Lemma 9: If (1) holds, then

$$\varphi_n(\lambda) = O\left(\frac{n}{\Delta_n}\right).$$

Proof: From (1) and $|i-j|\Delta_n \le |t_i - t_j|$, we get $|k(\lambda + |t_i - t_j|)| \le \alpha e^{-\beta\lambda} e^{-\beta|t_i-t_j|} \le \alpha e^{-\beta\lambda} e^{-|i-j|\beta\Delta_n}$. Hence

$$\begin{aligned}\left|\sum_{i=1}^{n}\sum_{j=1}^{n} k(\lambda + |t_i - t_j|)\right| &\le \alpha e^{-\beta\lambda}\sum_{i=1}^{n}\sum_{j=1}^{n} e^{-|i-j|\beta\Delta_n}\\ &= \alpha e^{-\beta\lambda}\sum_{i=1}^{n}(n-i)e^{-\beta i\Delta_n}.\end{aligned}$$

Since, for $|q|<1$, $\sum_{i=1}^{n}(n-i)q^i \le nq/(1-q)$, we obtain $\sum_{i=1}^{n}(n-i)e^{-\beta i\Delta_n} \le n\gamma(\Delta_n)$, where $\gamma(\Delta) = e^{-\beta\Delta}/(1-e^{-\beta\Delta})$. As $0 < \gamma(\Delta) \le 1/\beta\Delta$, we find that the quantity is of order $O(n/\Delta_n)$. Moreover, because $|k(t)| \le \alpha$

$$\begin{aligned}\left|\sum_{i=1}^{n}\sum_{j=1}^{n} k(\lambda - |t_i - t_j|)\right| &\le \sum_{i=1}^{n}\sum_{j=1}^{n} k(\lambda - |t_i - t_j|) I_{\{j;|t_i-t_j|\le\lambda\}}(j)\\ &\le \alpha\sum_{i=1}^{n}\sum_{j=1}^{n} I_{\{j;|t_i-t_j|\le\lambda\}}(j)\end{aligned}$$

where $I_A(j) = 1$ for $j \in A$ and $I_A(j) = 0$, otherwise. As

$$\sum_{j=1}^{n} I_{\{j;|t_i-t_j|\le\lambda\}}(j) \le \text{the number of } t_j'\text{s in the interval}$$
$$[t_i - \lambda, t_i + \lambda] \le \frac{2\lambda}{\Delta_n} + 1$$

the quantity is also of order $O(n/\Delta_n)$ which completes the proof. ■

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Robust Nonlinear Control Associating Robust Feedback Linearization and H_∞ Control

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***Abstract*—In this note, a robust nonlinear controller for a nonlinear system subject to model uncertainties is proposed. Such a controller associates a "robust feedback linearization" with a robust linear H_∞ controller. The robust feedback linearization exactly transforms the nonlinear system into a linear system equal to the linear approximation of the original nonlinear system around a nominal operating point. The robustness of the resulting overall nonlinear controller is proved by theoretical arguments and illustrated through an application example, the control of a magnetic bearing system.**

***Index Terms*—Feedback linearization, magnetic bearing system, nonlinear coprime factorization, nonlinear systems, robust control.**

I. INTRODUCTION

A common method for the control of nonlinear systems is to use a linear controller calculated for the linear approximation of the nonlinear system around an operating point. This method is largely used due to the fact that for linear systems the choice of control techniques is wider and the design can be done in a more systematic way than in the nonlinear case. Nevertheless, this kind of control works in general only in a small neighborhood of the operating point, since this is the region where the linear approximation is valid. Thus, when the system is far from this point, the linear controller will generally not behave as desired.

In this context, the feedback linearization seems to be interesting, because in this case the nonlinear system is exactly transformed into a linear system (which is valid for all the operating region) and only then the linear controller is applied. Therefore, a controller associating feedback linearization and a linear controller will work in any point, not only in a small neighborhood of the operating point. However, when using the classical feedback linearization [1], the linearized system obtained is in the Brunovsky form, a non-robust form whose dynamics is completely different from that of the original system, and the resulting closed-loop may be non robust in the presence of uncertainties.

A new form of feedback linearization, called robust feedback linearization, was proposed in [2]. This method gives a linearizing control law that transforms the nonlinear system into its linear approximation around an operating point. Thus, it causes only a small transformation in the natural behavior of the system, which is desired in order to obtain robustness. Hence, this method combines the advantages of the aforementioned methods and eliminates their drawbacks.

In this note, the robustness of the performance obtained with a robust feedback linearization associated with a McFarlane–Glover H_∞ controller [3], [4] is demonstrated theoretically and illustrated through an application example (whereas in [2] only the robustness of the stability was considered).

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The note is organized as follows. In Section II, the classical feedback linearizing method is briefly reviewed and the robust one is explained. In Section III, the dual case (normalized right coprime factorization) of the McFarlane–Glover method for the design of an H_∞ controller is presented. The robustness properties of the robust nonlinear controller (which associates the robust feedback linearization with a McFarlane–Glover H_∞ controller) are demonstrated in Section IV. The theory is illustrated by the application of the three aforementioned methods (linear control and classical and robust feedback linearization) to a magnetic bearing system, in Section V, where the design of the controllers is presented and simulation results are given.

A preliminary version of this note has been presented at an IEEE conference [5].

II. Feedback Linearization

Consider the nonlinear system with n states and m inputs described by the state–space equation

$$\dot{x} = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^m$ is the control input, and $f(x), g_1(x), \ldots, g_m(x)$ are smooth vector fields defined on an open subset of $\mathbb{R}^n$. Suppose that this system satisfies the well-known conditions for feedback linearization [1]: there exists a vector $\lambda(x) = [\,\lambda_1(x) \quad \cdots \quad \lambda_m(x)\,]^T$, formed by functions $\lambda_i(x)$ with relative degree r_i such that $r_1 + \cdots + r_m = n$, and the decoupling matrix of this system, given by

$$M(x) = \begin{bmatrix} L_{g_1}L_f^{r_1-1}\lambda_1(x) & \cdots & L_{g_m}L_f^{r_1-1}\lambda_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1}L_f^{r_m-1}\lambda_m(x) & \cdots & L_{g_m}L_f^{r_m-1}\lambda_m(x) \end{bmatrix} \tag{2}$$

is invertible. The output of system (1) is chosen as $y(x) = \lambda(x)$.

The objective here is to linearize this system by feedback in a neighborhood of an operating point x_0 chosen, without loss of generality, as $x_0 = 0$. Two different forms of feedback linearization are presented next. It is assumed that the state is available for control purposes.

A. Classical Feedback Linearization

The classical feedback linearization [1] is accomplished by using a linearizing control law of the form $u_c(x, w) = \alpha_c(x) + \beta_c(x)w$, where w is a linear control, and a diffeomorphism $x_c = \phi_c(x)$, with $\alpha_c(x) = -M^{-1}(x)[\,L_f^{r_1}\lambda_1(x) \quad \cdots \quad L_f^{r_m}\lambda_m(x)\,]^T$, $\beta_c(x) = M^{-1}(x)$, $\phi_c^T(x) = [\,\phi_{c_1}^T(x) \quad \cdots \quad \phi_{c_m}^T(x)\,]$, and $\phi_{c_i}^T(x) = [\,\lambda_i(x) \quad L_f\lambda_i(x) \quad \cdots \quad L_f^{r_i-1}\lambda_i(x)\,]$.

The linearized system is

$$\dot{x}_c = A_c x_c + B_c w \tag{3}$$

where A_c and B_c are the matrices of the Brunovsky canonical form [1]. It is a well-known fact that the Brunovsky form is extremely vulnerable to uncertainties and that the matrix A_c is ill conditioned [6]. Furthermore, this system has infinite equilibrium points and no physical meaning (two different nonlinear systems with the same dimensions will have the same Brunovsky form). For all these reasons, it is difficult to obtain a robust controller when using the classical feedback linearization.

B. Robust Feedback Linearization

The main difference between the robust feedback linearization [2] and the classical one is that the linearized system has the form

$$\dot{x}_r = A_r x_r + B_r v \tag{4}$$

with $A_r = \partial_x f(0)$ and $B_r = g(0)$, which corresponds to the linear approximation of the nonlinear system (1). In this case, little transformation is performed on the original nonlinear system, which makes it more probable that the properties of the linear design (including robustness) still hold for the nonlinear closed-loop.

The robust feedback linearization is accomplished by using a linearizing control law of the form $u(x, v) = \alpha(x) + \beta(x)v$, where v is a linear control, and a diffeomorphism $x_r = \phi(x)$, with $\alpha(x) = \alpha_c(x) + \beta_c(x)LT^{-1}\phi_c(x)$, $\beta(x) = \beta_c(x)R^{-1}$, $\phi(x) = T^{-1}\phi_c(x)$, $L = -M(0)\partial_x\alpha_c(0)$, $T = \partial_x\phi_c(0)$, and $R = M^{-1}(0)$. The functions $\alpha(x)$, $\beta(x)$, and $\phi(x)$ satisfy

$$\partial_x\alpha(0) = 0, \beta(0) = I, \text{and } \partial_x\phi(0) = I. \tag{5}$$

It is important to notice that applying the same linear control v to the nonlinear system (1), i.e., using $u = v$, corresponds to the first method discussed in the introduction (a linear control directly applied to the nonlinear system). As mentioned before, such control works only in a small neighborhood of the operating point x_0 and will not provide good results far from it.

III. H_∞ Robust Stabilization

In this section, the method used to calculate the linear controllers for the linearized systems (3) and (4) is presented.

The linear H_∞ controllers are obtained using the McFarlane–Glover method [3] with loop-shaping [4], i.e., applying the method of H_∞ robust stabilization to a system that has been previously "shaped" by a precompensator W_1 and/or a postcompensator W_2 to obtain a better performance. The McFarlane–Glover method [3], [4] deals with the H_∞ robust stabilization problem of perturbed linear plants, given by a normalized left coprime factorization. The case of a normalized right coprime factorization (needed for the nonlinear analysis in Section IV) is recalled below to clarify the theory in Section IV.

Lemma 1: Consider a strictly proper[1] linear system, with transfer matrix $G(s)$, and the augmented system $G_s(s) = W_2(s)G(s)W_1(s)$, assumed to be controllable and observable, where $W_1(s)$ and $W_2(s)$ are weighting matrices that "shape" the frequency response of $G(s)$. This augmented system has a normalized right coprime factorization given by $G_s(s) = N_r(s)M_r^{-1}(s)$ and a family of perturbed plants, also controllable and observable, with transfer matrices

$$G_p(s) = (N_r(s) + \Delta_{N_r}(s))(M_r(s) + \Delta_{M_r}(s))^{-1} \tag{6}$$

where $\Delta_{M_r}(s)$ and $\Delta_{N_r}(s)$ are stable unknown transfer matrices which represent the system uncertainty. A controller $K_s(s)$ such that

$$\|M_r^{-1}(s)(I - K_s(s)G_s(s))^{-1}[\,K_s(s) \quad I\,]\|_\infty \le \gamma \tag{7}$$

[1]Only the strictly proper case is of interest in this note, since it deals with systems whose output depends directly only on the state of the system, not on its input.

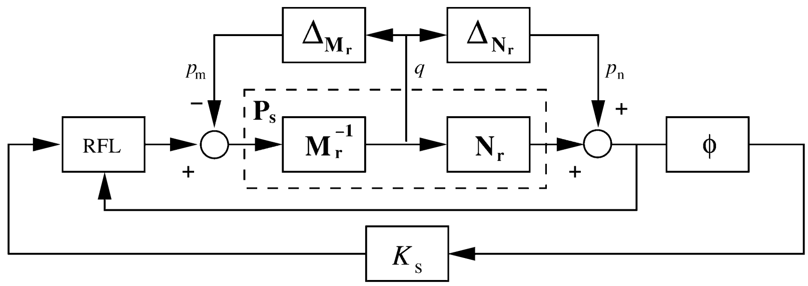

Fig. 1. Closed-loop system.

for a given $\gamma > \gamma_{min}$, guarantees that the closed-loop system is stable for all uncertainties such that

$$\left\| \begin{bmatrix} \Delta_{N_r}(s) \\ \Delta_{M_r}(s) \end{bmatrix} \right\|_\infty < \frac{1}{\gamma}. \tag{8}$$

Such a controller is given by

$$K_s = \left[\begin{array}{c|c} A_s - ZC_s^T C_s + \gamma^2 B_s B_s^T X L^{-1} & ZC_s^T \\ \hline \gamma^2 B_s^T X L^{-1} & 0 \end{array} \right] \tag{9}$$

with $L = (1-\gamma^2)I + ZX$, where (A_s, B_s, C_s) is a minimal state–space realization of $G_s(s)$ and Z and X are the unique positive–definite solutions of the algebraic Riccati equations

$$A_s Z + Z A_s^T - Z C_s^T C_s Z + B_s B_s^T = 0 \tag{10}$$
$$A_s^T X + X A_s - X B_s B_s^T X + C_s^T C_s = 0. \tag{11}$$

The controller for $G(s)$ is given by $K(s) = W_1(s) K_s(s) W_2(s)$.

Proof: The proof is obtained straightforwardly by "dualizing" the one given in [3] for the case of a left coprime factorization.

Remark 1: As shown in [7, Cor. 18.8], K_s is also an H_∞ controller for the normalized left coprime factorization of $G_s(s)$, but its above state space representation (9) is dual to the one usually obtained in the latter case.

IV. Robust Nonlinear Control

In this section, it is proved that the robustness properties of the controller obtained by the method of McFarlane–Glover for the linearized system are kept when this controller is applied, together with the robust feedback linearization, to the nonlinear system.

This demonstration uses the concept of "local W-stability," which allows the analysis of the local input-output stability of a nonlinear system by the means of a local version of the Small Gain Theorem. In what follows, W^n denotes the Sobolev space of functions $x : \mathbb{R} \to \mathbb{R}^n$ which are absolutely continuous and such that x and $\dot{x}$ belongs to L_2. This space is equipped with the norm $\|x\|_W = (\|x\|_2 + \|\dot{x}\|_2)^{1/2}$, where $\|\cdot\|_2$ is the L_2-norm. (See [8] and [9] for more details.)

Definition 1[Local W-Stability]: Let be $\mathbf{H} : W^n \to W^m$ and

$$\mathcal{K}_l = \{k > 0, \exists \epsilon > 0 : \|\mathbf{H}u\|_W \leq k\|u\|_W \text{ whenever } u \in W^n \text{ is such that} \|u\|_W < \epsilon\}. \tag{12}$$

If $\mathcal{K}_l$ is nonempty, then $\mathbf{H}$ is said to be locally W-stable and $\gamma_{W_l}(\mathbf{H}) = \inf(\mathcal{K}_l)$ is called the local W-gain of $\mathbf{H}$. If $\mathcal{K}_l$ is empty, we set $\gamma_{W_l}(\mathbf{H}) = +\infty$.

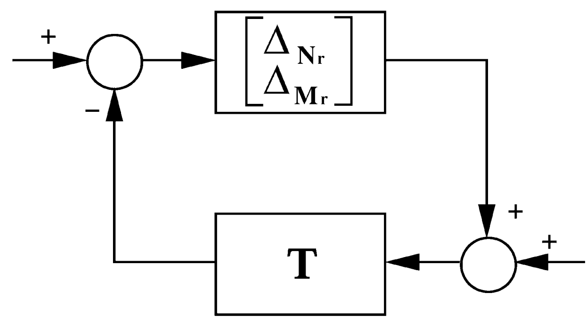

Fig. 2. Standard form for closed-loop system.

Property 1: Let $\mathbf{H}$ be a nonlinear system with state x, exponentially stable equilibrium $x_0 = 0$, and linear approximation $\mathbf{H}_1$ around $x_0 = 0$. Assuming that $\mathbf{H}_1$ is detectable and stabilizable and has a transfer matrix H, then the local W-gain of $\mathbf{H}$ around $x_0 = 0$ is such that $\gamma_{W_1}(\mathbf{H}) = \|H\|_\infty$.

To begin with, consider the loop-shaping of the nonlinear system $\mathbf{P}$, given by $\dot{x} = f(x) + g(x)u$, $y = x$, by the nonlinear weighting functions $\mathbf{W}_i$, given by $\dot{x}_{w_i} = f_{w_i}(x_{w_i}) + g_{w_i}(x_{w_i})u_{w_i}$, $y_{w_i} = h_{w_i}(x_{w_i}) + l_{w_i}(x_{w_i})u_{w_i}$, with $f_{w_i}(0) = h_{w_i}(0) = 0$, $i = 1, 2$. Then, the augmented nonlinear system $\mathbf{P}_s$ is

$$\underbrace{\begin{bmatrix} \dot{x}_{w_1} \\ \dot{x} \\ \dot{x}_{w_2} \end{bmatrix}}_{\dot{x}_s} = \underbrace{\begin{bmatrix} f_{w_1}(x_{w_1}) \\ f(x) + g(x) h_{w_1}(x_{w_1}) \\ f_{w_2}(x_{w_2}) + g_{w_2}(x_{w_2}) x \end{bmatrix}}_{f_s(x_s)} + \underbrace{\begin{bmatrix} g_{w_1}(x_{w_1}) \\ g(x) l_{w_1}(x_{w_1}) \\ 0 \end{bmatrix}}_{g_s(x_s)} u_s$$
$$y_s = \underbrace{h_{w_2}(x_{w_2}) + l_{w_2}(x_{w_2}) x}_{h_s(x_s)}. \tag{13}$$

Suppose that this system has a normalized right coprime factorization [10] and that it is subject to uncertainties $\mathbf{\Delta_{N_r}}$ and $\mathbf{\Delta_{M_r}}$ as shown in Fig. 1.

The perturbed nonlinear system has a normalized right coprime factorization given by [10]

$$\mathbf{N}_r : \dot{x}_s = \tilde{f}_s(x_s) + g_s(x_s) q$$
$$y_s = h_s(x_s) + p_n \tag{14}$$
$$\mathbf{M}_r^{-1} : \dot{x}_s = f_s(x_s) + g_s(x_s)(u_s - p_m)$$
$$q = (u_s - p_m) - \tilde{h}_s(x_s) \tag{15}$$

with $\tilde{f}_s(x_s) = f_s(x_s) - g_s(x_s)(g_s(x_s))^T (\partial_{x_s} V(x_s))^T$ and $\tilde{h}_s(x_s) = -(g_s(x_s))^T (\partial_{x_s} V(x_s))^T$ where $V(x_s)$ is a smooth proper

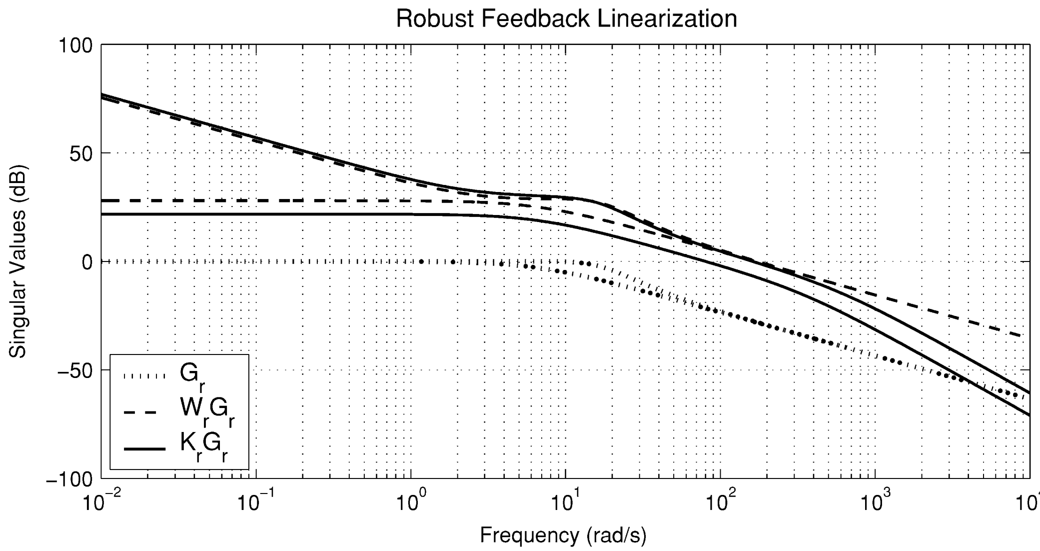

Fig. 3. Singular value plots of G_r, W_rG_r, and K_rG_r.

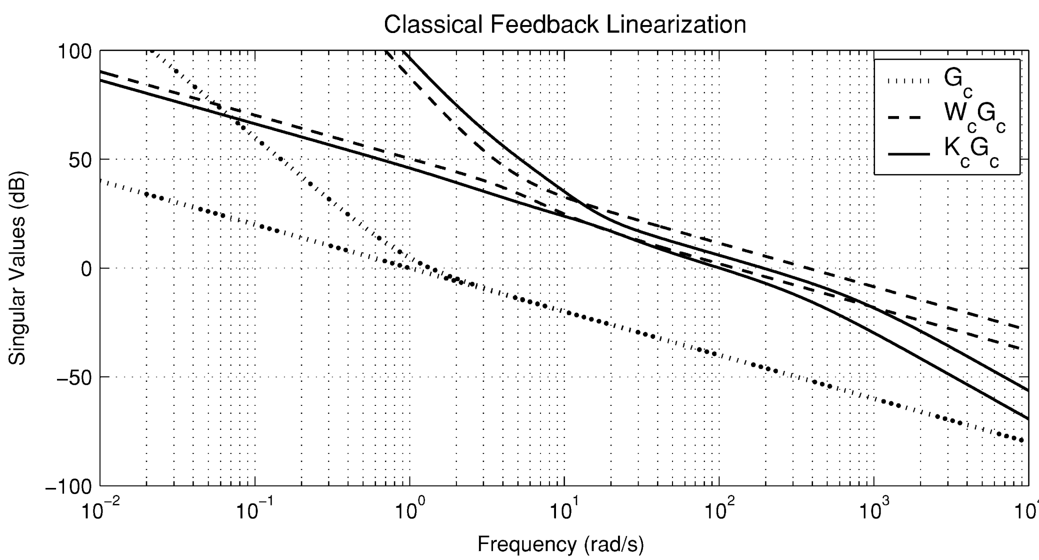

Fig. 4. Singular value plots of G_c, W_cG_c, and K_cG_c.

positive–definite solution of the Hamilton–Jacobi–Bellman (HJB) equation

$$2(\partial_{x_s}V)f_s - (\partial_{x_s}V)g_s g_s^T(\partial_{x_s}V)^T + h_s^T h_s = 0. \tag{16}$$

The nonlinear system $\mathbf{P}_s$ has a linear approximation with transfer matrix

$$G_s = \left[\begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array}\right] = \left[\begin{array}{ccc|c} A_{w_1} & 0 & 0 & B_{w_1} \\ BC_{w_1} & A & 0 & BD_{w_1} \\ 0 & B_{w_2} & A_{w_2} & 0 \\ \hline 0 & D_{w_2} & C_{w_2} & 0 \end{array}\right] \tag{17}$$

where $A_s = \partial_{x_s} f_s(0)$, $B_s = g_s(0)$, $C_s = \partial_{x_s} h_s(0)$, and $D_s = 0$, that is, where $A = \partial_x f(0)$, $B = g(0)$, $A_{w_i} = \partial_{x_{w_i}} f_{w_i}(0)$, $B_{w_i} = g_{w_i}(0)$, $C_{w_i} = \partial_{x_{w_i}} h_{w_i}(0)$ and $D_{w_i} = l_{w_i}(0)$ for $i = 1, 2$.

Theorem 1: The linear controller K_s given by (9), combined with the robust feedback linearization and applied to the nonlinear system $\mathbf{P}_s$ (as shown in Fig. 1), ensures that the closed-loop is locally W-stable for all nonlinear uncertainties $\boldsymbol{\Delta}_{\mathbf{N_r}}$ and $\boldsymbol{\Delta}_{\mathbf{M_r}}$ such that

$$\gamma_{W_l}\left(\begin{bmatrix} \boldsymbol{\Delta}_{\mathbf{N_r}} \\ \boldsymbol{\Delta}_{\mathbf{M_r}} \end{bmatrix}\right) < \frac{1}{\gamma}. \tag{18}$$

Proof: As seen in Section III, by using the McFarlane–Glover method, it is possible to obtain, for the linearized system G_s, a controller K_s that guarantees (7), where $M_r^{-1}(s) = I + B_s^T X(sI - A_s)^{-1}B_s$ and X is the unique positive–definite solution of (11).

For the controller K_s, the nonlinear system $\mathbf{T}$ (Fig. 2) with output q and inputs p_m and p_n is given by

$$\begin{aligned} \dot{x}_s &= f_s(x_s) + g_s(x_s)(u_s - p_m) \\ q &= (u_s - p_m) - \tilde{h}_s(x_s) \end{aligned} \tag{19}$$

with $u_s = \alpha \circ \psi(x_s, p_n) + \beta \circ \psi(x_s, p_n)v$ and $v = K_s \cdot \phi \circ \psi(x_s, p_n)$, where $\psi(x_s, p_n) = h_s(x_s) + p_n$ and $\psi(0, 0) = 0$.

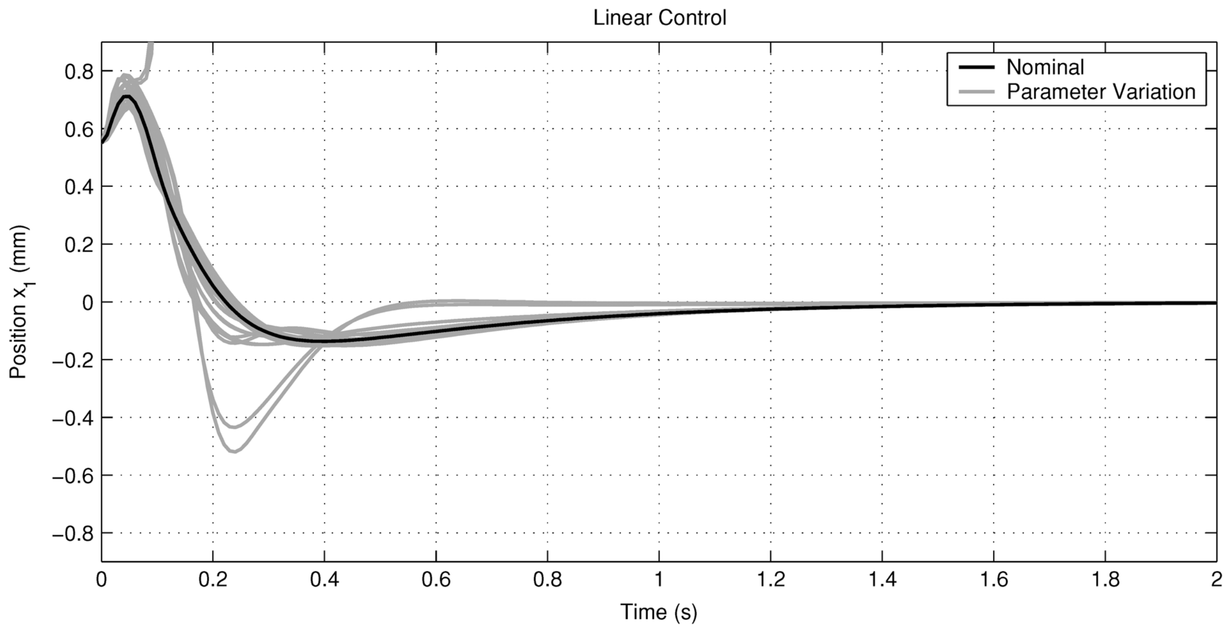

Fig. 5. Rotor position x_1 for the linear control.

Linearizing this system around the origin, $(x_s = u_s = p_m = p_n = 0)$, using the results in (5) and applying the chain rule to the composed functions, yields

$$\begin{aligned} \dot{x}_s &= A_s x_s + B_s(u_s - p_m) \\ q &= (u_s - p_m) - B_s^T \tilde{X} x_s \end{aligned} \tag{20}$$

with $u_s = v$ and $v = K_s(C_s x_s + p_n)$, where $\tilde{X}$ is the unique positive definite solution of the HJB equation associated with the linearized system, i.e.,

$$x_s^T (A_s^T \tilde{X} + \tilde{X} A_s - \tilde{X} B_s B_s^T \tilde{X} + C_s^T C_s) x_s = 0 \tag{21}$$

which is equivalent to the Riccati equation (11). By uniqueness of this solution, $\tilde{X} = X$.

After some algebraic manipulations, and using the Laplace transform, the transfer matrix of the linearized system is obtained as

$$T(s) = M_r^{-1}(s)(I - K_s(s)G_s(s))^{-1}[K_s(s) \quad I]. \tag{22}$$

From (7), it is known that $\|T(s)\|_\infty \leq \gamma$. Therefore, by *Property 1*, $\gamma_{W_1}(\mathbf{T}) = \|T(s)\|_\infty \leq \gamma$.

Considering the closed-loop standard form in Fig. 2, the local version of the Small Gain Theorem [8], [9] implies that this closed-loop is locally W-stable if

$$\gamma_{W_1}\left(\begin{bmatrix} \mathbf{\Delta_{N_r}} \\ \mathbf{\Delta_{M_r}} \end{bmatrix}\right) \gamma_{W_1}(\mathbf{T}) < 1 \tag{23}$$

that is, the closed-loop system is locally W-stable for all uncertainties $\mathbf{\Delta_{N_r}}$ and $\mathbf{\Delta_{M_r}}$ such that

$$\gamma_{W_1}\left(\begin{bmatrix} \mathbf{\Delta_{N_r}} \\ \mathbf{\Delta_{M_r}} \end{bmatrix}\right) < \frac{1}{\gamma}. \tag{24}$$

Furthermore, it is possible to see from (5), that the controller for $\mathbf{P}$ is $K = W_1 K_s W_2$.

Remark 2: The statement of the above theorem is not valid with the classical feedback linearization (because (22) is no longer true). Thus, there is no guarantee that the robustness obtained by a controller K for the linearized system in Brunovsky form is kept when this controller is applied, together with the classical feedback linearization, to the nonlinear system. In the case of the linear control directly applied to the nonlinear system, the robustness is kept only if the system state remains in a sufficiently small neighborhood of the operating point, thus this method is more limited than the linear control with robust feedback linearization.

These results are illustrated through an application example in the next section.

V. Application to a Magnetic Bearing System

Consider the magnetic bearing system described in [11]. Defining x_1 as the rotor position, x_2 as the rotor velocity, x_3 and x_4 as the currents, u as a vector of the input voltages u_1 and u_2, this system may be modelled in the state–space form

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{L_0}{m}\left(\frac{(x_3+I_0)^2}{(k-2x_1)^2} - \frac{(x_4+I_0)^2}{(k+2x_1)^2}\right) \\ -\frac{R_1(k-2x_1)x_3}{L_0} - \frac{2x_2(x_3+I_0)}{k-2x_1} \\ -\frac{R_2(k+2x_1)x_4}{L_0} + \frac{2x_2(x_4+I_0)}{k+2x_1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{k-2x_1}{L_0} & 0 \\ 0 & \frac{k+2x_1}{L_0} \end{bmatrix} u \tag{25}$$

where m is the rotor mass, I_0 is the premagnetization current, R_1 and R_2 are the resistances of the circuits and L_0 and k are positive constants depending on the system construction. The only equilibrium point is $x_0 = 0$. The nominal values of the system parameters are $m = 2$ kg, $k = 2.0125 \times 10^{-3}$ m, $L_0 = 3 \times 10^{-4}$ Hm, $R_1 = 1\Omega$, $R_2 = 1\Omega$, and $I_0 = 6 \times 10^{-2}$ A.

A. Controller Design

The conditions for the feedback linearization to be possible are satisfied, therefore, the nonlinear system (25) is linearized by feedback around its equilibrium point $x_0 = 0$. The output is chosen as $y(x) = \lambda(x) = [x_1 \quad x_3]^T$. Two different controllers are designed: The first one associates the classical feedback linearization with a linear H_∞ controller, and the second one associates the robust feedback linearization with a linear H_∞ controller.

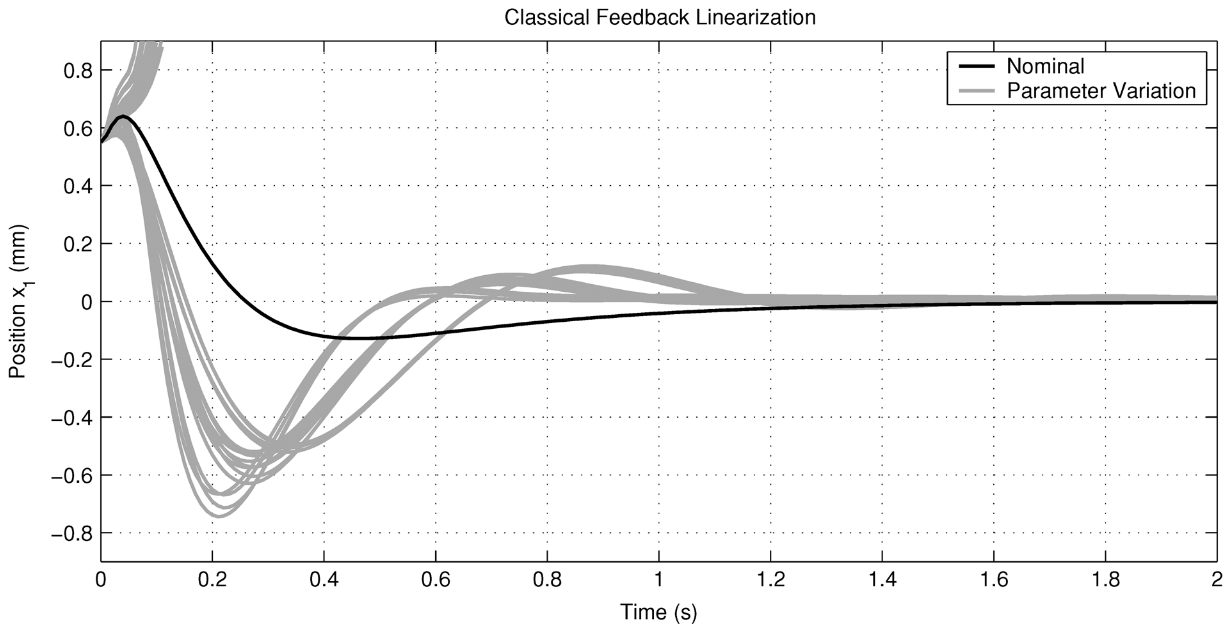

Fig. 6. Rotor position x_1 for the classical feedback linearization.

1) Feedback Linearization: The linearization of the nonlinear model (25) is done by using both the classical feedback linearization and the robust feedback linearization.

The functions for the classical linearizing feedback control law are

$$\alpha_c(x) = \begin{bmatrix} R_1 x_3 + \frac{2L_0 x_2 (x_3+I_0)}{(k-2x_1)^2} \\ R_2 x_4 + \frac{2x_2 L_0 (k+2x_1)(x_3+I_0)^2}{(k-2x_1)^3 (x_4+I_0)} \end{bmatrix} \tag{26}$$

$$\beta_c(x) = \begin{bmatrix} 0 & \frac{L_0}{k-2x_1} \\ \frac{-m(k+2x_1)}{2(x_4+I_0)} & \frac{L_0(k+2x_1)(x_3+I_0)}{(k-2x_1)^2(x_4+I_0)} \end{bmatrix} \tag{27}$$

$$\phi_c(x) = \begin{bmatrix} x_1 \\ x_2 \\ \frac{L_0}{m}\left(\frac{(x_3+I_0)^2}{(k-2x_1)^2} - \frac{(x_4+I_0)^2}{(k+2x_1)^2}\right) \\ x_3 \end{bmatrix}. \tag{28}$$

The classically linearized system (in the Brunovsky form) is then

$$\dot{x}_c = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A_c} x_c + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B_c} w \tag{29}$$

with transfer matrix $G_c(s) = (sI - A_c)^{-1} B_c$ and linear control law $w = K_c x_c$.

The functions for the robust linearizing feedback control law are $\alpha(x)$, $\beta(x)$, and $\phi(x)$ calculated using the functions $\alpha_c(x)$, $\beta_c(x)$, and $\phi_c(x)$ given before and the matrices

$$L = \begin{bmatrix} 0 & 0 & -\frac{2I_0 R_1}{mk} & \frac{2I_0 R_2}{mk} \\ 0 & -\frac{2I_0}{k} & -\frac{kR_1}{L_0} & 0 \end{bmatrix} \tag{30}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{8L_0 I_0^2}{mk^3} & 0 & \frac{2L_0 I_0}{mk^2} & -\frac{2L_0 I_0}{mk^2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{31}$$

$$R = \begin{bmatrix} 0 & \frac{L_0}{k} \\ -\frac{mk}{2I_0} & \frac{L_0}{k} \end{bmatrix}. \tag{32}$$

The robustly linearized system is then

$$\dot{x}_r = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{8L_0 I_0^2}{mk^3} & 0 & \frac{2L_0 I_0}{mk^2} & -\frac{2L_0 I_0}{mk^2} \\ 0 & -\frac{2I_0}{k} & -\frac{kR_1}{L_0} & 0 \\ 0 & \frac{2I_0}{k} & 0 & -\frac{kR_2}{L_0} \end{bmatrix}}_{A_r} x_r + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{k}{L_0} & 0 \\ 0 & \frac{k}{L_0} \end{bmatrix}}_{B_r} v \tag{33}$$

with transfer matrix $G_r(s) = (sI - A_r)^{-1} B_r$ and linear control law $v = K_r x_r$.

2) Linear H_∞ Controller Design: The controllers K_r and K_c are designed using the McFarlane–Glover method with loop-shaping. Since the systems G_r and G_c are different, two different, but "equivalent," weighting matrices W_r and W_c are chosen such that the frequency response of $K_r G_r$ (respectively, $W_r G_r$) is similar to the one of $K_c G_c$ (respectively, $W_c G_c$) and that the nominal performances for the two closed-loop systems are alike.

The loop-shaping $G_{sr} = W_r G_r$ is done with the weighting matrix W_r chosen as

$$W_r = \mathrm{diag}\left(\frac{200(s+25)}{s}, 115, 25, 25\right). \tag{34}$$

The singular value plots of G_r and $W_r G_r$ are shown in Fig. 3. The weighting matrix W_r adds an integrator to the first row of the transfer matrix G_{sr}, which is related to the rotor position x_1, to avoid steady-state errors, and a zero to better shape the position response. To the other lines of the transfer matrix G_{sr}, related to the velocity x_2 and the currents x_3 and x_4, only gains are added. The value $\gamma_r = 4.9$ is used to calculate the controller K_{sr} for the augmented system G_{sr}. This gives a robustness index of 20%. The controller K_r is given by

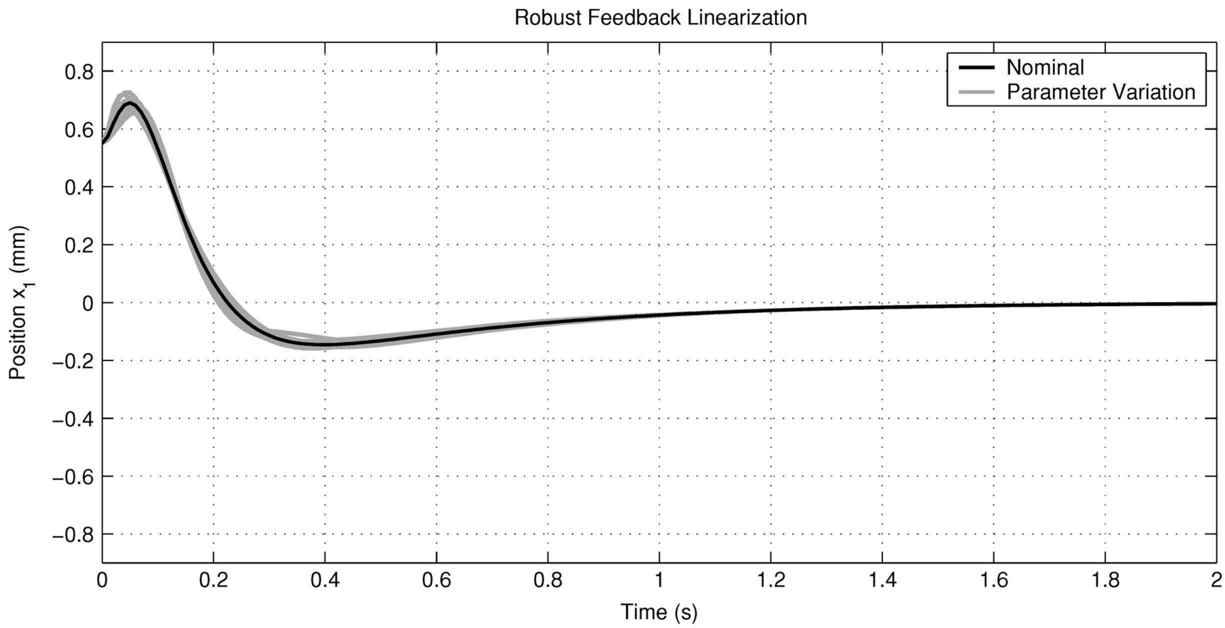

Fig. 7. Rotor position x_1 for the robust feedback linearization.

$K_r = K_{sr}W_r$, where K_{sr} is obtained from (9). The singular value plot of K_rG_r is also shown in Fig. 3.

Then, it is possible to perform a loop-shaping $G_{sc} = W_cG_c$ similar to G_{sr} by choosing W_c as

$$W_c = \begin{bmatrix} \frac{9000(s+2.5)}{s} & 0 & 0 & \frac{180(s+2.5)}{s} \\ 0 & 2200 & 0 & 300 \\ 0 & 0 & 125 & 2 \\ 0 & 0 & 2 & 125 \end{bmatrix}. \tag{35}$$

In Fig. 4, the singular value plots of G_c and W_cG_c are shown. As in the "robust case," the weighting matrix W_c adds an integrator to the first row of G_{sc}, to avoid steady-state errors (it is necessary to include an integrator because the integrators in G_c are not "real" in the physical system—they come from the Brunovsky form—and, in the case of parameter variations, they may disappear). The zeros and the gains are added to obtain for W_cG_c a "shape" similar to that of W_rG_r for medium and high frequencies (>15 rad/s). For low frequencies, it is not possible to obtain the same "shape," because this would imply adding slow poles to W_r or cancelling the poles at the origin of G_c and both solutions would result in bad performances for the closed-loop. The next step is to calculate the controller K_{sc} for the augmented system G_{sc} using (9). For this, the value $\gamma_c = 4.9$ is used, which gives a robustness index of 20%. The controller K_c is given by $K_c = K_{sc}W_c$. The singular value plot of K_cG_c is also shown in Fig. 4.

Hence, the two controllers provide similar "loop-shapes," similar nominal performances and the same values of γ_c and γ_r, thus the same robustness indexes with respect to G_c and G_r.

B. *Simulation With Parameter Variations*

In this subsection, a comparison of the robustness obtained for the nonlinear system with the three proposed controllers (K_c and K_r associated with the classical linearization and with the robust linearization, respectively, and K_r directly applied to the nonlinear system) is presented.

The simulations are carried out with Simulink/Matlab, using the Dormand–Prince algorithm with a maximal step size of 0.1 ms. For these simulations, it is supposed that the parameter variations may be $\pm 15\%$ for m, L_0, R_1 and R_2 and $\pm 5\%$ for k, yielding 32 different combinations of the extreme values (which are all tested). The initial condition for the position x_1 is 0.55 mm, which allows evaluation of the behavior of the controllers far from the equilibrium point. The results for the linear control (directly applied to the nonlinear system) are given in Fig. 5, for the classical feedback linearization in Fig. 6 and for the robust feedback linearization in Fig. 7.

These results show that with all the considered parameter variations the closed-loop system controlled by K_r associated with the robust feedback linearization behaves as desired, since the performance remains close to the nominal one. For the case where only K_r is used (without feedback linearization) the closed-loop system presents good performances for some combinations of the parameters, but is unstable or has big overshoots for some others. The closed-loop system controlled by K_c associated with the classical feedback linearization is unstable for some combinations of parameters and presents poor performances for the other ones.

VI. Concluding Remarks

As shown by the theory in Section IV and illustrated by the simulations in Section V-B, a *robust nonlinear controller* for the *nonlinear system* is obtained by using the robust feedback linearization associated with a McFarlane–Glover H_∞ controller. This does not hold when the classical feedback linearization is used or when a linear control is directly applied to the nonlinear system, as explained in Remark 2. In addition, the choice of the weighting matrix for the loop-shaping is much easier when using the robust linearization, as shown in Section V-A.

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Synchronization and Convergence of Linear Dynamics in Random Directed Networks

Chai Wah Wu

***Abstract*—Recently, methods in stochastic control are used to study the synchronization properties of a nonautonomous discrete-time linear system $x(k+1) = G(k)x(k)$ where the matrices $G(k)$ are derived from a random graph process. The purpose of this note is to extend this analysis to directed graphs and more general random graph processes. Rather than using Lyapunov type methods, we use results from the theory of inhomogeneous Markov chains in our analysis. These results have been used successfully in deterministic consensus problems and we show that they are useful for these problems as well. Sufficient conditions are derived that depend on the types of graphs that have nonvanishing probabilities. For instance, if a scrambling graph occurs with nonzero probability, then the system synchronizes.**

***Index Terms*—Directed graphs, dynamics, graph theory, Markov processes, synchronization.**

I. Introduction

Synchronizing dynamics among coupled nonlinear systems where the coupling topology is expressed as a graph is an active area of research [1]–[9]. In recent years, in the context of agreement and consensus problems, there is increased interest to study the case where the dynamics are linear [10]–[12]. Some of the applications include modeling the flocking of animals, communications in mobile autonomous robots and multivehicle control. In these studies the dynamical system is deterministic. In [13], a discrete-time nonautonomous linear system $x(k+1) = G(k)x(k)$ is studied where the matrices $G(k)$ are derived from a random graph process. It was found that $x(k)$ converges to the subspace spanned by $(1,\ldots,1)^T$ in probability if each edge is chosen with the same probability. The purpose of this note is to extend this to directed graphs and more general random graph models. In contrast to [13] where Lyapunov methods from stochastic control are used, we use results from the theory of inhomogeneous Markov chains. This approach allows us to easily obtain results for general random graph models.

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II. Problem Formulation

We consider the following nonautonomous discrete-time linear dynamical system:

$$x(k+1) = (G(k) \otimes D(k))x(k) + \mathbf{1} \otimes v(k) \tag{1}$$

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $x_i \in \mathbb{R}^m$, $v(k) \in \mathbb{R}^m$, and $\mathbf{1} = (1,\ldots,1)^T$. The matrix $G \otimes D$ is the Kronecker product or tensor product of the matrices G and D. We assume that $G(k)$ is an n by n stochastic matrix (i.e., $G(k)$ is a nonnegative matrix whose rows sum to 1). Most prior papers have studied the cases where $D(k)$ does not depend on k and is a scalar or the identity matrix. We say that the system in (1) *synchronizes* if $\|x_i(k) - x_j(k)\| \to 0$ as $k \to \infty$ for all i, j. If there exists $\varepsilon < 1$ such that $\|D(k)\| \le \varepsilon$ for all k and $G(k)$ are symmetric stochastic matrices, then $\|G \otimes D\| \le \varepsilon$ and synchronization occurs regardless of G. On the other hand, if $\|D(k)\| > 1$ for all k, then $x(k)$ can diverge. In this note, we will assume that $\|D(k)\| \le 1$ for each k. First, we show that the maximum distance between the x_i's is nonincreasing.

Theorem 2.1: Let $\kappa(k) = \max_{i,j} \|x_i(k) - x_j(k)\|$. Then, $\kappa(k+1) \le \kappa(k)$.

Proof: Let $S(k) = \{x_1(k),\ldots,x_n(k)\}$ and $T(k) = \{D(k)x_1(k) + v(k),\ldots,D(k)x_n(k) + v(k)\}$. Note that $\kappa(k)$ is the diameter of the convex hull of $S(k)$. Since $x_i(k+1)$ is a convex combination of elements of $T(k)$, it is in the convex hull of $T(k)$. Thus means that the convex hull of $S(k+1)$ is a subset of the convex hull of $T(k)$. Since $\|D(k)\| \le 1$, the diameter of the convex hull of $T(k)$ is less than or equal to the diameter of the convex hull of $S(k)$ and, thus, $\kappa(k+1) \le \kappa(k)$.

Note that synchronization is equivalent to $\lim_{k\to\infty} \kappa(k) = 0$.

III. Scrambling Stochastic Matrices and Hajnal's Inequality

We next summarize some results from the theory of inhomogeneous Markov chains which are useful in deriving sufficient conditions for synchronization. These results were first used in consensus problems by [10] and its use subsequently extended in [12], [14], and [15].

Definition 3.1: A matrix A is *scrambling* if for each pair of indexes (i,j) there exists k such that A_{ik} and A_{jk} are both nonzero.

Definition 3.2: For a real matrix A, the ergodicity coefficient $\mu(A)$ is defined as

$$\mu(A) = \min_{j,k} \sum_i \min(A_{ji}, A_{ki}).$$

For nonnegative matrices whose row sums is less than or equal to r, it is clear that $0 \le \mu(A) \le r$ with $\mu(A) > 0$ if and only if A is scrambling.