alpha_k^{-1}(H_{1,k} - \hat{K}_k H_{2,k})H_{2,k}^T \\ &+ 2\hat{K}_k(C_{s,k} P_k C_{s,k}^T + R_k) \\ &+ 2\,[\,A_k - \hat{K}_k C_k \quad \hat{K}_k C_k - \hat{A}_k\,] \\ &\cdot (\Sigma_k^{-1} - \alpha_k \tilde{E}_k^T \tilde{E}_k)^{-1}\,[\,-C_k \quad C_k\,]^T = 0. \end{aligned} \tag{43}$$

From (42) and (43), and through tedious but straightforward algebraic manipulation, we obtain the optimal filter parameters $\hat{A}_k$ and $\hat{K}_k$ as follows:

$$\begin{aligned} \hat{A}_k =& A_k + (A_k - \hat{K}_k C_k)\Theta_k E_k^T \\ &\cdot (\alpha_k^{-1} I - E_k \Theta_k E_k^T)^{-1} E_k, \end{aligned} \tag{44}$$

$$\begin{aligned} \hat{K}_k =& [\alpha_k^{-1} H_{1,k} H_{2,k}^T + A_k \\ &\cdot (\Theta_k^{-1} - \alpha_k E_k^T E_k)^{-1} C_k^T] R_{1,k}^{-1}. \end{aligned} \tag{45}$$

The filter parameters (44) and (45) are identical to (33) and (34), respectively. Therefore, we can conclude that the filter (10) with parameters (33) and (34) minimizes the bound Θ_k. This completes the proof. ■

Remark 3: Similar structure to Σ_k in (38) has been used in [5] and [19]. The choice of this special structure has been motivated by arguments related to the minimization of the upper bound of the state estimation error variance. Note that the difference Riccati equations (31) and (32) involve the scalar parameter α_k. Detailed discussions on the feasibility and convergent properties of such kind of difference Riccati equations can be found in [29] and [30]. ■

IV. A NUMERICAL EXAMPLE

The following uncertain system:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & -0.5 \\ 1 & 1 + 0.3\delta_k \end{bmatrix} x_k + \begin{bmatrix} -6 \\ 1 \end{bmatrix} w_k \\ y_k &= [\,-100 \quad 10\,]\,x_k + v_k \end{aligned} \tag{46}$$

has been considered in several papers [5], [19], [26], where the precise bound on the uncertain parameter δ_k has been assumed to be exactly known, i.e., $|\delta_k| \leq 1$. However, in practice, such a bound can often be obtained from the statistical estimation. It is usually the case that the probability of the uncertain parameter δ_k satisfying $|\delta_k| \leq 1$ is close to one, but *not equal to* one.

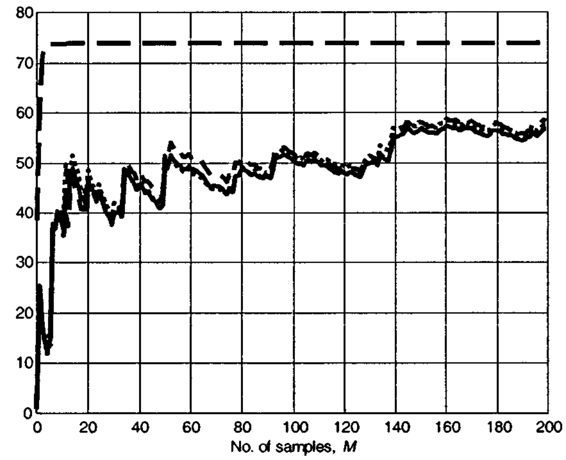

Fig. 1. MSE of the first-state estimate (MSE1) and its upper bound.

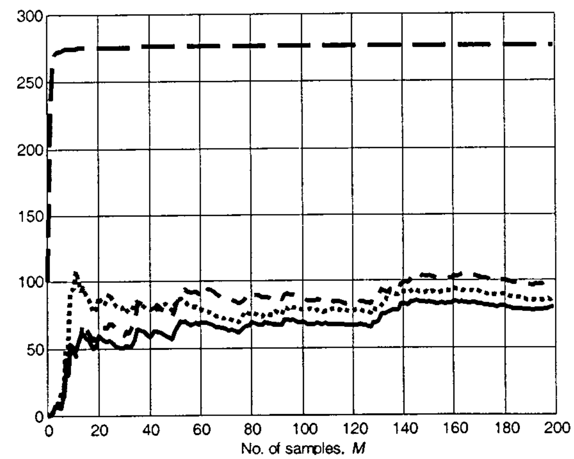

Fig. 2. MSE of the second-state estimate (MSE2) and its upper bound.

In our example, we assume that $P(|\delta_k| \leq 1) = 0.998$ where P means the probability and the statistical law for the uncertain parameter δ_k to satisfy $|\delta_k| > 1$ coincides the zero mean Gaussian white noise sequence with intensity 0.1. This leads to the following uncertain system with multiplicative noise:

$$\begin{aligned} x_{k+1} &= \left(\begin{bmatrix} 0 & -0.5 \\ 1 & 1 + 0.3\bar{\delta}_k \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}\eta_k\right) x_k + \begin{bmatrix} -6 \\ 1 \end{bmatrix} w_k, \\ y_k &= [\,-100 \quad 10\,]\,x_k + v_k \end{aligned} \tag{47}$$

where the deterministic uncertainty $\bar{\delta}_k$ satisfies $|\bar{\delta}_k| \leq 1$ and w_k, v_k and η_k are uncorrelated zero mean Gaussian white noise sequences with unity covariance. Note that the system (47) is of the form of system (1) with $H_{1,k} = [\,0 \quad 10\,]^T$, $H_{2,k} = 0$ and $E_k = [\,0 \quad 0.03\,]$. Suppose that the initial conditions are as follows:

$$x_0 = [\,1 \quad 0\,]^T \quad \hat{x}_0 = [\,0 \quad 0\,]^T \quad S_0 = 100 I_2 \text{ and } S_1 = 10 I_2.$$

The simulations have been conducted with $\alpha_k = 1.15$ for different deterministic uncertainties $\bar{\delta}_k = 0$, $\bar{\delta}_k = 1$ and $\bar{\delta}_k = -1$. Let MSE1 denote the mean square error (MSE) for the estimation of the first state, i.e., $(1/M)\sum_{k=1}^{M}\{[\,1 \quad 0\,](x_k - \hat{x}_k)\}^2$, where M is number of samples. Similarly, MSE2 is the mean square error for the estimation of the second state, i.e., $(1/M)\sum_{k=1}^{M}\{[\,0 \quad 1\,](x_k - \hat{x}_k)\}^2$. In Fig. 1, the solid (dotted, dashed, respectively) line plots the MSE1 for the case $\bar{\delta}_k = 0$ ($\bar{\delta}_k = 1$, $\bar{\delta}_k = -1$, respectively). The long dashed line is the diagonal element (1,1) of Θ_k, which is clearly shown to be the upper bound of MSE1 in all three cases. Analogously, in Fig. 2, the solid (dotted, dashed, respectively) line plots the MSE2 for the case $\bar{\delta}_k = 0$ ($\bar{\delta}_k = 1, = -1$, respectively), where the long dashed line is the diagonal element (2,2) of Θ_k.

We can see from the simulation results that our design goal is well achieved.

V. CONCLUSION

A robust finite-horizon Kalman filter has been designed in this note for the uncertain systems *with* multiplicative noises, which guarantees an upper bound on the state estimation error variance for admissible uncertainties. This bound has been minimized by the construction of filter parameters. Sufficient conditions for a finite-horizon filter to satisfy an upper bound on state estimation error variance for all admissible uncertainties have been given in terms of two discrete Riccati difference equations, which are of a form suitable for recursive computation.

The results obtained have plenty application potentials in many branches of control engineering. For example, in fault detection problems, multiplicative (also called state-dependent) faults could be viewed as a sort of stochastic multiplicative noise uncertainties and our task is to detect multiplicative faults in a finite evaluation window and establish a fault threshold that could be related to the upper bound of the estimation error variance. In such a case, the theoretical results of this note are directly applicable on a real-time basis, and this gives us one of the future research topics.

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