INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control 2004; **14**:1133–1168 (DOI: 10.1002/rnc.937)

Robust linear anti-windup synthesis for recovery of unconstrained performance

Gene Grimm[1,*,†,¶], Andrew R. Teel[2,‡] and Luca Zaccarian[3,§]

[1] *Raytheon Company, Space and Airborne Systems, 2000 E. El Segundo Blvd., El Segundo, CA 90245, U.S.A.*
[2] *Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106, U.S.A.*
[3] *Dipartimento di Informatica, Sistemi e Produzione, University of Rome, Tor Vergata 00133 Rome, Italy*

SUMMARY

Fixed order anti-windup augmentation is addressed by considering the dynamics of the mismatch between the constrained and unconstrained responses. The method utilizes LMIs, and the result parallels other recent LMI-based anti-windup synthesis methods. Robustness is directly addressed—which is not done in previous LMI work on anti-windup. An optimal LMI-based synthesis procedure is provided for static and plant-order linear anti-windup augmentation and the performance of the resulting design strategy is shown via a simulation example. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: anti-windup synthesis; input saturation; linear matrix inequalities

1. INTRODUCTION

Linear control synthesis and analysis tools have been long used for industrial applications and are very consolidated nowadays. Nevertheless, already in the 1950s, input saturation constituted one of the main obstacles to the applicability of linear designs [1], especially in high precision, high performance environments, where typically, high gains are involved in the linear designs. During the 1960s and in later years, a new control engineering problem, called 'windup', was identified and defined, arising from practical industrial needs, where linear designs would perform desirably during normal operation, but occasional situations would lead the actuators into saturation, thus 'winding up' the linear controller and causing performance and, possibly, stability loss. According to this nomenclature, the goal of 'anti-windup' designs was formulated

*Correspondence to: Gene Grimm, Raytheon Company, Space and Airborne Systems, 2000 E. El Segundo Blvd., El Segundo, CA 90245, U.S.A.
† E-mail: grimm@raytheon.com
‡ E-mail: teel@ece.ucsb.edu
§ E-mail: zack@disp.uniroma2.it
¶ Center for Control Engineering and Computation, University of California at Santa Barbara

Contract/grant sponsor: NSF; contract/grant number: ECS-9988813 and ECS-0324679
Contract/grant sponsor: AFOSR; contract/grant number: F49620-00-1-0106 and F49620-03-l-0203

Published online 27 May 2004

as the objective of preserving locally a prescribed linear behavior (at least, as long as saturation is not activated) and of reducing as much as possible the performance loss after saturation occurrence (see, e.g. Reference [2]).

Since those early years, many interesting results were accomplished in this research field. Early anti-windup schemes (see References [3–5] for interesting surveys) were very effective at solving certain instances of the anti-windup problem but, as pointed out in Reference [6], still unsuitable for generalization to a wide class of systems. During the 1990s, anti-windup research became a fully theoretical research field and many interesting solutions to the problem have been formally proven to induce desirable stability properties, in addition to some performance guarantees (see, e.g. References [7–16]). Within these modern approaches, an important paradigm is that of the so-called 'linear anti-windup design' where, as in Figure 1, a windup prone linear closed-loop system is augmented with a linear filter, called an anti-windup compensator, which (in the most general setting) has full authority toward injecting modifications on the controller state and output equations. An important result within the linear anti-windup field was stated in Reference [11], where a convex LMI-based formulation that optimizes the global (finite) $\mathscr{L}_2$ gain from w to z in Figure 1 provides a useful synthesis tool when the anti-windup compensator is chosen among all possible linear gains (namely, the anti-windup compensator is chosen as a static linear system). (See also Reference [14] where the same approach is used with relaxed sector bounds on the saturation nonlinearity.) The main limitation of this synthesis technique is that the LMI constraints are not always feasible, thus leaving the linear anti-windup design problem not completely solved. Later, in Reference [13], the set of all linear (possibly dynamic) selections of the anti-windup compensator was characterized by matrix inequalities that, in some notable cases, allowed convex minimization of the same global $\mathscr{L}_2$ gain. Moreover, in Reference [13], a nice system theoretic interpretation of the cases where static linear anti-windup design is feasible is given and it is proven that dynamic linear compensation of order equal to that of the plant is sufficient to induce the best possible performance level and is always feasible if the plant is Hurwitz (this condition is also necessary to achieve global finite $\mathscr{L}_2$ gain with bounded inputs). LMIs were also employed in References [15, 16] to solve the local anti-windup problem maximizing the operating region in the case where the plant is non-exponentially stable.

In this paper, following the ideas in Reference [17], we revisit the linear anti-windup paradigm of References [11, 13] by recovering the original goal of anti-windup designs (which consists in recovering as fast as possible the unconstrained response) and characterizing the performance of the anti-windup augmentation by the $\mathscr{L}_2$ norm of the deviation between the actual response of the augmented closed-loop system and the ideal response of the asymptotically stable unconstrained system. In particular, this $\mathscr{L}_2$ norm is related to the energy of the unconstrained

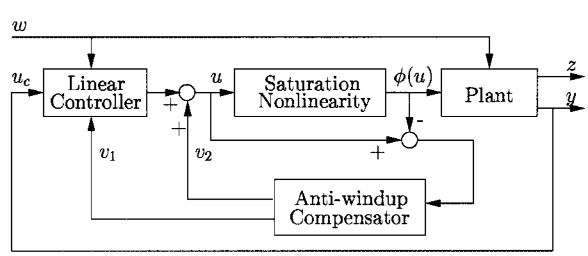

Figure 1. The linear anti-windup design paradigm.

(linear) plant input outside the saturation limits, so that unconstrained trajectories that are not reproducible at the steady-state by the saturated system are implicitly ruled out from our optimization problem. To account for modelling uncertainties, we also introduce robustness in our synthesis scheme. As expected, the feasibility conditions coincide with those of Reference [13], because they are not dependent on the performance characterization. Since we will require global quadratic stability, necessarily the plant must be asymptotically stable. Moreover, plant-order augmentation is shown to induce globally optimal performance level coinciding with the $\mathscr{L}_2$ gain from u to z of the open-loop plant (see Figure 1). Additional interesting properties, also related to the robustness of the anti-windup augmentation, are listed in Theorem 3 in Section 4.

The paper is organized as follows: In Section 2, we give some preliminaries. In Section 3, we define the anti-windup construction problem and formalize the properties that the augmented closed-loop system needs to accomplish. In Section 4, we formulate the anti-windup synthesis via matrix inequalities and prove important properties of this formulation. Then, we provide constructive LMI techniques for optimal construction in the static and plant-order case. In Section 5, we provide LMIs for determining the performance level induced by an existing anti-windup compensator. In Section 6, we compare our method to existing ones, also via a simulation example.

2. PRELIMINARIES

To assist in the system theoretic interpretation of the matrix inequalities in this paper, recall the well-known LMI formulation of the bounded real lemma for continuous time systems (for a complete proof see, e.g. Reference [18, p. 82]).

Lemma 1 (Bounded real lemma)
The following statements are equivalent:

(1) $\|D + C(sI - A)^{-1}B\|_\infty < \gamma$ and A is Hurwitz;
(2) there exists a symmetric positive definite solution $X = X^{\mathrm{T}} > 0$ to the LMI:

$$\begin{bmatrix} XA^{\mathrm{T}} + AX & B & XC^{\mathrm{T}} \\ \star & -\gamma I & D^{\mathrm{T}} \\ \star & \star & -\gamma I \end{bmatrix} < 0$$

(where the symbol '$\star$' represents the appropriate entry resulting in an overall symmetric matrix).

Definition 1
Let $u \in \mathbb{R}^{n_u}$, $\mathscr{U} \subset \mathbb{R}^{n_u}$, $M_1, M_2 \in \mathbb{R}^{n_u \times r}$, and a $W \in \mathbb{R}^{n_u \times n_u}$. Define $\sigma_{\max}(W)$ and $\sigma_{\min}(W)$ to be the maximum and minimum singular value of W, respectively. Suppose W is a symmetric positive definite matrix. Define the W-*product* of M_1 and M_2 as

$$\langle M_1, M_2 \rangle_W := M_1^{\mathrm{T}} W M_2$$

Define the *W-norm* as $|u|_W := \langle u, u \rangle_w^{1/2}$. We will use $|u| := |u|_I$. Define

$$\mathrm{dist}(u, \mathcal{U}) := \inf_{w_{\mathcal{U}} \in \mathcal{U}} |u - w_{\mathcal{U}}|_I$$

A function $f : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is said to *belong to the sector* $[0, I]_W$ if $\langle f(w), w - f(w) \rangle_W \geqslant 0$ for all $w \in \mathbb{R}^{n_u}$. A function $f : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is said[1] to *belong to the incremental sector* $[0, I]_W$ if f is locally Lipschitz and $\langle Jf(y), I - Jf(y) \rangle_W \geqslant 0$ for almost all $y \in \mathbb{R}^{n_u}$, where $Jf(y)$ denotes the Jacobian of f evaluated at y.

Definition 2

A function $\phi : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is said to *belong to* Φ_W ($\phi \in \Phi_W$) if the function ϕ is locally Lipschitz, belongs to the incremental sector $[0, I]_W$, and $\phi(0) = 0$. Furthermore, $\phi \in \Phi_W$ is said to be *bounded* if there exists $b \in \mathbb{R}$ such that $|\phi(s)|_W \leqslant b$ for all $s \in \mathbb{R}^{n_u}$.

Remark 1

If $\phi \in \Phi_W$ then $\phi(\cdot)$ belongs to the sector $[0, I]_W$. Furthermore, when $W = I$ the sector $[0, I]_W$ property coincides with the definition in Reference [21, p. 403].

Let M be such that $M^{\mathrm{T}} M = W$. The matrix M can be seen as a transformation that makes the function $\phi(\cdot)$ belong the Φ_I. In particular, $\phi(\cdot)$ belongs to Φ_W if and only if the function $s \mapsto M\phi(M^{-1}s)$ belongs to Φ_I.

Definition 3

The function ϕ is said to be a *standard decentralized saturation function* if

$$\mathrm{sat}(u) := [\mathrm{sat}_1(u_1) \mathrm{sat}_2(u_2) \cdots \mathrm{sat}_{n_u}(u_{n_u})]^{\mathrm{T}} \quad \text{where } \mathrm{sat}_i(u_i) := \frac{u_i}{\max\{1, |u_i|/M_i\}}$$

$M_i \in \mathbb{R}$, $M_i > 0$ for $i = 1, \ldots, n_u$.

Any standard decentralized saturation function belongs to Φ_W if W is diagonal positive definite. Furthermore, if $\phi(\cdot)$ is a standard decentralized saturation function then $|u - \phi(u)|_I = \mathrm{dist}(u, \mathcal{U})$.

Lemma 2

Let $\phi \in \Phi_W$ and $\mathcal{U} \subset \{u \in \mathbb{R}^{n_u} : \phi(u) = u\}$ be compact. Then $|u - \phi(u)|_I \leqslant \sigma_{\max}(W)/\sigma_{\min}(W) \,\mathrm{dist}(u, \mathcal{U})$.

Proof

Since $\mathcal{U}$ is compact there is some element $u_1 \in \mathcal{U}$ such that $\mathrm{dist}(u, \mathcal{U}) = \mathrm{dist}(u, \{u_1\})$. Define the function $f(u_2) := \phi(u_2 + u_1) - \phi(u_1)$. Then $f \in \Phi_W$ because $f(0) = 0$ and

$$\langle Jf(y), I - Jf(y) \rangle_W = \langle J\phi(y + u_1), I - J\phi(y + u_1) \rangle_W \geqslant 0$$

for all y, u_1. Define $u_2 := u - u_1$. Then $u_1 \in \mathcal{U}$ implies $|u - \phi(u)|_W = |u_2 + u_1 - \phi(u_1) - f(u_2)|_W = |u_2 - f(u_2)|_W$ since $u_1 - \phi_r(u_1) = 0$. Since f belongs to the sector $[0, I]_W$ then

$$0 \leqslant \langle f(u_2), u_2 - f(u_2) \rangle_W = \langle u_2 - f(u_2), u_2 - (u_2 - f(u_2)) \rangle_W$$

[1] See also References [19, 20] for an analogous characterization of incremental sector bound.

which implies $|u_2 - f(u_2)|_W^2 = \langle u_2 - f(u_2), u_2 - f(u_2)\rangle_W \leqslant \langle u_2 - f(u_2), u_2\rangle_W \leqslant |u_2 - f(u_2)|_W |u_2|_W$. Hence, $\sigma_{\min}(W)|u - \phi(u)| \leqslant |u - \phi(u)|_W = |u_2 - f(u_2)|_W \leqslant |u_2|_W \leqslant \sigma_{\max}(W)|u_2| = \sigma_{\max}(W)\mathrm{dist}(u, \mathcal{U})$. □

Definition 4
Let $C_{p,\mu}$ and $D_{p,\mu u}$ be given. The dynamic system θ is said to have *finite incremental gain g* (*from* $\mu := C_{p,\mu}x + D_{p,\mu u}u$ *to* θ) if

$$\|\theta(\sigma_0, x_1, u_1, w) - \theta(\sigma_0, x_2, u_2, w)\|_2 \leqslant g \left\| \begin{bmatrix} C_{p,\mu}(x_1 - x_2) \\ D_{p,\mu u}(u_1 - u_2) \end{bmatrix} \right\|_2 \tag{1}$$

for all σ_0 and signals x_1, x_2, u_1, u_2 and w.

3. PROBLEM DEFINITION

3.1. The desirable unconstrained closed-loop behaviour

In this section, we will characterize the desirable behaviour of the closed-loop system when the control input is unconstrained. The performance of the system with control input constraints and anti-windup compensation will be measured in terms of the deviation from the unconstrained trajectory generated by this unconstrained closed-loop system. Consider an *unconstrained plant*, with dynamic nonlinear perturbation θ included to investigate robustness properties, described by

$$\mathscr{P}\begin{cases} \dot{\bar{x}}_p = A_p\bar{x}_p + B_{p,u}\bar{u} + B_{p,w}w + B_{p,\theta}\theta(\sigma_0, \bar{x}_p, \bar{u}, w) \\ \bar{y} = C_{p,y}\bar{x}_p + D_{p,yu}\bar{u} + D_{p,yw}w + D_{p,y\theta}\theta(\sigma_0, \bar{x}_p, \bar{u}, w) \\ \bar{z} = C_{p,z}\bar{x}_p + D_{p,zu}\bar{u} + D_{p,zw}w + D_{p,z\theta}\theta(\sigma_0, \bar{x}_p, \bar{u}, w) \\ \bar{\mu} = C_{p,\mu}\bar{x}_p + D_{p,\mu u}\bar{u} \end{cases} \tag{2}$$

where the dynamical system θ is finite dimensional, forward complete, stable, its output belongs to $\mathbb{R}^{n_\theta}$ and it has continuous right-hand side and initial state σ_0. The variable $\bar{x}_p \in \mathbb{R}^{n_p}$ is the *unconstrained plant state*, $\bar{u} \in \mathbb{R}^{n_u}$ is the *control input*, $w \in \mathbb{R}^{n_w}$ is the *exogenous input* (possibly containing disturbances, references and measurement noise), $\bar{y} \in \mathbb{R}^{n_y}$ is the *plant output* available for measurement, $\bar{z} \in \mathbb{R}^{n_z}$ is the *performance output*, $\bar{\mu} \in \mathbb{R}^{n_\mu}$ and A_p, $B_{p,u}$, $B_{p,w}$, $B_{p,\theta}$, $C_{p,y}$, $D_{p,yu}$, $D_{p,yw}$, $D_{p,y\theta}$, $C_{p,z}$, $D_{p,zu}$, $D_{p,zw}$, $D_{p,z\theta}$, $C_{p,\mu}$ and $D_{p,\mu u}$ are matrices of suitable dimensions.

Assume that an *unconstrained controller* has been designed:

$$\mathscr{C}\begin{cases} \dot{\bar{x}}_c = A_c\bar{x}_c + B_{c,y}\bar{y} + B_{c,w}w \\ \bar{u} = C_c\bar{x}_c + D_{c,y}\bar{y} + D_{c,w}w \end{cases} \tag{3}$$

(where $\bar{x}_c \in \mathbb{R}^{n_c}$ is the *unconstrained controller state* and A_c, $B_{c,y}$, $B_{c,w}$, C_c, $D_{c,y}$ and $D_{c,w}$ are matrices of suitable dimensions) in such a way that its interconnection to the unconstrained plant with dynamic perturbation is well-posed and guarantees internal stability of the resulting *unconstrained closed-loop system*. This unconstrained closed-loop system is depicted on the left-hand side of Figure 2[2] and can be concisely written with state $\bar{\xi} \in \mathbb{R}^{n_\xi}$, $n_\xi := n_p + n_c$, $\bar{\xi} := [\bar{x}_p^{\mathrm{T}} \; \bar{x}_c^{\mathrm{T}}]^{\mathrm{T}}$ as

$$\begin{aligned}
\dot{\bar{\xi}} &= A_{CL}\bar{\xi} + B_{CL,w}w + B_{CL,\theta}\theta(\sigma_0, [I \;\; 0]\bar{\xi}, \bar{u}, w) \\
\bar{u} &= C_{CL,u}\bar{\xi} + D_{CL,uw}w + D_{CL,u\theta}\theta(\sigma_0, [I \;\; 0]\bar{\xi}, \bar{u}, w) \\
\bar{z} &= C_{CL,z}\bar{\xi} + D_{CL,zw}w + D_{CL,z\theta}\theta(\sigma_0, [I \;\; 0]\bar{\xi}, \bar{u}, w) \\
\bar{\mu} &= C_{CL,\mu}\bar{\xi} + D_{CL,\mu w}w + D_{CL,\mu\theta}\theta(\sigma_0, [I \;\; 0]\bar{\xi}, \bar{u}, w)
\end{aligned} \tag{4}$$

where A_{CL}, $B_{CL,w}$, $B_{CL,\theta}$, $C_{CL,u}$, $D_{CL,uw}$, $D_{CL,u\theta}$, $C_{CL,z}$, $D_{CL,zw}$, $D_{CL,z\theta}$, $C_{CL,\mu}$, $D_{CL,\mu w}$, and $D_{CL,\mu\theta}$ are matrices of suitable dimensions. For explicit expressions, see Appendix A; in particular, see (A3).

3.2. Input saturation and anti-windup augmentation

Paralleling Reference [13], instead of considering a particular plant input nonlinearity, we consider a class of input nonlinearities Φ_W (see Definition 2) in order to state necessary and sufficient conditions for stability and performance.

Suppose the plant actually has a nonlinearity $\phi \in \Phi_W$ at the control input; in particular suppose the *saturated plant* is given by

$$\begin{aligned}
\dot{x}_p &= A_p x_p + B_{p,u}\phi(u) + B_{p,w}w + B_{p,\theta}\theta(\sigma_0, x_p, \phi(u), w) \\
y &= C_{p,y}x_p + D_{p,yu}\phi(u) + D_{p,yw}w + D_{p,y\theta}\theta(\sigma_0, x_p, \phi(u), w) \\
z &= C_{p,z}x_p + D_{p,zu}\phi(u) + D_{p,zw}w + D_{p,z\theta}\theta(\sigma_0, x_p, \phi(u), w) \\
\mu &= C_{p,\mu}x_p + D_{p,\mu u}\phi(u)
\end{aligned} \tag{5}$$

In general, $\phi(u) \neq u$ and the unconstrained controller no longer selects the control input for which it was designed. Thus we suppose that the unconstrained controller is allowed to be modified to cope with the input nonlinearity of the plant. In particular, we introduce a *modified controller*

$$\mathscr{C}_{\mathscr{M}} \begin{cases} \dot{x}_c = A_c x_c + B_{c,y}y + B_{c,w}w + v_1 \\ u = C_c x_c + D_{c,y}y + D_{c,w}w + v_2 \end{cases} \tag{6}$$

where the signal $v^{\mathrm{T}} = [v_1^{\mathrm{T}} \; v_2^{\mathrm{T}}]$ is allowed to modify the controller state and output equations. By introducing the function $\psi : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$, defined as

$$\psi(s) := s - \phi(s), \quad \forall s \in \mathbb{R}^{n_u} \tag{7}$$

[2]In Figure 2, the line pointing from the unconstrained plant to the dynamical system θ is dashed to emphasize the following: The output $\bar{\mu}$ is not the input to θ, but the finite incremental gain assumed on the system θ can be used to relate incremental changes in $\bar{\mu}$ to incremental changes in θ. (When introduced in the following section, the dashed line associated with the output μ is understood in the same fashion.)

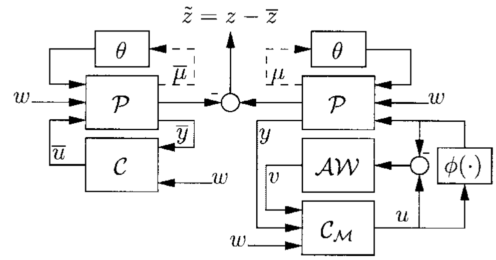

Figure 2. The unconstrained and anti-windup closed-loop systems.

and the state $\xi := [x_p^{\mathrm{T}} \ x_c^{\mathrm{T}}] \in \mathbb{R}^{n_\xi}$, we can concisely write the interconnection of (5), (6) using (7) as

$$\begin{aligned}
\dot{\xi} &= A_{CL}\xi + B_{CL,w}w + B_{CL,\psi}\psi(u) + B_{CL,v}v + B_{CL,\theta}\theta(\sigma_0, [I \ \ 0]\xi, \phi(u), w)\\
u &= C_{CL,u}\xi + D_{CL,uw}w + D_{CL,u\psi}\psi(u) + D_{CL,uv}v + D_{CL,u\theta}\theta(\sigma_0, [I \ \ 0]\xi, \phi(u), w)\\
z &= C_{CL,z}\xi + D_{CL,zw}w + D_{CL,z\psi}\psi(u) + D_{CL,zv}v + D_{CL,z\theta}\theta(\sigma_0, [I \ \ 0]\xi, \phi(u), w)\\
\mu &= C_{CL,\mu}\xi + D_{CL,\mu w}w + D_{CL,\mu\psi}\psi(u) + D_{CL,\mu v}v + D_{CL,\mu\theta}\theta(\sigma_0, [I \ \ 0]\xi, \phi(u), w)
\end{aligned} \tag{8}$$

where A_{CL}, $B_{CL,w}$, $B_{CL,\psi}$, $B_{CL,v}$, $B_{CL,\theta}$, $C_{CL,u}$, $D_{CL,uw}$, $D_{CL,u\psi}$, $D_{CL,uv}$, $D_{CL,u\theta}$, $C_{CL,z}$, $D_{CL,zw}$, $D_{CL,z\psi}$, $D_{CL,zv}$, $D_{CL,z\theta}$, $C_{CL,\mu}$, $D_{CL,\mu w}$, $D_{CL,\mu\psi}$, $D_{CL,\mu v}$ and $D_{CL,\mu\theta}$ are matrices of suitable dimensions. Note, many of these matrices appeared in (4). For explicit expressions, see Appendix A; in particular, see (A3).

Given an integer $n_{aw} \geq 0$, in this paper we address the problem of designing an order n_{aw} linear *anti-windup compensator*

$$\mathscr{AW}\begin{cases}
\dot{x}_{aw} = A_{aw}x_{aw} + B_{aw}\psi(u) = A_{aw}x_{aw} + B_{aw}(u - \phi(u))\\
v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = C_{aw}x_{aw} + D_{aw}\psi(u) = C_{aw}x_{aw} + D_{aw}(u - \phi(u))
\end{cases} \tag{9}$$

where $x_{aw} \in \mathbb{R}^{n_{aw}}$ is the *anti-windup state*, $v \in \mathbb{R}^{n_v}$ (with $n_v := n_c + n_u$) is the *anti-windup output*, and the matrices A_{aw}, B_{aw}, C_{aw} and D_{aw} are of suitable dimensions. The interconnection of (5), (6), (9) will be called the *anti-windup closed-loop system* and is depicted on the right-hand side of Figure 2.[3] Occasionally, we will also consider the realization the anti-windup closed-loop system described by interconnection (8), (9).

3.3. Optimizing the unconstrained response recovery

The scope of this section is to formalize the requirement that the output response z of the anti-windup closed-loop system (5), (6), (9) (equivalently (8), (9)) be as close as possible to the response $\bar{z}$ of the unconstrained closed-loop system (2), (3) (equivalently (4)). (Note that the deviation between the two outputs is characterized in Figure 2 and therein denoted by $\tilde{z}$). To this aim, we recall the $\mathscr{L}_2$ anti-windup problem definition given in Reference [17]:[4]

[3] Recall the meaning of the dotted lines in Figure 2, which was previously discussed in Footnote 2.
[4] The anti-windup problem defined in Reference [17] also allowed for nonlinear anti-windup compensation schemes and nonlinear $\mathscr{L}_2$ gain.

Definition 5
Given a bounded function $\phi \in \Phi_W$ and a compact set $\mathcal{U} \subset \{u \in \mathbb{R}^{n_u} : \phi(u) = u\}$, an anti-windup compensator is said to solve the *robust global $\mathcal{L}_2$ anti-windup problem for $\mathcal{U}$* if the interconnection of (5), (6), (9) compared to the interconnection of (2), (3) is such that if θ has sufficiently small finite incremental gain, then

1. assuming the initial conditions $x_{aw}(0) = 0$, $x_p(0) = \bar{x}_p(0)$, and $x_c(0) = \bar{x}_c(0)$, if $\bar{u}(\cdot) \equiv \phi(\bar{u}(\cdot))$ then $z(\cdot) \equiv \bar{z}(\cdot)$;
2. if $\mathrm{dist}(\bar{u}(\cdot), \mathcal{U}) \in \mathcal{L}_2$ then $(z - \bar{z})(\cdot) \in \mathcal{L}_2$.
   The anti-windup compensator is said to solve the problem with *performance recovery level* γ if the inter-connection is such that if θ has sufficiently small finite incremental gain, then items 1 and 2 are satisfied and
3. with initial conditions $x_{aw}(0) = 0$, $x_p(0) = \bar{x}_p(0)$, and $x_c(0) = \bar{x}_c(0)$, then

$$\|(z - \bar{z})(\cdot)\|_2 \leqslant \gamma \|\mathrm{dist}(\bar{u}(\cdot), \mathcal{U})\|_2$$

Furthermore, the anti-windup compensator is said to solve the problem with *performance recovery level γ and guaranteed robustness level g* if the compensator solves the robust global $\mathcal{L}_2$ anti-windup problem for $\mathcal{U}$ with performance recovery level γ for all θ with finite incremental gain smaller than g.

If an anti-windup compensator solves the robust global $\mathcal{L}_2$ anti-windup problem for $\mathcal{U}$, then implicitly the anti-windup compensator is such that the anti-windup closed-loop system is well-posed (since all signals within the anti-windup closed-loop must be well-defined). With the aim to guarantee the properties in Definition 5, a key representation of the unconstrained closed-loop system with state variable $(\bar{x}_p, \bar{x}_c)$ and of the anti-windup closed-loop system with state variable (x_p, x_c, x_{aw}) can be expressed in terms of the unconstrained closed-loop system, whose state variable is $(\bar{x}_p, \bar{x}_c)$ and of the 'mismatch system' whose state variable is given by

$$x := \begin{bmatrix} x_p - \bar{x}_p \\ x_c - \bar{x}_c \\ x_{aw} \end{bmatrix}$$

Note that the idea of defining a mismatch system which captures the difference between the actual and unconstrained responses can be found in previous work on anti-windup design (see, e.g. References [17, 22–24]). To simplify the subsequent notation, consider the introduction of the following definitions:

$$\varphi(u - \bar{u}, \bar{u}) := \psi(u) - \psi(\bar{u}) \tag{10a}$$

$$\tilde{u} := u - \bar{u} \tag{10b}$$

$$\tilde{z} := z - \bar{z} \tag{10c}$$

$$\tilde{\mu} := \mu - \bar{\mu} \tag{10d}$$

$$\tilde{\theta}(\sigma_0, x_p, \tilde{u}, \bar{x}_p, \bar{u}, w) := \theta(\sigma_0, x_p, \tilde{u} + \bar{u} - \psi(\tilde{u} + \bar{u}), w) - \theta(\sigma_0, \bar{x}_p, \bar{u}, w) \tag{10e}$$

Then, by way of (10) and using the identity $\phi(u) - \bar{u} = \tilde{u} - (\psi(\bar{u}) + \varphi(\tilde{u}, \bar{u}))$ (derived from (7) and (10)), the overall dynamics can be written as the cascade connection of two systems, the first one

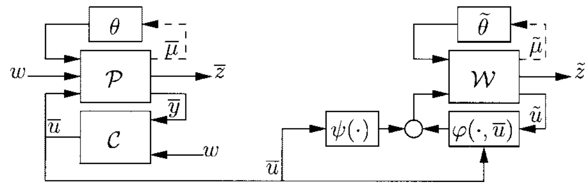

Figure 3. The cascade interconnection of the unconstrained closed-loop system and of the mismatch system.

being the unconstrained closed-loop (4) and the second one being the following *mismatch system*:

$$\mathscr{W} \begin{cases} \dot{x} = Ax + B_\psi(\psi(\bar{u}) + \varphi(\tilde{u}, \bar{u})) + B_\theta \tilde{\theta} \\ \tilde{u} = C_u x + D_{u\psi}(\psi(\bar{u}) + \varphi(\tilde{u}, \bar{u})) + D_{u\theta}\tilde{\theta} \\ \tilde{z} = C_z x + D_{z\psi}(\psi(\bar{u}) + \varphi(\tilde{u}, \bar{u})) + D_{z\theta}\tilde{\theta} \\ \tilde{\mu} = C_\mu x + D_{\mu\psi}(\psi(\bar{u}) + \varphi(\tilde{u}, \bar{u})) + D_{\mu\theta}\tilde{\theta} \end{cases} \tag{11}$$

This cascade representation is shown in Figure 3 with the unconstrained closed-loop shown once again on the left-hand side and the mismatch system shown on the right-hand side.[5] Note that, as compared to the equivalent representation of Figure 2, the performance output $\tilde{z}$ now corresponds to an output of the mismatch system. As we will further discuss, any and all of the properties of Definition 5 (including the implicitly required well-posedness property) can be guaranteed by the properties of the mismatch system. One such property is given in the following Definition 6 and much of the rest of the paper is dedicated to showing that it is computationally useful.

Definition 6

Given the plant $\mathscr{P}$ in (2) and the unconstrained controller $\mathscr{C}$ in (3), a linear anti-windup compensator (9) of order n_{aw} is said to guarantee *quadratic robust performance recovery* (γ, g) if the mismatch system (11) is such that

1. for all $\phi \in \Phi_W$ the mismatch system is well-posed, and
2. there exist real scalars $\varepsilon > 0$ and $\rho \geqslant 0$ and a quadratic function $x \mapsto V(x) := x^T P x$ (with $P = P^T > 0$) such that

$$\frac{\partial V}{\partial x}(Ax + B_\psi(\psi_0 + \varphi_0) + B_\theta\tilde{\theta}) < -\varepsilon x^T x - \frac{1}{\gamma}\tilde{z}^T\tilde{z} + \gamma\psi_0^T\psi_0 + \rho(\tilde{\theta}^T\tilde{\theta} - g^2\tilde{\mu}^T\tilde{\mu}) \tag{12}$$

[5] Although $\tilde{\mu}$ a not an Input to $\tilde{\theta}$, the finite incremental gain of θ essentially leads to a finite gain from $\tilde{\mu}$ to $\tilde{\theta}$. Thus $\tilde{\mu}$ is a pseudo-input to $\tilde{\theta}$ and represented as a dashed line in Figure 3 (the same argument was previously used in Figure 2 and discussed in Footnote 2).

for all $(x, \psi_0, \varphi_0, \tilde{\theta}) \neq 0$ such that $\langle \varphi_0, \tilde{u} - \varphi_0 \rangle_W \geqslant 0$ where

$$\tilde{u} = C_u x + D_{u\psi}(\psi_0 + \varphi_0) + D_{u\theta}\tilde{\theta}$$
$$\tilde{z} = C_z x + D_{z\psi}(\psi_0 + \varphi_0) + D_{z\theta}\tilde{\theta}$$
$$\tilde{\mu} = C_\mu x + D_{\mu\psi}(\psi_0 + \varphi_0) + D_{\mu\theta}\tilde{\theta}$$

Theorem 1

Given $\phi \in \Phi_W$ and a compact set $\mathscr{U} \subset \{u \in \mathbb{R}^{n_u} : \phi(u) = u\}$, if a linear anti-windup compensator guarantees quadratic robust performance recovery (γ, g), then it solves the $\mathscr{L}_2$ anti-windup problem for $\mathscr{U}$ with performance recovery level $\sigma_{\max}(W)/\sigma_{\min}(W)\gamma$ and guaranteed robustness level g. Moreover, if ϕ is a standard decentralized saturation function, then the linear anti-windup compensator solves the $\mathscr{L}_2$ anti-windup problem for $\mathscr{U}$ with performance recovery level γ and guaranteed robustness level g.

Proof

Assume a linear anti-windup compensator guarantees quadratic robust performance recovery (γ, g). Then item 1 guarantees well-posedness of the mismatch system and (since the desirable closed-loop system is well-posed by assumption) the anti-windup closed-loop system is also well-posed. Assume the function $\phi \in \Phi_W$. Then for any $\bar{u} \in \mathbb{R}^{n_u}$ the function $\varphi(\cdot, \bar{u})$ defined via (7), (10a) belongs to Φ_W because for fixed $\bar{u}$ we have $\varphi(0, \bar{u}) = 0$, $\varphi(\cdot, \bar{u})$ is locally Lipschitz and

$$\langle J\varphi(\tilde{u}, \bar{u}), I - J\varphi(\tilde{u}, \bar{u}) \rangle_W = \langle J\psi(u), I - J\psi(u) \rangle_W = \langle I - J\phi(u), J\phi(u) \rangle_W \geqslant 0 \tag{13}$$

almost everywhere implies $\varphi(\cdot, \bar{u})$ belongs to the incremental sector $[0, I]_W$. Let $\psi_0 = \psi(\bar{u})$ and $\varphi_0 = \phi(\tilde{u}, \bar{u})$. Since $\varphi(\cdot, \bar{u}) \in \Phi_W$ then $\langle \varphi_0, \tilde{u} - \varphi_0 \rangle_W \geqslant 0$ for any input to and trajectory of the mismatch system. Since the left-hand side of (12) corresponds to $\partial V/\partial x\, \dot{x}$, it follows that integration of (12) with respect to time shows that item 2 guarantees

$$V(x(\infty)) - V(x(0)) \leqslant -\varepsilon \|x\|_2^2 - \frac{1}{\gamma}\|\tilde{z}\|_2^2 + \gamma \|\psi(\bar{u})\|_2^2 + \rho(\|\tilde{\theta}\|_2^2 - g^2\|\tilde{\mu}\|_2^2)$$

Since for any θ that has finite incremental gain g the inequality $(\|\tilde{\theta}\|_2^2 - g^2\|\tilde{\mu}\|_2^2) \leqslant 0$ holds and $V(x(\infty)) \geqslant 0$, this implies

$$\|\tilde{z}\|_2^2 \leqslant \gamma^2 \|\psi(\bar{u})\|_2^2 + \gamma V(x(0))$$

whenever θ has finite incremental gain g. Hence if $x(0) = 0$ and $\bar{u}(\cdot) \equiv \phi(\bar{u}(\cdot))$ then $\psi(\bar{u}(\cdot)) \equiv 0$ and $\tilde{z}(\cdot) \equiv 0$ for any θ with finite incremental gain g. Additionally, Lemma 2 guarantees $|\psi(\bar{u})| \leqslant \sigma_{\max}(W)/\sigma_{\min}(W)\,\mathrm{dist}(\bar{u}, \mathscr{U})$, thus if $\mathrm{dist}(\bar{u}(\cdot), \mathscr{U}) \in \mathscr{L}_2$ then $\psi(\bar{u}(\cdot)) \in \mathscr{L}_2$ which implies $\tilde{z}(\cdot) \in \mathscr{L}_2$ for any θ with finite incremental gain g. Again using Lemma 2, if $x(0) = 0$ then $\|\tilde{z}\|_2 \leqslant \gamma \|\psi(\bar{u})\|_2 \leqslant \sigma_{\max}(W)/\sigma_{\min}(W)\gamma \|\mathrm{dist}(\bar{u}(\cdot), \mathscr{U})\|_2$ for any θ with finite incremental gain g. Thus the compensator solves the $\mathscr{L}_2$ anti-windup problem for $\mathscr{U}$ with performance recovery level $\sigma_{\max}(W)/\sigma_{\min}(W)\gamma$ and guaranteed robustness level g. Moreover, if $\phi(\cdot)$ is a standard decentralized saturation function $\|\psi(\bar{u})\|_2 = \|\mathrm{dist}(\bar{u}(\cdot), \mathscr{U})\|_2$ which is used to imply the $\mathscr{L}_2$ gain from $\mathrm{dist}(\bar{u}, \mathscr{U})$ to $\tilde{z}$ is less than γ. □

4. LMI-BASED ANTI-WINDUP SYNTHESIS

4.1. Feasibility of the anti-windup synthesis problem

If an anti-windup compensator guarantees quadratic robust performance recovery (γ, g) then performance and robustness are guaranteed for all input nonlinearities $\phi \in \Phi_W$. There are two special extreme input nonlinearities within this set Φ_W that we may wish to consider.

1. Choose the particular input nonlinearity $\phi(\cdot) \equiv 0$. Then $\psi(\bar{u}) \equiv \bar{u}$ and for any anti-windup compensator and any desired controller. From (2) and (5) with $\tilde{x}_p := x_p - \bar{x}_p$ it can be shown that

$$\begin{aligned} \dot{\tilde{x}}_p &= A_p \tilde{x}_p - B_{p,u}\bar{u} + B_{p,\theta}\tilde{\theta} \\ \tilde{z} &= C_{p,z}\tilde{x}_p - D_{p,zu}\bar{u} + D_{p,z\theta}\tilde{\theta} \\ \tilde{\mu} &= C_{p,\mu}\tilde{x}_p - D_{p,\mu u}\bar{u} \end{aligned} \tag{14}$$

2. Choose the particular input nonlinearity $\phi(\cdot) \equiv \mathrm{Id}(\cdot)$, the identity operator. Then $\psi(\bar{u}) \equiv 0$. Moreover, for any anti-windup compensator if $x_{aw}(0) = 0$ then $x_{aw} \equiv 0$. From (4) and (8), with $\tilde{\xi} := \xi - \bar{\xi}$ it can be shown that

$$\begin{aligned} \dot{\tilde{\xi}} &= A_{CL}\tilde{\xi} + B_{CL,\theta}\tilde{\theta} \\ \tilde{z} &= C_{CL,z}\tilde{\xi} + D_{CL,z\theta}\tilde{\theta} \\ \tilde{u} &= C_{CL,u}\tilde{\xi} + D_{CL,u\theta}\tilde{\theta} \\ \tilde{\mu} &= C_{CL,\mu}\tilde{\xi} + D_{CL,\mu\theta}\tilde{\theta} \end{aligned} \tag{15}$$

Systems (14) and (15) are interesting because a necessary and sufficient condition for the existence of a suitable anti-windup compensator, in the sense of Definition 6, can be given in terms of these two systems.

Definition 7

Given the plant $\mathscr{P}$ in (2), the controller $\mathscr{C}$ in (3), an integer $n_{aw} \geqslant 0$ and the scalars $\tilde{\gamma} > 0$, and $\tilde{g} \geqslant 0$, define the matrix condition $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$ in the unknown variables $(R, S, \gamma, \pi, \tau) \in (\mathbb{R}^{n_\xi \times n_\xi}, \mathbb{R}^{n_\xi \times n_\xi}, \mathbb{R}, \mathbb{R}, \mathbb{R})$ as

$$\begin{bmatrix} R_{11}A_p^{\mathrm{T}} + A_p R_{11} & B_{p,u} & R_{11}C_{p,z}^{\mathrm{T}} & \tau B_{p,\theta} & \tilde{g}R_{11}C_{p,\mu}^{\mathrm{T}} \\ \star & -\gamma I_{n_u} & D_{p,zu}^{\mathrm{T}} & 0 & \tilde{g}D_{p,\mu u}^{\mathrm{T}} \\ \star & \star & -\gamma I_{n_z} & \tau D_{p,z\theta} & 0 \\ \star & \star & \star & -\tau I_{n_\theta} & 0 \\ \star & \star & \star & \star & -\tau I_{n_\mu} \end{bmatrix} < 0 \tag{16a}$$

$$\begin{bmatrix} A_{CL}S + SA_{CL}^{\mathrm{T}} & SC_{CL,u}^{\mathrm{T}} & SC_{CL,z}^{\mathrm{T}} & \tau B_{CL,\theta} & \tilde{g}SC_{CL,\mu}^{\mathrm{T}} \\ \star & -\pi I_{n_u} & 0 & \tau D_{CL,u\theta} & 0 \\ \star & \star & -\gamma I_{n_z} & \tau D_{CL,z\theta} & 0 \\ \star & \star & \star & -\tau I_{n_\theta} & \tau\tilde{g}D_{CL,\mu\theta} \\ \star & \star & \star & \star & -\tau I_{n_\mu} \end{bmatrix} < 0 \tag{16b}$$

$$\gamma \leqslant \tilde{\gamma} \tag{16c}$$

$$R = R^{\mathrm{T}} = \begin{bmatrix} R_{11} & R_{12} \\ \star & R_{22} \end{bmatrix} > 0 \tag{16d}$$

$$S = S^{\mathrm{T}} > 0 \tag{16e}$$

$$R - S \geqslant 0 \tag{16f}$$

$$\mathrm{rank}(R - S) \leqslant n_{aw} \tag{16g}$$

Moreover, $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$ is said to be *feasible* if there exists a solution $(R, S, \gamma, \pi, \tau)$ that satisfies (16).

Theorem 2

Given the plant $\mathscr{P}$ in (2), the unconstrained controller $\mathscr{C}$ in (3), an integer $n_{aw} \geqslant 0$, and real scalars $\tilde{g} \geqslant 0$ and $\tilde{\gamma} > 0$, there exists a linear anti-windup compensator of order n_{aw} that guarantees quadratic robust performance recovery $(\tilde{\gamma}, \tilde{g})$ if and only if $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$ is feasible.

Proof

The technique is very similar to the proof used in Reference [13] and relies on the $\mathscr{S}$-procedure and the projection lemma. For a complete proof see Section 7. □

By way of Theorem 2, through the matrix conditions $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$ of Definition 7, the following theorem establishes important properties of the optimal anti-windup construction proposed in this paper.

Theorem 3

The following properties hold:

1. There exist real scalars $\tilde{g} \geqslant 0$ and $\tilde{\gamma} > 0$ such that $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, 0, \tilde{g}, \tilde{\gamma})$ is feasible if and only if there exists a matrix R such that

$$\begin{aligned} R_{11}A_p^{\mathrm{T}} + A_p R_{11} &< 0 \\ RA_{CL}^{\mathrm{T}} + A_{CL}R &< 0 \\ R = R^{\mathrm{T}} = \begin{bmatrix} R_{11} & R_{12} \\ \star & R_{22} \end{bmatrix} &> 0 \end{aligned} \tag{17}$$

2. The matrix condition $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_p, 0, \tilde{\gamma})$ is feasible if and only if the matrices A_p and A_{CL} are Hurwitz and $\tilde{\gamma} > \|D_{p,zu} + C_{p,z}(sI - A_p)^{-1}B_{p,u}\|_\infty$.
3. If there exists a $\tilde{\gamma}$ such that $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_p, \tilde{g}, \tilde{\gamma})$ is feasible, then the matrices A_p and A_{CL} are Hurwitz, $\tilde{g}\|C_{p,\mu}(sI - A_p)^{-1}B_{p,\theta}\|_\infty < 1$ and $\tilde{g}\|D_{CL,\mu\theta} + C_{CL,\mu}(sI - A_{CL})^{-1}B_{CL,\theta}\|_\infty < 1$.
4. If $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_1, \tilde{g}, \tilde{\gamma})$ is feasible and $n_1 \leqslant n_2$ then $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_2, \tilde{g}, \tilde{\gamma})$ is feasible.
5. If $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_1, \tilde{g}, \tilde{\gamma})$ is feasible and $n_1 \geqslant n_p$ then $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_p, \tilde{g}, \tilde{\gamma})$ is feasible.
6. If $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, 0, \tilde{\gamma})$ has a feasible solution $(R, S, \gamma, \pi, \tau)$ then there exists a scalar $\tilde{g} > 0$ such that $(R, S, \gamma, \pi, \tau)$ is also a feasible solution to $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$.

Proof

Item 1: If $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, 0, \tilde{g}, \tilde{\gamma})$ is feasible then (16g) implies $R = S$, thus (16a), (16b), and (16d) (with $R = S$) imply (17).

Assume there exists a symmetric positive definite matrix R that satisfies (17). Since (17) corresponds to the top left block diagonal entries of conditions (16a) and (16b), then with $\tilde{g} = 0$ there exists a large enough $\gamma = \bar{\gamma} > 0$, large enough $\pi = \bar{\pi}$, and small enough $\tau = \bar{\tau}$ such that (16a) and (16b) are satisfied. Thus, $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, 0, \tilde{g}, \tilde{\gamma})$ is feasible.

Item 2: If $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_p, 0, \tilde{\gamma})$ is feasible then the intersection of the first three columns and rows in (16a) are seen to imply A_p is Hurwitz and $\tilde{\gamma} > \|D_{p,zu} + C_{p,z}(sI - A_p)^{-1}B_{p,u}\|_\infty$ by Lemma 1. Also the top left block entry in (16b) is negative definite, hence A_{CL} is also Hurwitz.

Assume the matrices A_p and A_{CL} are Hurwitz, and $\tilde{\gamma} > \|D_{p,zu} + C_{p,z}(sI - A_p)^{-1}B_{p,u}\|_\infty$. Then Lemma 1 guarantees the existence of a matrix $R_{11} = R_{11}^{\mathrm{T}} > 0$ such that the upper left three by three block matrix in (16a) is negative definite with $\gamma = \tilde{\gamma}$. Since A_{CL} is Hurwitz, there exists a matrix

$$\bar{S} = \bar{S}^{\mathrm{T}} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} \\ \star & \bar{S}_{22} \end{bmatrix} > 0$$

such that $A_{CL}\bar{S} + \bar{S}A_{CL}^{\mathrm{T}} < 0$. Then there exists a small enough $\varepsilon > 0$ are large enough π such that with $S = \varepsilon\bar{S}$ the upper left three by three block matrix in (16b) is negative definite and $R_{11} - S_{11} \geqslant 0$. Since $g = 0$, there exists τ such that both (16a) and (16b) is negative definite. Then

$$\left(\begin{bmatrix} R_{11} & S_{12} \\ \star & S_{22} \end{bmatrix}, S, \gamma, \pi, \tau \right)$$

is a solution to $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_p, 0, \tilde{\gamma})$.

Item 3: The proof parallels the first part of the proof of item 2.

Item 4: The result follows from Definition 7 since if the rank condition (16g) holds for $n_{aw} = n_1$ then it also holds for $n_{aw} = n_2 \geqslant n_1$.

Item 5: If there exists a solution $(\bar{R}, \bar{S}, \bar{\gamma}, \bar{\pi}, \bar{\tau})$ to $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_1, \tilde{g}, \tilde{\gamma})$ and $n_1 \geqslant n_p$, partition $\bar{S}$ as

$$\bar{S} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} \\ \star & \bar{S}_{22} \end{bmatrix}, \quad \text{then} \quad \left(\begin{bmatrix} \bar{R}_{11} & \bar{S}_{12} \\ \star & \bar{S}_{22} \end{bmatrix}, \bar{S}, \bar{\gamma}, \bar{\pi}, \bar{\tau} \right)$$

is a solution to $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_p, \tilde{g}, \tilde{\gamma})$.

Item 6: Since both (16a) and (16b) are negative definite, there exists a sufficiently small $g > 0$ such that both constraints remain negative definite. □

Remark 2
Note that item 2 of Theorem 3 implies that the $\mathscr{L}_2$ gain of the plant constitutes the optimal performance level achievable by dynamic anti-windup compensation. This was already noted in Reference [25, Remark 2], where it is also stressed that IMC anti-windup is a possible selection among the many optimal solutions to this anti-windup construction problem. On the other hand, the IMC construction leads in some cases to unsatisfactory solutions to the anti-windup problem, while optimal solutions resulting from our construction typically performed very desirably in simulation studies. This observation suggests that further investigation is worth pursuing, about a selection criterion among the anti-windup compensators that induce an optimal (or almost optimal) performance level. To this aim, the LMIs reported in this paper provide a useful parametrization of a family of such compensators that may be augmented with additional LMIs that characterize such a selection criterion. Note that using the notation of this paper, the IMC solution would correspond to selecting the compensator (9) with $A_{aw} = A_p$, $B_{aw} = B_{p,u}$, $C_{aw} = \begin{bmatrix} B_{c,y} \\ D_{c,y} \end{bmatrix} C_{p,y}$ and $D_{aw} = \begin{bmatrix} B_{c,y} \\ D_{c,y} \end{bmatrix} D_{p,yu}$.

4.2. LMI formulations of the feasibility condition

Theorem 2 shows that a necessary and sufficient condition for the existence of a linear anti-windup compensator of order n_{aw} that guarantees quadratic robust performance recovery $(\tilde{\gamma}, \tilde{g})$ is a non-convex coupling of LMIs due to a rank constraint. In this section, we will show that the nonlinear matrix condition $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$ can be transformed into a linear one, when the anti-windup compensator of interest is static ($n_{aw} = 0$) or has order at least as large as the plant order ($n_{aw} \geqslant n_p$).

Proposition 1 ($n_{aw} = 0$)
Given the plant $\mathscr{P}$ in (2), the unconstrained controller $\mathscr{C}$ in (3), and real scalars $\tilde{g} \geqslant 0$ and $\tilde{\gamma} > 0$, $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, 0, \tilde{g}, \tilde{\gamma})$ is feasible if and only if there exists a solution (R, γ, π, τ) to the LMI (16a)–(16d) with $S = R$. Moreover, given a solution $(\bar{R}, \bar{\gamma}, \bar{\pi}, \bar{\tau})$ to this LMI, then $(R, S, \gamma, \pi, \tau) = (\bar{R}, \bar{S}, \bar{\gamma}, \bar{\pi}, \bar{\tau})$ is a feasible solution to $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, 0, \tilde{g}, \tilde{\gamma})$.

Proof
If $n_{aw} = 0$ then for $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, 0, \tilde{g}, \tilde{\gamma})$ to be feasible, by (16g), we must have $S = R$. Thus (16f) is satisfied and (16e) is redundant. Hence, the proof follows by rewriting the remaining inequalities in (16) with $S = R$. □

Proposition 2 ($n_{aw} \geqslant n_p$)
Given the plant $\mathscr{P}$ in (2), the unconstrained controller $\mathscr{C}$ in (3), an integer $n_{aw} \geqslant n_p$, and real scalars $\tilde{g} \geqslant 0$ and $\tilde{\gamma} > 0$, $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$ is feasible if and only if there exists a solution $(R, S, \gamma, \pi, \gamma, \tau)$ to the LMI (16a)–(16f).

Moreover, given a solution

$$(\bar{R}, \bar{S}, \bar{\gamma}, \bar{\pi}, \bar{\tau}) = \left(\begin{bmatrix} \bar{R}_{11} & \bar{R}_{12} \\ \star & \bar{R}_{22} \end{bmatrix}, \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} \\ \star & \bar{S}_{22} \end{bmatrix}, \bar{\gamma}, \bar{\pi}, \bar{\tau} \right)$$

to the LMI (16a)–(16f), then

$$(R, S, \gamma, \pi, \tau) = \left(\begin{bmatrix} \bar{R}_{11} & \bar{S}_{12} \\ \star & \bar{S}_{22} \end{bmatrix}, \bar{S}, \bar{\gamma}, \bar{\pi}, \bar{\tau} \right)$$

is a feasible solution to $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$.

Proof

Items 4 and 5 of Theorem 3 combine to guarantee that $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_p, \tilde{g}, \tilde{\gamma})$ is feasible if and only if $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_p + n_c, \tilde{g}, \tilde{\gamma})$ is feasible. Since $R,\ S \in \mathbb{R}^{(n_p+n_c)\times(n_p+n_c)}$, $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_p + n_c, \tilde{g}, \tilde{\gamma})$ is feasible if and only if (16a)–(16f) has a feasible solution $(\bar{R}, \bar{S}, \bar{\gamma}, \bar{\pi}, \bar{\tau})$. Moreover, the proof of item 5 of Theorem 3 shows that

$$(R, S, \gamma, \pi, \tau) = \left(\begin{bmatrix} \bar{R}_{11} & \bar{S}_{12} \\ \star & \bar{S}_{22} \end{bmatrix}, \bar{S}, \bar{\gamma}, \bar{\pi}, \bar{\tau} \right)$$

is a feasible solution to $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$. □

4.3. *LMI-based anti-windup synthesis*

Although the results in Section 4.1 provide a condition for the existence of an anti-windup compensator achieving a certain robust quadratic performance recovery level $(\tilde{\gamma}, \tilde{g})$ for the anti-windup closed-loop system, they do not provide tools for the construction of such a compensator. In this section, based on a solution $(R, S, \gamma, \pi, \tau)$ to $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$ arising from Theorem 2 or Propositions 1 or 2, we give a procedure to construct a state-space representation of an anti-windup compensator that guarantees robust quadratic performance recovery level $(\tilde{\gamma}, \tilde{g})$. The effectiveness of the procedure is then formally stated in Theorem 4.

Based on the matrices of system (8), we can formalize a procedure for the construction of the anti-windup compensator.

Procedure 1 (Construction of the anti-windup compensator)

Step 1: *Solve the feasibility condition.*

Given the plant $\mathscr{P}$, the controller $\mathscr{C}$, an integer $n_{aw} \geqslant 0$ and real scalars $\tilde{g} \geqslant 0$ and $\tilde{\gamma} > 0$, determine a solution $(R, S, \gamma, \pi, \tau)$ that satisfies the condition $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$.

Step 2: *Construct the matrix Q.*

Using R and S from the solution $(R, S, \gamma, \pi, \tau)$ found in Step 1, define the matrix $N \in \mathbb{R}^{n_\xi \times n_{aw}}$ as a solution of the following equation:

$$RS^{-1}R - R = NN^{\mathrm{T}} \tag{18}$$

Since R and S are non-singular and conditions (16f) and (16g) of Definition 7 are satisfied, then $RS^{-1}R - R$ is positive semidefinite and of rank n_{aw}, so there always exists a matrix N satisfying Equation (18). Define the matrix $M \in \mathbb{R}^{n_{aw}\times n_{aw}}$ as

$$M := I + N^{\mathrm{T}} R^{-1} N \tag{19}$$

Finally, let $n := n_\xi + n_{aw}$ and define the matrix $Q \in \mathbb{R}^{n\times n}$ as

$$Q := \begin{bmatrix} R & N \\ \star & M \end{bmatrix} \tag{20}$$

Step 3: *Construct the matrix U*.

Using π and γ from the solution $(R, S, \gamma, \pi, \tau)$ found in Step 1, find a solution $\delta \in \mathbb{R}$ to the LMI

$$\begin{bmatrix} -2W^{-1} - \gamma\delta W^{-2} & I \\ \star & -\delta\pi^{-1} I \end{bmatrix} < 0, \quad \delta > 0 \tag{21}$$

and define $U := \delta W^{-1}$.

Step 4: *Construct other required matrices*.

Construct the matrices $A_0 \in \mathbb{R}^{n\times n}$, $B_{\psi 0} \in \mathbb{R}^{n\times n_u}$, $C_{u0} \in \mathbb{R}^{n_u\times n}$, $D_{u\psi 0} \in \mathbb{R}^{n_u\times n_u}$, $C_{z0} \in \mathbb{R}^{n_z\times n}$, $D_{z\psi 0} \in \mathbb{R}^{n_z\times n_u}$, $C_{\mu 0} \in \mathbb{R}^{n_\mu\times n}$, $D_{\mu\psi 0} \in \mathbb{R}^{n_\mu\times n_u}$, $H_1^{\mathrm{T}} \in \mathbb{R}^{n\times(n_{aw}+n_v)}$, $G_1 \in \mathbb{R}^{(n_{aw}+n_u)\times n}$, $G_2 \in \mathbb{R}^{(n_{aw}+n_u)\times n_u}$, $H_2^{\mathrm{T}} \in \mathbb{R}^{n_u\times(n_{aw}+n_v)}$, $H_3^{\mathrm{T}} \in \mathbb{R}^{n_z\times(n_{aw}+n_v)}$, $H_4^{\mathrm{T}} \in \mathbb{R}^{n_\mu\times(n_{aw}+n_v)}$, $B_\theta \in \mathbb{R}^{n\times n_\theta}$, $D_{u\theta} \in \mathbb{R}^{n_u\times n_\theta}$, $D_{z\theta} \in \mathbb{R}^{n_z\times n_\theta}$ and $D_{\mu\theta} \in \mathbb{R}^{n_\mu\times n_\theta}$ as follows:

$$\begin{gathered} A_0 = \begin{bmatrix} A_{CL} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_{\psi 0} = \begin{bmatrix} B_{CL,\psi} \\ 0 \end{bmatrix}, \quad C_{u0} = [C_{CL,u} \;\; 0] \\ D_{u\psi 0} = D_{CL,u\psi}, \quad C_{z0} = [C_{CL,z} \;\; 0], \quad D_{z\psi 0} = D_{CL,z\psi} \\ C_{\mu 0} = [C_{CL,\mu} \;\; 0], \quad D_{\mu\psi 0} = D_{CL,\mu\psi} \end{gathered} \tag{22a}$$

$$\begin{gathered} H_1^{\mathrm{T}} = \begin{bmatrix} 0 & B_{CL,v} \\ I_{n_{aw}} & 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & I_{n_{aw}} \\ 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} \\ H_2^{\mathrm{T}} = [0 \;\; D_{CL,uv}], \quad H_3^{\mathrm{T}} = [0 \;\; D_{CL,zv}], \quad H_4^{\mathrm{T}} = [0 \;\; D_{CL,\mu v}] \end{gathered} \tag{22b}$$

$$B_\theta = \begin{bmatrix} B_{CL,\theta} \\ 0 \end{bmatrix}, \quad D_{u\theta} = D_{CL,u\theta}, \quad D_{z\theta} = D_{CL,z\theta}, \quad D_{\mu\theta} = D_{CL,\mu\theta} \tag{22c}$$

Step 5: *Construct and solve the anti-windup compensator LMI*.

Stack the matrices of the anti-windup compensator (9) in a single matrix $\Lambda \in \mathbb{R}^{(n_{aw}+n_v)\times(n_{aw}+n_u)}$ as follows:

$$\Lambda := \begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} \tag{23}$$

Based on the matrices determined in Steps 2–4 of this procedure, construct the matrices $\Psi \in \mathbb{R}^{(n+n_u+n_w+n_z)\times(n+n_u+n_w+n_z)}$, $H \in \mathbb{R}^{(n_{aw}+n_v)\times(n+n_u+n_w+n_z)}$ and $G \in \mathbb{R}^{(n_{aw}+n_u)\times(n+n_u+n_w+n_z)}$ as follows:

$$\Psi = \begin{bmatrix} QA_0^{\mathrm{T}} + A_0 Q & B_{\psi 0} U + QC_{u0}^{\mathrm{T}} & B_{\psi 0} & QC_{z0}^{\mathrm{T}} & \tau B_\theta & \tilde{g} QC_{\mu 0}^{\mathrm{T}} \\ \star & D_{u\psi 0} U + UD_{u\psi 0}^{\mathrm{T}} - 2U & D_{u\psi 0} & UD_{z\psi 0}^{\mathrm{T}} & \tau D_{u\theta} & \tilde{g} UD_{\mu\psi 0}^{\mathrm{T}} \\ \star & \star & -\gamma I & D_{z\psi 0}^{\mathrm{T}} & 0 & \tilde{g} D_{\mu\psi 0}^{\mathrm{T}} \\ \star & \star & \star & -\gamma I & \tau D_{z\theta} & 0 \\ \star & \star & \star & \star & -\tau I & \tau \tilde{g} D_{\mu\theta}^{\mathrm{T}} \\ \star & \star & \star & \star & \star & -\tau I \end{bmatrix} \quad (24a)$$

$$H = [H_1 \quad H_2 \ \ 0 \ \ H_3 \ \ 0 \ \ \tilde{g} H_4] \quad (24b)$$

$$G = [G_1 Q \ \ G_2 U \ \ G_2 \ \ 0 \ \ 0 \ \ 0] \quad (24c)$$

Finally, compute the matrix Λ associated with the desired anti-windup compensator by solving the LMI

$$\Psi + G^{\mathrm{T}} \Lambda^{\mathrm{T}} H + H^{\mathrm{T}} \Lambda G < 0 \quad (25)$$

Theorem 4
Given the plant $\mathscr{P}$, the unconstrained controller $\mathscr{C}$, an integer n_{aw}, real scalars $\tilde{g} \geqslant 0$ and $\tilde{\gamma} > 0$ and a solution $(R, S, \gamma, \pi, \tau)$ to $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$, LMI (25) constructed according to Procedure 1 is guaranteed to be solvable for Λ. Furthermore, the solution Λ defines the matrices of a linear anti-windup compensator (9) of order n_{aw} that guarantees quadratic robust performance recovery of level $(\tilde{g}, \tilde{\gamma})$.

Proof
See Section 7. □

5. LMI-BASED ANTI-WINDUP PERFORMANCE ANALYSIS

Assume that the plant, controller and anti-windup compensator are given. Then, for analysis purposes, the performance recovery level can be determined by solving an LMI eigenvalue problem.

Theorem 5
Given the mismatch system (11) (derived from the plant, controller and anti-windup compensator), and real scalars $\tilde{g} \geqslant 0$ and $\tilde{\gamma} > 0$, the anti-windup closed-loop system guarantees quadratic robust performance recovery $(\tilde{\gamma}, \tilde{g})$ if and only if there exists a solution $(Q, \gamma, \delta, \tau) \in \mathbb{R}^{n\times n}$,

$\mathbb{R}, \mathbb{R}, \mathbb{R}$) where $n := n_\xi + n_{aw}$ to the LMI

$$\begin{bmatrix} QA^{\mathrm{T}} + AQ & B_\psi U + QC_u^{\mathrm{T}} & B_\psi & QC_z^{\mathrm{T}} & \tau B_\theta & \tilde{g} QC_\mu^{\mathrm{T}} \\ \star & D_{u\psi} U + UD_{u\psi}^{\mathrm{T}} - 2U & D_{u\psi} & UD_{z\psi}^{\mathrm{T}} & \tau D_{u\theta} & \tilde{g} UD_{\mu\psi}^{\mathrm{T}} \\ \star & \star & -\gamma I_{n_u} & D_{z\psi}^{\mathrm{T}} & 0 & \tilde{g} D_{\mu\psi}^{\mathrm{T}} \\ \star & \star & \star & -\gamma I_{n_z} & \tau D_{z\theta} & 0 \\ \star & \star & \star & \star & -\tau I_{n_\theta} & \tau \tilde{g} D_{\mu\theta}^{\mathrm{T}} \\ \star & \star & \star & \star & \star & -\tau I_{n_\mu} \end{bmatrix} < 0 \tag{26a}$$

$$\tau > 0 \tag{26b}$$

$$Q = Q^{\mathrm{T}} > 0 \tag{26c}$$

$$U = \delta W^{-1} > 0 \tag{26d}$$

$$\gamma \leqslant \tilde{\gamma} \tag{26e}$$

Proof
See Section 7.1. □

Corollary 1
Given the mismatch system (11) and a scalar $\tilde{\gamma} > 0$, if the anti-windup closed-loop system guarantees quadratic robust performance recovery $(\tilde{\gamma}, 0)$ then there exists $\tilde{g} > 0$ such that quadratic robust performance recovery $(\tilde{\gamma}, \tilde{g})$ is guaranteed.

Proof
If there exists a solution $(Q, \gamma, \delta, \tau)$ to (26) with $\tilde{g} = 0$, then by taking the Schur compliment of (26a) with respect to the last column and row, there exists $\tilde{g} > 0$ sufficiently small such that (26) is still satisfied for some $\tilde{g} > 0$. □

6. COMPARISON TO OTHER LMI-BASED METHODS

In this section, we will first make connections to previous results and then illustrate the proposed anti-windup construction and compare it with some similar architectures on a simple example.

An important connection should be stated between our approach and that of References [11, 13]. Indeed, the two constructions are characterized by the same architecture, although the performance goal and, consequently, the arising LMI formulation is extremely different. Nevertheless, the architecture of the two schemes being the same, we expect (as it is confirmed by the results in Theorem 3) that the feasibility of the two constructions share the same system theoretic conditions. On the other hand, the optimal anti-windup constructions resulting from the two different performance characterizations, are very different. Indeed, in References [11, 13], the performance is chosen as the $\mathscr{L}_2$ gain γ_{wz} from w to z in the anti-windup closed-loop system of Figure 1. Although there are examples that exhibit very satisfactory responses when using an anti-windup compensator constructed with such methods, other examples show that the performance index may not capture the essence of what is most desirable to optimize. For instance, the example reported in the following section shows that the response with the optimal anti-windup compensation designed according to References [11, 13] is very sluggish. On the other hand, the construction proposed here captures within our performance index (that will be referred to as γ_{TK} in the following) the goal of recovery of the unconstrained response and leads to a highly improved anti-windup action.

Interesting connections can be also made with the IMC approach introduced in Reference [26] (see also References [3, 8]), and the solution proposed in Reference [17], which can be seen as a (nonlinear, in general) generalization of the IMC approach (IMC is recovered in Reference [17] by choosing $k(\cdot) \equiv 0$). For comparison purposes, the responses and performance level achieved by using IMC will also be computed for the example of the following section.[6] Note that the necessary condition of asymptotic stability of the plant derived in this paper (see item 2 in Theorem 3) is confirmed by the results in Reference [17], where it is shown that this same condition is necessary to guarantee any level of robustness.

6.1. Simulation example

Consider a damped mass-spring system whose equations of motion are given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -k/m & -f/m \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u \\ y &= [1 \;\; 0]\, x \\ z &= y - r \end{aligned} \tag{27}$$

where $x := [q \;\, \dot{q}]^{\mathrm{T}}$ represents position and speed of the body connected to the spring, m is the mass of the body, k is the elastic constant of the spring, f is the damping coefficient, and u represents a force exerted on the mass. We choose the following values for the parameters:

$$m = 0.1 \text{ kg}, \quad k = 1\,\frac{\text{kg}}{\text{s}^2}, \quad f = 0.005\,\frac{\text{kg}}{\text{s}} \tag{28}$$

Assume that $r \in \mathbb{R}$ is a reference input corresponding to the desired mass position, and consider the following linear controller:

$$u = C_{fb}(s)(C_{ff}(s)r - y) \tag{29a}$$

with

$$C_{fb}(s) := 200\,\frac{(s+5)^2}{s(s+80)}, \quad C_{ff}(s) := \frac{5}{s+5} \tag{29b}$$

This controller, has been determined with the aim of guaranteeing a fast response with zero steady state error to step reference changes, robust to parameter uncertainties. The state space

[6] Note however, that IMC anti-windup is aimed at guaranteeing stability of the closed-loop, without any concern for the resulting performance.

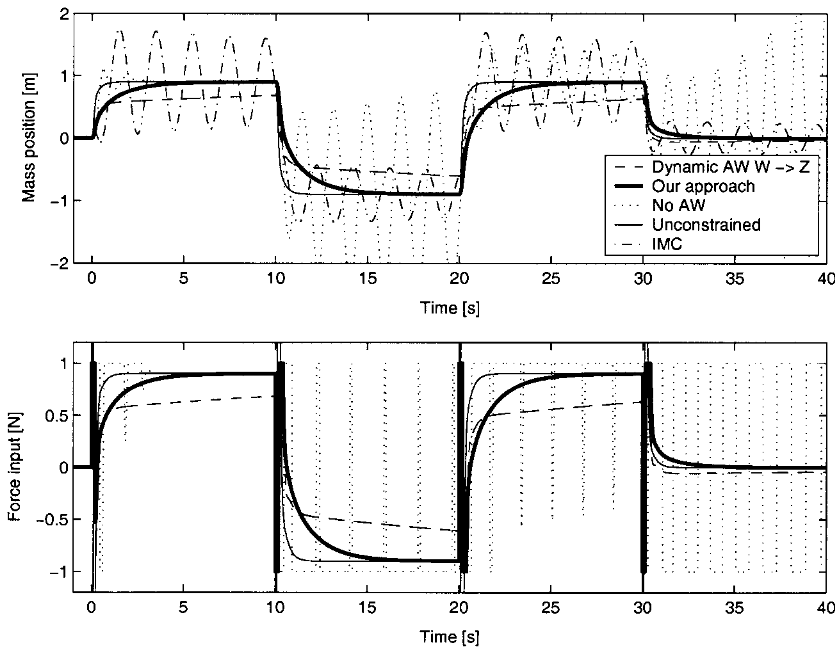

Figure 4. Time responses of the saturated (dotted), unconstrained (solid), and anti-windup closed-loop systems using the IMC construction (dash–dotted), the dynamic construction of Reference [13] (dashed) and our technique (bold).

realization that we will use is given by (3) with matrices

$$A_c = \begin{bmatrix} -5 & 0 & 0 \\ 320 & -80 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_{c,y} = \begin{bmatrix} 0 \\ -128 \\ 0 \end{bmatrix}, \quad B_{c,w} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$C_c = [500 \quad -109.3750 \quad 39.0625], \quad D_{c,y} = -200, \quad D_{c,w} = 0.$$

The response of the unconstrained closed-loop system (27), (29), starting from the rest position and with the reference switching between ± 0.9 m every 10 s and going back to zero permanently after 30 s, is shown by the solid curve in Figure 4. If the force exerted at the plant's input u is limited between ± 1 kg m/s^2, the closed-loop response corresponds to the dotted curve in Figure 4, which converges to a limit cycle where the output persistently oscillates between positive and negative peaks $q_{\mathrm{PEAK}} \approx 35.7$. The windup effect shown by this saturated response is associated with a complicated compensation problem. Indeed, the following attempts for anti-windup design all lead to unacceptable results:

1. *IMC/model-based anti-windup* leads to very large oscillations decaying at a very slow rate (corresponding to the slow modes of the open-loop plant dynamics) and is represented by the dash–dotted curve in the Figure 4; the corresponding values of the performance indexes γ_{wz} and γ_{TK}, determined using the techniques described in Reference [13] and in Section 5, respectively, are reported in Table I;

Table I. Performance levels induced by the different anti-windup constructions

	IMC	[13]	This paper
γ_{wz}	476.46	21.00	97.90
γ_{TK}	63.25	64.57	63.25

2. *linear static anti-windup compensation*, is not feasible (namely, the LMI conditions (17) are unfeasible) for this example;
3. *optimal dynamic linear anti-windup compensation*, which minimizes the gain γ_{wz}, following the construction in Reference [13] gives the sluggish response represented by the dashed curve in Figure 4. This response has been determined using the following gains

$$A_{aw\ wz} = \begin{bmatrix} -1.0337 & -18.6516 \\ -23.2234 & -439.0613 \end{bmatrix}, \quad B_{aw\ wz} = \begin{bmatrix} -22.5096 \\ -0.6577 \end{bmatrix}$$

$$C_{aw\ wz} = \begin{bmatrix} 0.0241 & 0.4512 \\ 11.1751 & 0.2642 \\ -0.1904 & -3.5996 \\ 17.3118 & -4.0514 \end{bmatrix}, \quad D_{aw\ wz} = \begin{bmatrix} -0.1545 \\ -1.0668 \\ 0.7789 \\ 0.9847 \end{bmatrix}$$

the corresponding values of the performance indexes γ_{wz} and γ_{TK} are reported in Table I.

When applying our construction to this example, since static anti-windup compensation is not feasible, we employ plant-order dynamic compensation and following Procedure 1 we obtain the following gains:

$$A_{awTK} = \begin{bmatrix} -0.6511 & 1.1922 \\ 0.9420 & -1.7249 \end{bmatrix} 10^8, \quad B_{awTK} = \begin{bmatrix} 428.8273 \\ -620.4231 \end{bmatrix}$$

$$C_{awTK} = \begin{bmatrix} -0.0573 & 0.1049 \\ -0.2445 & 0.4477 \\ 0.3912 & -0.7164 \\ 2.0416 & -3.7384 \end{bmatrix} 10^7, \quad D_{awTK} = \begin{bmatrix} 3.7746 \\ 16.1051 \\ -25.7677 \\ -133.4671 \end{bmatrix} \tag{30}$$

which induce the performance levels listed in Table I. As already noted in Remark 2, IMC anti-windup induces globally optimal performance level, as seen from the performance index γ_{TK}.

The response corresponding to gains (30) is represented in Figure 4 by the bold curve. This response performs satisfactorily in recovering the unconstrained response (which corresponds to the solid curve in Figure 4), thus confirming the effectiveness of the proposed anti-windup construction.

7. PROOF OF THE MAIN RESULTS

A key step in the proof of Theorems 2, 4, and 5 is to establish the connection between the matrix condition $\mathrm{MC}_{Rr}(\mathscr{P},\mathscr{C},n_{aw},\tilde{g},\tilde{\gamma})$ in Definition 7, the LMIs for analysis (26) in Theorem 5, and the LMI (25) in the final step of Procedure 1. The LMIs (26a) and (25) coincide but are in different unknowns; the LMI (26a) is in the unknowns (Q,γ,δ,τ) whereas the LMI (25) is in the unknown Λ. To verify this, it is easy to check that the matrices B_θ, $D_{u\theta}$, $D_{z\theta}$ and $D_{\mu\theta}$ in (11) coincide with the definitions in Equations (22c) and the remaining matrices in (11) satisfy

$$\begin{aligned} &A = A_0 + H_1^{\mathrm{T}}\Lambda G_1, \quad C_u = C_{u0} + H_2^{\mathrm{T}}\Lambda G_1, \quad C_z = C_{z0} + H_3^{\mathrm{T}}\Lambda G_1 \\ &B_\psi = B_{\psi 0} + H_1^{\mathrm{T}}\Lambda G_2, \quad D_{u\psi} = D_{u\psi 0} + H_2^{\mathrm{T}}\Lambda G_2, \quad D_{z\psi} = D_{z\psi 0} + H_3^{\mathrm{T}}\Lambda G_2 \\ &C_\mu = C_{\mu 0} + H_4^{\mathrm{T}}\Lambda G_1, \quad D_{\mu\psi} = D_{\mu\psi 0} + H_4^{\mathrm{T}}\Lambda G_2 \end{aligned} \tag{31}$$

For a detailed computation of the matrices in Equations (31), the reader is referred to Appendix A.

The following theorem establishes the equivalence between the feasibility of the matrix condition $\mathrm{MC}_{Rr}(\mathscr{P},\mathscr{C},n_{aw},\tilde{g},\tilde{\gamma})$ in Definition 7 and the feasibility of the matrix constraints (26) (equivalently (25)), which constitute a key result used to prove Theorems 2, 4 and 5.

Theorem 6

(1) Given the plant $\mathscr{P}$ in (2), the unconstrained controller $\mathscr{C}$ in (3), an integer $n_{aw} \geqslant 0$ and real scalars $\tilde{g} \geqslant 0$ and $\tilde{\gamma} > 0$, there exist matrices Q, Λ and real scalars γ, δ, τ satisfying (26) (with the definitions (22), (31)) if and only if the matrix condition $\mathrm{MC}_{Rr}(\mathscr{P},\mathscr{C},n_{aw},\tilde{g},\tilde{\gamma})$ is feasible.

(2) Given a feasible solution (R,S,γ,π,τ) to $\mathrm{MC}_{Rr}(\mathscr{P},\mathscr{C},n_{aw},\tilde{g},\tilde{\gamma})$, the matrix Q constructed in (18)–(20) and the matrix U and scalar δ selected according to (21) guarantees that LMI (25) in the unknown Λ is solvable. Moreover, given such Q,γ,δ,τ and a feasible solution Λ to LMI (25), we have that (Q,γ,δ,τ) is a feasible solution to the matrix inequalities (26) (with definitions (22), (31)).

Proof

The proof of Theorem 6 requires quite cumbersome notation and a few preliminary lemmas. Since the proof technique is very similar to the parallel technical result of Reference [13, Theorem 4], we do not consider this proof central for this paper and defer it to the later Section 7.2. □

Proof of Theorem 2

The composition of Theorem 5 and item 1 in Theorem 6 imply Theorem 2. □

Proof of Theorem 4

Step 1 of Procedure 1 is assumed to be solvable. Steps 2–4 are constructive. As far as Step 5 is concerned, matrices (24) can always be constructed based on the matrices computed at the preceding steps. Moreover, by item 2 in Theorem 6, the matrix Q constructed in Step 2 guarantees that LMI (25) is solvable for Λ and any feasible solution Λ to LMI (25) is such that $(Q, \Lambda, \delta, \gamma, \tau)$ satisfies (26). Hence, by Theorem 5, the anti-windup closed-loop system (8), (9) corresponding to Λ guarantees quadratic robust performance recovery $(\tilde{\gamma}, \tilde{g})$. $\square$

7.1. Proof of Theorem 5

To prove Theorem 5, the following lemmas will be useful. (The first of these lemmas was proven in Reference [24].)

Lemma 3
Consider a locally Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^n$ and assume that the Jacobian of F satisfies

$$JF(x) \in \mathcal{M} \quad \textit{for almost all } x \in \mathbb{R}^n$$

where the set $\mathcal{M}$ is compact, convex, and each matrix in $\mathcal{M}$ is non-singular. Then there exists a (unique) globally Lipschitz function $G : \mathbb{R}^n \to \mathbb{R}^n$ such that $F(G(x)) = x$ for all $x \in \mathbb{R}^n$. Equivalently, F is a homeomorphism with globally Lipschitz inverse.

The following result is a generalization of Lemma 3, whose proof is carried out along the lines of Reference [27, Remark 7.1.3]. (The proof is included for completeness.)

Lemma 4
Given a locally Lipschitz function $H : \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_u}$, assume that there exists a compact set $\mathcal{M}_w \subset \mathbb{R}^{n_u \times n_w}$ and a compact convex set $\mathcal{M}_u \subset R^{n_u \times n_w}$ of non-singular matrices, such that

$$\begin{array}{l} J_u H(u, w) \in \mathcal{M}_u \\ J_w H(u, w) \in \mathcal{M}_w \end{array} \quad \text{wherever they exist}$$

where $J_u H(u, w)$ and $J_w H(u, w)$ denote the Jacobian of H with respect to u and w, respectively. Then there exists a (unique) globally Lipschitz function $\zeta : \mathbb{R}^{n_w} \to \mathbb{R}^{n_u}$ such that $H(\zeta(w), w) = 0$.

Proof
Define the function

$$F(Z) := \begin{bmatrix} H(u, w) \\ w \end{bmatrix} \tag{32}$$

(where $Z := (u, w)$), and note that, wherever the Jacobian of F exists, it is of the form:

$$JF(Z) = \begin{bmatrix} M_u & M_w \\ 0 & I \end{bmatrix}$$

where, by assumption, $M_u \in \mathcal{M}_u$ and $M_w \in \mathcal{M}_w$.

By the block triangular structure of the Jacobian of F, there exists a compact and convex set $\mathcal{M}$ such that $JF(Z) \in \mathcal{M}$ wherever it exists and all the matrices in $\mathcal{M}$ are non-singular. Then, Lemma 3 guarantees the existence of a unique inverse function G for F, which is also globally Lipschitz. Denote by $G(f_1, f_2) =: (g_1, g_2)$ the partition of G corresponding to the partition of F.

Based on the structure of F (see Equation (32)), the following relation holds:

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = F(G(f_1, f_2)) = \begin{bmatrix} H(g_1(f_1, f_2), w) \\ f_2 \end{bmatrix}$$

where, by uniqueness, necessarily, $f_2 = w$. Defining $\zeta(w) := g_1(0, w)$, we get

$$H(\zeta(w), w) = 0$$

where $\zeta(\cdot)$ is globally Lipschitz. □

Lemma 5
Let

$$E := \begin{bmatrix} I - \frac{D_{u\psi}}{2} & 0 \\ -\frac{D_{\mu\psi}}{2} & I \end{bmatrix}, \quad D := \begin{bmatrix} D_{u\psi} & D_{u\theta} \\ D_{\mu\psi} & D_{\mu\theta} \end{bmatrix}, \quad W_1 := \begin{bmatrix} 2\sigma W & 0 \\ \star & \rho I \end{bmatrix}, \quad W_2 := \begin{bmatrix} \frac{\sigma W}{2} & 0 \\ \star & \rho g^2 I \end{bmatrix}$$

If

$$L := \begin{bmatrix} \sigma(W D_{u\psi} + D_{u\psi}^{\mathrm{T}} W - 2W) + \rho g^2 D_{\mu\psi}^{\mathrm{T}} D_{\mu\psi} & \sigma W D_{u\theta} + \rho g^2 D_{\mu\psi}^{\mathrm{T}} D_{\mu\theta} \\ \star & -\rho I + \rho g^2 D_{\mu\theta}^{\mathrm{T}} D_{\mu\theta} \end{bmatrix} < 0 \tag{33}$$

then $E - DD_J$ is non-singular for all D_J such that $D_J^{\mathrm{T}} W_1 D_J \leqslant W_2$.

Proof
Let

$$F := \begin{bmatrix} I - \frac{D_{u\psi}}{2} & -\frac{D_{u\theta}}{2} \\ 0 & I \end{bmatrix}.$$

Note that with this selection $DF = ED$. The negative definite upper left entry of inequality (33) can be rearranged to see that

$$\sigma(W D_{u\psi} + D_{u\psi}^{\mathrm{T}} W - 2W) + \rho g^2 D_{\mu\psi}^{\mathrm{T}} D_{\mu\psi} = \sigma\left(-2\left(I - \frac{D_{u\psi}}{2}\right)^{\mathrm{T}} W \left(I - \frac{D_{u\psi}}{2}\right) + \frac{1}{2} D_{u\psi}^{\mathrm{T}} W D_{u\psi}\right) + \rho g^2 D_{\mu\psi}^{\mathrm{T}} D_{\mu\psi} < 0$$

From this expression, we note two useful properties. First the matrix $I - D_{u\psi}/2$ is non-singular since $\sigma(I - D_{u\psi}/2)^{\mathrm{T}} W (I - D_{u\psi}/2) > 0$, hence E and F are non-singular. Second, this expression makes it straightforward to verify $L = D^{\mathrm{T}} W_2 D - F^{\mathrm{T}} W_1 F < 0$. This, combined with the fact $E^{-1} D = D F^{-1}$, implies that

$$D^{\mathrm{T}} E^{-\mathrm{T}} W_2 E^{-1} D < W_1 \tag{34}$$

Suppose by contradiction that there exists D_J such that $D_J^{\mathrm{T}} W_1 D_J \leqslant W_2$ and $E - DD_J$ is singular. Then there exists $w \neq 0$ such that $w = E^{-1} D D_J w$. From this and from (34), it follows that $w^{\mathrm{T}} W_2 w = w^{\mathrm{T}} D_J^{\mathrm{T}} D^{\mathrm{T}} E^{-\mathrm{T}} W_2 E^{-1} D D_J w < w^{\mathrm{T}} D_J^{\mathrm{T}} W_1 D_J w \leqslant w^{\mathrm{T}} W_2 w$. Thus we have a contradiction. □

Lemma 6
Suppose f and h are continuous and the finite dimensional forward complete system

$$\begin{aligned}\dot{x}_\delta &= f(x_\delta, x, u, w)\\ \theta &= h(x_\delta, x, u, w)\end{aligned}$$

has finite incremental gain g from $\mu := C_{p,\mu}x + D_{p,\mu u}u$ to θ. Then

$$|h(x_\delta, x+\delta_x, u+\delta_u, w) - h(x_\delta, x, u, w)| \leqslant g\left|[C_{p,\mu} \;\; D_{p,\mu u}]\begin{bmatrix}\delta_x\\ \delta_u\end{bmatrix}\right|$$

for all $x_\delta, x, \delta_x, u, \delta_u$, and w.

Proof
Suppose not. Then there exist $x_\delta, x, \delta_x, u, \delta_u, w$ and $\varepsilon > 0$ such that

$$|h(x_\delta, x+\delta_x, u+\delta_u, w) - h(x_\delta, x, u, w)| \geqslant \varepsilon + g\left|[C_{p,\mu} \;\; D_{p,\mu u}]\begin{bmatrix}\delta_x\\ \delta_u\end{bmatrix}\right|$$

Since h is continuous, then there exists δ such that for all z_1, z_2 where $|z_1 - x_\delta| < \delta$ and $|z_2 - x_\delta| < \delta$ then

$$|h(z_1, x+\delta_x, u+\delta_u, w) - h(x_\delta, x+\delta_x, u+\delta_u, w)| \leqslant \frac{\varepsilon}{3}$$

and

$$|h(x_\delta, x, u, w) - h(z_2, x, u, w)| \leqslant \frac{\varepsilon}{3}$$

Since f is assumed continuous and forward complete, solutions are guaranteed to exist and there exists a $t^* > 0$ such that the solution of $\dot{z}_1 = f(z_1, x+\delta_x(t), u+\delta_u(t), w)$, $\dot{z}_2 = f(z_2, x, u, w)$ (where $z_1(0) = z_2(0) = x_\delta$) satisfies $|z_1(t) - x_\delta| < \delta$ and $|z_2(t) - x_\delta| < \delta$ for all $t \in [0, t^*]$ where

$$\delta_x(t) := \begin{cases}\delta_x & \text{if } t \in [0, t^*]\\ 0 & \text{otherwise}\end{cases} \quad \text{and} \quad \delta_u(t) := \begin{cases}\delta_u & \text{if } t \in [0, t^*]\\ 0 & \text{otherwise}\end{cases}$$

Then

$$\begin{aligned}&\|h(z_1(\cdot), x+\delta_x(\cdot), u+\delta_u(\cdot), w) - h(z_2(\cdot), x+\delta_x, u+\delta_u, w)\|_2\\ &\quad\geqslant \|h(z_1(\cdot), x+\delta_x, u+\delta_u, w) - h(z_2(\cdot), x+\delta_x, u+\delta_u, w)\|_2\\ &\quad\geqslant t^*|h(x_\delta, x+\delta_x, u+\delta_u, w) - h(x_\delta, x, u, w)|\\ &\qquad - \|h(z_1(\cdot), x+\delta_x, u+\delta_u, w) - h(x_\delta, x+\delta_x, u+\delta_u, w)\|_{2,[0,t^*]}\\ &\qquad - \|h(x_\delta, x, u, w) - h(z_2(\cdot), x, u, w)\|_{2,[0,t^*]}\\ &\quad\geqslant g\left\|\begin{bmatrix}C_{p,\mu}\delta_x(\cdot)\\ D_{p,\mu u}\delta_u(\cdot)\end{bmatrix}\right\|_2 + \frac{\varepsilon t^*}{3}\end{aligned}$$

Thus,

$$\|\theta(x_\delta, x+\delta_x(\cdot), u+\delta_u(\cdot), w) - \theta(x_\delta, x, u, w)\|_2 > g\left\|\begin{bmatrix} C_{p,\mu}(x+\delta_x(\cdot)-x) \\ D_{p,\mu u}(u+\delta_u(\cdot)-u)\end{bmatrix}\right\|_2$$

which contradicts the assumption that θ has finite incremental gain g from μ to θ. □

The following facts will also be useful for the proof of Theorem 5.

Fact 1: By Schur complements [28, p. 7] and rearranging columns and rows, it can be shown that

$$2x^{\mathrm{T}}P(Ax + B_\psi(\psi+\varphi) + B_\theta\tilde{\theta}) < -\frac{1}{\gamma}\tilde{z}^{\mathrm{T}}\tilde{z} + \gamma\psi^{\mathrm{T}}\psi + \rho(\tilde{\theta}^{\mathrm{T}}\tilde{\theta} - g^2\tilde{\mu}^{\mathrm{T}}\tilde{\mu}) - 2\sigma\varphi^{\mathrm{T}}W(\tilde{u}-\varphi)$$

for all $(x, \psi, \varphi, \tilde{\theta}) \neq 0$ if and only if

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & PB_\psi + \sigma C_u^{\mathrm{T}}W & PB_\psi & C_z^{\mathrm{T}} & PB_\theta & \rho g C_\mu^{\mathrm{T}} \\ \star & \sigma(WD_{u\psi} + D_{u\psi}^{\mathrm{T}}W - 2W) & \sigma WD_{u\psi} & D_{z\psi}^{\mathrm{T}} & \sigma WD_{u\theta} & \rho g D_{\mu\psi}^{\mathrm{T}} \\ \star & \star & -\gamma I & D_{z\psi}^{\mathrm{T}} & 0 & \rho g D_{\mu\psi}^{\mathrm{T}} \\ \star & \star & \star & -\gamma I & D_{z\theta} & 0 \\ \star & \star & \star & \star & -\rho I & \rho g D_{\mu\theta}^{\mathrm{T}} \\ \star & \star & \star & \star & \star & -\rho I \end{bmatrix} < 0 \tag{35}$$

Fact 2: Recall from (11) that $\tilde{u}$, $\tilde{z}$ and $\tilde{\mu}$ are functions of x, $\psi(\bar{u})$, $\varphi(\tilde{u}, \bar{u})$ and $\tilde{\theta}$. Given any symmetric positive definite matrix W, the following statements are equivalent:

1. there exists a scalar $\sigma \geqslant 0$ such that

$$2x^{\mathrm{T}}P(Ax + B_\psi(\psi+\varphi) + B_\theta\tilde{\theta}) < -\frac{1}{\gamma}\tilde{z}^{\mathrm{T}}\tilde{z} + \gamma\psi^{\mathrm{T}}\psi + \rho(\tilde{\theta}^{\mathrm{T}}\tilde{\theta} - g^2\tilde{\mu}^{\mathrm{T}}\tilde{\mu}) - 2\sigma\varphi^{\mathrm{T}}W(\tilde{u}-\varphi)$$

   for all $(x, \psi, \varphi, \tilde{\theta}) \neq 0$;
2.

$$2x^{\mathrm{T}}P(Ax + B_\psi(\psi+\varphi) + B_\theta\tilde{\theta}) < -\frac{1}{\gamma}\tilde{z}^{\mathrm{T}}\tilde{z} + \gamma\psi^{\mathrm{T}}\psi + \rho(\tilde{\theta}^{\mathrm{T}}\tilde{\theta} - g^2\tilde{\mu}^{\mathrm{T}}\tilde{\mu})$$

   for all $(x, \psi, \varphi, \tilde{\theta}) \neq 0$ such that

$$\varphi^{\mathrm{T}}W(\tilde{u}-\varphi) = \varphi^{\mathrm{T}}W(C_u x + (D_{u\psi} - I)\varphi + D_{u\psi}\psi + D_{u\theta}\tilde{\theta}) \geqslant 0$$

Proof

By the $\mathscr{S}$-procedure [28, p. 24], item 1 implies item 2 and if there exists at least one selection $(x^\star, \psi^\star, \varphi^\star, \tilde{\theta}^\star)$ such that

$$\varphi^{\star\mathrm{T}}W(C_u x^\star + (D_{u\psi} - I)\varphi^\star + D_{u\psi}\psi^\star + D_{u\theta}\tilde{\theta}^\star) > 0 \tag{36}$$

then item 2 implies item 1. To show that there exists a selection $(x^\star, \psi^\star, \varphi^\star, \tilde{\theta}^\star)$ that satisfies (36), we consider two cases. If $[C_u \ \ D_{u\psi} \ \ D_{u\theta}] = 0$ then (11) says $\tilde{u} = 0$, hence $u = \bar{u}$. Thus the anti-windup construction problem is non-existent since the closed-loop response is independent of Λ. In the case when $[C_u \;\; D_{u\psi} \;\; D_{u\theta}]\neq 0$, there exist $x^\star$, $\psi^\star$ and $\tilde{\theta}^\star$ such that $[C_u x^\star + D_{u\psi}\psi^\star + D_{u\theta}\tilde{\theta}^\star]\neq 0$. Then pick $\varphi^\star = \varepsilon[C_u x^\star + D_{u\psi}\psi^\star + D_{u\theta}\tilde{\theta}^\star]$ with ε sufficiently small to satisfy (36). □

Fact 3: Consider a finite dimensional continuous function h such that

$$|h(x_\delta, x+\delta_x, u+\delta_u, w) - h(x_\delta, x, u, w)| \leqslant g\left|[C_{p,\mu} \;\; D_{p,\mu u}]\begin{bmatrix}\delta_x \\ \delta_u\end{bmatrix}\right|$$

for all x_δ, x, δ_x, u, δ_u, and w. Then there exists a Lipschitz function $\hat{h}$ such that

$$h(x_\delta, x, u, w) = \hat{h}(x_\delta, C_{p,\mu}x + D_{p,\mu u}u, w)$$

and $|J_\mu \hat{h}(x_\delta, \mu, w)| \leqslant g$ for almost all (x_δ, μ, w).

Proof
Let $W_1 = [C_{p,\mu} \;\; D_{p,\mu u}]$. Let W_2^{T} be any full rank matrix that spans the null space of W_1. Let $M := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$. Define $\bar{h}$ according to

$$\bar{h}(x_\delta, \mu, \mu_\perp, w) := h\left(x_\delta, [I \;\; 0]M^\dagger\begin{bmatrix}\mu \\ \mu_\perp\end{bmatrix}, [0 \quad I]M^\dagger\begin{bmatrix}\mu \\ \mu_\perp\end{bmatrix}, w\right)$$

where $M^\dagger$ is the pseudo-inverse of M and it follows that $\bar{h}$ is continuous and $h(x_\delta, x, u, w) = \bar{h}(x_\delta, C_{p,\mu}x + D_{p,\mu u}u, 0, w)$. Lemma 6 implies $\bar{h}(x_\delta, \mu, \mu_\perp + \delta_{\mu\perp}, w) = \bar{h}(x_\delta, \mu, \mu_\perp, w)$ for all x_δ, μ, $\mu_\perp$, $\delta_{\mu\perp}$ and w. Hence $\bar{h}(x_\delta, \mu, \mu_\perp, w) = \bar{h}(x_\delta, \mu, 0, w) =: \hat{h}(x_\delta, \mu, w)$. Moreover, Lemma 6 also implies

$$|\hat{h}(x_\delta, \mu + \delta_\mu, w) - \hat{h}(x_\delta, \mu, w)| \leqslant g|\delta_\mu|$$

for all x_δ, μ, δ_μ and w. Hence $|J_\mu \hat{h}(x_\delta, \mu, w)| < g$ for almost all (x_δ, μ, w). □

Proof of Theorem 5
Necessity. Assume that for a given plant, controller and anti-windup compensator of order n_{aw}, quadratic robust performance recovery $(\tilde{\gamma}, \tilde{g})$ is guaranteed in the sense of Definition 6. Then, there exist $P = P^{\mathrm{T}} > 0$ and $\rho \geqslant 0$ such that item 2 in Fact 2 is satisfied with $\gamma = \tilde{\gamma}$ and $g = \tilde{g}$. Then, Fact 2 implies that there exists a constant $\sigma \geqslant 0$ that satisfies item 1 of Fact 2. Finally, by Fact 1, inequality (35) holds (with $\gamma = \tilde{\gamma}$, $g = \tilde{g}$ and for some $\sigma \geqslant 0$). Moreover, since all block diagonal terms in (35) must be negative definite, then $\sigma > 0$. Defining $Q := P^{-1}$, $U := \sigma^{-1}W^{-1}$ and $\tau := \rho^{-1}$ and premultiplying and postmultiplying (35) by the symmetric block diagonal matrix $\mathrm{diag}(Q, U, I, I, \tau I, \tau I)$, it follows that there exists $Q = Q^{\mathrm{T}} > 0$, $\delta := \sigma^{-1} > 0$, and $\tau \geqslant 0$ that satisfy (26a), as desired.

Sufficiency. If there exists a solution $(Q, \delta, \gamma, \tau)$ that satisfies (26), define $P := Q^{-1}$, $\sigma := \delta^{-1}$ and $\rho := \tau^{-1}$ and premultiply and postmultiply (26a) by the symmetric block diagonal matrix diag $(P, \sigma W, I, I, \rho I, \rho I)$. The resulting inequality guarantees (35) is satisfied (with $g = \tilde{g}$ and $\gamma = \tilde{\gamma}$) because $\tilde{\gamma} \geqslant \gamma$. Then, Facts 1 and 2 guarantee that P satisfies item 2 in Fact 2 with $\rho = \tau^{-1}$. Since the first inequality in item 2 of Fact 2 is satisfied for all $(x, \psi, \varphi, \tilde{\theta}) \neq 0$ with strict inequality, there exists a small enough $\varepsilon > 0$ such that inequality (12) in item 2 of Definition 6 is guaranteed.

Next, we prove well-posedness of the mismatch system (item 1 of Definition 6). Let a realization of the finite dimensional, forward complete, stable system $\theta(\sigma_0, \bar{x}_p, \bar{u}, w)$ with finite incremental gain g (from $\bar{\mu} := C_{p,\mu}\bar{x}_p + D_{p,\mu u}\bar{u}$ to θ) be given by

$$\begin{aligned}\dot{\bar{x}}_\delta &= f(\bar{x}_\delta, \bar{x}_p, \bar{u}, w)\\ \theta &= h(\bar{x}_\delta, \bar{x}_p, \bar{u}, w)\end{aligned}$$

where f and h are continuous. Then Lemma 6 and Fact 3 imply that a realization of the system $\theta(\sigma_0, \bar{x}_p, \bar{u}, w)$ is also

$$\begin{aligned}\dot{\bar{x}}_\delta &= f(\bar{x}_\delta, \bar{x}_p, \bar{u}, w)\\ \theta &= \hat{h}(\bar{x}_\delta, \bar{\mu}, w)\end{aligned}$$

where $|J_{\bar{\mu}}\hat{h}(x_\delta, \bar{\mu}, w)| \leqslant g$ for almost all $(x_\delta, \bar{\mu}, w)$. Based on this realization of the system $\theta(\sigma_0, \bar{x}_p, \bar{u}, w)$, the signal $\tilde{\theta}(\sigma_0, x_p, \tilde{u}, \bar{x}_p, \bar{u}, w)$ in (10e) is the output of the following system:

$$\begin{aligned}\dot{\bar{x}}_\delta &= f(\bar{x}_\delta, \bar{x}_p, \bar{u}, w)\\ \dot{x}_\delta &= f(x_\delta, x_p, \phi(\tilde{u} + \bar{u}), w)\\ \tilde{\theta} &= \hat{h}(x_\delta, \tilde{\mu} + \bar{\mu}, w) - \hat{h}(\bar{x}_\delta, \bar{\mu}, w)\end{aligned}$$

where $\hat{h}$ is globally Lipschitz and μ and $\bar{\mu}$ (defined by (2), (5), (10b)) are functions of x_p, $\tilde{u}$, $\bar{x}_p$ and $\bar{u}$. Thus the $\tilde{u}$ and $\tilde{\mu}$ equations of (11) (using (10)) are equivalent to

$$H(\tilde{u}, \tilde{\mu}, x, \bar{u}, \bar{\mu}, \bar{x}_\delta, x_\delta, w) := \begin{bmatrix} \tilde{u} - D_{u\psi}\psi(\tilde{u} + \bar{u}) - D_{u\theta}(\hat{h}(x_\delta, \tilde{\mu} + \bar{\mu}, w) - \hat{h}(\bar{x}_\delta, \bar{\mu}, w)) - C_u x \\ -D_{\mu\psi}\psi(\tilde{u} + \bar{u}) + \tilde{\mu} - D_{\mu\theta}(\hat{h}(x_\delta, \tilde{\mu} + \bar{\mu}, w) - \hat{h}(\bar{x}_\delta, \bar{\mu}, w)) - C_\mu x \end{bmatrix} = 0$$

where for given $(\bar{u}, \bar{\mu}, \bar{x}_\delta, x_\delta, w)$ the function $H(\cdot, \cdot, \cdot, \bar{u}, \bar{\mu}, \bar{x}_\delta, x_\delta, w)$ is globally Lipschitz. Then the Jacobian of H with respect to its first two arguments and its third wherever they exist can be written as

$$J_{\tilde{u},\tilde{\mu}}H = \begin{bmatrix} I - D_{u\psi}J_{\tilde{u}}\psi(\tilde{u} + \bar{u}) & -D_{u\theta}J_{\tilde{\mu}}\hat{h}(x_\delta, \tilde{\mu} + \bar{\mu}, w) \\ -D_{\mu\psi}J_{\tilde{u}}\psi(\tilde{u} + \bar{u}) & I - D_{\mu\theta}J_{\tilde{\mu}}\hat{h}(x_\delta, \tilde{\mu} + \bar{\mu}, w) \end{bmatrix}, \quad J_x H = \begin{bmatrix} -C_u \\ -C_\mu \end{bmatrix} \tag{37}$$

where $J_{\tilde{u}}\psi(\tilde{u} + \bar{u})$ denotes the Jacobian of ψ evaluated at $\tilde{u} + \bar{u}$ and $J_{\tilde{\mu}}\hat{h}(x_\delta, \tilde{\mu} + \bar{\mu}, w)$ is the Jacobian of the second argument of $\hat{h}$ evaluated at $(x_\delta, \tilde{\mu} + \bar{\mu}, w)$. By inequality (13), it follows that almost everywhere $J_{\tilde{u}}\psi(\tilde{u} + \bar{u})$ satisfies $\langle J_{\tilde{u}}\psi(\tilde{u} + \bar{u}), I - J_{\tilde{u}}\psi(\tilde{u} + \bar{u})\rangle_W \geqslant 0$. Then it follows that

$$J_{\tilde{u}}\psi(\tilde{u} + \bar{u}) \in \{J : -2J^{\mathrm{T}}WJ + WJ + J^{\mathrm{T}}W \geqslant 0\} \quad \text{for almost all } \tilde{u} \in \mathbb{R}^{n_u},\ \bar{u} \in \mathbb{R}^{n_u}$$

By loop transformation, we can state

$$J_{\tilde{u}}\left(\psi(\tilde{u} + \bar{u}) - \frac{\tilde{u}}{2}\right) \in \left\{J : J^{\mathrm{T}}WJ \leqslant \frac{1}{4}W\right\} \quad \text{for almost all } \tilde{u} \in \mathbb{R}^{n_u},\ \bar{u} \in \mathbb{R}^{n_u} \tag{38}$$

Additionally, the incremental sector property and Lemma 6 imply

$$\langle J_{\tilde{\mu}}\hat{h}(x_\delta, \tilde{\mu} + \bar{\mu}, w), J_{\tilde{\mu}}\hat{h}(x_\delta, \tilde{\mu} + \bar{\mu}, w)\rangle_I \leqslant g^2$$

almost everywhere, which we also write as

$$J_{\tilde{\mu}}(\hat{h}(x_\delta, \tilde{\mu}+\bar{\mu}, w)) \in \{J : J^{\mathrm{T}} J \leqslant g^2 I\} \quad \text{almost everywhere} \tag{39}$$

Then, combining (37)–(39), almost everywhere, the Jacobian of H with respect to its first two arguments satisfies $J_{\tilde{u},\tilde{\mu}} H \in J_{\tilde{u},\tilde{\mu}} \mathscr{H}$ where

$$J_{\tilde{u},\tilde{\mu}}\mathscr{H} := \left\{ \begin{bmatrix} I - \dfrac{D_{u\psi}}{2} - D_{u\psi}\tilde{D}_{J\tilde{u}} & -D_{u\theta}D_{J\tilde{\mu}} \\ -\dfrac{D_{\mu\psi}}{2} - D_{\mu\psi}\tilde{D}_{J\tilde{u}} & I - D_{\mu\theta}D_{J\tilde{\mu}} \end{bmatrix}, \tilde{D}_{J\tilde{u}} : \tilde{D}_{J\tilde{u}}^{\mathrm{T}} W \tilde{D}_{J\tilde{u}} \leqslant \frac{1}{4} W, D_{J\tilde{\mu}} : D_{J\tilde{\mu}}^{\mathrm{T}} D_{J\tilde{\mu}} \leqslant g^2 I \right\}$$

and where the set $J_{\tilde{u},\tilde{\mu}}\mathscr{H}$ is compact and convex because it is a linear function of a compact and convex set. Furthermore, since (26a) is negative definite, then the submatrix composed of the intersection of the second, fifth and sixth rows and columns is also negative definite. By applying a Schur complement to this submatrix, it follows that inequality (33) holds. By Lemma 5, each matrix in the set $J_{\tilde{u},\tilde{\mu}}\mathscr{H}$ is non-singular. Thus by applying Lemma 4, for any given $(\bar{u}, \bar{\mu}, \bar{x}_\delta, x_\delta, w)$ there exist globally Lipschitz functions (of x) $\eta_{\tilde{u},(\bar{u},\bar{\mu},\bar{x}_\delta,x_\delta,w)}$ and $\eta_{\tilde{\mu},(\bar{u},\bar{\mu},\bar{x}_\delta,x_\delta,w)}$ such that $\tilde{u} = \eta_{\tilde{u},(\bar{u},\bar{\mu},\bar{x}_\delta,x_\delta,w)}(x)$ and $\tilde{\mu} = \eta_{\tilde{\mu},(\bar{u},\bar{\mu},\bar{x}_\delta,x_\delta,w)}(x)$. Finally, the dynamics of the mismatch system (11) are governed by

$$\dot{x} = Ax + B_\psi(\psi(\bar{u}) + \varphi(\eta_{\tilde{u},(\bar{u},\bar{\mu},\bar{x}_\delta,x_\delta,w)}(x), \bar{u})) + B_\theta(\hat{h}(x_\delta, \eta_{\tilde{\mu},(\bar{u},\bar{\mu},\bar{x}_\delta,x_\delta,w)}(x) + \bar{\mu}, w) - \hat{h}(\bar{x}_\delta, \bar{\mu}, w)) \tag{40}$$

Since the unconstrained closed-loop system is well-posed and the system θ is forward complete, the signals $(\bar{u}, \bar{\mu}, \bar{x}_\delta, x_\delta, w)$ are well-defined. Thus, solutions of (40) are guaranteed to exist since the right-hand side is Lipschitz in x and measurable in $(\bar{u}, \bar{\mu}, \bar{x}_\delta, x_\delta, w)$ and the signals $(\bar{u}, \bar{\mu}, \bar{x}_\delta, w)$ are assumed to be well-defined. Hence, we conclude that the mismatch system is well-posed. □

7.2. *Proof of Theorem 6*

The following lemmas, proven in References [29–31], respectively, will be useful for the proof of Theorem 6.

Lemma 7 (Projection lemma [29, Lemma 3.1])
Given a symmetric matrix $\Psi \in \mathbb{R}^{m\times m}$ and two matrices G, H of column dimension m, consider the problem of finding some matrix Λ of compatible dimensions such that

$$\Psi + G^{\mathrm{T}}\Lambda^{\mathrm{T}} H + H^{\mathrm{T}}\Lambda G < 0 \tag{41}$$

Denote by W_G, W_H any matrices whose columns form bases of the null space of G and H, respectively. Then (41) is solvable for Λ if and only if

$$W_H^{\mathrm{T}}\Psi W_H < 0 \tag{42a}$$

$$W_G^{\mathrm{T}}\Psi W_G < 0 \tag{42b}$$

Lemma 8 (Packard [31])
Let $R, Z \in \mathbb{R}^{n\times n}$ be symmetric positive definite matrices. Then the two conditions

$$Z - R^{-1} \geqslant 0$$

$$\operatorname{rank}(Z - R^{-1}) \leqslant n_{aw}$$

hold if and only if there exist $N \in \mathbb{R}^{n \times n_{aw}}$ and $M \in \mathbb{R}^{n_{aw} \times n_{aw}}$, with $M = M^{\mathrm{T}} > 0$ such that

$$\begin{bmatrix} R & N \\ \star & M \end{bmatrix} > 0, \quad \begin{bmatrix} R & N \\ \star & M \end{bmatrix}^{-1} = \begin{bmatrix} Z & ? \\ \star & ? \end{bmatrix}$$

where the symbol '?' represents a matrix that we do not care to label.

Proof of Theorem 6

According to definitions (22)–(24) and (31), inequality (26a) coincides with inequality (25). We will apply Lemma 7 to inequality (25) to show that there exists a feasible solution $(Q, \Lambda, \delta, \gamma, \tau)$ to (26) if and only if the condition $\mathrm{MC}_{Rr}(\mathscr{P}, \mathscr{C}, n_{aw}, \tilde{g}, \tilde{\gamma})$ in Definition 1 is feasible. In particular, we will show that (42a) alone is equivalent to (16a) and that (42b) alone is equivalent to (16b). However, the matrix inequalities (42a) and (42b) are coupled since each has some components of the Q matrix. This coupling will be shown to be (16d)–(16g).

Condition (16a): According to (22b), (24b), H can be written as

$$H = \begin{bmatrix} 0 & 0 & I_{n_{aw}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_c} & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta_u^{\mathrm{T}} B_{p,u}^{\mathrm{T}} & D_{p,yu}^{\mathrm{T}} \Delta_y^{\mathrm{T}} B_{c,y}^{\mathrm{T}} & 0 & \Delta_u^{\mathrm{T}} & 0 & \Delta_u^{\mathrm{T}} D_{p,zu}^{\mathrm{T}} & 0 & \tilde{g} \Delta_u^{\mathrm{T}} D_{p,\mu u}^{\mathrm{T}} \end{bmatrix} \tag{43}$$

According to this special structure, a matrix that spans the null space of H is

$$W_H = \begin{bmatrix} I_{n_p} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -B_{p,u}^{\mathrm{T}} & 0 & -D_{p,zu}^{\mathrm{T}} & 0 & -\tilde{g} D_{p,\mu u}^{\mathrm{T}} \\ 0 & -I_{n_u} & 0 & 0 & 0 \\ 0 & 0 & I_{n_z} & 0 & 0 \\ 0 & 0 & 0 & I_{n_\theta} & 0 \\ 0 & 0 & 0 & 0 & I_{n_\mu} \end{bmatrix} \tag{44}$$

Indeed, by the assumption of well-posedness of the desirable closed-loop system, Δ_u is full rank, hence, according to (43), the dimension of the null space of H is necessarily $n_p + n_u + n_z + n_\theta + n_\mu$. Moreover, the rank of W_H is $n_p + n_u + n_z + n_\theta + n_\mu$ and it can be verified by computation that $HW_H = 0$.

Assume that, according to (20), the matrix Q is partitioned as follows:

$$\begin{bmatrix} R & N \\ \star & M \end{bmatrix} = Q, \quad \begin{bmatrix} R_{11} & R_{12} \\ \star & R_{22} \end{bmatrix} = R \tag{45}$$

then inequality (42a) and be computed explicitly based on Equations (44) and (24a) with (22a), (22c). After some computations it is seen that

$$W_H^{\mathrm{T}} \Psi W_H = \begin{bmatrix} R_{11}A_p^{\mathrm{T}} + A_p R_{11} & B_{p,u} & R_{11}C_{p,z}^{\mathrm{T}} & \tau B_{p,\theta} & \tilde{g} R_{11} C_{p,\mu}^{\mathrm{T}} \\ \star & -\gamma I & D_{p,zu}^{\mathrm{T}} & 0 & \tilde{g} D_{p,\mu u}^{\mathrm{T}} \\ \star & \star & -\gamma I & \tau D_{p,z\theta} & 0 \\ \star & \star & \star & -\tau I & 0 \\ \star & \star & \star & \star & -\tau I \end{bmatrix} \tag{46}$$

Condition (16b): According to (24c), the matrix G can be factorized as follows:

$$\begin{aligned} G = G_O \bar{T} &= [G_1 Q \;\; G_2 U \;\; G_2 \;\; 0 \;\; 0 \;\; 0] \\ &= \underbrace{[G_1 \;\; G_2 \;\; G_2 \;\; 0 \;\; 0 \;\; 0]}_{G_O} \underbrace{\mathrm{diag}(Q, U, I, I, I, I)}_{\bar{T}} \end{aligned}$$

where $G_O \in \mathbb{R}^{(n_{aw}+n_u)\times(n+n_u+n_u+n_z+n_\theta+n_\mu)}$ and $\bar{T} \in \mathbb{R}^{(n+n_u+n_u+n_z+n_\theta+n_\mu)\times(n+n_u+n_u+n_z+n_\theta+n_\mu)}$ and $n = n_\xi + n_{aw}$. Since $\bar{T}$ is non-singular (indeed, $Q > 0$ and $U > 0$ by assumption), we can write

$$W_G^{\mathrm{T}} \Psi W_G = W_G^{\mathrm{T}} \bar{T} \underbrace{\bar{T}^{-1} \Psi \bar{T}^{-1}}_{\Phi} \underbrace{\bar{T} W_G}_{W_{G_O}} = W_{G_O}^{\mathrm{T}} \Phi W_{G_O} \tag{47}$$

where W_{G_O} spans the null space of G_O and, according to the definitions $P = Q^{-1}$ and $U = W^{-1}$,

$$\Phi = \begin{bmatrix} PA_0 + A_0^{\mathrm{T}} P & PB_{\psi 0} + C_{u0}^{\mathrm{T}} W & PB_{\psi 0} & C_{z0}^{\mathrm{T}} & \tau PB_\theta & \tilde{g} C_{\mu 0}^{\mathrm{T}} \\ \star & WD_{u\psi 0} + D_{u\psi 0}^{\mathrm{T}} W - 2W & WD_{u\psi 0} & D_{z\psi 0}^{\mathrm{T}} & \tau WD_{u\theta} & \tilde{g} D_{\mu\psi 0}^{\mathrm{T}} \\ \star & \star & -\gamma I & D_{z\psi 0}^{\mathrm{T}} & 0 & \tilde{g} D_{\mu\psi 0}^{\mathrm{T}} \\ \star & \star & \star & -\gamma I & \tau D_{z\theta} & 0 \\ \star & \star & \star & \star & -\tau I & \tau \tilde{g} D_{\mu\theta}^{\mathrm{T}} \\ \star & \star & \star & \star & \star & -\tau I \end{bmatrix} \tag{48}$$

Based on (22b), we can write explicitly the entries of G_O as

$$G_O = [G_1 | G_2 \;\; G_2 \;\; 0 \;\; 0 \;\; 0] = \left[\begin{array}{ccc|ccccc} 0 & 0 & I_{n_{aw}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_u} & I_{n_u} & 0 & 0 & 0 \end{array}\right]$$

Hence, a matrix $W_{G_O} \in \mathbb{R}^{(n_p+n_c+n_{aw}+n_u+n_u+n_z+n_\theta+n_\mu)\times(n_p+n_c+n_u+n_z+n_\theta+n_\mu)}$ that spans the null space of G_O is

$$W_{G_O} := \left[\begin{array}{cccccc} I_{n_p} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{n_c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{n_u} & 0 & 0 & 0 \\ 0 & 0 & -I_{n_u} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_z} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_\mu} \end{array}\right] \tag{49}$$

Based on the following partition of the matrix P:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ \star & P_{22} \end{bmatrix} \tag{50}$$

we can compute explicitly inequality (42b) based on (47) and on the definitions (48) and (49), and substituting (22a) and (22c) into the entries of Φ. After some computations the following inequality is obtained:

$$W_G^{\mathrm{T}}\Psi W_G = \begin{bmatrix} P_{11}A_{CL} + A_{CL}^{\mathrm{T}}P_{11} & C_{CL,u}^{\mathrm{T}}W & C_{CL,z}^{\mathrm{T}} & \tau P_{11}B_{CL,\theta} & \tilde{g}C_{CL,\mu}^{\mathrm{T}} \\ \star & -\gamma I - 2W & 0 & \tau W D_{CL,u\theta} & 0 \\ \star & \star & -\gamma I & \tau D_{CL,z\theta} & 0 \\ \star & \star & \star & -\tau I & \tau\tilde{g}D_{CL,\mu\theta} \\ \star & \star & \star & \star & -\tau I \end{bmatrix} \tag{51}$$

By premultiplying and postmultiplying by the symmetric matrix diag(S, U, I, I, I) (where $S = P_{11}^{-1}$ and $U = W^{-1}$), then $W_G^{\mathrm{T}}\Psi W_G < 0$ if and only if

$$\begin{bmatrix} A_{CL}S + SA_{CL}^{\mathrm{T}} & SC_{CL,u}^{\mathrm{T}} & SC_{CL,z}^{\mathrm{T}} & \tau B_{CL,\theta} & \tilde{g}SC_{CL,\mu}^{\mathrm{T}} \\ \star & -\gamma U^2 - 2U & 0 & \tau D_{CL,u\theta} & 0 \\ \star & \star & -\gamma I & \tau D_{CL,z\theta} & 0 \\ \star & \star & \star & -\tau I & \tau\tilde{g}D_{CL,\mu\theta} \\ \star & \star & \star & \star & -\tau I \end{bmatrix} < 0 \tag{52}$$

Conditions (16f) (16g): Since $P = Q^{-1}$, and $S = P_{11}^{-1}$, then from (45) and (50) we have

$$Q = \begin{bmatrix} R & N \\ \star & M \end{bmatrix} > 0, \quad Q^{-1} = P = \begin{bmatrix} S^{-1} & P_{12} \\ \star & P_{22} \end{bmatrix} \tag{53}$$

which can be rewritten as follows:

$$\begin{bmatrix} R & N \\ \star & M \end{bmatrix} > 0, \quad \begin{bmatrix} R & N \\ \star & M \end{bmatrix}^{-1} = \begin{bmatrix} S^{-1} & P_{12} \\ \star & P_{22} \end{bmatrix} \tag{54}$$

By virtue of Lemma 8 expressions (54) are equivalent to

$$S^{-1} - R^{-1} \geqslant 0 \tag{55a}$$

$$\operatorname{rank}(S^{-1} - R^{-1}) \leqslant n_{aw} \tag{55b}$$

Premultiplying and postmultiplying the matrices in Equation (55b) by S and R, respectively, and performing a Cholesky factorization (see, e.g. Reference [32, p. 195]) on (55a), we get conditions (16f) and (16g), thus completing the proof of the necessity part of item 1. To prove the sufficiency in item 1, the above reasoning can be reversed. In particular, conditions (16f), (16g) imply (55), which by Lemma 8 imply the existence of M, N satisfying (53). Finally, by (16a) and (16b) with U constructed according to Step 3 of Procedure 1, inequalities (46) and (52) hold with $\gamma \leqslant \tilde{\gamma}$. Hence, $W_H^{\mathrm{T}} \Psi W_H < 0$ and $W_G^{\mathrm{T}} \Psi W_G < 0$ and by Lemma 7, inequality (41) holds too. This, in turn, implies that (26) is solvable.

Finally, we prove item 2 of the theorem. Since (25) coincides with (26) with the selection for Q (18)–(20), then provided the matrix Q satisfies expression (53) the proof of the sufficiency of item 1 can be followed verbatim to show that (25) is solvable with (18)–(20). To show that construction (18)–(20) for Q satisfies (53), note that by the formulae for the inversion of block matrices [33, p. 23], the upper left block of P needs to satisfy

$$P_{11} = S^{-1} = R^{-1} + R^{-1}N(M - N^{\mathrm{T}}R^{-1}N)^{-1}N^{\mathrm{T}}R^{-1}$$

which, when premultiplied and postmultiplied by R and substituting selection (19) for M, becomes

$$R + NN^{\mathrm{T}} = RS^{-1}R$$

which, by (18), is always satisfied.

8. CONCLUSIONS

In this paper we proposed an LMI-based robust construction for optimal linear anti-windup compensation for linear systems. We showed that the problem has a solution (of order equal to that of the plant) if and only if the plant and the unconstrained closed-loop are asymptotically stable. We also proved other interesting properties of this anti-windup design approach. By selecting the performance level as the $\mathscr{L}_2$ norm of the deviation of the actual response from the unconstrained response, we provided LMIs that characterize a family of compensators inducing optimal performance. Future work may include choosing from this family compensators that also optimize other performance measures. The effectiveness of the proposed construction is shown via a simulation example, by comparing the results to other existing techniques.

APPENDIX A: EXPLICIT EXPRESSIONS OF THE CLOSED-LOOP MATRICES

For completeness, we compute the explicit matrices appearing in systems (4), (8), (11) and (15).

Since the interconnection of the unconstrained closed-loop system is well-posed we know that $\Delta_y := (I - D_{p,yu}D_{c,y})^{-1}$ and $\Delta_u := (I - D_{c,y}D_{p,yu})^{-1}$ are well defined (namely the matrices in parentheses are non-singular). We consider the interconnection between the saturated plant (5) and the modified controller (6) where the function $\phi(u) = u - \psi(u)$ is replaced by $u - q$, and now q, v and w are considered external inputs. Then the y and u output equations in (5) and (6) can be written as

$$\begin{aligned} y &= \Delta_y(C_{p,y}x_p + D_{p,yu}C_c x_c - D_{p,yu}q + (D_{p,yu}D_{c,w} + D_{p,yw})w + D_{p,y\theta}\theta + D_{p,yu}v_2) \\ u &= \Delta_u(D_{c,y}C_{p,y}x_p + C_c x_c - D_{c,y}D_{p,yu}q + (D_{c,y}D_{p,yw} + D_{c,w})w + D_{c,y}D_{p,y\theta}\theta + v_2) \end{aligned} \tag{A1}$$

Equations (A1) can be used to rewrite the remaining equations in (5) and (6) as

$$\begin{aligned} \dot{x}_p &= B_{p,u}\Delta_u(C_c x_c + D_{c,y}C_{p,y}x_p - D_{c,y}D_{p,yu}q + (D_{c,y}D_{p,yw} + D_{c,w})w + D_{c,y}D_{p,y\theta}\theta + v_2) \\ &\quad + A_p x_p - B_{p,u}q + B_{p,w}w + B_{p,\theta}\theta \\ \dot{x}_c &= B_{c,y}\Delta_y(C_{p,y}x_p + D_{p,yu}C_c x_c + D_{p,yu}v_2 - D_{p,yu}q + (D_{p,yu}D_{c,w} + D_{p,yw})w + D_{p,y\theta}\theta) \\ &\quad + A_c x_c + B_{c,w}w + v_1 \\ z &= D_{p,zu}\Delta_u(C_c x_c + D_{c,y}C_{p,y}x_p - D_{c,y}D_{p,yu}q + (D_{c,y}D_{p,yw} + D_{c,w})w + v_2 + D_{c,y}D_{p,y\theta}\theta) \\ &\quad + C_{p,z}x_p - D_{p,zu}q + D_{p,zw}w + D_{p,z\theta}\theta \\ \mu &= D_{p,\mu u}\Delta_u(D_{c,y}C_{p,y}x_p + C_c x_c - D_{c,y}D_{p,yu}q + (D_{c,y}D_{p,yw} + D_{c,w})w + D_{c,y}D_{p,y\theta}\theta + v_2) \\ &\quad + C_{p,\mu}x_p - D_{p,\mu u}q \end{aligned} \tag{A2}$$

Based on these equations it becomes clear that the saturated plant and modified controller can be written as the system in (8) with

$$A_{CL} = \begin{bmatrix} A_p + B_{p,u}\Delta_u D_{c,y}C_{p,y} & B_{p,u}\Delta_u C_c \\ B_{c,y}\Delta_y C_{p,y} & A_c + B_{c,y}\Delta_y D_{p,yu}C_c \end{bmatrix}, \quad B_{CL,\psi} = \begin{bmatrix} -B_{p,u}\Delta_u \\ -B_{c,y}\Delta_y D_{p,yu} \end{bmatrix}$$

$$B_{CL,w} = \begin{bmatrix} B_{p,w} + B_{p,u}\Delta_u(D_{c,y}D_{p,yw} + D_{c,w}) \\ B_{c,w} + B_{c,y}\Delta_y(D_{p,yu}D_{c,w} + D_{p,yw}) \end{bmatrix}, \quad B_{CL,v} = \begin{bmatrix} 0 & B_{p,u}\Delta_u \\ I & B_{c,y}\Delta_y D_{p,yu} \end{bmatrix}$$

$$B_{CL,\theta} = \begin{bmatrix} B_{p,\theta} + B_{p,u}\Delta_u D_{c,y}D_{p,y\theta} \\ B_{c,y}\Delta_y D_{p,y\theta} \end{bmatrix}$$

$$\begin{aligned} C_{CL,z} &= [D_{p,zu}\Delta_u D_{c,y}C_{p,y} + C_{p,z} \quad D_{p,zu}\Delta_u C_c], \quad D_{CL,z\psi} = -D_{p,zu}\Delta_u \\ D_{CL,zw} &= D_{p,zw} + D_{p,zu}\Delta_u(D_{c,y}D_{p,yw} + D_{c,w}), \quad D_{CL,zv} = [0 \;\; D_{p,zu}\Delta_u] \\ D_{CL,z\theta} &= D_{p,z\theta} + D_{p,zu}\Delta_u D_{c,y}D_{p,y\theta} \end{aligned} \tag{A3}$$

$$\begin{aligned} C_{CL,u} &= [\Delta_u D_{c,y}C_{p,y} \quad \Delta_u C_c], \quad D_{CL,u\psi} = I - \Delta_u \\ D_{CL,uw} &= \Delta_u(D_{c,w} + D_{c,y}D_{p,yw}), \quad D_{CL,uv} = [0 \quad \Delta_u] \\ D_{CL,u\theta} &= \Delta_u D_{c,y}D_{p,y\theta} \end{aligned}$$

$$C_{CL,\mu} = [C_{p,\mu} + D_{p,\mu u}\Delta_u D_{c,y} C_{p,y} \quad D_{p,\mu u}\Delta_u C_c], \quad D_{CL,\mu\psi} = -D_{p,\mu u}\Delta_u$$
$$D_{CL,\mu w} = D_{p,\mu u}\Delta_u(D_{c,w} + D_{c,y}D_{p,yw}), \quad D_{CL,\mu v} = [0 \;\; D_{p,\mu u}\Delta_u]$$
$$D_{CL,\mu\theta} = D_{p,\mu u}\Delta_u D_{c,y} D_{p,y\theta}$$

If $\phi \equiv I$ and $v \equiv 0$ and $q \equiv 0$, then the system in (8) produces the same trajectories as the system in (4). Thus the matrices in (A3) are also matrices that appear in (4).

As an intermediate step to construct the system in (11) define a system with state $\tilde{\xi} := \xi - \bar{\xi}$, define $\tilde{\theta}$ according to (10e) and continue to use the definition $q := \psi(u)$. This system can be written by taking the difference between the systems in (4) and in (8) resulting in

$$\mathscr{H} \begin{cases} \dot{\tilde{\xi}} = A_{CL}\tilde{\xi} + B_{CL,\psi}q + B_{CL,v}v + B_{CL,\theta}\tilde{\theta} \\ \tilde{u} = C_{CL,u}\tilde{\xi} + D_{CL,u\psi}q + D_{CL,uv}v + D_{CL,u\theta}\tilde{\theta} \\ \tilde{z} = C_{CL,z}\tilde{\xi} + D_{CL,z\psi}q + D_{CL,zv}v + D_{CL,z\theta}\tilde{\theta} \\ \tilde{\mu} = C_{CL,\mu}\tilde{\xi} + D_{CL,\mu\psi}q + D_{CL,\mu v}v + D_{CL,\mu\theta}\tilde{\theta} \end{cases} \tag{A4}$$

System (A4) can be combined with the anti-windup compensator in (9) to construct the system in (11) with matrices

$$A = \begin{bmatrix} A_{CL} & B_{CL,v}C_{aw} \\ 0 & A_{aw} \end{bmatrix}, \quad B_\psi = \begin{bmatrix} B_{CL,\psi} + B_{CL,v}D_{aw} \\ B_{aw} \end{bmatrix}, \quad B_\theta = \begin{bmatrix} B_{CL,\theta} \\ 0 \end{bmatrix}$$
$$C_u = [C_{CL,u} \quad D_{CL,uv}C_{aw}], \quad D_{u\psi} = (D_{CL,u\psi} + D_{CL,uv}D_{aw}), \quad D_{u\theta} = D_{CL,u\theta} \tag{A5}$$
$$C_z = [C_{CL,z} \quad D_{CL,zv}C_{aw}], \quad D_{z\psi} = (D_{CL,z\psi} + D_{CL,zv}D_{aw}), \quad D_{z\theta} = D_{CL,z\theta}$$
$$C_\mu = [C_{CL,\mu} \;\; D_{CL,\mu v}C_{aw}], \quad D_{\mu\psi} = (D_{CL,\mu\psi} + D_{CL,\mu v}D_{aw}), \quad D_{\mu\theta} = D_{CL,\mu\theta}$$

ACKNOWLEDGEMENTS

Grimm and Teel were supported by NSF under Grants ECS-9988813 and ECS-0324679, and AFOSR under Grants F49620-00-1-0106 and F49620-03-1-0203. Zaccarian was supported in part by ASI and MIUR through PRIN project MATRICS and FIRB project TIGER.

REFERENCES

1. Lozier JC. A steady-state approach to the theory of saturable servo systems. *IRE Transactions on Automatic Control* 1956; **1**:19–39.
2. Kramer LC, Jenkins KW. A new technique for preventing direct digital control windup. In *Joint Automatic Control Conference*, St. Louis, MO, August 1971; 571–577.
3. Hanus R. Antiwindup and bumpless transfer: a survey. In *Proceedings of the 12th IMACS World Congress*, vol. 2, Paris, France, July 1988; 59–65.
4. Åström KJ, Rundqwist L. Integrator windup and how to avoid it. In *ACC*, vol. 2, Pittsburgh, PA, June 1989; 1693–1698.
5. Kothare MV, Campo PJ, Morari M, Nett N. A unified framework for the study of anti-windup designs. *Automatica* 1994; **30**(12):1869–1883.
6. Doyle JC, Smith RS, Enns DF. Control of plants with input saturation nonlinearities. In *ACC*, Minneapolis, MN, U.S.A., June 1987; 1034–1039.

7. Gilbert EG, Kolmanovsky I, Tan KT. Discrete-time reference governors and the nonlinear control of systems with state and control constraints. *International Journal of Robust and Nonlinear Control* 1995; **5**(5):487–504.
8. Zheng A, Kothare MV, Morari M. Anti-windup design for internal model control. *International Journal of Control* 1994; **60**(5):1015–1024.
9. Miyamoto A, Vinnicombe G. Robust control of plants with saturation nonlinearity based on coprime factor representation. In *36th CDC*, Kobe, Japan, December 1996; 2838–2840.
10. Edwards C, Postlethwaite I. An anti-windup scheme with closed-loop stability considerations. *Automatica* 1999; **35**(4):761–765.
11. Mulder EF, Kothare MV, Morari M. Multivariable anti-windup controller synthesis using linear matrix inequalities. *Automatica* September 2001; **37**(9):1407–1416.
12. Shamma JS. Anti-windup via constrained regulation with observers. *Systems and Control Letters* 2000; **40**: 1869–1883.
13. Grimm G, Hatfield J, Postlethwaite I, Teel AR, Turner MC, Zaccarian L. Anti-windup for stable linear systems with input saturation: an LMI-based synthesis. *IEEE Transactions on Automatic Control* 2003; **48**(9):1509–1525.
14. Saeki M, Wada N. Synthesis of a static anti-windup compensator via linear matrix inequalities. *International Journal of Robust and Nonlinear Control* 2002; **12**:927–953.
15. Cao Y-Y, Lin Z, Ward DG. An anti-windup approach to enlarging the domain of attraction for linear systems subject to actuator saturation. *IEEE Transactions on Automatic Control* 2002; **47**(1):140–145.
16. Gomes da Silva Jr JM, Tarbouriech S, Reginatto R. Analysis of regions of stability for linear systems with saturating inputs through an anti-windup scheme. In *Proceedings of the IEEE Conference on Control Applications*, Glasgow, Scotland, September 2002; 1106–1111.
17. Teel AR, Kapoor N. The $\mathscr{L}_2$ anti-windup problem: its definition and solution. In *Proceedings of the 4th ECC*, Brussels, Belgium, July 1997.
18. Scherer C. The Riccati inequality and state-space H_∞-optimal control. *Ph.D. Thesis*, Universitat Wurzburg, Germany, 1990.
19. Kulkarni VV, Safonov MG. Incremental positivity non-preservation by stability multipliers. In *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, FL, U.S.A., December 2001; 33–38.
20. Chen X, Wen JT. Robustness analysis of LTI systems with structured incrementally sector bounded nonlinearities. In *Proceedings of the American Control Conference*, Seattle, WA, U.S.A., June 1995; 3883–3887.
21. Khalil HK. *Nonlinear Systems* (2nd edn). Prentice-Hall: Englewood Cliffs, New Jersey, U.S.A., 1996.
22. Kapoor N, Teel AR, Daoutidis P. An anti-windup design for linear systems with input saturation. *Automatica* 1998; **34**(5):559–574.
23. Teel AR. Anti-windup for exponentially unstable linear systems. *International Journal of Robust and Nonlinear Control* 1999; **9**:701–716.
24. Zaccarian L, Teel AR. A common framework for anti-windup, bumpless transfer and reliable designs. *Automatica* 2002; **38**(10):1735–1744.
25. Crawshaw S, Vinnicombe G. Anti-windup synthesis for guaranteed $\mathscr{L}_2$ performance. In *Proceedings of the Conference on Decision and Control*, Sydney, Australia, December 2000; 1063–1068.
26. Morari M, Zafiriou E. *Robust Process Control*. Prentice Hall: Englewood Cliffs, NJ, 1989.
27. Clarke FH. *Optimization and Nonsmooth Analysis*. Society for Industrial and Applied Mathematics: Philadelphia, 1990.
28. Boyd S, El Ghaoui L, Feron E, Balakrishnan V. *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics: Philadelphia, 1994.
29. Gahinet P, Apkarian P. A linear matrix inequality approach to $\mathscr{H}_\infty$ control. *International Journal of Robust and Nonlinear Control* 1994; **4**:421–448.
30. Iwasaki T, Skelton RE. All controllers for the general $\mathscr{H}_\infty$ control problem: LMI existence conditions and state space formulas. *Automatica* 1994; **30**(8):1307–1317.
31. Packard A. Gain scheduling via linear fractional transformations. *Systems and Control Letters* 1994; **22**(2):79–92.
32. Strang G. *Linear Algebra and its Applications*. Harcourt Brace Jovanovich: New York, 1998.
33. Zhou K, Doyle JC, Glover K. *Robust and Optimal Control*. Prentice Hall: Englewood Cliffs, NJ, 1995.

