

ROBUST OPTIMAL CONTROL FOR AN INSURER WITH REINSURANCE AND INVESTMENT UNDER HESTON'S STOCHASTIC VOLATILITY MODEL

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ABSTRACT. This paper considers a robust optimal reinsurance and investment problem under Heston's Stochastic Volatility (SV) model for an Ambiguity-Averse Insurer (AAI), who worries about model misspecification and aims to find robust optimal strategies. The surplus process of the insurer is assumed to follow a Brownian motion with drift. The financial market consists of one risk-free asset and one risky asset whose price process satisfies Heston's SV model. By adopting the stochastic dynamic programming approach, closed-form expressions for the optimal strategies and the corresponding value functions are derived. Furthermore, a verification result and some technical conditions for a well-defined value function are provided. Finally, some of the model's economic implications are analyzed by using numerical examples and simulations. We find that ignoring model uncertainty leads to significant utility loss for the AAI. Moreover we propose an alternate model and associated investment strategy which would can be considered more adequate under certain finance interpretations, and which leads to significant improvements in our numerical example.

Keywords: Reinsurance and investment strategy, Stochastic volatility, Robust optimal control, Utility maximization, Ambiguity-Averse Insurer.

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1. INTRODUCTION

Reinsurance and investment are two significant issues for insurers: reinsurance is an effective risk-management approach (risk-spreading), while investment is an increasingly important way to utilize the surplus of insurers. Recently, optimal reinsurance and/or investment problems for insurers have attracted great interest. For example, Bai & Guo (2008), Luo (2009), Azcue & Muler (2009) and Chen et al. (2010) investigated the optimal reinsurance and/or investment strategies for insurers to minimize the ruin probability under different market assumptions; Bäuerle (2005), Delong & Gerrard (2007), Bai & Zhang (2008), Zeng et al. (2010) and Zeng & Li (2011) studied the optimal reinsurance and investment strategies for insurers with mean-variance criteria. In addition, some scholars have recently studied the optimal reinsurance and/or investment strategies for insurers with constant absolute risk aversion (CARA) utility, see among Browne (1995), Yang & Zhang (2005), Wang (2007), Xu et al. (2008) and so on.

Although many scholars have investigated optimal reinsurance and investment strategies for insurers, we think that two aspects ought to be explored further. On the one hand, most of the above-mentioned literature assumes that the volatilities of risky assets' prices are constants or deterministic functions. This simplification goes against well-documented evidence to support the existence of stochastic volatility (SV), as far back as French et al. (1987) and Pagan & Schwert (1990), with detailed studies of SV and empirical evidence of continuing to this day (see Viens (2012)). In particular, SV can be seen as an explanation of many well-known empirical findings, for example, the volatility smile and the volatility clustering implied by option prices. To study more practical financial market, Heston (1995) assumed that the volatility of the risky asset was driven by a Cox-Ingersoll-Ross (CIR) process; this model has some computational and empirical advantages. Since then, numerous scholars have investigated the optimal portfolio choice for investors under Heston's SV model. For instance, Liu & Pan (2003), Chacko & Viceira (2005), Kraft (2005) and Liu (2007) considered the optimal investment and/or consumption problems under Heston's SV model by adopting the stochastic dynamic programming approach; Pham & Quenez (2001), Viens (2002), and Kim & Viens (2012) focused on optimization portfolio problem under SV model with partially observed information using particle filter theory.

Recently, some papers have appeared investigating the optimal reinsurance and investment strategies for insurers with various stochastic investment opportunities, such as Constant Elasticity of Variance (CEV) models, stochastic risk premium models and stochastic interest rate models. Gu et al. (2012) considered the optimal reinsurance and investment strategies for an insurer under a CEV model, where the volatility of the risky asset was dependent on the price of the risky asset. Liang et al. (2011) assumed that the risk premium satisfies an Ornstein-Uhlenbeck (OU) process, and derived the explicit expression for an optimal strategy with reinsurance and investment. Note that none of these models contain full-fledged SV assumptions, the CEV model being a local volatility one. The question of optimal reinsurance and investment under Heston's SV model was just introduced in the paper Li et al. (2012), which pioneers the investigation of an optimal time-consistent strategy for insurers, and includes a closed-form solution by solving an extended Hamilton-Jacobi-Bellman (HJB) equation under a mean-variance criterion.

Even this paper suffers from being unable to account for model uncertainty at levels beyond the volatility. However, it is a notorious fact in the practice of portfolio management that return levels for risky assets are difficult to estimate with precision. In the context of insurance and reinsurance, the same uncertainty is true regarding expected surpluses. As a consequence, the Ambiguity-Averse Insurer (AAI) will look for a methodology to handle this uncertainty. Rather than make ad-hoc decisions about how much error is contained in the estimates return levels for risky assets and surpluses, the AAI may instead consider some alternative models which are close to the estimated model. This more systematic method has been successfully implemented over the last 15 years in quantitative investment finance, for portfolio selection and asset pricing with model uncertainty or model misspecification, and has seen some recent applications in insurance. We review some of the prominent results.

Anderson et al. (1999) introduced ambiguity-aversion into the Lucas model, and formulated alternative models. Uppal & Wang (2003) extended Anderson et al. (1999), and developed a framework which allows investors to consider the level of ambiguity. Maenhout (2004) optimized an intertemporal consumption problem with ambiguity, and derived closed-form expressions for the optimal strategies under "homothetic robustness". Liu et al. (2005) studied the role of ambiguity-aversion in options pricing under an equilibrium model with rare-event premia. Maenhout (2006) found the optimal portfolio choice under model uncertainty and stochastic premia, and provided a methodology to measure the quantitative effect of model uncertainty. Xu et al. (2010) considered a robust equilibrium pricing model under Heston's SV model. In recent

years, some papers focused on optimal reinsurance and investment strategies with ambiguity. Zhang & Siu (2009) investigated a reinsurance and investment problem with model uncertainty, and formulated the problem as a zero-sum stochastic differential game. Lin et al. (2012) discussed an optimal portfolio selection problem for an insurer who faces model uncertainty in a jump-diffusion model by using a game-theory approach.

Among the very few papers studying optimal reinsurance and investment strategies with Heston SV, only Li et al. (2012) found an optimal time-consistent strategy for insurers, and did so with a mean-variance criterion. In our paper, we take up this general question with Heston's SV model as well, but chose to use an AAI, and look for a mathematically tractable framework under model uncertainty. Consequently, we settle on a CARA utility criterion (Constant Absolute Risk Aversion, exponential utility).

It is known that Heston's SV model may result in an infinite value function if the insurer's utility is CARA (power-function utilities also have this deficiency, see Taksar & Zeng (2009)). One gets around this problem by imposing some technical conditions on the model parameters to guarantee that the value function is well-defined. With such a model, it is possible for the AAI to allow for model uncertainty, and to seek robust decision rules, i.e. investment strategies that are insensitive to these uncertainties to a large extent. In summary, in this paper, we investigate the robust optimal reinsurance and investment strategy for an AAI with CARA utility in a SV financial market.

Specifically, the surplus process of the insurer is assumed to follow a Brownian motion with drift; the financial market consists of one risk-free asset and one risky asset whose price is described by Heston's SV model. To incorporate the model uncertainty, we assume that the insurer is ambiguity-averse, and we model the level of ambiguity by weighing it with a preference parameter that is state-dependent: following Menhout (2004, 2006), this ambiguity level is chosen as inversely proportional to the optimization's value function, which is consistent with the economically correct interpretation of high value function implying high levels of risk aversion (so high aversion to uncertainty). With this model for the market and surplus, and ambiguity quantification, we formulate a robust problem with alternative models. Secondly, we derive the explicit closed-form expressions for the optimal reinsurance and investment strategy for the AAI with CARA utility, as well as the corresponding value function. Convenient sufficient conditions for a verification result are provided. Finally, some economic implications of our results and numerical illustrations are presented.

Summarizing and comparing with the existing literature, we think our paper proposes four main innovations:

- (i): A robust optimal reinsurance and investment problem under Heston's SV model with CARA utility is considered, and at the mathematical level, the verification result for this model has distinct differences with the result in Li et al. (2012): in particular, sufficient conditions are proposed in our paper, which ensure the optimal strategies and corresponding value functions may satisfy the verification theorem in Kraft (2004).
- (ii): The levels of ambiguity in a time-varied investment opportunity set are investigated for the AAI, which Zhang & Siu (2009) and Lin et al. (2012) did not consider. The different ambiguity levels give more flexibility to model the individual attitudes to misspecification.
- (iii): The utility losses from ignoring model uncertainty and prohibiting reinsurance for the AAI are disclosed: our numerics clearly show the wisdom in not ignoring the impacts of model misspecification, and the importance of risk management via reinsurance.
- (iv): An alternative and effective robust model is proposed, in which one assumes the AAI has the full confidence in the parameters associated with SV, but not with those relative to mean rates. Such an assumption is consistent with the current trends by which volatility is ever more closely monitored and recorded¹, while mean rates are still considered as exceedingly difficult to pin down. Our numerics show that small changes in the optimal strategy greatly enhance the value function, when SV is no longer uncertain. This improved model should be of direct practical significance for those insurers who choose to invest in S&P500 index funds, which the VIX tracks explicitly.

The rest of this paper is organized as follows. The economy and assumptions are described in Section 2. In Section 3, a robust control problem for an AAI with CARA utility is presented. Section 4 derives the closed-form expressions for the optimal strategy and the corresponding value function with some technical conditions, and explores some economic implications of our results. Section 5 analyzes our results with numerical illustration, and recommends an improved model for applications. Section 6 provides our conclusions, and proposes some promising extensions of our work.

¹For instance, the VIX index is regarded so highly as an accurate measure of volatility on the Chicago Board of Options Exchange (CBOE), that the derivative products based on the VIX have produced, over the past 5 years, some of the highest trade volume on the CBOE.

2. ECONOMY AND ASSUMPTIONS

We consider a continuous-time financial model with the following standard assumptions: an insurer can trade continuously in time, and trading in the financial market or the insurance market involves no extra costs or taxes. Let (Ω, \mathcal{F}, P) be a complete probability space with filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ generated by three standard one-dimension Brownian motions $\{Z^S(t)\}$, $\{Z^M(t)\}$ and $\{Z^R(t)\}$, where $\{Z^R(t)\}$ is independent of $\{Z^S(t)\}$ and $\{Z^M(t)\}$, T is a positive finite constant representing the terminal time. Any decision made at time t is based on \mathcal{F}_t which can be interpreted as the information available until time t . Thus $T - t$ can be understood as the horizon at time t (time to maturity).

2.1. Surplus process. Following the assumption in Promislow & Young (2005), we formulate the claim process $C(t)$ of the insurer as

$$dC(t) = adt - bdZ^R(t), \quad (2.1)$$

where $a > 0$ is the rate of the claim and $b > 0$ can be regarded as the volatility of the claim process. Note that the diffusion model of the claim process is an approximation of the classical Cramér-Lundberg model (see, e.g., Grandell (1991) and Zeng & Li (2012)). As stated in Promislow & Young (2005), in actuarial practice one uses the model (2.1) only when the ratio a/b is large enough ($a/b > 3$) so that the probability of realizing negative claims in any one period is small.

The premium is paid continuously at the constant rate $\varsigma_0 = (1 + \mu)a$ with safety loading $\mu > 0$. When both reinsurance and investment are absent, the dynamics of the surplus is given by (see e.g. Emanuel et al. (1975), Grandell (1991) and Promislow & Young (2005))

$$dR_0(t) = \varsigma_0 dt - dC(t) = \mu adt + bdZ^R(t). \quad (2.2)$$

If the insurer can purchase proportional reinsurance or acquire new business (by acting as a reinsurer for other insurers, for example) to manager her or his insurance business risk, the reinsurance level at any time t , is associated with the value $1 - q(t)$, where $q(t) \in [0, +\infty)$ can be regarded as the value of risk exposure. When $q(t) \in [0, 1]$, it corresponds to a proportional reinsurance cover. In this case, reinsurance premia will be paid continuously by the cedent at the constant rate $\varsigma_1 = (1 + \eta)(1 - q(t))a$ with safety loading $\eta \geq \mu > 0$ as the cost of reinsurance; at the same time the reinsurer pays $100(1 - q(t))\%$ of each claim occurring at time t while the insurer pays $100q(t)\%$. When $q(t) \in (1, +\infty)$, it corresponds to acquiring new business (see

Bäuerle (2005)). The process of risk exposure $\{q(t) : t \in [0, T]\}$ is called a reinsurance strategy, and the surplus process with such a reinsurance strategy $\{q(t) : t \in [0, T]\}$ is given by

$$dR(t) = \varsigma_0 dt - q(t)dC(t) - \varsigma_1 dt = [\lambda + \eta q(t)]adt + bq(t)dZ^R(t),$$

where $\lambda = \mu - \eta$.

2.2. Financial Market. We consider a financial market consisting of one risk-free asset (e.g., a bond) and one risky asset (e.g., a stock). The price process $S_0(t)$ of the risk-free asset evolves according to the ordinary differential equation (ODE)

$$dS_0(t) = r_0 S_0(t)dt, \quad (2.3)$$

where $S_0(0) = s_0 > 0$ and $r_0 > 0$ is the risk-free interest rate. The price process $S_1(t)$ of the risky asset follows Heston's SV model

$$\begin{cases} dS_1(t) = S_1(t) \left[(r_0 + \xi V(t))dt + \sqrt{V(t)}dZ^S(t) \right], \\ dV(t) = \kappa(\theta - V(t))dt + \sigma \sqrt{V(t)}dZ^M(t), \end{cases} \quad (2.4)$$

where $S_1(0) = s_1 > 0$; $V(0) = v_0 > 0$; $\xi > 0$ is the premium for volatility; the three parameters $\kappa > 0$, $\theta > 0$ and $\sigma > 0$ denote the mean-reversion rate, the long-run mean and the volatility of the volatility ("vol-vol") parameter, respectively. Note that σ is sometimes called the volatility of the volatility, but since we are using a CIR model, it is more correct, in analogy with the Black-Scholes model, to say that the vol-vol is the state dependent expression $\sigma \sqrt{V(t)}$, just as we would have for a leveraged CEV model with power " γ " = 1/2. We require $2\kappa\theta \geq \sigma^2$ to ensure that $V(t)$ is almost surely non-negative. $Z^S(t)$ and $Z^M(t)$ are two one-dimensional standard Brownian motions with $\text{Cov}(Z^S(t), Z^M(t)) = \rho_0 t$ in which $\rho_0 \in [-1, 1]$. In addition, by standard Gaussian linear regression, $Z^M(t)$ can be rewritten as

$$dZ^M(t) = \rho_0 dZ^S(t) + \rho dZ^V(t),$$

where $\rho = \sqrt{1 - \rho_0^2}$ and $Z^V(t)$ is a standard Brownian motion which is independent of $Z^S(t)$ and $Z^R(t)$.

2.3. Wealth process. In this paper, we assume that the self-financing insurer is allowed to purchase proportional reinsurance, acquire new business and invest the surplus in the financial market over $t \in [0, T]$, and the trading strategy is represented by a pair of stochastic processes $\pi = \{q(t), l(t)\}_{t \in [0, T]}$, where $q(t)$ represents the value of risk exposure and $l(t)$ denotes the dollar amount invested in the risky asset at time t . The remainder $W^\pi(t) - l(t)$ is invested in the risk-free

asset, where $W^\pi(t)$ is the wealth process associated with strategy π . Thus, the wealth process $W^\pi(t)$ can be presented by the following stochastic differential equation (SDE)

$$\begin{aligned} dW^\pi(t) = & [a\lambda + a\eta q(t) + \xi l(t)V(t) + W^\pi(t)r_0]dt + bq(t)dZ^R(t) \\ & + l(t)\sqrt{V(t)}dZ^S(t), \end{aligned} \quad (2.5)$$

where $W^\pi(0) = w_0$ is the initial wealth.

3. ROBUST CONTROL PROBLEM FOR AN AAI

The insurer is assumed to have CARA utility

$$U(x) = -\frac{1}{m} \exp(-mx), \quad (3.1)$$

where $m > 0$ is a constant representing the absolute risk aversion coefficient. In traditional reinsurance and investment models, the insurer is assumed to be ambiguity-neutral with objective function

$$\sup_{\pi \in \tilde{\Pi}} E^P[U(W^\pi(T))] = \sup_{\pi \in \tilde{\Pi}} E^P \left[-\frac{1}{m} \exp(-mW^\pi(T)) \right], \quad (3.2)$$

where $\tilde{\Pi}$ is the set of admissible strategies π in a given market, and E^P is the expectation under the single model's probability measure P . See Yang & Zhang (2005) for the traditional models.

To incorporate information with model uncertainty into the optimal reinsurance and investment problem for the AAI, we assume that, in our economy, the insurer's knowledge with ambiguity is described by allowing changes in the probability measure P , namely uncertainty in the model's parameters. This is consistent with the fact that it is often the case that P is the result of some estimation process with misspecification errors. The fundamental assumption is that the AAI can not precisely know whether the reference model is the true model. Hence, she or he will consider some alternative models in the decision process, allowing, for instance, for a range of each possible parameter in the model for which there is uncertainty. Loosely speaking, robustness is then achieved by guarding against all adverse alternative models that are reasonably similar to the reference one. Parallel to Anderson et al. (1999), one may broadly define the alternative models by a class of probability measures which are equivalent to P :

$$\mathcal{Q} := \{Q | Q \sim P\}. \quad (3.3)$$

Definition 3.1. A trading strategy $\pi = \{q(t), l(t)\}_{t \in [0, T]}$ is said to be admissible, if

- (i) π is progressively measurable and $E^Q \left[\int_0^T \|\pi\|^4 dt \right] < \infty, \forall Q \in \mathcal{Q}$;
- (ii) $\forall (w_0, v_0) \in \mathbb{R} \times \mathbb{R}^+$, the SDE (2.5) has a pathwise unique solution $\{W^\pi(t)\}_{t \in [0, T]}$, and

$$E^Q \left[\sup_{t \in [0, T]} \exp \left(-2me^{r_0(T-t)} W^\pi \right) \right] < +\infty, \forall Q \in \mathcal{Q}.$$

Denote by Π the set of all admissible strategies.

According to Girsanov's theorem, for each $Q \in \mathcal{Q}$ there exists progressively measurable process $\varphi(t) = (h(t), g(t), f(t))$ such that

$$\frac{dQ}{dP} = \nu(T),$$

where

$$\nu(t) = \exp \left\{ \int_0^t h(s) dZ^S + g(s) dZ^V + f(s) dZ^R - \frac{1}{2} \int_0^t h(s)^2 + g(s)^2 + f(s)^2 ds \right\}$$

is a P -martingale. The reference Karatzas & Shreve (1988) can be consulted for this theorem.

That reference also contains the well-known fact that, if $\varphi(t) = (h(t), g(t), f(t))$ satisfies Novikov's condition²

$$E^P \left[\exp \left(\frac{1}{2} \int_0^T \|\varphi(s)\|^2 ds \right) \right] < \infty,$$

with $\|\varphi(t)\|^2 = h(t)^2 + g(t)^2 + f(t)^2$, then $\nu(t)$ is a P -martingale with filtration $\{\mathcal{F}_t\}_{t \in (0, T)}$. Furthermore, by Girsanov's theorem, Brownian motions under $Q \in \mathcal{Q}$ can be defined as

$$dZ_Q^S(t) = dZ^S(t) - h(t)dt, \tag{3.4}$$

$$dZ_Q^V(t) = dZ^V(t) - g(t)dt, \tag{3.5}$$

$$dZ_Q^R(t) = dZ^R(t) - f(t)dt. \tag{3.6}$$

Girsanov's theorem is also known as the method of "removal of drift", since it allows one to compute the law of a multidimensional semimartingale by comparing it to the same model with no drift components.

We formulate a robust control problem inspired by Maenhout (2004) to modify problem (3.2) as following

$$\sup_{\pi \in \Pi} \inf_{Q \in \mathcal{Q}} E^Q \left\{ \int_0^T \frac{1}{\phi(s)} R(s) ds + U(W^\pi(T)) \right\}, \tag{3.7}$$

where alternative models are defined by each probability measure $Q \in \mathcal{Q}$, $\phi(t)$ stands for a preference parameter for ambiguity-aversion, which measures the degree of confidence in the reference model P at time t , and $R(t)$ measures the relative entropy between P and Q . Along the line of Hansen & Sargent (2001), by defining $R(t) := \frac{1}{2} [h(t)^2 + g(t)^2 + f(t)^2]$, then $E^Q \left[\int_0^T R(s) ds \right]$ measures the discrepancy between P and Q (see, e.g., Dupuis & Ellis (1997)).

²The technical requirements will be stated later.

With this specification, penalties are incurred for alternative models when they deviate from the reference model. According to Maenhout (2004), $\int_0^T \frac{1}{\phi(s)} R(s) ds$ effects penalty in decision and the magnitude of the penalty depends on the preference parameter. Choosing this penalty in the optimization problem (3.7) allows one to indeed be able to find an interior point maximizer, i.e. an optimal strategy, even if the parameter ranges are unbounded.

In the case $\phi \equiv 0$, the insurer is extremely convinced that the true model is the reference model P , any deviation from P will be penalized heavily by $\frac{1}{\phi} R$. Thus, $R \equiv 0$ must be satisfied to guarantee $\frac{1}{\phi} R \equiv 0$ and problem (3.7) reverts to problem (3.2), where no model uncertainty is allowed, as it should. At the other extreme, if $\phi \equiv \infty$, the insurer has no information about the true model. Since the term $\frac{1}{\phi} R$ vanishes, the scenario will degenerate to the situation discussed in Zhang & Siu (2009) and Lin et al. (2011), in which the interpretation of robustness is arguably less quantitatively clear. (For reference to this literature, see Anderson et al. (1999), Uppal & Wang (2003) and Maenhout (2004)).

Inserting (3.4) and (3.5) into (2.4), Heston's SV model under the alternative model Q can be described as

$$\begin{cases} dS_1(t) = S_1(t) \left[(r_0 + \xi V(t) + \sqrt{V(t)} h(t)) dt + \sqrt{V(t)} dZ_Q^S(t) \right], \\ dV(t) = \left[\kappa(\theta - V(t)) + \sigma \sqrt{V(t)} \rho_0 h(t) + \sigma \sqrt{V(t)} \rho g(t) \right] dt \\ \quad + \sigma \sqrt{V(t)} \rho_0 dZ_Q^S(t) + \sigma \sqrt{V(t)} \rho dZ_Q^V(t). \end{cases} \quad (3.8)$$

We notice that the alternative models in class Q only differ in the drift terms, which is natural since we are using Girsanov's theorem to define them. Due to (2.4) and (3.6), the wealth process can be rewritten as

$$\begin{aligned} dW^\pi(t) = & \left[a\lambda + a\eta q(t) + l(t)\xi V + W^\pi(t)r_0 + bq(t)f(t) \right. \\ & \left. + l(t)\sqrt{V}h(t) \right] dt + bq(t)dZ_Q^R(t) + l(t)\sqrt{V}dZ_Q^S(t). \end{aligned} \quad (3.9)$$

With the Girsanov's shift, only changes to drift parameters are allowed, which means we are unable to consider robustness on the vol-vol parameter σ . Fortunately, as mentioned in the introduction, and as explored in detail in Section 5, we see that allowing robustness on $dV(t)$ does not noticeably change the optimal strategy, compared to only allowing for robustness on the coefficients in the drift part of $dS_1(t)$.

4. OPTIMAL STRATEGY

To solve problem (3.7), we define the value function J as

$$J(t, W, V) = \sup_{\pi \in \Pi} \inf_{Q \in \mathcal{Q}} E_{t,W,V}^Q \left[\int_t^T \frac{1}{\phi(s)} R(s) ds + U(W^\pi(T)) \right], \quad (4.1)$$

where $E_{t,W,V}^Q[\cdot] = E^Q[\cdot \mid W(t) = W, V(t) = V]$. This means that we consider that the volatility $V(t)$ is observed, an assumption which, as we commented in the introduction, is becoming more and more reasonable, particularly on indexes (such as the S&P500) where a good volatility tracker exists (the CBOE's VIX for the S&P500).

According to the principle of dynamic programming, the HJB equation can be derived as (see Uppal & Wang (2005), Liu et al. (2005) and Maenhout (2006)):

$$\begin{aligned} \sup_{\pi \in \Pi} \inf_{h,g,f} \left\{ \mathcal{D}^{(\pi)} J + (bqf + l\sqrt{V}h) J_W + (\sigma\sqrt{V}\rho_0h + \sigma\sqrt{V}\rho g) J_V \right. \\ \left. + \frac{1}{\phi} \left(\frac{1}{2}h^2 + \frac{1}{2}g^2 + \frac{1}{2}f^2 \right) \right\} = 0 \end{aligned} \quad (4.2)$$

with the boundary condition $J(T, W, V) = -\frac{1}{m} \exp(-mW)$, where $\mathcal{D}^{(\pi)} J$ is the infinitesimal generator of the Markov diffusion (W, V) under Q applied to the value function J , and is defined by (or computes as)

$$\begin{aligned} \mathcal{D}^{(\pi)} J = J_t + [a\lambda + a\eta q + l\xi V + Wr_0] J_W + \kappa(\theta - V) J_V \\ + \frac{1}{2}(l^2V + b^2q^2) J_{WW} + l\sigma V\rho_0 J_{WV} + \frac{1}{2}\sigma^2 V J_{VV}. \end{aligned} \quad (4.3)$$

Here $J_t, J_W, J_V, J_{WV}, J_{WW}$ and J_{VV} represent the value function's partial derivatives with respect to (w.r.t) the corresponding variables. Furthermore, for analysis convenience, we choose a suitable form³ of preference parameter $\phi(t)$ proposed by Menhout (2004, 2006) as

$$\phi(t) = -\frac{\beta}{mJ(t, W, V)}, \quad (4.4)$$

which is state-dependent. Here $\beta \geq 0$ is the ambiguity-aversion coefficient describing individual attitude to model uncertainty.

Next we aim to derive the explicit solution to the HJB equation (4.2) with preference parameter (4.4). Firstly, we propose an ansatz for the structure of the value function. Then, under the ansatz we determine the drift corresponding to the worst scenario (the drifts that realize the

³This form of $\phi(t)$ has following reasonable properties: it increases w.r.t β, W and V and decreases w.r.t m , which implies the insurer with higher β, W, V or lower m prefers more robustness. Also see footnote in the introduction on page 5.

minimum in (4.2)), and derive the optimal strategy. At last, plugging the drift and optimal strategy into (4.2), we hope that the ansatz will allow to separate the variables in the HJB equation, and obtain the explicit solution to (4.2).

Step 1: Propose the ansatz. Let $c(t)$, $d(t)$ and $e(t, V)$ be three functions to be determined. Conjecturing that the value function has the following form

$$J(t, W, V) = -\frac{1}{m} \exp \{-m [c(t)(W - d(t)) + e(t, V)]\}, \quad (4.5)$$

a direct calculation yields

$$\begin{cases} J_t = -m[c_t(W - d) - d_t c + e_t]J, & J_W = -mcJ, & J_V = -me_V J, \\ J_{WV} = m^2 c e_V J, & J_{WW} = m^2 c^2 J, & J_{VV} = (m^2 e_V^2 - m e_{VV})J. \end{cases} \quad (4.6)$$

Step 2: Derive the optimal control. According to the first-order conditions, the functions h^* , g^* and f^* which realize the minimum in (4.2) are given by

$$h^*(t) = [-l(t) \sqrt{V} J_W(t, W, V) - J_V(t, W, V) \sigma \sqrt{V} \rho_0] \phi(t), \quad (4.7)$$

$$g^*(t) = -J_V(t, W, V) \sigma \sqrt{V} \rho \phi(t), \quad (4.8)$$

$$f^*(t) = -J_W(t, W, V) b q(t) \phi(t). \quad (4.9)$$

Substituting (4.5) and (4.6) into (4.7)-(4.9), we have

$$h^*(t) = -\beta [c(t) l(t) \sqrt{V} + e_V(t, V) \sigma \sqrt{V} \rho_0], \quad (4.10)$$

$$g^*(t) = -\beta e_V(t, V) \sigma \sqrt{V} \rho, \quad (4.11)$$

$$f^*(t) = -\beta b c(t) q(t). \quad (4.12)$$

The drift terms h^* , g^* and f^* correspond to a worst scenario and the optimal strategy will be derived under this worst scenario. Inserting (4.5) and (4.10)-(4.12) into HJB equation (4.2) yields

$$\begin{aligned} \sup_{\pi \in \Pi} \Big\{ & -c_t(W - d) + d_t c - e_t - c(a\lambda + a\eta q + l\xi V + W r_0) \\ & + clV\beta e_V \sigma \rho_0 - e_V \kappa(\theta - V) + \frac{1}{2} m c^2 (l^2 V + b^2 q^2) + m e_V c l \sigma V \rho_0 \\ & + \frac{1}{2} (m e_V^2 - e_{VV}) \sigma^2 V + \frac{1}{2} (e_V^2 \sigma^2 V \beta + \beta c^2 b^2 q^2 + \beta c^2 l^2 V) \Big\} = 0. \end{aligned} \quad (4.13)$$

According to the first-order conditions for $\pi = \{q(t), l(t)\}_{t \in [0, T]}$, we have

$$q^*(t) = \frac{a\eta}{\gamma c b^2}, \quad l^*(t) = \frac{\xi - \gamma e_V \sigma \rho_0}{\gamma c}, \quad (4.14)$$

where $\gamma = \beta + m$.

Step 3: Separate the variables. Plugging (4.14) into (4.13) implies

$$\begin{aligned} (-c_t - cr_0)W + (c_t dc^{-1} + d_t - a\lambda)c - e_t - \frac{a^2\eta^2}{2\gamma b^2} - \frac{\xi^2 V}{2\gamma} \\ + \xi V e_V \sigma \rho_0 + \frac{1}{2} e_V^2 \sigma^2 \rho^2 V \gamma - e_V \kappa (\theta - V) - \frac{1}{2} e_{VV} \sigma^2 V = 0. \end{aligned} \quad (4.15)$$

Equation (4.15) is ensured if the following equations are satisfied:

$$-c_t - cr_0 = 0, \quad (4.16)$$

$$c_t dc^{-1} + d_t - a\lambda = 0, \quad (4.17)$$

$$-e_t - \frac{a^2\eta^2}{2\gamma b^2} - \frac{\xi^2 V}{2\gamma} + \xi V e_V \sigma \rho_0 + \frac{1}{2} e_V^2 \sigma^2 \rho^2 V \gamma - e_V \kappa (\theta - V) - \frac{1}{2} e_{VV} \sigma^2 V = 0. \quad (4.18)$$

Taking into account the boundary conditions $c(T) = 1$ and $d(T) = 0$, the solutions to (4.16) and (4.17) are

$$c(t) = \exp[r_0(T - t)], \quad (4.19)$$

$$d(t) = -a\lambda \frac{1 - \exp[-r_0(T - t)]}{r_0}. \quad (4.20)$$

To solve equation (4.18) with the boundary conditions $e(T, V) = 0$, we may propose a further ansatz: we assume that $e(t, V)$ has a linear structure as follows

$$e(t, V) = u(t) + n(t)V, \quad (4.21)$$

and we will show that a solution of this type exists. Substituting (4.21) into (4.18) and separating the variables with and without V , respectively, we can derive the following system of ODEs:

$$-u_t - \frac{a^2\eta^2}{2\gamma b^2} - n\kappa\theta = 0, \quad (4.22)$$

$$-n_t - \frac{\xi^2}{2\gamma} + (\xi\sigma\rho_0 + \kappa)n + \frac{1}{2}n^2\sigma^2\rho^2\gamma = 0 \quad (4.23)$$

with the boundary conditions $u(T) = 0$ and $n(T) = 0$. By solving (4.22) and (4.23), we can obtain the solutions as

$$u(t) = \frac{a^2\eta^2}{2\gamma b^2}(T - t) - \frac{2\kappa\theta}{\sigma^2\rho^2\gamma} \ln \left(\frac{2k_2 \exp((k_1 + k_2)(T - t)/2)}{2k_2 + (k_1 + k_2)(\exp(k_2(T - t)) - 1)} \right), \quad (4.24)$$

$$n(t) = \frac{\exp(k_2(T - t)) - 1}{2k_2 + (k_1 + k_2)(\exp(k_2(T - t)) - 1)} k_3, \quad (4.25)$$

where

$$k_1 = \xi\sigma\rho_0 + \kappa, \quad k_2 = \sqrt{\kappa^2 + 2\xi\sigma\rho_0\kappa + \xi^2\sigma^2}, \quad k_3 = \frac{\xi^2}{\gamma}.$$

Inserting (4.21) into (4.14) and combining (4.19), (4.20), (4.24) and (4.25), we derive the optimal strategy of problem (3.7). In other words, modulo verification, we have proved the following theorem.

Theorem 4.1. *For problem (3.7) with preference parameter $\phi(t) = -\frac{\beta}{mJ(t,W,V)}$, if the parameters satisfy certain conditions ⁴, the optimal strategy is given by*

$$\pi^* = \{(q^*(t), l^*(t))\}_{t \in [0, T]},$$

where

$$q(t)^* = \frac{a\eta}{\gamma b^2} \exp(-r_0(T-t)), \quad l(t)^* = \left(\frac{\xi}{\gamma} - n(t)\sigma\rho_0 \right) \exp(-r_0(T-t)), \quad (4.26)$$

and the corresponding value function is given by

$$J(t, W, V) = -\frac{1}{m} \exp \{-m[c(t)(W - d(t)) + u(t) + n(t)V]\}, \quad (4.27)$$

where $c(t)$, $d(t)$, $u(t)$ and $n(t)$ are given by (4.19), (4.20), (4.24) and (4.25), respectively.

Although the optimal strategy is derived, we should guarantee that the Radon-Nikodym derivative $\nu(t)^*$ of Q with respect to P corresponding to the optimal (worst-case scenario) drifts h^* , g^* and f^* , i.e. the expression $\nu(t)$ with h^* , g^* , f^* instead of h , g , f , is indeed a P -martingale to ensure a well-defined Q^* . The following corollary states a sufficient condition for this, based on Novikov's condition and Theorem 5.1 in Taksar & Zeng (2009).

Corollary 4.2. *Novikov's condition holds for h^* , g^* , f^* , if the parameters satisfy the following condition:*

$$\frac{\beta^2 \xi^2}{\gamma^2} + \beta^2 N^2 \sigma^2 \rho^2 < \frac{\kappa}{\sigma^2}, \quad (4.28)$$

where $N = \frac{\xi^2}{\gamma(k_1 + k_2)}$.

Proof. Putting (4.19)-(4.21), (4.24) and (4.25) into (4.10)-(4.12), we have

$$h^* = -\frac{\beta\xi}{\gamma} \sqrt{V}, \quad g^* = -\beta n \sigma \rho \sqrt{V}, \quad f^* = -\frac{a\eta\beta}{\gamma b}. \quad (4.29)$$

With condition (4.28), we can verify that $\varphi^* = (h^*, g^*, f^*)$ satisfies the Novikov's condition as follows.

⁴The verification result and sufficient conditions for it will be stated later.

Note that $n < N$ for all $t \in [0, T]$ and $T \in [0, +\infty)$, we have

$$\begin{aligned} E^P \left[\exp \left(\frac{1}{2} \int_0^T \|\varphi^*\|^2 ds \right) \right] &= E^P \left[\exp \left(\int_0^T \frac{1}{2} h^{*2} + \frac{1}{2} g^{*2} + \frac{1}{2} f^{*2} ds \right) \right] \\ &\leq K E^P \left[\exp \left(\int_0^T \frac{1}{2} (h^{*2} + g^{*2}) ds \right) \right] \end{aligned} \quad (4.30)$$

with appropriate constant $K > 0$, since f^* is deterministic and bounded on $[0, T]$. Thus

$$\begin{aligned} E^P \left[\exp \left(\frac{1}{2} (h^{*2} + g^{*2}) ds \right) \right] &= E^P \left[\exp \left(\frac{1}{2} \int_0^T \left(\beta^2 n^2 \sigma^2 \rho^2 + \frac{\beta^2 \xi^2}{\gamma^2} \right) V(s) ds \right) \right] \\ &\leq E^P \left[\exp \left(\frac{\kappa^2}{2\sigma^2} \int_0^T V(s) ds \right) \right] < \infty. \end{aligned} \quad (4.31)$$

The first estimate in (4.31) follows from the condition (4.28) and the second is from Theorem 5.1 in Taksar & Zeng (2009). \square

Subsequently, we apply the result of Corollary 1.2 in Kraft (2004) to verify the candidate value function (4.27) for problem (3.7). The verification result is summarized in the following proposition.

Proposition 4.3. *For problem (3.7), if there exists a function J , which is a solution to HJB equation (4.2), the parameters satisfy technical condition (4.28) and*

$$\begin{cases} 32\gamma^2\sigma^2\rho_0^2N^2 - 8\xi\gamma\sigma\rho_0(8-\gamma)N + 32\xi^2 - 8\gamma\xi^2 \leq \frac{\kappa^2\gamma^2}{2m^2\sigma^2}, \\ 32\xi^2 - 8\gamma\xi^2 \leq \frac{\kappa^2\gamma^2}{2m^2\sigma^2}, \end{cases} \quad (4.32)$$

then π^ is an optimal strategy for problem (3.7), and J is the corresponding value function.*

The proof of this proposition is given in the Appendix. The dollar amount $l^*(t)$ invested in the risky asset for the optimal strategy, is state-independent, and has the following structure when it is expressed in units of time T :

$$D(t) := l^*(t) \exp(r_0(T-t)) = \frac{\xi}{\gamma} - n(t)\sigma\rho_0. \quad (4.33)$$

Here, $\exp(r_0(T-t))$ stands for an accumulation factor (see Henderson (2005)). As stated in Chacko & Viceira (2005), we decomposed $D(t)$ into two components: $\frac{\xi}{\gamma}$ (the myopic demand) and $-n(t)\sigma\rho_0$ (the intertemporal demand). The first component coincides with the optimal investment strategy when the volatility is deterministic. As seen in (4.25), the second is a function of the horizon $T-t$; it is used to hedge SV, and thus is often referred to as a “hedging device”). For further detail on myopic and intertemporal demands, see Liu & Pan (2003) and Chacko & Viceira (2005).

Remark 4.4. (1) Since $n(t) > 0$, the insurer's intertemporal demand (or hedging device) $-n(t)\sigma\rho_0$ for the risky asset becomes positive if and only if $\rho_0 < 0$, i.e., the insurer only invests more in the risky asset than under the myopic demand if the shock to the risky asset price and the shock to the volatility are negatively correlated.

(2) Intertemporal demand will be affected by the horizon $T - t$. As time passes, the positive intertemporal demand ($\rho_0 < 0$) shrinks, which means that the hedging device is more useful for the long-horizon insurer.

(3) The myopic demand $\frac{\xi}{\gamma}$ is affected by an aggregate risk-aversion $\gamma = m + \beta$. We thus see that the myopic demand will shrink when the AAI has either a large m or a large β . This provides the same insight as the intuition that increase in either aspect of risk causes the insurer-investor to decrease her position in the risky asset.

In fact, comparing with the Ambiguity-Neutral Insurer (ANI) who ignores the model uncertainty, Remark 4.4(3) can be extended to following proposition.

Proposition 4.5. The AAI with preference parameter (4.4) and CARA utility function $U(W^\pi(T)) = -\frac{1}{m} \exp(-mW^\pi(T))$ has the same optimal strategy as the ANI with CARA utility function $U(W^\pi(T)) = -\frac{1}{\gamma} \exp(-\gamma W^\pi(T))$, where $\gamma = m + \beta$.

Proof. If the AAI with ambiguity-aversion coefficient β_0 extremely confirms the true model is the preference model, the preference parameter $\phi \equiv 0$ leads to $\beta_0 = 0$. As we discussed above, the AAI with $\beta_0 = 0$ insists P is the true model and becomes an ANI. Thus, the optimal strategy for the ANI with CARA utility $U(W^\pi(T)) = -\frac{1}{m+\beta} \exp(-(m+\beta)W^\pi(T))$ can be shown as $\pi_0^* = \{(q_0^*(t), l_0^*(t))\}_{t \in [0, T]}$, where

$$q_0^*(t) = \frac{a\eta}{(m+\beta)b^2} \exp(-r_0(T-t)), \quad l_0^*(t) = \left(\frac{\xi}{m+\beta} - n(t)\sigma\rho_0 \right) \exp(-r_0(T-t)).$$

According to Theorem 4.1, π_0^* coincides with the optimal strategy for the AAI with β and utility $U(W^\pi(T)) = -\frac{1}{m} \exp(-mW^\pi(T))$. \square

Remark 4.6. The AAI with high γ is prone to purchasing reinsurance. Besides, the ratio $\frac{a\eta}{b^2}$ affects the possible profit of being a reinsurer. In addition, the horizon is another key factor to influence the optimal reinsurance strategy for the insurer. If $\frac{a\eta}{b^2\gamma} \geq 1$, when $T \in \left(\frac{1}{r_0} \ln\left(\frac{a\eta}{\gamma b^2}\right), +\infty\right)$, the optimal reinsurance strategy is to acquire new business only if $T - t < \frac{1}{r_0} \ln\left(\frac{a\eta}{\gamma b^2}\right)$; when $T \in \left(0, \frac{1}{r_0} \ln\left(\frac{a\eta}{\gamma b^2}\right)\right)$, the insurer's optimal strategy is to acquire new business over the entire horizon of insurance-investment. On the other hand, if $\frac{a\eta}{b^2\gamma} < 1$, the insurer may consider that

the risk of acquiring new business is too high to be accepted. As a result, she or he purchases reinsurance to spread the risk throughout the investment-insurance horizon.

5. ANALYSIS OF OUR RESULTS AND AND NUMERICAL ILLUSTRATION

a	μ	η	b	r_0	ξ	κ	σ	θ	ρ_0	T	β	m
4	0.2	0.4	1	0.05	4	5	0.25	$(0.13)^2$	-0.4	4	1	1.2

This section is devoted to illustrating the impact of parameters and model uncertainty on optimal reinsurance and investment strategy by some numerical examples. In the following numerical illustrations, unless otherwise stated, the basic parameters are given in the above table. The basic insurance parameters are set up as in Promislow & Young (2005), and the financial market parameters for this section are from existing empirical studies, see Liu & Pan (2003). Setting $\xi = 4$ with other parameters, the average equity risk premium per year is 6.76%. Moreover, the technical conditions (4.28), (4.32) and $a/b > 3$ are met with these parameters.

As states in Section 3, only in the extreme case ($\phi \equiv 0$), would the model uncertainty fail to affect the optimal strategy. Specifically, due to (4.7)-(4.9), $\phi \equiv 0$ results in $h^* = g^* = f^* \equiv 0$, which implies that the AAI considers that the reference model is the true model, and thus the AAI is equivalent to an ANI in this case. Consequently, in the numerical examples, we need not distinguish between the ANI and the AAI with $\beta = 0$.

5.1. Robustness on the optimal strategy. In this subsection, we analyze the effects of parameters on the optimal investment strategy. For the sake of simplifying our exposition, we omit the effect of the accumulation factor, and therefore by formula (4.33), it is sufficient for us to investigate how the resulting rescaled wealth $D(t)$ invested in the risky asset changes when we modify some of the preference and other parameters. We perturb the values of parameters from 80% to 120% of their base values. Figure 1(a) and Figure 1(b) show that the wealth invested in the risky asset is reduced as the ambiguity aversion β becomes larger; on the other hand, higher premium for volatility ξ leads to invest more in the risky asset. Comparing these two subfigures, the robust optimal strategy reduces the sensitivity on ξ . Figure 1(c) and Figure 1(d) indicate an interesting feature that, for an ANI, $D(t)$ lacks sensitivity w.r.t σ and κ . This corroborates our intuition that it is not necessary to allow for robustness w.r.t. uncertainty on the vol-vol and volatility mean reversion rate parameters. More detailed discussions on these two parameters will be provided in subsection 5.4.

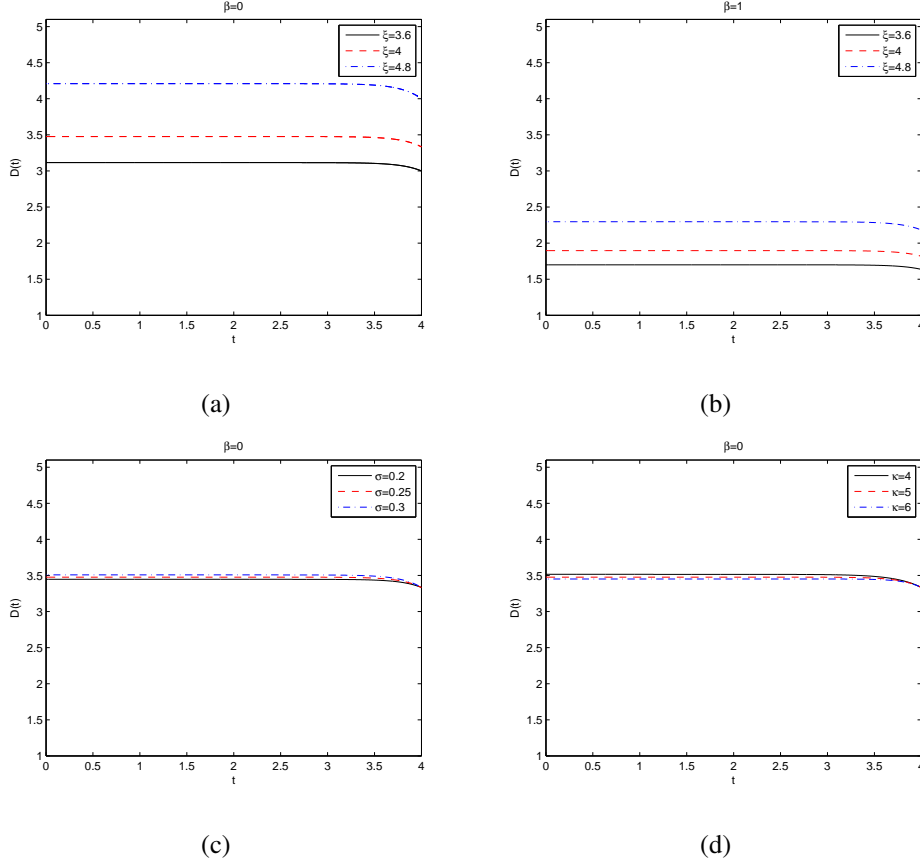


FIGURE 1. The impact of parameters on $D(t)$.

Figure 2 illustrates that the AAI has lower risk exposure in the insurance market than the ANI: specifically, the use of a robust optimal strategy decreases the sensitivity to both the drift and diffusion parameters a and b in the claims process.

In conclusion, the AAI cuts down their risk exposure significantly both in the insurance market and the investment market. An explanation for this phenomenon is that the AAI has a higher aggregate risk aversion due to the model uncertainty; this was already illustrated in Proposition 4.5.

5.2. Impact of reinsurance on the value function. To analyze the impact of reinsurance on the value function, we consider a special case in which the insurer neither acquires new business nor purchases reinsurance, i.e., $q(t) \equiv 1, \forall t \in [0, T]$. It is called an investment-only problem, and the set of all admissible strategies is denoted by $\Pi' = \{l(t) | (1, l(t)) \in \Pi\}$. For the investment-only problem, the wealth process can be written as

$$dW^\pi(t) = [a\lambda + a\eta + \xi l(t)V(t) + W^\pi(t)r_0]dt + b dZ^R(t) + l(t) \sqrt{V(t)} dZ^S(t). \quad (5.1)$$

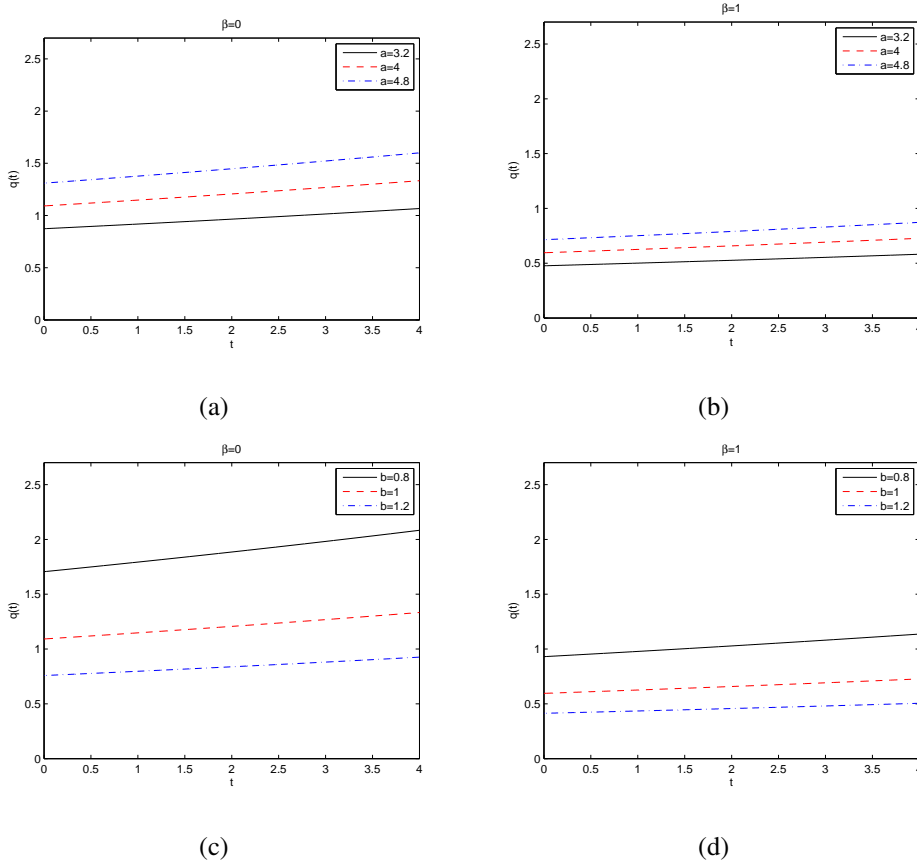


FIGURE 2. The impact of parameters on $q(t)$.

Theorem 5.1. *For the investment-only problem, the robust optimal investment strategy $l^*(t)_{t \in [0, T]}$ is given by*

$$l^*(t) = \left(\frac{\xi}{\gamma} - n(t)\sigma\rho_0 \right) \exp(-r_0(T-t)), \quad (5.2)$$

and the corresponding value function is

$$\tilde{J}(t, W, V) = -\frac{1}{m} \exp \left\{ -m \left[c(t) (W - \tilde{d}(t)) + \tilde{u}(t) + n(t)V \right] \right\}, \quad (5.3)$$

where $c(t)$ and $n(t)$ are given by (4.19) and (4.25), respectively, and

$$\tilde{d}(t) = \frac{\gamma b^2}{4r_0} \exp(r_0(T-t)) + \left(\frac{a\mu}{r_0} - \frac{\gamma b^2}{4r_0} \right) \exp(-r_0(T-t)) - \frac{a\mu}{r_0}, \quad (5.4)$$

$$\tilde{u}(t) = -\frac{2\kappa\theta}{\sigma^2\rho^2\gamma} \ln \left(\frac{2k_2 \exp((k_1 + k_2)(T-t)/2)}{2k_2 + (k_1 + k_2)(\exp(k_2(T-t)) - 1)} \right). \quad (5.5)$$

Proof. Conjecture that the value function has the form of (5.3) and substitute (5.3) into HJB equation (4.2). Using the similar approach to Theorem 4.1, the optimal investment strategy and the corresponding value function can be obtained as (5.2) and (5.3), respectively. \square

Furthermore, we introduce utility loss to measure the impact of reinsurance on the value function, and have the following proposition.

Proposition 5.2. *The utility loss of the investment-only problem, compared with the reinsurance and investment problem, is given by*

$$\begin{aligned}
L(t) &:= 1 - \frac{J(t, W, V)}{\tilde{J}(t, W, V)} = 1 - \exp \left\{ -m \left[c(t)(\tilde{d}(t) - d(t)) + u(t) - \tilde{u}(t) \right] \right\} \\
&= 1 - \exp \left\{ -m \left[\frac{\gamma b^2}{4r_0} \exp(2r_0(T-t)) + \frac{a^2 \eta^2}{2b^2 \gamma} (T-t) \right. \right. \\
&\quad \left. \left. + \left(\frac{a\eta}{r_0} - \frac{\gamma b^2}{4r_0} \right) - \frac{a\eta}{r_0} \cdot \exp(r_0(T-t)) \right] \right\}. \tag{5.6}
\end{aligned}$$

Proof. (5.6) can be derived by substituting (4.5) and (5.3) into the definition of $L(t)$. □

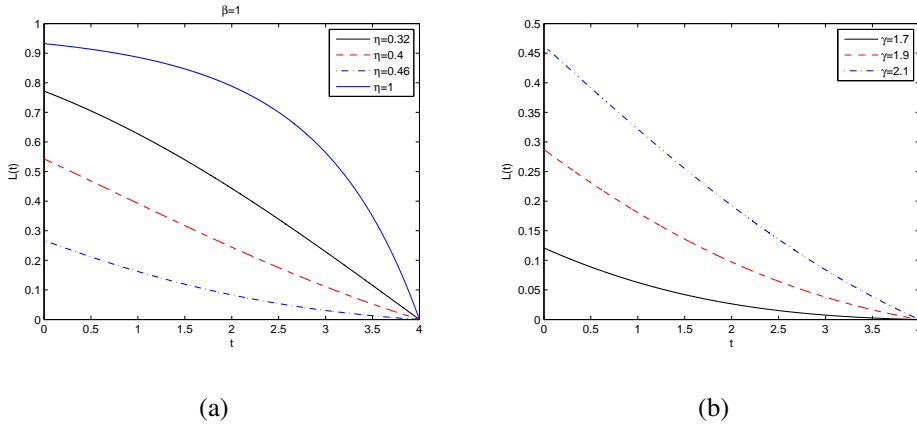


FIGURE 3. The utility loss from prohibition of reinsurance

The utility loss function $L(t)$ is independent of the state variables W and V . Figure 3 illustrates that the utility loss is caused by prohibiting reinsurance with the horizon $T - t \in [0, 4]$. As the time passes, the utility loss drops. A reasonable explanation is that the long-horizon insurer relies on reinsurance more significantly to divert potential risk. Figure 3(a) shows that the utility loss decreases with reinsurance premium η at the beginning, and later increases after η reaches a threshold. Intuitively, an extremely high reinsurance premium provides a favored position for acquiring business. Thus it will causes utility loss if acquiring new business is prohibited. Figure 3(b) indicates that the utility loss increases w.r.t γ , which indicates that the AAI with higher aggregate risk-aversion demands much more for reinsurance for risk-spreading.

5.3. Model uncertainty robustness. If we understand J_β as the value function for the AAI with β and J_0 as the value function for the ANI, the utility deviation for model uncertainty can

be defined as

$$L_1^\beta := 1 - \frac{J_0}{J_\beta}.$$

Given $t \in [0, T]$, base wealth $W(t) = 1$ and base volatility $V(t) = 0.225$, Figure 4(a) shows the utility deviations for three levels of model uncertainty. With higher β , the AAI would be willing to give up higher utility to seek a much more conservative strategy. This implies that the more suspicion about preference model she or he has, the more utility deviation will be accepted. Moreover, L_1^β is increasing as the horizon extends, which indicates that the utility deviation of model uncertainty in long-horizon is much larger than the deviation in short-horizon.

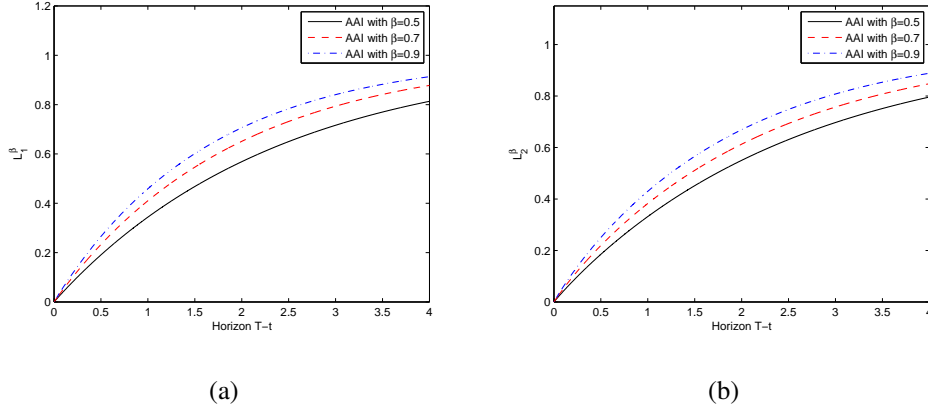


FIGURE 4. Model uncertainty robustness on the value function

Now we turn to consider a suboptimal strategy. Assume the AAI with ambiguity-aversion coefficient $\beta > 0$ does not take the optimal strategy (4.26), but makes decisions as if she or he is an ANI. In other words, the AAI follows the strategy $\pi_0 = \{q_0(t), l_0(t)\}_{t \in [0, T]}$ given by (4.26) with $\beta = 0$ as

$$q_0(t) = \frac{a\eta}{mb^2} \exp(-r_0(T-t)), \quad l_0(t) = \left(\frac{\xi}{m} - n(t)\sigma\rho_0 \right) \exp(-r_0(T-t)).$$

where $n(t)$ is given by (4.25). The value function for the AAI following the given strategy π_0 is defined by

$$J^{\pi_0}(t, W, V) = \inf_{Q \in \mathcal{Q}} E_{t, W, V}^Q \left[\int_t^T \frac{1}{\phi^{\pi_0}}(s) R(s) ds + U(W^{\pi_0}(T)) \right],$$

where $\phi^{\pi_0}(t) = -\frac{\beta}{mJ(t, W^{\pi_0}, V)}$ is the preference parameter with π_0 . Via a calculation that is parallel to previous ones, we arrive at

$$J^{\pi_0}(t, W, V) = -\frac{1}{m} \exp \{ -m [c(t)(W - d(t)) + \bar{u}(t) + \bar{n}(t)V] \},$$

where $\bar{u}(t)$ and $\bar{n}(t)$ satisfy the following ODEs

$$\begin{aligned} -\bar{u}_t(t) - \frac{a^2 \eta^2}{mb^2} \left(1 - \frac{\gamma}{2m} - \bar{u}(t) \kappa \theta \right) &= 0, \quad \bar{u}(0) = 0, \\ -\bar{n}_t(t) + \frac{1}{2} \gamma \sigma^2 \bar{n}(t)^2 + \left(\frac{\gamma}{m} \sigma \rho_0 \xi - \gamma \sigma^2 \rho_0^2 n(t) + \kappa \right) \bar{n}(t) &+ \left(\frac{\gamma}{2m^2} - \frac{1}{m} \right) \xi^2 \\ + \frac{1}{2} \gamma \sigma^2 \rho_0^2 n(t)^2 + \xi \sigma \rho_0 n(t) - \frac{\gamma}{m} \xi \sigma \rho_0 n(t) &= 0, \quad \bar{n}(0) = 0, \end{aligned}$$

with $c(t)$, $d(t)$ and $n(t)$ are given by (4.19), (4.20) and (4.25), respectively. Define the utility loss for the suboptimal strategy π_0 as

$$L_2^\beta := 1 - \frac{J_\beta}{J_\beta^{\pi_0}}.$$

Figure 4(b) discloses the utility losses from ignoring model uncertainty for the AAI as a function of horizon $T - t$. For the AAI who has less information about the model P (higher β), the utility loss is higher than for the one with more information. Moreover, the utility losses has a remarkable upward trend as the horizon $T - t$ extends.

Overall, the impact of model uncertainty on the optimal strategy and the corresponding value function is highly significant. The AAI is willing to abandon a certain proportion of utility to guard against some adverse models which are not too far away from the reference model. Furthermore, the AAI suffers significant utility loss if she or he ignores model uncertainty, especially in long-horizon cases.

5.4. An improvement in finance applications. Although we have studied the impact of robustness along the line of Anderson et al. (1999), we uncovered an interesting feature in Subsection 5.1 by which the optimal investment strategy for the ANI seems to lack sensitivity to the SV parameters σ and κ . In addition to this, there is a possible drawback seen in our robustness calculations: the deviation for the AAI attains a surprising rate above 80% with a long horizon $T - t = 4$, which means that the AAI will abandon 80% of her or his utility to guard against model uncertainty. In real markets, it is doubtful whether an insurer will abandon so much utility to acquire a robust optimal strategy. The culprit could be the fact that, as illustrated in Figures 1(c) and 1(d), model uncertainty on the volatility model is largely irrelevant in terms of the actual strategy followed, but is costly in terms of utility which is then presumably more sensitive to the strategy followed.

Therefore, we propose abandoning robustness hedging on $dV(t)$ in (2.4), in an effort to obtain a lower utility deviation in finance applications. Since κ , σ are the parameters of SV, we assume

that the AAI has full knowledge on $dV(t)$, and does not consider the robustness on $dV(t)$. Technically, to eliminate the robustness on SV, we simply require that the drift terms h and g in (3.8) satisfy $\rho_0 h(t) + \rho g(t) = 0$; the Heston SV model under the reference model $Q \in \mathcal{Q}$ can then be shown to be

$$\begin{cases} dS_1(t) = S_1(t) \left[(r_0 + \xi V(t) + \sqrt{V(t)}h(t)) dt + \sqrt{V(t)}dZ_Q^S(t) \right], \\ dV(t) = [\kappa(\theta - V(t))] dt + \sigma \sqrt{V(t)}\rho_0 dZ_Q^S(t) + \sigma \sqrt{V(t)}\rho dZ_Q^V(t). \end{cases}$$

Using a similar approach to Theorem 4.1, we obtain the following theorem without the robustness on SV.

Theorem 5.3. *For the optimal problem without robustness on SV, the robust optimal reinsurance strategy $q^*(t)$ keeps the same form in (4.26). However, the robust optimal investment strategy $l^*(t)$ is now has the form*

$$l^*(t) = \frac{\xi - mn_2(t)\sigma\rho_0}{c\gamma_2},$$

where $\gamma_2 = m + \beta\rho^2$, and the corresponding value function is given by

$$J(t, W, V) = -\frac{1}{m} \exp \{ -m [c(t)(W - d(t)) + u_2(t) + n(t)V] \},$$

where $c(t)$, $d(t)$ and $n(t)$ are given by (4.19), (4.20) and (4.25) respectively, u_2 has the forms as

$$u_2(t) = \frac{a^2\eta^2}{2\gamma b^2}(T - t) - \frac{2\kappa\theta\gamma_2}{m\sigma^2\rho^2\gamma} \ln \left(\frac{2k_2 \exp((k_1 + k_2)(T - t)/2)}{2k_2 + (k_1 + k_2)(\exp(k_2(T - t)) - 1)} \right).$$

Here, k_1 , k_2 and k_3 in $n(t)$ and $u_2(t)$ are given by

$$k_1 = \frac{m\xi^2\sigma^2}{\gamma_2} + \kappa, \quad k_2 = \sqrt{\kappa^2 + \frac{2\xi\sigma\rho_0\kappa m}{\gamma_2} + \frac{\xi^2\sigma^2 m}{\gamma_2}}, \quad k_3 = \frac{\xi^2}{\gamma_2}.$$

With the assumption of full SV model knowledge and observation, Proposition 4.5 is still correct iff $\rho = 1$. In fact, β can not be seen as a direct replacement for risk aversion m if $\rho \neq 1$. Instead, ρ plays another important role in the robust decision under this situation. Figure 5(a) and Figure 5(b) show the great impact of ρ on robust optimal investment strategy, in the range of standard and extreme levels of leverage ($\rho < 0$). Since we have full knowledge on SV, the AAI has more information about the price of the risky asset when $V(t)$ and $S_1(t)$ are highly correlated. This compensates the model uncertainty robustness. Figure 5(b) indicates that the model uncertainty robustness completely loses its impact on the investment strategy $D(t)$ with extreme negative correlation $\rho_0 = -1$.

This assumption of full model and observation information on SV may influence the robust optimal reinsurance strategy only mildly, but makes a big difference in terms of value function:

it enhances the performance for expected utility remarkably. Figure 5(c) gives us the significant result that the utility deviation L_1^β for the AAI with $\beta = 0.5$ compared to the ANI, remains below 20% even with a long horizon $T - t = 4$. This compares extremely favorably with the same levels reported in Figure 4(a) in subsection 5.3 when volatility parameter uncertainty is hedged.

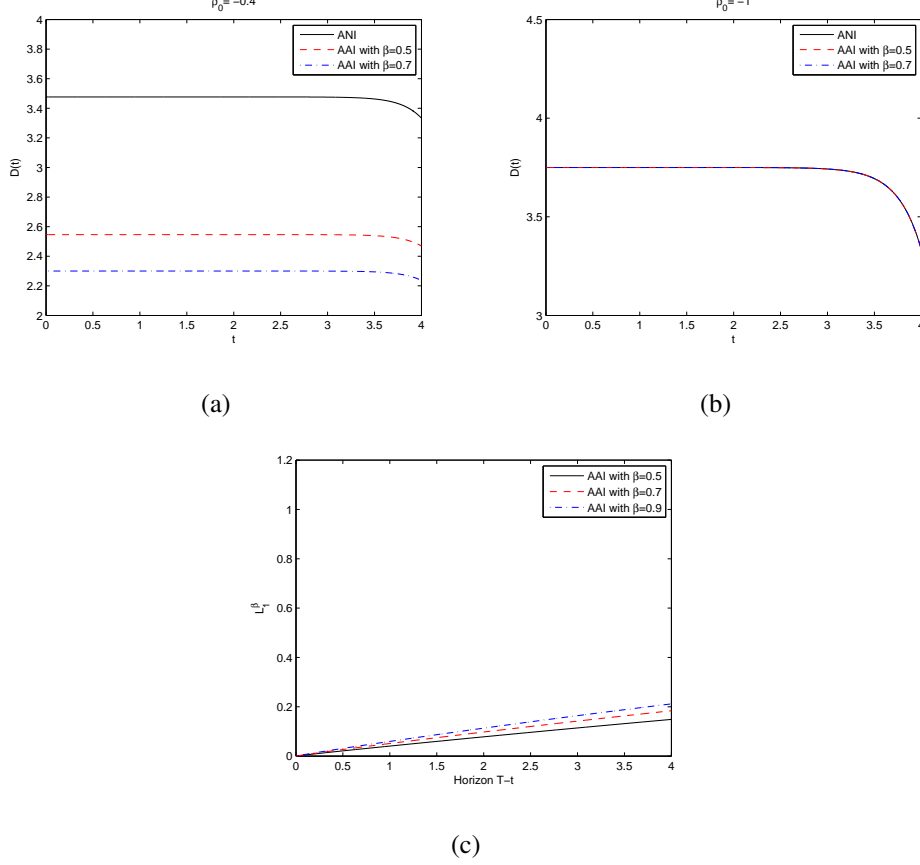


FIGURE 5. Results without robustness on SV

6. CONCLUSIONS

In this paper, we have investigated optimal reinsurance and investment strategies for an AAI under Heston's SV model. The claims process of the insurer is approximated by a standard diffusion model. She or he can invest wealth in a time-dependent financial environment, modeled by Heston's SV model. At the same time, the insurer may lack full confidence in the model describing the economy. We propose a systematic analysis of the impact of reinsurance and model uncertainty on optimal strategies for the AAI.

We have formulated a general problem and derived the optimal strategy, and the corresponding well-defined value function have been derived based on the technical assumptions. In addition, we have defined utility losses to analyze the impact of reinsurance and robust optimal

strategy on the value functions. We also have investigated the model uncertainty robustness, and explored some economic implications. The main findings are as follows. (i) The optimal strategy and the corresponding value function for the AAI are well-defined within certain parameter ranges, and the well-defined optimal strategy is state-independent. (ii) The optimal strategy for the AAI is affected by the attitude toward ambiguity, and the AAI facing model uncertainty has a safer optimal strategy. In addition, much utility loss is experienced if the AAI operates like an ANI who does not lend heed to the ambiguity. (iii) Abandoning the robustness on SV is preferable in financial applications, particularly when investing in markets such as the S&P500, where excellent volatility tracking exists. In fact, the parameters in the SV model have moderate influence on the optimal investment strategy, but a significant utility enhancement occurs when one assumes full knowledge of these parameters.

Our conclusions lead to some promising directions for future works. (i) For other non-utility criteria, such as the mean-variance criterion, would the qualitative conclusions for the AAI in this paper still be satisfied? (ii) Our basic premise is that the SV in financial markets can be observed completely; while this is becoming increasingly true in certain markets with highly liquid options trading, other authors point out the limitations of this assumption: Pham & Quenez (2001), Viens (2002) and Kim & Viens (2012), particularly when transaction costs must be taken into account. In such cases, one ought to investigate the robust control problem for the insurer under partially observed SV.

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APPENDIX

In this appendix, we prove Proposition 4.3.

Proof. We are able to prove Proposition 4.3 by using Corollary 1.2 in Kraft (2004) if π^* and the corresponding candidate value function $J(t, W, V)$ has the following properties:

- (1). π^* is an admissible strategy;
- (2).

$$E^{\mathcal{Q}^*} \left(\sup_{t \in [0, T]} |J(t, W^{\pi^*}, V)|^4 \right) < \infty \quad (6.1)$$

where \mathcal{Q}^* is defined by $\nu(t)^*$ with h^* , f^* and g^* ;

- (3).

$$E^{\mathcal{Q}^*} \left(\sup_{t \in [0, T]} \left| \frac{1}{\phi(t)} R^*(t) \right|^2 \right) < \infty \quad (6.2)$$

with $W(0) = w_0$, $V(0) = v_0$ and $R^*(t) = \frac{1}{2}(h^*(t)^2 + f^*(t)^2 + g^*(t)^2)$;

Subsequently, we verify the properties (1)-(3), respectively.

Proof of (1). Optimal strategy π^* is deterministic and state-independent, thus condition (i) in Definition 3.1 is met. Condition (ii) in Definition 3.1 can be obtained by Property (2).

Proof of (2). Substituting (4.26) and (4.29) into (3.9), we have

$$\begin{aligned} W^{\pi^*}(t) = & w_0 \exp(r_0 t) + \int_0^t a \lambda \exp(r_0(t-s)) ds + \frac{1}{c(t)} \left\{ \int_0^t \frac{ma^2 \eta^2}{\gamma b^2} + \right. \\ & \left. + \underbrace{\frac{m\xi(\xi - \gamma \sigma \rho_0 n(s))}{\gamma}}_A V ds + \underbrace{\int_0^t \frac{a\eta}{\gamma b} dZ_{\mathcal{Q}^*}^R}_B + \underbrace{\int_0^t \frac{\xi - \gamma \sigma \rho_0 n(s)}{\gamma} \sqrt{V} dZ_{\mathcal{Q}^*}^S}_C \right\}. \end{aligned} \quad (6.3)$$

Inserting (6.3) into candidate value function (4.27), we obtain the following estimate with appropriate constants $K > 0$:

$$\begin{aligned} |J(t, W^{\pi^*}, V)^4| &= \left| \frac{1}{m^4} \exp(-4mcW^{\pi^*} + 4mcd - 4mu - 4mnV) \right| \leq K \exp[-4mcW^{\pi^*}] \\ &\leq K \exp \left[-4m \left(\int_0^t AV ds + \int_0^t B dZ_{\mathcal{Q}^*}^R + \int_0^t C \sqrt{V} dZ_{\mathcal{Q}^*}^S \right) \right]. \end{aligned} \quad (6.4)$$

The first estimate in (6.4) is valid, since m , c , d and u are deterministic and bounded on $[0, T]$ with $n(t) > 0, \forall t \in [0, T]$. The second follows from deterministic and bounded $w_0 \exp(r_0 t)$, $a \lambda \exp(r_0(t-s))$ and $\frac{ma^2 \eta^2}{\gamma b^2}$.

Now we consider the integral $\exp(\int_0^t -4mBdZ_{Q^*}^R)$. Note that $mB = \frac{a\eta m}{\gamma b}$ is bounded on $[0, T]$, we find

$$\exp\left(\int_0^t -4mBdZ_{Q^*}^R\right) = \underbrace{\exp\left(\int_0^t 8m^2B^2ds\right)}_{const.} \cdot \underbrace{\exp\left(-\int_0^t 8m^2B^2ds + \int_0^t -4mBdZ_{Q^*}^R\right)}_{martingale}.$$

Consequently,

$$E^{\mathcal{Q}^*}\left(\exp\left(\int_0^t -4mBdZ_{Q^*}^R\right)\right) < \infty. \quad (6.5)$$

Then, we aim to find an estimate for $\exp\left(\int_0^t -4mAVds + \int_0^t -4mC\sqrt{V}dZ_{Q^*}^S\right)$:

$$\begin{aligned} & \exp\left(\int_0^t -4mAVds + \int_0^t -4mC\sqrt{V}dZ_{Q^*}^S\right) \\ &= \underbrace{\exp\left[\int_0^t (16m^2C^2 - 4mA)Vds\right]}_G \cdot \underbrace{\exp\left(-\int_0^t 16m^2C^2Vds - \int_0^t 4mC\sqrt{V}dZ_{Q^*}^S\right)}_F. \end{aligned}$$

For the term F , we can find an estimate as

$$E^{\mathcal{Q}^*}(F^2) = E^{\mathcal{Q}^*}\left[\exp\left(-\int_0^t 32m^2C^2Vds - \int_0^t 8mC\sqrt{V}dZ_{Q^*}^S\right)\right] < \infty, \quad (6.6)$$

since F^2 is a nonnegative local martingale, and thus it is a supermartingale. In fact, F^2 is a martingale due to bounded function $-8mC$ on $[0, T]$ (see Lemma 4.3 in Taksar & Zeng (2009)).

For the term G , we estimate $E^{\mathcal{Q}^*}(G^2)$ as

$$E^{\mathcal{Q}^*}(G^2) = E^{\mathcal{Q}^*}\left\{\exp\left[\int_0^t (32m^2C^2 - 8mA)Vds\right]\right\}. \quad (6.7)$$

Again applying Theorem 5.1 in Taksar & Zeng (2009), we obtain the following sufficient condition for $E^{\mathcal{Q}^*}(G^2) < \infty$:

$$32m^2C^2 - 8mA \leq \frac{\kappa^2}{2\sigma^2}, \quad (6.8)$$

which is equivalent to the inequality

$$32\gamma^2\sigma^2\rho_0^2n^2 - 8\xi\gamma\sigma\rho_0(8 - \gamma)n + 32\xi^2 - 8\gamma\xi^2 \leq \frac{\kappa^2\gamma^2}{2m^2\sigma^2}. \quad (6.9)$$

Note $\forall T \in \mathbb{R}^+$, $0 < n(t) < N, t \in [0, T]$ and the technical condition (4.32), (6.9) holds for $\forall T \in \mathbb{R}^+$ and $\forall t \in [0, T]$ because of the property of quadratic function.⁵ Hence, applying

⁵Our conditions exclude the unstable situation in the sense that value function might be infinite when the horizon exceeds a certain level.

(6.4)-(6.6) and $E^{\mathcal{Q}^*}(G^2) < \infty$, we can arrive at

$$\begin{aligned}
E^{\mathcal{Q}^*} |J(t, W^{\pi^*}, V)^4| &\leq KE^{\mathcal{Q}^*} \left\{ \exp \left[-4m \left(\int_0^t AV ds + \int_0^t BdZ_{\mathcal{Q}^*}^R + \int_0^t C\sqrt{V}dZ_{\mathcal{Q}^*}^S \right) \right] \right\} \\
&= KE^{\mathcal{Q}^*} \left(\exp \left(\int_0^t -4mBdZ_{\mathcal{Q}^*}^R \right) \right) \cdot E^{\mathcal{Q}^*} \left[\exp \left(\int_0^t -4mAV ds + \int_0^t -4mC\sqrt{V}dZ_{\mathcal{Q}^*}^S \right) \right] \\
&\leq KE^{\mathcal{Q}^*}(GF) \leq K \left(E^{\mathcal{Q}^*}(G^2) E^{\mathcal{Q}^*}(F^2) \right)^{\frac{1}{2}} < \infty.
\end{aligned}$$

The first estimate follows from (6.4). The second equation is satisfied since $Z_{\mathcal{Q}^*}^R(t)$ is independent of $Z_{\mathcal{Q}^*}^V(t)$ and $Z_{\mathcal{Q}^*}^S(t)$. The second estimate is valid due to (6.5). The third estimate follows from Cauchy-Schwarz inequality while the last from $E^{\mathcal{Q}^*}(G^2) < \infty$ and (6.6). Therefore, property (2) is proved.

Proof of (3). Inserting (4.4) into (6.2), we obtain

$$E^{\mathcal{Q}^*} \left(\sup_{t \in [0, T]} \left| \frac{1}{\phi(t)} R^*(t) \right|^2 \right) = E^{\mathcal{Q}^*} \left(\sup_{t \in [0, T]} \frac{m^2}{\beta^2} |J(t, W^{\pi^*}, V)|^2 |R^*|^2 \right) \quad (6.10)$$

Property (3) can be proved if (6.10) is well-defined. Since $\forall M > 0, E^{\mathcal{Q}^*}(V^M) < \infty$, we have $E^{\mathcal{Q}^*}(R^{*4}) < \infty$. From property (2), we obtain $E^{\mathcal{Q}^*} |J(t, W^{\pi^*}, V)|^4 < \infty$. Hence, (6.10) is well-defined and thus property (3) is verified.

With all the properties are satisfied, we can simply apply Corollary 1.2 in Kraft (2004) to prove Proposition 4.1 which guarantees that π^* is the optimal strategy for problem (3.7) and $J(t, W, V)$ is the corresponding value function. \square