Open access • Journal Article • DOI:10.1007/S11081-009-9086-2

Robust production management — Source link

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Published on: 24 Apr 2009 - Optimization and Engineering (Springer US)

Topics: Stochastic programming, Robust optimization and Linear programming

ROBUST PRODUCTION MANAGEMENT

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Abstract

The problem of production management can often be cast in the form of a linear program with uncertain parameters and risk constraints. Typically, such problems are treated in the framework of multi-stage Stochastic Programming. Recently, a Robust Counterpart (RC) approach has been proposed, in which the decisions are optimized for the worst realizations of problem parameters. However, an application of the RC technique often results in very conservative approximations of uncertain problems. To tackle this drawback, an Adjustable Robust Counterpart (ARC) approach has been proposed in (Ben-Tal et al. 2003). In ARC, some decision variables are allowed to depend on past values of uncertain parameters. A restricted version of ARC, introduced in (Ben-Tal et al. 2003), which can be efficiently solved, is referred to as Affinely Adjustable Robust Counterpart (AARC).

In this paper, we consider an application of the ARC and AARC methodologies to the problem of yearly electricity production management in France. We provide tractable formulations for the AARC of quadratic and of some conic quadratic optimization problems, as well as for the ARC and AARC of the electricity production problem. We then give the quality of robust solutions obtained by using different uncertainty sets estimated using simulated and historical data. Our methodology is finally compared with other management methods.

Keywords: Uncertain linear programs; Affinely Adjustable Robust Counterpart; Robust Optimization; Stochastic Programming; Mid-term generation problem.

Mathematics Subject Classification: 90C31, 90C20, 90C22, 91B28, 62G05.

1. Introduction

Given a fixed mix of electric power plants (nuclear, thermal, hydroelectric, and demand side management contracts modelled as virtual production plants) we need to minimize the production costs over the management horizon while satisfying the demand and some operational constraints at each time step. In practice, the modelling approach is highly dependent on the time horizon of the optimization problem: for short time horizons, typically of one day or of one week, the problem is generally assumed to be deterministic (Batut and Renaud 1992), whereas for longer management horizons, a special emphasis is done on the stochastic nature of data. In particular, on a yearly scale, reservoir inflows, demand, as well as availability of the plants cannot be considered deterministic.

Production management problems (and particularly electricity production management problems) have been widely studied both concerning the modelling and the solution methods (Brignol and Renaud 1997; Dentcheva and Römisch 1998; Philpott et al. 2000; Gröwe-Kuska and Römisch 2005 for instance). Generally, the evolution of the uncertain parameters over the management period is modelled by a scenario tree and the goal is to minimize the expected production cost over this set of scenarios (Gröwe-Kuska and Römisch 2005). Recent solution methods (e.g. Bacaud et al. 2001), use Lagrangian relaxation and various nondifferentiable optimization methods and tools to solve the associated local subproblems.

From the modelling point of view, we present an alternative to the use of scenario trees. Our objective is to propose robust management methods. Little attention has been paid so far to this preoccupation while non-robust policies could lead to large financial losses if difficult scenarios occur. To our knowledge, a robust approach for the electricity production management problem was first introduced in (Brignol and Rippault 1999) where the scenarios of the scenario tree are ranked in order to find the most unfavorable. The deterministic optimization problem corresponding to the worst scenario $\mathcal{S}_{det}$ with optimal value $\mathcal{C}_{det}$ is then solved. A stochastic optimization over the scenario tree is then done adding a constraint ensuring that the cost on the scenario $\mathcal{S}_{det}$ must be close to $\mathcal{C}_{det}$ i.e., $\mathcal{C}_{sto} \leq \mathcal{C}_{det} + \varepsilon$, where $\mathcal{C}_{sto}$ is the cost on the scenario $\mathcal{S}_{det}$ resulting from the global optimization on the scenario tree and ε a tolerance level. One critical step in this approach is to determine a procedure for ranking the scenarios between them. More recently, a robust methodology also based on a scenario tree was proposed in [13] while risk measures were used in (Eichhorn et al. 2004).

The paper is organized as follows. We introduce the physical model and its mathematical formulation in Section 2. In Section 3, we introduce ARC and AARC, and give some tractability results for ARC and AARC of uncertain optimization problems (from (Ben-Tal et al. 2003) for linear programming problems). We then explain in Section 4 how to apply the ARC and AARC methodologies to the electricity production management problem. In particular, in this section, we provide closed-form expressions for AARC of this problem and comment on the calibration of the uncertainty sets. Finally, in Section 5, simulated and real data are used to compare the robust methods of this paper with other management methods.

2. Modelling of the electricity power generation problem

The goal is to decide the production levels of the plants composing the mix in such a way that the demand and the operational constraints are satisfied at each time step, and the production cost is minimized. The physical model we consider is a stochastic dynamical system for which the uncertain parameters are the electric consumption, the availability rates of thermal plants and the water inflows into reservoirs. The modelling uses some aspects of (Dentcheva and Römisch 1998) and (Gröwe-Kuska and Römisch 2005), introduces availability rates for the thermal plants, and value functions for hydroelectric plants and EJP contracts ("Effacement Jours de Pointe" in French, or "demand side management" contracts in English).

Let T be the number of subintervals obtained by partitioning the time horizon. This partition can be chosen uniformly (daily, weekly, monthly) or adaptively. Let $\mathcal{L} = \mathcal{L}_T \cup \mathcal{L}_H \cup \mathcal{L}_J$ be a partition of the set of production units in which $\mathcal{L}_T$ represents the set of thermal plants, $\mathcal{L}_H$ the set of hydroelectric plants and $\mathcal{L}_J$ the set of EJP contracts. We denote the duration (in hours) of time step t by $Duration(t)$ and for every $\ell \in \mathcal{L}$, the minimal and maximal theoretical production powers for unit ℓ are respectively denoted by $P_{\min}^{\ell}$ and $P_{\max}^{\ell}$ (in MW).

2.1. **Thermal plant modelling.**

For every thermal plant $\ell \in \mathcal{L}_T$, we denote the production level (in MWh) or control at time step t by p_t^{ℓ}. Each thermal plant ℓ is made of a certain number of thermal groups, some of which can be out of commission at time step t. Thus, for each time step t and each thermal plant ℓ, the minimal and maximal theoretical powers $P_{\min}^{\ell}$ and $P_{\max}^{\ell}$ must be corrected taking into account the availability rate τ_t^{ℓ} of thermal plant ℓ at time step t. The real minimal and maximal powers for time step t and thermal plant ℓ are then respectively $\tau_t^{\ell} P_{\min}^{\ell}$ and $\tau_t^{\ell} P_{\max}^{\ell}$. The constraints on the production levels of the thermal plants may thus be expressed as

$$\tau_t^{\ell}\, P_{\min}^{\ell}\, Duration(t) \leq p_t^{\ell} \leq \tau_t^{\ell}\, P_{\max}^{\ell}\, Duration(t), \quad \ell \in \mathcal{L}_T, \quad t = 1, \ldots, T. \tag{1}$$

The production cost is a linear function of the production level with a fixed unit production cost c_{ℓ} (in Francs[1]/MWh) for plant ℓ. For mid or long-term management, nuclear power plants can be modelled similarly, (Brignol and Renaud 1997).

2.2. **Hydroelectric power station modelling.**

The hydroelectric network is made up of a set of interconnected hydroelectric plants and reservoirs. Each plant can have one or several turbines and each turbine can receive water from different reservoirs. For the simplicity of the exposure, we suppose that each hydroelectric plant only has one upstream reservoir. The index of station $\ell \in \mathcal{L}_H$ and of its upstream reservoir are the same. For every station $\ell \in \mathcal{L}_H$, let $(p_t^{\ell}, sp_t^{\ell})$ be the

[1] 1 Euro=6.55 957 Francs

control applied to station ℓ at time step t, where p_t^ℓ is the production (in MWh) of station ℓ at time step t and sp_t^ℓ the spillage at time step t for reservoir ℓ if there is some overflow. State x_t^ℓ of hydroelectric plant ℓ at time step t corresponds to the volume (in MWh) of reservoir ℓ at the beginning of this time step. Two kinds of inflows tend to increase the reservoir levels. On the one hand, the natural inflows due to rainwater: we denote the natural inflow (in MWh) of reservoir ℓ at time step t by $\mathcal{I}_t^\ell$. On the other hand, for each reservoir ℓ, water can come from upstream hydroelectric plants. For every station $\ell \in \mathcal{L}_H$, we denote the index of the reservoir downstream station ℓ (if it exists) by $f(\ell)$; the time (in time steps) for the water to go from station ℓ to reservoir $f(\ell)$ being $d(\ell, f(\ell))$. We assume that the stations are always available. The operational constraints of the stations are of two kinds: (i) box constraints on the production levels and the reservoir volumes and (ii) flow balance equations for each reservoir. These constraints read

$$\begin{array}{ll} x_{t+1}^\ell = x_t^\ell + \mathcal{I}_t^\ell - p_t^\ell + \sum_{m \,\mid\, f(m)=\ell} p_{t-d(m,f(m))}^m - sp_t^\ell, & \forall\, \ell \in \mathcal{L}_H, \forall\, t = 1, \ldots, T, \\ Duration(t) P_{\min}^\ell \le p_t^\ell \le Duration(t)\, P_{\max}^\ell, & \forall\, \ell \in \mathcal{L}_H, \forall\, t = 1, \ldots, T, \\ x_{\min}^\ell \le x_{t+1}^\ell \le x_{\max}^\ell,\ sp_t^\ell \ge 0, & \forall\, \ell \in \mathcal{L}_H, \forall\, t = 1, \ldots, T, \end{array} \tag{2}$$

where $x_{\min}^\ell$ and $x_{\max}^\ell$ are the minimal and maximal levels (in MWh) of reservoir ℓ volume. The initial stock of each reservoir is known. To use the water in a parsimonious way (the more water we have at the end of the year, the more we can use the next period), two strategies are frequently used. The first one (Gröwe-Kuska and Römisch 2005) consists of constraining the level of the reservoir at the end of the management period to be greater than the level it had at the beginning of this period. An alternative choice consists of associating to each reservoir ℓ, value function $V_H^\ell(\cdot)$ for the water stock at the last time step. This function associates a value (in Francs) to each admissible value of the water stock. It is an increasing function which is quadratic in (Brignol and Rippault 1999) and linear in (Philpott et al. 2000). We suppose this function is concave and piecewise affine. We then wish to maximize the value $\sum_{\ell \in \mathcal{L}_H} V_H^\ell(x_{T+1}^\ell)$ of the hydro energy stock at the end of the period.

2.3. **EJP contract modelling.** The EJP contracts can be seen as independent reservoirs, having a limited production capacity. They cannot be used more than a certain number of time steps fixed by the contract and for each time step either the full production capacity is used or this capacity is not used at all. For every EJP contract $\ell \in \mathcal{L}_J$, let p_t^ℓ be the production (in MWh) or control applied to this contract ℓ for time step t. If the EJP contract is used for time step t, $p_t^\ell = Duration(t) P_{\max}^\ell$ and $p_t^\ell = 0$ otherwise. The state of EJP contract ℓ for time step t is represented by the variable x_t^ℓ giving the energy stock still available on this contract at the beginning of time step t. The energy stock x_1^ℓ available on contract ℓ over the optimization period is given. The equations ruling the evolution of the controls and states is thus

$$(p_t^\ell - Duration(t) P_{\max}^\ell)\, p_t^\ell = 0,\ x_{t+1}^\ell = x_t^\ell - p_t^\ell,\ x_{t+1}^\ell \ge 0, \ell \in \mathcal{L}_J,\ 1 \le t \le T.$$

In this article, we use flexible EJP contracts convexifying the constraints, in such a way that the states and controls of the EJP contracts satisfy

$$0 \le p_t^\ell \le Duration(t)\ P_{\max}^\ell,\ \ x_{t+1}^\ell = x_t^\ell - p_t^\ell,\ \ \ x_{t+1}^\ell \ge 0,\ \ \ell \in \mathcal{L}_J,\ \ 1 \le t \le T. \tag{3}$$

This means that for each EJP contract, we have a certain amount of energy, which, every day, and as long as this reserve is not finished, can be used all or part of the day. This modelling amounts to considering that an EJP contract is a particular hydro reservoir without inflows. Indeed, similar to hydro reservoirs, we associate to each contract ℓ a value function $V_J^\ell(\cdot)$ for the energy stock at the end of the horizon. This function is concave and piecewise affine. We then maximize the sum $\sum_{\ell \in \mathcal{L}_J} V_J^\ell(x_{T+1}^\ell)$ of the energy stock at the end of the horizon.

2.4. **Deterministic formulation of the problem.** The production units have to be used in order to satisfy the demand. If $\mathcal{D}_t$ is the electric consumption for time step t, the demand satisfaction constraints read

$$\sum_{\ell \in \mathcal{L}} p_t^\ell = \sum_{\ell \in \mathcal{L}_T} p_t^\ell + \sum_{\ell \in \mathcal{L}_H} p_t^\ell + \sum_{\ell \in \mathcal{L}_J} p_t^\ell \ge \mathcal{D}_t,\ \ t = 1, \ldots, T. \tag{4}$$

The electricity production management problem then consists of minimizing

$$\sum_{\ell \in \mathcal{L}_T} \sum_{t=1}^{T} c_\ell \, p_t^\ell - \sum_{\ell \in \mathcal{L}_H} V_H^\ell(x_{T+1}^\ell) - \sum_{\ell \in \mathcal{L}_J} V_J^\ell(x_{T+1}^\ell),$$

under constraints (1),(2), (3), and (4). Function $V_H^\ell(\cdot)$ can be expressed as

$$V_H^\ell(x) = \min_{0 \le k \le m_H^\ell - 1} f_{H,k}^\ell(x),$$

where functions $f_{H,k}^\ell(\cdot)$ are affine. More precisely, between values denoted by $g_{H,k}^\ell$ and $g_{H,k+1}^\ell$ $(0 \le k \le m_H^\ell - 1)$, function $V_H^\ell(\cdot)$ coincides with function $f_{H,k}^\ell(\cdot)$. Thus, we can replace the contribution $-\sum_{\ell \in \mathcal{L}_H} V_H^\ell(x_{T+1}^\ell)$ of the hydroelectric plants to the objective by $-\sum_{\ell \in \mathcal{L}_H} a_\ell$ adding the constraints

$$a_\ell \le f_{H,k}^\ell(x_{T+1}^\ell), \;\; \ell \in \mathcal{L}_H, \;\; 0 \le k \le m_H^\ell - 1. \tag{5}$$

If we define $c_{H,k}^\ell = \frac{V_H^\ell(g_{H,k+1}^\ell) - V_H^\ell(g_{H,k}^\ell)}{g_{H,k+1}^\ell - g_{H,k}^\ell}$ and $d_{H,k}^\ell = V_H^\ell(g_{H,k}^\ell) - c_{H,k}^\ell \, g_{H,k}^\ell$, then inequality (5) becomes

$$a_\ell \le c_{H,k}^\ell \, x_{T+1}^\ell + d_{H,k}^\ell, \;\; \ell \in \mathcal{L}_H, \;\; 0 \le k \le m_H^\ell - 1. \tag{6}$$

For each EJP contract $\ell \in \mathcal{L}_J$, function $V_J^\ell(\cdot)$ is also concave, piecewise affine and can be written as $V_J^\ell(x) = \min_{0 \le k \le m_J^\ell - 1} f_{J,k}^\ell(x)$, where functions $f_{J,k}^\ell(\cdot)$ are affine. We can then introduce the quantities $c_{J,k}^\ell$ and $d_{J,k}^\ell$ defined replacing H by J in $c_{H,k}^\ell$ and $d_{H,k}^\ell$. The electricity production management problem then consists of minimizing $\sum_{\ell \in \mathcal{L}_T} \sum_{t=1}^{T} c_\ell \, p_t^\ell - \sum_{\ell \in \mathcal{L}_H} a_\ell - \sum_{\ell \in \mathcal{L}_J} b_\ell$, under constraints (1), (2), (3), (4), (6), and $b_\ell \le c_{J,k}^\ell \, x_{T+1}^\ell + d_{J,k}^\ell$, $\ell \in \mathcal{L}_J$, $0 \le k \le m_J^\ell - 1$.

In what follows, we assume that the hydroelectric network is made up of a set of independent hydroelectric plant-reservoir pairs. The flow balance equations for the hydro and EJP reservoirs can then be written

$$x_{t+1}^\ell = x_1^\ell + \sum_{k=1}^{t} (\mathcal{I}_k^\ell - p_k^\ell - sp_k^\ell), \; \ell \in \mathcal{L}_H, \;\; x_{t+1}^\ell = x_1^\ell - \sum_{k=1}^{t} p_k^\ell, \; \ell \in \mathcal{L}_J,$$

for all $t = 1, \ldots, T$. Plugging these equality constraints into constraints $x_{\min}^\ell \le x_{t+1}^\ell \le x_{\max}^\ell$, $\ell \in \mathcal{L}_H$, and $x_{t+1}^\ell \ge 0$, $\ell \in \mathcal{L}_J$, on the states of the hydro reservoirs and EJP contracts, we can express the electricity production management problem as

$$\left\{ \begin{array}{l} \min \displaystyle\sum_{t=1}^{T} \sum_{\ell \in \mathcal{L}_T} c_\ell \, \mathbf{p}_\mathbf{t}^\ell - \sum_{\ell \in \mathcal{L}_H} \mathbf{a}_\ell - \sum_{\ell \in \mathcal{L}_J} \mathbf{b}_\ell \\ \mathbf{a}_\ell + c_{H,k}^\ell \displaystyle\sum_{t=1}^{T} (\mathbf{p}_\mathbf{t}^\ell + \mathbf{sp}_\mathbf{t}^\ell) \le c_{H,k}^\ell \, (x_1^\ell + \sum_{t=1}^{T} \mathcal{I}_t^\ell) + d_{H,k}^\ell, \;\; \ell \in \mathcal{L}_H, 0 \le k \le m_H^\ell - 1, \\ \mathbf{b}_\ell + c_{J,k}^\ell \displaystyle\sum_{t=1}^{T} \mathbf{p}_\mathbf{t}^\ell \le c_{J,k}^\ell x_1^\ell + d_{J,k}^\ell, \;\; \ell \in \mathcal{L}_J, 0 \le k \le m_J^\ell - 1, \\ Duration(t) \, \tau_t^\ell \, P_{\min}^\ell \le \mathbf{p}_\mathbf{t}^\ell \le Duration(t) \, \tau_t^\ell \, P_{\max}^\ell, \;\; \ell \in \mathcal{L}_T, 1 \le t \le T, \\ x_1^\ell + \displaystyle\sum_{k=1}^{t} \mathcal{I}_k^\ell - x_{\max}^\ell \le \sum_{k=1}^{t} \mathbf{p}_\mathbf{k}^\ell + \mathbf{sp}_\mathbf{k}^\ell \le x_1^\ell + \sum_{k=1}^{t} \mathcal{I}_k^\ell - x_{\min}^\ell, \;\; \ell \in \mathcal{L}_H, 1 \le t \le T, \\ Duration(t) \, P_{\min}^\ell \le \mathbf{p}_\mathbf{t}^\ell \le Duration(t) \, P_{\max}^\ell, \;\; 0 \le \mathbf{sp}_\mathbf{t}^\ell, \;\; \ell \in \mathcal{L}_H, 1 \le t \le T, \\ 0 \le \mathbf{p}_\mathbf{t}^\ell \le Duration(t) P_{\max}^\ell, \; \ell \in \mathcal{L}_J, \; \forall t, \;\; \displaystyle\sum_{k=1}^{T} \mathbf{p}_\mathbf{k}^\ell \le x_1^\ell, \; \ell \in \mathcal{L}_J, \;\; \sum_{\ell \in \mathcal{L}} \mathbf{p}_\mathbf{t}^\ell \ge \mathcal{D}_t, \; \forall t. \end{array} \right. \tag{7}$$

In this problem, the demand satisfaction constraint is active. Variables sp have been introduced to make the problem feasible (if the production capacity is important enough), whatever the natural inflows.

Besides, this modelling supposes that all the parameters of the system (availability rates, reservoir inflows and electric consumption) are known. In this case, the deterministic optimization problem (7) indeed gives an optimal generation schedule. However, in practice, these parameters are uncertain. The goal is then to determine management strategies, that is to say, adaptive production schedules (which will depend on the realizations of the parameters) and allowing us, following this strategy, to satisfy the consumption, whatever this consumption may be.

3. Adjustable and Affinely Adjustable Robust Counterpart

In the case where the parameters are uncertain, we intend to use the methodology of robust optimization and more particularly the concepts of *Adjustable Robust Counterpart* (ARC) and *Affinely Adjustable Robust Counterpart* (AARC) which are introduced in this section. This needs an adapted modelling of the uncertainty.

3.1. **Modelling of the uncertainty.** We consider that the unknown parameters of the system belong to a nonempty convex, compact and known uncertainty set $\mathcal{Z}$. No statistical assumption is made on the parameters. Notice that from a practical point of view, the set $\mathcal{Z}$ will be estimated and the hypotheses made in this section will not necessarily be satisfied. This means that the values of the parameters over the management period will not necessarily belong to the set $\mathcal{Z}$. A compromise will thus have to be found to determine uncertainty sets $\mathcal{Z}$ which will contain the realizations of the parameters with a large probability without obtaining too conservative a solution.

3.2. **Presentation of the ARC.** Consider an uncertain optimization problem of the kind

$$\mathcal{P}_{\mathcal{Z}} = \left\{ \min_x \{c^\top x \ : \ x \in X, \ f_i(x, \xi_i) \geq 0, \ i = 1, \ldots, m\} \right\}_{(\xi_1, \ldots, \xi_m) \in (\mathcal{Z}_1 \times \ldots \times \mathcal{Z}_m)} \tag{8}$$

which is a family of optimization problems parameterized by vector $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{Z} = (\mathcal{Z}_1 \times \ldots \times \mathcal{Z}_m)$ of uncertain parameters, where $c \in \mathbb{R}^n$ is fixed, X is a compact and convex set and sets $\mathcal{Z}_i$ are convex. The Robust Counterpart of problem $\mathcal{P}_{\mathcal{Z}}$ (Ben-Tal and Nemirovski 1998) is defined as follows:

$$(RC) \ \min_x \left\{ c^\top x \ : \ x \in X, \ \inf_{\xi_i \in \mathcal{Z}_i} f_i(x, \xi_i) \geq 0, \ \ i = 1, \ldots, m \right\}. \tag{9}$$

The solution of (RC) problem thus provides the best possible solution which satisfies the constraints for every realization of the parameter ξ in the uncertainty set $\mathcal{Z}$. In the above problem (RC), it is supposed that all decision variables, grouped in vector x, have to be determined before the realizations of the uncertain parameters are known. If this is the case, such modelling is an adapted robust modelling. However, in the majority of uncertain optimization problems, only part of the decision variables must be determined before the realization of the uncertain parameters. The other variables can *adjust* to the uncertain data when this data becomes known. These adjustable variables are of two kinds. On the one hand, auxiliary variables such as slack variables or the variables introduced to present the problem in a simplified form (eliminating piecewise linear functions like $|x_i|$ or $\max(x_i, 0)$ for instance). On the other hand, variables that can be determined when part of the uncertain data becomes known (uncertain when the decision has to be taken). These variables are called the "wait and see" variables. We can thus partition the variables into two groups: variables that have to be determined before the realizations of the uncertain parameters ("here and now" decision variables) and variables that can adjust to all or part of the uncertain data ("wait and see" decision variables). Generally, each adjustable variable may have its own information, that is to say that it depends on a specific part of the data. In order to simplify the notation and the presentation of the results, we suppose that part of the variables, grouped in vector u, is not adjustable, and the remaining part, grouped in vector v, can adjust to all the uncertain data. The results can easily be extended to the case where each adjustable variable has its own information. Consequently, decision vector x is partitioned into $x = (u^\top, v^\top)^\top$. We can then always write generic uncertain optimization problem $\mathcal{P}_{\mathcal{Z}}$ under the form

$$\mathcal{P}_{\mathcal{Z}} = \left\{ \min_{u,v} \{c^\top u \ \ : \ (u, v) \in X, \ f_i(u, v, \xi_i) \geq 0, \ i = 1, \ldots, m\} \right\}_{\xi \in \mathcal{Z}}.$$

Its Robust Counterpart then reads

$$(RC)\ \min_{u,v} \left\{ c^\top u\ :\ (u,v) \in X,\ \inf_{\xi_i \in \mathcal{Z}_i} f_i(u,v,\xi_i) \geq 0,\ \ i=1,\ldots,m \right\};$$

which is the same as

$$(RC)\ \min_u \{ c^\top u\ : \exists\, v \,|\, (u,v) \in X,\ \forall i=1,\ldots,m,\ \forall (\xi_i \in \mathcal{Z}_i),\ f_i(u,v,\xi_i) \geq 0 \}\,. \tag{10}$$

A more flexible robust model for uncertain problem $\mathcal{P}_{\mathcal{Z}}$ is then the *Adjustable Robust Counterpart* (ARC) recently introduced in (Ben-Tal et al. 2003) as follows

$$(ARC)\ \min_u \{ c^\top u : \forall \xi \in \mathcal{Z},\ \exists\, v | (u,v) \in X, \forall i=1,\ldots,m,\ f_i(u,v,\xi_i) \geq 0 \}\,.$$

The Adjustable Robust Counterpart provides a more flexible policy than the Robust Counterpart defined by (10). Indeed, the ARC feasible set is larger, which allows us to obtain a better optimal value while getting a solution satisfying the constraints for every realization of the uncertain parameters. In (Ben-Tal et al. 2003) , the simple example of an uncertain linear equality constraint $u+v=a$ is given to illustrate this, the uncertainty set being $\mathcal{Z}=\{a \mid a \in [0,1]\}$. In this case, the feasibility set of the RC is empty while the feasibility set of the ARC is $\mathbb{R}$ (for every $u \in \mathbb{R}$, for every $0 \leq a \leq 1$, $v=a-u$ satisfies the constraint). However, Adjustable Robust Counterparts generally yield problems that are much more difficult to solve than the Robust Counterparts for which efficient numerical treatments exist for a wide range of problems and of uncertainty sets. In the following section, we recall the results of (Ben-Tal et al. 2003) and focus on the Adjustable Robust Counterparts of uncertain linear programming problems.

3.3. **ARC of linear programs.** Partitioning the decision vector in $x=(u^\top, v^\top)^\top$ as in the previous section, we can write an uncertain linear optimization problem under the form

$$LP_{\mathcal{Z}} = \left\{ \min_{u,v} \{ c^\top u\ :\ Uu + Vv \leq b \} \right\}_{\xi=[U,V,b] \in \mathcal{Z}}, \tag{11}$$

where $\mathcal{Z}$ is a nonempty convex and compact set and the uncertain parameter ξ is made up of the matrices U and V and the right hand side b. Using the terminology of (Ben-Tal et al. 2003), we call V the recourse matrix. The Adjustable Robust Counterpart of such a problem is given by

$$(ARC)\ \min_u \{ c^\top u\ :\ \forall\ (\xi=[U,V,b]) \in \mathcal{Z},\ \exists\ v \mid Uu+Vv \leq b \}, \tag{12}$$

and its Robust Counterpart by

$$(RC)\ \min_u \{ c^\top u\ :\ \exists v \mid \forall\ (\xi=[U,V,b]) \in \mathcal{Z},\ Uu+Vv \leq b \}. \tag{13}$$

Under quite restrictive hypotheses, the RC and the ARC of an uncertain linear programming problem are equivalent (Ben-Tal et al. 2003). One of these hypotheses is that the constraints are constraint-wise (the uncertain parameters of a given constraint do not appear in the other constraints). Nevertheless, we can easily construct examples showing that the ARC is more flexible than the RC. Beyond the case of uncertain equality constraints previously mentioned, let us take an example with two uncertain inequality constraints to illustrate this phenomenon. Consider the uncertain optimization problem $\min_{u,v}\ u$ subject to $u \geq -1,\ v \leq (\xi-1)u+1,\ v \geq \xi u + 1$, where $\xi \in [-1,1]$ is an uncertain parameter. The only feasible point for the RC is $(u,v)=(0,1)$ so the RC optimal value is 0. The ARC optimal value is -1 so it is lower. Indeed, the ARC feasibility set is [-1,0] : for every $u \in [-1,0]$ and every $\xi \in [-1,1]$, $v=\xi u$ (for instance) is feasible. Besides, notice that the uncertainty is not constraint-wise.

In the case of an uncertain linear programming problem, the RC is a tractable problem as soon as the uncertainty set is tractable (see (Ben-Tal and Nemirovski 1999) for a definition of tractability). A simple case where the ARC (12) of problem (11) is tractable (and is in fact a linear programming problem) is the case where the recourse matrix V is fixed and the uncertainty set $\mathcal{Z}$ is a convex hull of scenarios $\mathcal{Z}=\text{Conv}\{[U_1,V,b_1],\ldots,[U_S,V,b_S]\}$. In this case, the ARC, given in (Ben-Tal et al. 2003), is the linear program $\min\limits_{u,v_1,\ldots,v_S} c^\top u$ subject to $U_\ell u + V v_\ell \leq b_\ell,\ \ell=1,\ldots,S$. In the case when the recourse matrix is not fixed and when $\mathcal{Z}$ is also given by a convex hull of scenarios i.e., $\mathcal{Z} = \mathrm{Conv}\{[U_1, V_1, b_1], \ldots, [U_S, V_S, b_S]\}$, the ARC can be NP hard (Ben-Tal et al. 2003). Similarly, if the recourse matrix is fixed and if $\mathcal{Z}$ is a polytope defined by a list of linear inequalities, the ARC can also be NP hard (Ben-Tal et al. 2003). In this case, we look for tractable approximations of the ARC for a wider range of uncertainty sets. This motivates the introduction of *Affinely Adjustable Robust Counterparts* (AARC) described in the next section.

3.4. **AARC of linear programs.** In many applications of the control theory, we look for optimal controls which are linear functions of the observations of the state of the system. This restriction on the link between the decision variables and the state of the system often allows an efficient computation of the control. In the same spirit, we can impose, for fixed u, that adjustable variables v of problem $LP_{\mathcal{Z}}$ are affine functions of the data

$$v = w + W\xi, \quad \text{where } \xi = [U, V, b]. \tag{14}$$

In this setting, a non-adjustable vector u can be completed in a robust solution (u, w, W) iff $Uu + V(w + W\xi) \leq b$, $\forall\, (\xi = [U, V, b]) \in \mathcal{Z}$. The *Affinely Adjustable Robust Counterpart* (AARC) of the uncertain linear programming problem (11) is then defined in (Ben-Tal et al. 2003) as the optimization problem

$$(AARC) \ \min_{u,w,W} \{c^\top u \ : \ Uu + V(w + W\xi) \leq b, \quad \forall(\xi \ = \ [U, V, b]) \in \mathcal{Z}\}. \tag{15}$$

Notice that the RC feasibility set is contained in the AARC feasibility set, itself contained in the feasibility set of the ARC. Consequently, the AARC is more conservative than the ARC but less conservative than the RC.

If recourse matrix V is fixed and if uncertainty set $\mathcal{Z}$ is tractable then the AARC (15) of problem $LP_{\mathcal{Z}}$ is tractable (Ben-Tal et al. 2003). If the recourse matrix is not fixed and if the uncertainty set is defined by a finite intersection of concentric ellipsoids, we can build an SDP problem which is a good approximation of the AARC (Ben-Tal et al. 2003).

The hypothesis of affine dependence in (14), whose essential motivation is the tractability of the resulting AARC, can be discussed. Indeed, the dependency of the adjustable variables as a function of the uncertain data is not necessarily affine. However, in some cases, such modelling can be justified (see Section 4.2). Finally, this approach can also be justified if it provides satisfying robust solutions in practice (in particular better than those obtained by applying the Robust Counterpart (13)).

3.5. **AARC of quadratic and conic quadratic optimization problems.** Consider an uncertain quadratic optimization problem of the kind

$$QP_{\mathcal{Z}} \ = \ \left\{\min_{u,v} \{c^\top u \ : \ U^i u + V^i v + u^\top A_i u + v^\top B_i v + u^\top C_i v \leq b^i,\, i = 1, \ldots, m\}\right\}_{\xi \in \mathcal{Z}};$$

where $\xi = [U, V, A_i, B_i, C_i, b]$, $\mathcal{Z}$ is a nonempty convex compact set and U^i and V^i are the ith rows of matrices U and V. Suppose that symmetric matrices B_i are known and let us reparametrize the problem writing the uncertain parameters as

$$[U, V, A_i, C_i, b] = [U_0, V_0, A_{i,0}, C_{i,0}, b_0] + \sum_{\ell=1}^{p} \zeta_\ell [U_\ell, V_\ell, A_{i,\ell}, C_{i,\ell}, b_\ell],$$

parameter ζ belonging to a nonempty convex compact set χ. We can then write the adjustable variables under the form $v = v_0 + \sum_{\ell=1}^{p} \zeta_\ell\, v_\ell$. If $x = [u, v_0, \ldots, v_p]$, then the constraints of the AARC of $QP_{\mathcal{Z}}$ read

$$\forall \zeta \in \chi, \ \ \alpha_i(x) + 2\zeta^\top \beta_i(x) + \zeta^\top \Gamma_i(x)\zeta \geq 0, \ i = 1, \ldots, m,$$

where for all $i = 1, \ldots, m$

- $\alpha_i(x) = -U_0^i u - V_0^i v_0 - u^\top A_{i,0} u - v_0{}^\top B_i v_0 - u^\top C_{i,0} v_0 + b_0^i$,
- $\beta_i^\ell(x) = \frac{1}{2}(-U_\ell^i u - V_\ell^i v_0 - V_0^i v_\ell - u^\top A_{i,\ell} u - 2v_0{}^\top B_i v_\ell - u^\top C_{i,0} v_\ell - u^\top C_{i,\ell} v_0 + b_\ell^i)$, $\ \ell = 1, \ldots, p$,
- $\Gamma_i^{\ell,k}(x) = -\frac{1}{2}(V_k^i v_\ell + V_\ell^i v_k + u^\top C_{i,k} v_\ell + u^\top C_{i,\ell} v_k + 2v_\ell{}^\top B_i v_k)$, $\ \ell, k = 1, \ldots, p$.

Theorem 3.1. *Let $Q \succ 0$ and $\rho > 0$. The AARC of problem $QP_{\mathcal{Z}}$ with uncertainty set $\chi = \{\zeta \mid \zeta^\top Q\, \zeta \le \rho^2\}$ is the semidefinite problem*

$$\left\{\begin{array}{l} \min \;\; c^\top u \\ \begin{pmatrix} \Gamma_i(x) + \frac{\lambda_i}{\rho^2} Q & \beta_i(x) \\ \beta_i^\top(x) & \alpha_i(x) - \lambda_i \end{pmatrix} \succeq 0, \quad i = 1, \ldots, m, \\ \lambda_i \ge 0. \end{array}\right.$$

Proof. It suffices to follow the proof of Theorem 4.2 in (Ben-Tal et al. 2003) which deals with the AARC of linear program. □

Lemma 3.1. *Consider a conic quadratic problem of the form*

$$\min c^\top u \quad : \quad \|A_i u + b_i\|_2 \le p_i^\top u + q_i^\top v + d_i, \;\; i = 1, \ldots, m, \tag{16}$$

where u (resp. v) is the vector of unadjustable (resp. adjustable) variables and for $i = 1, \ldots, m$, the vectors $(p_i; d_i)$ are uncertain. If the uncertainty set for the vectors $(p_i; d_i)$ is the convex hull of S scenarios $((p_i^1; d_i^1), \ldots, (p_i^S; d_i^S))$, the ARC of problem (16) *is the optimization problem*

$$\min_{u, v_1, \ldots, v_S} c^\top u \quad : \quad \|A_i u + b_i\|_2 \le {p_i^\ell}^\top u + q_i^\top v_\ell + d_i^\ell, \;\; i = 1, \ldots, m, \; \ell = 1, \ldots, S. \tag{17}$$

Proof. Let u belong to the feasibility set of the ARC of (16). For every α such that $\alpha \ge 0, \;\; \sum_i \alpha_i = 1$, $\exists\, v(\alpha) \mid$

$$\|A_i u + b_i\|_2 \le \sum_{\ell=1}^{S} \alpha_\ell u^\top p_i^\ell + q_i^\top v(\alpha) + \sum_{\ell=1}^{S} \alpha_\ell \, d_i^\ell, \;\; i = 1, \ldots, m. \tag{18}$$

For $\ell = 1, \ldots, S$, choosing $\alpha = e_\ell$ for e_ℓ a canonical vector and $v_\ell = v(e_\ell)$, we see that u is feasible for problem (17). However, if u is feasible for (17), then for every α such that $\alpha \ge 0, \;\; \sum_i \alpha_i = 1$, and for all $1 \le i \le m$, multiplying term by term $\|A_i u + b_i\|_2 \le {p_i^\ell}^\top u + q_i^\top v_\ell + d_i^\ell$ by α_ℓ (for $1 \le \ell \le S$) and adding term by term these S inequalities, we see that there exists $v(\alpha) = \sum_{\ell=1}^{S} \alpha_\ell \, v_\ell$ such that $(u, v(\alpha))$ satisfies constraints (18) and so u is feasible for the ARC of (16). □

3.6. **Comparison with other modellings in stochastic optimization.** In this section, we comment on the advantages and disadvantages of using the methodology of *Robust Optimization* (*RO*) over *Stochastic Programming* (*SP*) and *Stochastic Dynamic Programming* (*SDP*) to deal with uncertainty in stochastic optimization. We also introduce another robust model that we call *Extended Robust Counterpart* that enjoys the same tractability properties as the RC and that we compare to the RC and ARC.

Comparison between *SP*, *SDP* and *RO*. We first compare the computational effort needed when using *SP*, *SDP*, and *RO*. To that end, we provide for the following T-stage uncertain optimization problem

$$P(\omega) \left\{\begin{array}{l} \displaystyle\min_{x_{t+1}(\omega), u_t(\omega)} \sum_{t=1}^{T} f_t(u_t(\omega)), \\ x_{t+1}(\omega) = g_t(x_t(\omega), u_t(\omega), \xi_t(\omega)), t = 1, \ldots, T, \\ x_{t+1}(\omega) \in \chi_t, \;\; u_t(\omega) \in \mathcal{U}(\xi_t(\omega)), t = 1, \ldots, T; \end{array}\right.$$

the number, type, and size of the optimization problems solved when using *SP*, *SDP*, and *RO* to deal with the T uncertain parameters $\xi_1(\omega), \ldots, \xi_T(\omega)$ (realizations of random vectors $\xi_1, \ldots, \xi_T$) in $\mathbb{R}^M$ in the above problem. To fix the ideas, we then also give this piece of information when problem $P(\omega)$ is the stochastic counterpart of problem (7) when using data detailed in Section 5.

For the problem $P(\omega)$ above, we suppose that for each time step $t = 1, \ldots, T$, there are p constraints, and that the sizes of the state vector x_{t+1} (state of the system at the beginning of time step $t+1$) and of the vector of controls u_t are respectively n_x and n_u. To use *SP* or *SDP*, we consider problem EP in (19) below, which amounts to choosing the expectation of the total cost as a criterion to minimize. For the simplicity of the exposure, we suppose that $\xi_t,\ t=1,\dots,T$, are independent and that for every t, the M components of ξ_t are also independent.

In *SP*, the evolution of the uncertain parameters over the optimization period is organized in a tree. For time step t, there are k_t nodes in this tree and the realization of ξ_t at node number n (for $n=1,\dots,k_t$) is ξ_t^n. The probability to be at node n of time step t is π_t^n and we denote by $F(n)$ the father node of node n. With this convention and this notation, an optimal solution to problem SP in (19) below provides a solution to EP, i.e., optimal states $(x_{t+1}^n)_{t,n}$ and controls $(u_t^n)_{t,n}$ for time steps t and nodes n:

$$(19)\qquad EP\left\{\begin{array}{l}\displaystyle\min_{x_{t+1},u_t}\ \mathbb{E}_{\xi_1,\dots,\xi_T}[\sum_{t=1}^{T} f_t(u_t)],\\ x_{t+1}=g_t(x_t,u_t,\xi_t),\forall\, t,\\ x_{t+1}\in\chi_t, u_t\in\mathcal{U}(\xi_t),\forall\, t,\end{array}\right. \qquad SP\left\{\begin{array}{l}\displaystyle\min_{x_{t+1}^n,u_t^n}\sum_{t,n}\pi_t^n f_t(u_t^n),\\ x_{t+1}^n=g_t(x_t^{F(n)},u_t^n,\xi_t^n),\forall\, t,n,\\ x_{t+1}^n\in\chi_t,\ u_t^n\in\mathcal{U}(\xi_t^n),\forall\, t,n.\end{array}\right.$$

If there are N nodes in the tree, the number of variables in SP is $N(n_x+n_u)$ and the number of constraints Np. This problem has the same analytical structure as any instance $P(\omega)$. If K realizations are possible for each of the components at each time step, we end up with $N=K^{MT}$ nodes. The size of problem SP thus exponentially increases with the number of time steps and components. Even if scenario tree reduction techniques can be used to reduce drastically the size of the tree, this size (and thus the size of SP) must remain large when there are many time steps and components for ξ_t. Moreover, once problem SP is solved and if the scenarios of the scenario tree do not reproduce all the possible evolutions of the parameters over the optimization period, some dynamic programming technique may be used (as in (Guigues et al. 2009)) to compute Bellman functions. For a given realization $(\xi_1(\omega),\dots,\xi_T(\omega))$, these Bellman functions then allow us to compute adaptive feasible strategies. For the problem considered in (Guigues et al. 2009), if D_i is the number of discretization points for the ith component of the state vector x_{t+1}, then we have to solve $N\displaystyle\sum_{i=1}^{n_x} D_i$ optimization problems. Even if the size of the problems are small, the considerable number of optimization problems to be solved makes the procedure computationally heavy.

Another alternative to solve problem EP is to use *SDP*. To that end, we introduce for $t=1,\dots,T$, the following future cost-to-go functions $\alpha_t(\cdot)$ defined by

$$(20)\qquad \alpha_t(x_t)=\mathbb{E}_{\xi_t}\left\{\begin{array}{l}\displaystyle\min_{x_{t+1},u_t} f_t(u_t)+\alpha_{t+1}(x_{t+1}),\\ x_{t+1}=g_t(x_t,u_t,\xi_t),\\ x_{t+1}\in\chi_t,\ u_t\in\mathcal{U}(\xi_t),\end{array}\right.$$

with $\alpha_{T+1}\equiv 0$. The optimal value of EP is given by $\alpha_1(x_1)$. *SDP* allows us to obtain approximations for the cost-to-go functions as follows. For each time step $t=2,\dots,T$, we fix D^{n_x} states $(x_t^i)_i$ obtained discretizing each component of state vector x_t in D admissible values (state x_1 being given). We then calculate iteratively for $t=T$ down to $t=2$, the $D^{n_x}T$ approximate values $(\hat\alpha_t(x_t^i))_{t,i}$ for $(\alpha_t(x_t^i))_{t,i}$. To compute the approximate value $\hat\alpha_t(x_t^i)$ for $\alpha_t(x_t^i)$, we need an approximation for $\alpha_{t+1}(\cdot)$. The previous step provides approximate values $(\hat\alpha_{t+1}(x_{t+1}^i))_i$ for $(\alpha_{t+1}(x_{t+1}^i))_i$. A parametric form is then used for $\alpha_{t+1}(\cdot)$ that interpolates these values $(\hat\alpha_{t+1}(x_{t+1}^i))_i$. To compute the expectation in (20) at time step t, we sample K values from the distribution of ξ_t. With this method, we thus have to solve $D^{n_x}KT$ optimization problems with n_x+n_u variables and p constraints.

If constraints $x_{t+1}\in\chi_t$, and $u_t\in\mathcal{U}(\xi_t)$ are inequality constraints then we may also form the Robust Counterpart of problem $P(\omega)$ which amounts to minimizing $\sum_t f_t(u_t)$, under constraints $u_t\in\mathcal{U}(\xi_t),\ \forall\, t,\ \forall\,\xi_t\in\mathcal{Z}_t$ and $g_t(g_{t-1}(\dots(g_2(g_1(x_1,u_1,\xi_1),u_2,\xi_2),\dots,u_{t-1},\xi_{t-1}),u_t,\xi_t)\in\chi_t,\ \forall\, t,\ \forall\,\xi_t\in\mathcal{Z}_t$. In this case, we only have one optimization problem to solve. We can provide uncertainty sets such that the RC of a Linear Program (LP) is an LP, a Conic Quadratic Program (CQP) or a Semidefinite Program (SDP), the RC of a CQP is a CQP or an SDP, and the RC of an SDP is an SDP. We can also give uncertainty sets for which only a tractable SDP approximation of the RC of a CQP or of an SDP can be given. The analytical structure of the RC is thus not necessarily the same as that of problem instances $P(\omega)$ and only tractable approximations are available in some cases. However, for LP, CQP, and SDP, we can define uncertainty sets such that the RC respectively remains an LP, a CQP, or an SDP, i.e., a convex optimization problem efficiently solved with interior point methods for instance. A simple particular case when the structure of the original problem is conserved is when functions $f_i(x, \xi)$ in $\mathcal{P}_{\mathcal{Z}}$ are affine in ξ and the uncertainty set is a convex hull of scenarios. We can also add that the AARC allows us to provide an adaptive solution without having to solve additional optimization problems as is the case in *SDP* and in *SP*.

However, the size of the RC, ARC, or AARC may be much larger than the size of any problem instance $P(\omega)$. In that case, it is possible to define an aggregation of the time steps (and uncertainty sets accordingly) to end up with lower size problems. In Table 1 below, we provide the computational effort for solving problem (7) using the data from Section 5 (in this case, each instance has 387 variables and 828 constraints) with *SP*, *SDP*, and *RO*.

Method	Nb. of opt. problems	Nb. of variables	Nb. of constraints
SP	13 681	Between 2 and 1425	Between 25 and 3819
SDP	3.84×10^6	Between 19 and 22	Between 37+I and 72+I
RC	1	387	828
AARC	1	17 913	30 086

TABLE 1. Computational effort needed when using *SP*, *SDP*, the *RC* and the *AARC* using box constrained uncertainty sets, to deal with uncertainty in the stochastic counterpart of problem (7). In this table, I stands for the number of constraints needed to represent cost-to-go functions.

The above considerations show that for large values of the number T of time steps, of the number of components M and of the size n_x of the state vector, the SDP and SP methodologies would entail prohibitive computational time and are thus not applicable, contrary to *RO* (eventually after aggregating the time steps).

Now if the *SP*, *SDP*, and *RO* are all applicable, it remains to choose which model is the most appropriate.

RO methodology is appropriate in one of the three following cases. First, if the domain of variation of the uncertain parameters is bounded and known and the constraints are hard, i.e., they have to be satisfied for every value of the parameters in this domain. This is the case of load constraints for a bridge for instance. Second, *RO* is also appropriate when the parameters, though uncertain, are not random by nature (the *RO* does not need any statistical assumption contrary to SP and SDP). This holds for instance when these parameters are physical measures (temperature, pressure,...) obtained with a device only able to provide approximate measures. Finally, one important fact about *RO* is that it is a *worst case* oriented methodology. Indeed, if the objective function of the uncertain problem we consider is of the form $f_0(x, \xi)$ with $\xi \in \mathcal{Z}$ an uncertain parameter, then writing the problem as (8) and forming the RC (9), we see that the objective function of the RC is $f_{wc}(x) = \max_{\xi \in \mathcal{Z}} f_0(x, \xi)$. Under mild assumptions (Ben-Tal and Nemirovski 1998), the optimal cost of the RC is the supremum of the optimal cost of the instances. Thus if the system can recover from the worst case scenario (the scenario associated to the instance of highest optimal cost) only with an optimal strategy on this scenario, then the *RO* is necessary.

However an objection that can be formulated towards *RO* is that precisely, for the objective function, it only deals with the worst case scenario. This robust strategy can be far from the optimal strategy in mean or give a far from optimal cost when the scenario is not the worst scenario. If it is not so critical to design an optimal strategy on the worst case scenario, but if we however wish to limit the cost on 100p% of the worst scenarios (with $p \in (0, 1)$), other approaches could be chosen instead. We could add to the objective function of model EP in (19) the $CVaR_p$ of the cost for instance (mean cost of the 100p% worst scenarios), and then use *SP* to solve the problem. The ARC and AARC also appear as ways of designing less conservative robust solutions than the RC even if the problem of calibration of the uncertainty sets remains for these methods.

Finally from the practical point of view, the *RO* needs the delicate step of calibration of uncertainty sets instead of probability distributions for *SP*.

Extended Robust Counterpart. An Extended Robust Counterpart (ERC) for uncertain problem $\mathcal{P}_{\mathcal{Z}}$ could be defined penalizing in the objective the decisions that are not feasible for the RC. We define the ERC of uncertain problem $\mathcal{P}_{\mathcal{Z}}$ as follows:

$$\min_{u,w}\{c^\top u + \eta\, e^\top w \ : \exists\, v, w \,|\, f_i(u, v, \xi_i) \geq w_i,\ \forall\, \xi_i \in \mathcal{Z}_i,\ i = 1, \ldots, m,\ w \geq 0\},$$

for some $\eta > 0$. When $\eta = +\infty$, the ERC is the RC. For finite values of η, the feasibility set of ERC is always nonempty. An advantage of the ERC with respect to the ARC is also its tractability. Indeed, the ERC is tractable as soon as the RC is, so for LP the ERC is tractable as soon as the uncertainty set $\mathcal{Z}$ is tractable which is not the case of the ARC. However, we believe that the ERC changes the modelling more notably than the ARC. For the ARC, the objective function and the constraints are robustified versions of the objective and of the constraints in $\mathcal{P}_{\mathcal{Z}}$. The ERC, on the other hand, changes the objective function and poses the problem of calibration of the parameter η. When using a robust approach to deal with uncertainty in $\mathcal{P}_{\mathcal{Z}}$, we want to satisfy the constraints for all possible values of ξ in $\mathcal{Z}$. When using the ERC modelling, we should thus take sufficiently large values of η to obtain sufficiently small values of w_i, and thus obtain a pair (u, v) that will nearly satisfy all the constraints for every value of $\xi \in \mathcal{Z}$. But the larger the η, the more the objective function differs from the initial objective function $c^\top u$.

4. ARC and AARC of the electricity generation management problem

The goal of this section is to explain how to apply the methodology described in the previous section to the stochastic counterpart of (7). Notice that (7) is a convex optimization problem and thus the results of the previous section apply.

4.1. **ARC.** We first have to decide which variables are adjustable and which variables are not. This decision is a modelling choice and different answers can be given. The control variables can be seen as non-adjustable and the remaining variables (especially the state variables) as adjustable. Nevertheless, at the beginning of the management horizon, no uncertain parameter has been observed and only the controls applicable at the first time step are really not adjustable. We thus adopt the following modelling choice: we consider that the control variables of the first time step are not adjustable and that all the other variables are adjustable. If the uncertainty set is the convex hull of S scenarios $(\mathcal{D}_t(s), \mathcal{I}_t^\ell(s), \tau_t^\ell(s))_{s=1}^S$, the Adjustable Robust Counterpart of the electricity generation management problem is the linear programming problem given in (Guigues 2005).

We can then wonder if this ARC is equivalent to the RC of problem (7) obtained choosing the same polytopic uncertainty set or if the ARC allows us to have a lower cost. Since the uncertainty is not constraint-wise, the hypotheses in (Ben-Tal et al. 2003) giving conditions ensuring the equivalence of the ARC and the RC are not satisfied. In fact, if the uncertainty set is polytopic, we can give examples of electricity generation management problems for which the RC and the ARC are equivalent and other examples for which they are not, see (Guigues 2005). However, even when the RC and ARC optimal values are the same, the solution of the ARC provides a solution, which, on average, can be better than the robust solution which is not adaptive. Indeed, if the uncertainty set is the convex hull of the scenarios $(\xi_1, \ldots, \xi_S)$ and if we denote the solution of the ARC by $(u, v_1, \ldots, v_S)$, when the scenario $\sum_{i=1}^S \ \alpha_i\ \xi_i$ occurs, then the solution of the ARC provides a feasible point $(u, \sum_{i=1}^S \ \alpha_i\ v_i)$ whose associated cost is lower than or equal to the optimal value of the ARC. On the other hand, the use of the ARC sets a problem from the practical point of view. Indeed, for a given time step, the dependency of the controls as a function of past realizations of the parameters is not known. In order to know the controls provided by the ARC, we should know values $(\alpha_1, \ldots, \alpha_S)$ such that the vector of parameters over the management period is $\sum_{i=1}^{S} \alpha_i \xi_i$. In this case, we can use controls $(u, \sum_{i=1}^{S} \alpha_i v_i)$ which will probably yield a cost lower than the RC cost.

However, as long as the management period is not over, we cannot determine such α_i. A way to use, from a practical point of view, the methodology of the ARC to robustify problem (7), consists of determining an ARC at each time step with shorter and shorter management horizons. We call this method "sliding" ARC. More precisely, to determine the controls to apply at time step $t \leq T$ with this method, we solve the ARC of problem (7) written for the last $T - t + 1$ time steps using a polytopic uncertainty set. The non-adjustable variables of this problem correspond to the controls to apply at time step t.

4.2. **AARC.** In order to know the dependency of the adjustable variables as a function of past realizations of the parameters we can choose a particular parametric form for this dependency. When the adjustable variables are affine functions of the uncertain parameters, we get the concept of AARC introduced in Section 3. In this approach, we choose vector of controls $(p_t^\ell)_{t,\ell}$ for the vector v of adjustable variables. Potentially, all the uncertain parameters can influence these adjustable variables. Nevertheless, to limit the size of the AARC, for a given adjustable variable, we choose the uncertain parameters which could most influence it. This step is part of the modelling process and tests will be necessary to see if a given choice leads to a significant improvement compared to the use of a more traditional RC. Parameters which most influence the controls are electric consumption $\mathcal{D}_t$ for different time steps t. For time step t, we can thus reasonably suppose that thermal controls $p_t^\ell, \ell \in \mathcal{L}_T$, hydro controls $p_t^\ell, \ell \in \mathcal{L}_H$, and EJP contract controls, $p_t^\ell, \ell \in \mathcal{L}_J$, affinely depend on electric consumption $\mathcal{D}_j$ observed for time steps $j \in I_t$, where I_t is a given subset of $\{1, \ldots, t\}$, i.e.,

$$p_t^\ell = q_t^{\ell,0} + \sum_{j \in I_t} q_t^{\ell,j} \mathcal{D}_j, \quad \ell \in \mathcal{L}. \tag{21}$$

Remark 4.1. *The above dependency scheme may appear naturally in some situations, when process D_t ($\mathcal{D}_t$ is a realization of D_t) is an $AR(m)$ model for instance. In this case, we have $D_t = \sum_{i=1}^{m} \alpha_i D_{t-i} + \eta_t$, where $\eta_t \sim \mathcal{N}(0, \sigma^2)$. A $(1-\varepsilon)$-quantile of the distribution of $D_t | D_{t-1} = \mathcal{D}_{t-1}, \ldots, D_{t-m} = \mathcal{D}_{t-m}$,(that could be a "robust" value for $\mathcal{D}_t$) would be of the form $\sum_{i=1}^{m} \alpha_i \mathcal{D}_{t-i} + \sigma \Phi^{-1}(1-\varepsilon)$. Since demand satisfaction constraints are active, $\sum_{\ell \in \mathcal{L}} p_t^\ell = \sum_{i=1}^{m} \alpha_i \mathcal{D}_{t-i} + \sigma\Phi^{-1}(1-\varepsilon)$. In this case, we have $I_t = \{t-1, \ldots, t-m\}$ and $\sum_\ell q_t^{\ell,0} = \sigma\Phi^{-1}(1-\varepsilon)$.*

Going back to (21), the new non-adjustable variables are now $(q_t^{\ell,j})_{t,\ell,j}$. We then look for the best linear combinations defined by (21), choosing different uncertainty sets for the demand. It also seems reasonable to think that the controls applied to hydroelectric plant ℓ depend on the upstream reservoir inflows. If this leads to an AARC of reasonable size, we can thus also increase the information from which the hydro controls are computed under the form

$$p_t^\ell = q_t^{\ell,0} + \sum_{j \in I_t} q_t^{\ell,j} \mathcal{D}_j + \sum_{j \in I_t} s_t^{\ell,j} \mathcal{I}_j^\ell, \quad \ell \in \mathcal{L}_H. \tag{22}$$

We can of course combine these dependency schemes. We can decide that only some controls are adjustable and ruled by equations (21) or (22).

Set I_t depends on the available information at time step t. In practice, we know, for each time step, all the past realizations of the problem parameters, so the most natural choice for I_t is $I_t = \{1, \ldots, t-1\}$. If the information on the values of the parameters arrives with m time steps delay, we have at best, $I_t = \{1, \ldots, t-m\}$. To reduce the size of the AARC and to favor recent information, we can choose to use only the values of the parameters for the last m time steps taking $I_t = \{t-1, \ldots, t-m\}$. We can also imagine dynamical choices of the subset I_t whose size may depend on t and on the data collected until time step t.

Contrary to the ARC previously introduced, the AARC provides a solution directly usable from the practical point of view. Indeed, for time step t, the values of the parameters for all time steps $j \in I_t$ are known. We can thus use the solution of the AARC and equations (21) and/or (22) to determine controls $(p_t^\ell)_{\ell \in \mathcal{L}}$ provided by the AARC. Even in the case where the AARC and the

RC optimal values are the same, the use of the controls given by (21) and/or (22) is preferable because it has great chances of providing a better solution on average (see Section 5).

The above dependency schemes (21) and (22) imply a particular form for matrix W introduced in the previous section when decomposing v as $v = w + W\xi$. To explicit the form of this matrix W when $I_t = \{1, \dots, t-1\}$, let us suppose that $\mathcal{L}_T = \{1, \dots, m\}$, $\mathcal{L}_H = \{m+1, \dots, m+h\}$, and $\mathcal{L}_J = \{m+h+1, \dots, m+h+nbJ\}$. For $\ell \in \mathcal{L}$, let $p^\ell_\bullet = (p^\ell_1, \dots, p^\ell_T)$, and let us order vectors v and ξ as

$$\begin{aligned} v &= (p^1_\bullet, \dots, p^m_\bullet, p^{m+1}_\bullet, \dots, p^{m+h}_\bullet, p^{m+h+1}_\bullet, \dots, p^{m+h+nbJ}_\bullet)^\top, \\ \xi &= (\tau^1_1, \dots, \tau^1_T, \dots, \tau^m_1, \dots, \tau^m_T, \mathcal{I}^1_1, \dots, \mathcal{I}^1_T, \dots, \mathcal{I}^h_1, \dots, \mathcal{I}^h_T, \mathcal{D}_1, \dots, \mathcal{D}_T)^\top. \end{aligned}$$

Let S_ℓ (for $1 \le \ell \le |\mathcal{L}_H|$), and Q_ℓ (for $1 \le \ell \le |\mathcal{L}|$), be the (T, T) matrices defined by

$$Q_\ell(i,j) = \begin{cases} 0 & \text{if } j \ge i, \\ q^{\ell,j}_i & \text{if } j < i, \end{cases} \quad \text{and} \quad S_\ell(i,j) = \begin{cases} 0 & \text{if } j \ge i, \\ s^{\ell,j}_i & \text{if } j < i. \end{cases}$$

With this notation, we can express matrix W as

$$W = \begin{pmatrix} 0 & 0 & Q_T \\ 0 & S & Q_H \\ 0 & 0 & Q_J \end{pmatrix}, \quad \text{where} \quad S = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_{|\mathcal{L}_H|} \end{pmatrix}, \tag{23}$$

$$Q_T = \begin{pmatrix} Q_1 \\ \vdots \\ Q_{|\mathcal{L}_T|} \end{pmatrix}, \quad Q_H = \begin{pmatrix} Q_{|\mathcal{L}_T|+1} \\ \vdots \\ Q_{|\mathcal{L}_T|+|\mathcal{L}_H|} \end{pmatrix}, \quad \text{and} \quad Q_J = \begin{pmatrix} Q_{|\mathcal{L}_T|+|\mathcal{L}_H|+1} \\ \vdots \\ Q_{|\mathcal{L}_T|+|\mathcal{L}_H|+|\mathcal{L}_J|} \end{pmatrix}.$$

We finally recover variables $q^{\ell,0}_t$ in w using $w((\ell-1)T + t) = q^{\ell,0}_t$, for $1 \le \ell \le |\mathcal{L}|$ and $1 \le t \le T$.

We now intend to provide the AARC of the stochastic counterpart of problem (7) for box constrained, polytopic and ellipsoidal uncertainty sets. This problem (7) is of the form

$$\mathcal{P}_\xi \begin{cases} \min\ c^\top u \\ Uu + Vv \le A\xi + b, \end{cases} \tag{24}$$

where U and V are fixed and ξ is the uncertain parameter.

Lemma 4.1. *When the uncertainty set for ξ is the box $\chi = \{\xi \mid \xi_{\inf} \le \xi \le \xi_{\sup}\}$, then the AARC of uncertain problem (24) can be expressed by one of the two following linear programming problems*

$$LP_1 \begin{cases} \min\limits_{u,w,W,\theta_i,\tilde{\theta}_i} c^\top u \\ U^i u + V^i w - b^i + \xi_{\sup}^\top \theta_i + \xi_{\inf}^\top \tilde{\theta}_i \le 0, \\ \theta_i \ge 0,\ \ \theta_i{}^\top \ge V^i W - A^i, \\ \tilde{\theta}_i \le 0,\ \ \tilde{\theta}_i^\top \le V^i W - A^i, \end{cases}$$

$$LP_2 \begin{cases} \min\limits_{u,w,W,\theta_i} c^\top u \\ 2[U^i u + V^i w - b^i] + [V^i W - A^i][\xi_{\inf} + \xi_{\sup}] + [\xi_{\sup} - \xi_{\inf}]^\top \theta_i \le 0, \\ -[V^i W - A^i] \le \theta_i^\top,\ \theta_i^\top \ge V^i W - A^i. \end{cases}$$

Proof. The AARC of (24) consists in minimizing $c^\top u$ under the constraints in (u, w, W)

$$U^i u + V^i w - b^i + [V^i W - A^i]\xi \le 0, \quad \forall\, \xi \in [\xi_{inf}, \xi_{\sup}].$$

An equivalent representation of these constraints is

$$U^i u + V^i w - b^i + \max_{\xi_{inf} \le \xi \le \xi_{\sup}} [V^i W - A^i]\xi \le 0.$$

We then express $\max\limits_{\xi_{inf} \leq \xi \leq \xi_{\sup}} [V^i W - A^i]\xi$ as

$$\sum_{[V^i W - A^i](j) \geq 0} [V^i W - A^i](j)\xi_{\sup}(j) + \sum_{[V^i W - A^i](j) < 0} [V^i W - A^i](j)\xi_{\inf}(j)$$

$$= \sum_j \max([V^i W - A^i](j), 0)\xi_{\sup}(j) + \sum_j \min([V^i W - A^i](j), 0)\xi_{\inf}(j) \tag{25}$$

$$= \frac{1}{2}\Big([V^i W - A^i](\xi_{\inf} + \xi_{\sup}) + |V^i W - A^i|(\xi_{\sup} - \xi_{\inf})\Big), \tag{26}$$

where vector $|x|$ is defined by $|x|^i = |x^i|$, and where we have used for the last equality that for $x \in \mathbb{R}$, $\max(x, 0) = \frac{1}{2}(x + |x|)$ and $\min(x, 0) = \frac{1}{2}(x - |x|)$. Adding slack variables and using (25) (resp. (26)), we obtain representation LP_1 (resp. LP_2). □

Lemma 4.2. *When the uncertainty set for ξ is the polytope $\chi = Conv(\bar{\xi}_1, \ldots, \bar{\xi}_k)$, then the AARC of uncertain problem (24) is the linear programming*

$$LP_3 \begin{cases} \min\limits_{u,w,W,\theta_i} \; c^\top u \\ U^i u + V^i w - b^i + \theta_i \leq 0, \\ \theta_i \geq [V^i W - A^i]\bar{\xi}_\ell, \;\; \ell = 1, \ldots, k. \end{cases}$$

Proof. The AARC of (24) consists in minimizing $c^\top u$ under the constraints in (u, w, W)

$$U^i u + V^i w - b^i + [V^i W - A^i]\xi \leq 0, \quad \forall\, \xi \in \text{Conv}(\bar{\xi}_1, \ldots, \bar{\xi}_k).$$

It suffices to observe that $\max\limits_{\xi \in \chi} [V^i W - A^i]\xi = \max\limits_{1 \leq \ell \leq k} [V^i W - A^i]\bar{\xi}_\ell$ and to add slack variables to conclude. □

Lemma 4.3. *Let K be a definite positive matrix. When the uncertainty set for ξ is the ellipsoid $\chi = \{\xi \mid (\xi - \bar{\xi})^\top K^{-1}(\xi - \bar{\xi}) \leq \kappa^2\}$, then the AARC of uncertain problem (24) is the following conic quadratic optimization problem*

$$CQ_1 \begin{cases} \min\limits_{u,w,W} \; c^\top u \\ U^i u + V^i w - b^i + [V^i W - A^i]\bar{\xi} + \kappa\sqrt{[V^i W - A^i]K^{-1}[V^i W - A^i]^T} \leq 0. \end{cases}$$

Proof. The AARC of (24) consists in minimizing $c^\top u$ under the constraints in (u, w, W)

$$U^i u + V^i w - b^i + \phi_\chi([V^i W - A^i]^\top) \leq 0,$$

where $\phi_\chi(\cdot)$ is the support function of the ellipsoid χ given for any $x \in \mathbb{R}^M$ by $\phi_\chi(x) = \max\limits_{\xi \in \chi} x^\top \xi = x^\top \bar{\xi} + \kappa\sqrt{x^\top K^{-1} x}$. □

Lemmas 4.1, 4.2 and 4.3 thus provide the AARC of the stochastic counterpart of (7) for respectively box constrained, polytopic, and ellipsoidal uncertainty sets. When the thermal and EJP controls are determined by equations (21), the hydro controls by equations (22) and when $I_t = \{1, \ldots, t-1\}$, matrix W adopts the particular form given in (23). The detailed formulation of these AARC (writing the constraints with the elements of U, V, A, b, c, and W) can be found in (Guigues 2005). Notice that the number of variables is in general much larger in the AARC. For instance, if $\mathcal{P}_\xi$ has n_u (size of u) + n_v (size of v) variables and p constraints then LP_1 has $n_u + n_v + N_W + 2pMT$ variables and $p(4MT + 1)$ constraints where $M = |\mathcal{L}_T| + |\mathcal{L}_H| + 1$ and N_W is the number of non-zero elements in W. Thus, to obtain an AARC of reasonable size, we can aggregate the time steps (which provides a smaller value of T) and diminish N_W by taking smaller sets I_t.

4.3. **Calibration of uncertainty sets.** The robust methods developed in the previous sections need uncertainty sets for the uncertain parameters over the optimization period. From the practical point of view, the determination of uncertainty sets for robust optimization is a statistical problem of great importance. The uncertainty sets for our application are calibrated using two different approaches.

In the first approach, we suppose we know the 456 possible scenarios for the uncertain parameters over the optimization period. These scenarios of equal probability are provided by EDF (the company providing electricity in France) after a statistical analysis. In this case, the uncertainty sets are the convex hull of the scenarios or a box constrained uncertainty set containing these scenarios.

In the second approach, we only consider uncertainty in the demand and calibrate a box constrained uncertainty set for the demand using historical data.

The determination of uncertainty sets in robust optimization is in fact made in one of the three following contexts.

- (i) The domain of variation D of the uncertain parameters (support of the underlying random vector if the uncertainty is of stochastic nature) is bounded and known. In this case, we take as uncertainty set a nonempty closed and convex set containing domain D. This is the case in our first approach. The two remaining situations (ii) and (iii) below hold when the uncertainty is of stochastic nature.
- (ii) Support $\mathcal{S}$ of the underlying random vector is bounded but not known. In this case, it first has to be estimated by a bounded set before determining a nonempty closed, and convex set containing this support.
- (iii) Finally, in the case where $\mathcal{S}$ is not bounded, we do not look for an estimation of $\mathcal{S}$ but for bounded, nonempty, closed, and convex prediction areas. If X has n components, we look for a set $E \subseteq \mathbb{R}^n$, such that $\mathbb{P}(X \in E) \geq 1 - \varepsilon$, or sets $E_i \subseteq \mathbb{R}, 1 \leq i \leq n$, such that $\mathbb{P}(X_i \in E_i) \geq 1 - \varepsilon$, ε being a given confidence level. The difficulty of these problems depends on the structure of sets E and E_i chosen and on the probability distribution of X. This probability distribution can be known or estimated from available historical data as is the case for the second approach we consider.

In case (i) and thus in our first approach, the assumption (used to build the Robust Counterparts) that the uncertain parameters over the optimization period belong to the uncertainty set is satisfied.

In case (iii) and when uncertainty sets have to be determined using historical data, the applications presented in the literature so far propose in general simple nonparametric approaches that do not rely on a statistical study. For instance, given a sample $(\xi_1, \ldots, \xi_T)$ of random vector ξ, we can choose as uncertainty set for ξ the convex hull of the scenarios $(\xi_1, \ldots, \xi_T)$ or a shrinkage of this convex hull. More precisely, if g is the barycenter of $(\xi_1, \ldots, \xi_T)$ and if $\xi_i' = g + \alpha(\xi_i - g)$, with $0 < \alpha < 1$, a shrinkage of the convex hull of $(\xi_1, \ldots, \xi_T)$ could be the convex hull of $(\xi_1', \ldots, \xi_T')$. Using this contraction of the convex hull as an uncertainty set instead of the convex hull itself, induces a less conservative Robust Counterpart. We can also take the ellipsoid of smallest volume (or the sphere of smallest volume) which contains all points $(\xi_1, \ldots, \xi_T)$, which is an SDP problem. We can also look for the closest ellipsoid or sphere to the cloud of points $(\xi_1, \ldots, \xi_T)$, (Calafiore 2002). Our second approach corresponds to case (iii) with unknown probability distribution for the vector of demands over the optimization period. In this case, instead of using one of the nonparametric calibration methods mentioned above, we do a statistical analysis of the underlying random process for our stochastic optimization problem. This aspect has often been neglected so far but is of crucial importance for the application of robust optimization methodology on real-life problems. The methods used in such a study depend on the practical situations met. In the case of the process of electricity consumption, we study different methods taking into account certain specificities of the time series. These methods intend to provide lower bounds $\mathcal{D}_t^{\min}$ and upper bounds $\mathcal{D}_t^{\max}$ for the electricity consumption at time step t over the optimization period. The uncertainty sets are prediction intervals of level 95% for each demand D_t, i.e., satisfying $\mathbb{P}(D_t \in [\mathcal{D}_t^{\min}, \mathcal{D}_t^{\max}]) \geq 0.95$.

In what follows, we briefly mention the two models that are tested for determining these prediction intervals. The historical data of demands we use are the weekly observations in France from 1996 to 2003 (the largest set of data available on the RTE website when this study was done). We refer to (Guigues 2005) for a detailed treatment of this study.

Electricity consumption is essentially influenced by the temperature (for heating in winter and air conditioning in summer) and the cloud cover (for lightning). The first model is a *regression model* that uses as regressors the temperature and the cloud cover (measured in octa by a number between 0 and 8). We introduce gradients of temperatures for heating (used below a given temperature) and for the air conditioning (used above a given temperature). The increasing tendency in the demand is modelled by an affine function $\alpha t + \beta$ of time t and the seasonal effect by a periodic process with a one-year period. The prediction intervals are computed by adjusting this model to the available data of demands and after testing and accepting the normality of the residuals. In this approach, the values of the regressors need to be known over the optimization period (in practice estimations of these values are used).

The second model fitted to the data is a seasonal *SARIMA* model (Box et al. 1976) of 52 week seasonality and prediction intervals are computed following the lines of (Box et al. 1976).

In (Guigues 2005), we propose another seasonal model (affine tendency plus seasonal process) where a hidden Markov chain model is fitted to the noise.

5. Numerical simulations

We now compare the robust management policies introduced in this article with other management policies. The management horizon is one year and each time step lasts 15 days. Two data sets are used.

On the one hand, a set of 456 simulated scenarios (provided by EDF) giving different possible evolutions of the uncertain parameters over the year.

On the other hand, we use the electric consumption in France from 1996 to 2003. For $3 \leq x \leq 7$, using the historical data from 1996 to $1995 + x$, we compare the costs when using the RC and the AARC methodology over the year $1996 + x$. These costs will be compared with the optimal management cost (obtained when solving (7) using the demands, availability rates and inflows of the year $1996 + x$).

There are eleven thermal plants. Each thermal plant is described by its unit production cost, its maximal and minimal power, the number of thermal groups and the probability that a thermal group works a given day. There are two independent hydroelectric plants. Each hydroelectric plant is connected to a different reservoir. We know the maximal and initial stock (in GWh) of each reservoir and the maximal power (in MW) of each hydroelectric plant. There is finally one EJP contract of 22 days.

The optimization problems are solved using Matlab and the Mosek optimization library. The absolute error in the computed objective is at most 10^{-8}.

5.1. **Simulated data.**

5.1.1. *ARC.* We implement the ARC of the stochastic counterpart of (7) with 24 time steps, choosing for uncertainty set the convex hull of the 456 scenarios or a subset of these scenarios.

We show that the greater the number of scenarios, the more the solution of the ARC is interesting compared to the solution of the RC. To that end, we implement the RC and the ARC obtained choosing as uncertainty set the convex hull of the first x scenarios, x successively taking the values 5, 200 and 456. The number of variables of the ARC linearly increases with the number of scenarios. Using the 456 scenarios, we end up with an ARC with 170 103 variables and 248 991 constraints[2]. The ARC provides, for each scenario ξ_k, a solution (u, v_k) adapted to this scenario. In Table 2 which follows, "m(ARC)" is the mean cost (over the first x scenarios) obtained using these adaptive solutions and "s(ARC)" is the standard deviation of these costs. In this table, we also indicate the mean costs "m(R)", "m(S)", "m(Opt)" and the standard deviations "s(R)", "s(S)" and "s(Opt)" when using respectively (i) the RC, (ii) the solution of the problem where each uncertain parameter is replaced by its mean over the set of the first x scenarios and finally (iii) the optimal strategy (choosing on each scenario the optimal controls on this scenario).

[2]solved in about one hour with a 1.6GHz processor and 256MBytes of RAM.

x	m(ARC)	s(ARC)	m(S)	s(S)	m(R)	s(R)	m(Opt)	s(Opt)
5	4.23	1.11	4.17	1.41	4.54	2.65	4.15	5.53
200	4.40	1.17	4.14	1.70	7.02	3.4	4.12	12.1
456	4.54	1.01	4.14	1.80	8.14	3.5	4.12	12.7

TABLE 2. Average cost (divided by 10^8) and standard deviation (divided by 10^6) of the cost over a set of x scenarios for the uncertain parameters using the ARC and RC techniques.

The costs obtained by replacing each uncertain parameter by an estimation of its mean over the set of scenarios are very close to the optimal costs. Nevertheless, this policy cannot be used in practice because for quite a number of time steps, the demand is not satisfied. The average cost with the ARC is, as expected, less than with the RC. When the number of scenarios increases, the use of the RC yields a far from optimal solution (nearly twice as costly if we use 456 scenarios).

The RC solution is the same for all the scenarios. If the inflows were fixed, the standard deviation of the cost using the RC would be null. However, the differences between the inflows from a scenario to another explain the standard deviation obtained using the RC. We even notice that the standard deviation of the cost obtained using the RC is greater than the one obtained using the ARC. The ARC thus doubly satisfies our objective: reduction of the mean cost and reduction of the standard deviation of the cost. Choosing the controls of the first time step as the only adjustable variables, we could however have expected the cost "m(ARC)" to be closer to the optimal cost "m(Opt)". The large number of scenarios and the presence of certain hard scenarios could explain this gap.

These results on the use of the ARC are theoretical. In practice, to determine the controls for time step t, we use the non-adjustable controls of the ARC of the electricity generation management problem corresponding to the last $T - t + 1$ time steps. Using this approach and taking as uncertainty set the convex hull of the first 5 scenarios, the mean cost is 4.30×10^8 and the standard deviation of the cost is 1.75×10^6. The sliding "ARC" thus yields a mean cost not too far from the optimal mean cost and a small standard deviation of the cost.

5.1.2. *AARC.* **Polytopic uncertainty sets.** We first consider the AARC obtained by choosing for the uncertainty set the convex hull of a subset of the 456 scenarios.

In this paragraph, we confine ourselves to choosing a restricted number of scenarios (6 and 20) to show the influence of the choice of interval I_t and of the dependency of the adjustable variables as a function of the uncertain parameters on the quality of the solutions of the AARC.

First, we suppose that the uncertainty set is the convex hull of the first 6 scenarios and that the adjustable variables are given by (21). For each time step, the use of the solution of the AARC and of (21) gives the controls provided by the AARC for this time step. We then use different sizes of interval $I_t : I_t = \{1, \ldots, t-m\}$ if $t > m$ and $I_t = \emptyset$ otherwise; for different values of m. We report in Table 3 below, for each choice of interval I_t, the mean cost and the standard deviation of the cost over the 6 scenarios.

Value of m	$m = 1$	$m = 2$	$m = 4$	$m = 6$	$m = 12$
Mean	4.26×10^8	4.261×10^8	4.288×10^8	4.33×10^8	4.57×10^8
S.d	1.04×10^5	3.95×10^4	7.66×10^4	1.71×10^5	4.05×10^6

TABLE 3. Influence of the quantity of information used on the performance of the AARC.

As expected, the mean cost is an increasing function of m. Indeed, the larger m is, the less information is used to explain the adjustable controls and the smaller the feasibility set. When m is 24 (the number of time steps), we get the solution of the RC.

We now fix $I_t = \{1, \ldots, t - 1\}$, and we observe the influence of the choice of the adjustable variables on the performances of the AARC. We propose 4 dependency schemes for the adjustable variables as a function of the uncertain parameters.

- For the first method (`Method 1`), the thermal controls, the hydro and EJP controls are adjustable and given by (21). For the second method (`Method 2`), the thermal and hydro controls are adjustable and given by (21). The EJP controls are not adjustable.
- For the third method (`Method 3`), all the controls are adjustable; the thermal and EJP controls being given by (21), and the hydro controls by (22). For the last method (`Method 4`), the thermal and hydro controls are adjustable and given respectively by (21) and (22). The EJP controls are not adjustable.

Choosing the convex hull of the first 6 and first 20 scenarios for the uncertainty set, we get the results reported in Tables 4 and 5 which follow.

Method	`Method 1`	`Method 2`	`Method 3`
Mean	4.26×10^8	4.26×10^8	4.25×10^8
Standard deviation	1.04×10^5	9.00×10^4	6.87×10^4

TABLE 4. Influence of the choice of the adjustable variables and dependency schemes on the performances of the AARC. First 6 scenarios.

Method	`Method 1`	`Method 2`	`Method 3`	`Method 4`
Mean	4.471×10^8	4.472×10^8	4.456×10^8	4.456×10^8
Standard deviation	2.391×10^6	2.387×10^6	6.223×10^4	6.394×10^4

TABLE 5. Influence of the choice of the adjustable variables and dependency schemes on the performances of the AARC. First 20 scenarios.

As expected, the richer the information, the less the average cost. It seems interesting to use both electric consumption and inflows to explain hydro controls. Besides, it does not seem necessary, regarding these simulations, to consider the EJP variables as adjustable.

Box constrained uncertainty sets. We now choose an uncertainty set defined by box constraints: $\mathcal{D}_t^{\min} \leq \mathcal{D}_t \leq \mathcal{D}_t^{\max}$ for the demand, $\mathcal{I}_t^{\ell,\min} \leq \mathcal{I}_t^{\ell} \leq \mathcal{I}_t^{\ell,\max}$ for the inflows, and $\tau_t^{\ell,\min} \leq \tau_t^{\ell} \leq \tau_t^{\ell,\max}$ for the availability rates. We simply calibrate $\mathcal{D}^{\min}$ and $\mathcal{D}^{\max}$ estimating for every time step t, $\mathcal{D}_t^{\min}$ and $\mathcal{D}_t^{\max}$ by the minimal and maximal values of the consumption for time step t over the set of scenarios. The vectors $\mathcal{I}^{\ell,\min}$,$\mathcal{I}^{\ell,\max}$, $\tau^{\ell,\min}$ and $\tau^{\ell,\max}$ are calibrated in the same way and we choose $I_t = \{1, \dots, t-1\}$. Two AARCs are implemented. The adjustable variables are functions of the uncertain parameters as in (21) for the first one (denoted by $\texttt{AARC}_1$). For the second one (denoted by $\texttt{AARC}_2$), the thermal and EJP controls are given by (21) and the hydro controls by (22). We denote by $\texttt{SAARC}_2$ the sliding version of method $\texttt{AARC}_2$. Problem $\texttt{AARC}_2$ has 17 913 variables and 30 086 constraints.[3] The mean cost and the standard deviation of the cost over the set of 456 scenarios for these different *AARCs* are given in Table 6 which follows.

Method	`Optimal cost`	$\texttt{AARC}_1$	$\texttt{AARC}_2$	$\texttt{SAARC}_2$
Mean	4.12×10^8	5.01×10^8	4.70×10^8	4.75×10^8
Standard deviation	1.27×10^7	2.4×10^6	3.57×10^6	1.2×10^7

TABLE 6. Performances of the AARC. Uncertainty set defined by box constraints.

The use of inflows to explain hydro controls seems here particularly interesting. If we were not in a "back-testing" context, we would expect the sliding version $\texttt{SAARC}_2$ to perform better than $\texttt{AARC}_2$.

[3]Solved in about 10 min with a 1.6GHz processor and 256MBytes of RAM.

Now to highlight the importance of the size of uncertainty sets on the quality of robust methods, we finally consider an AARC using a contraction of the box constrained uncertainty set used with $\mathtt{AARC}_2$. To that end, if $\bar{\mathcal{D}} = \frac{1}{2}(\mathcal{D}^{\min} + \mathcal{D}^{\max})$ and if we had used uncertainty set $[\bar{\mathcal{D}} - \frac{1}{8}(\bar{\mathcal{D}} - \mathcal{D}^{\min}), \bar{\mathcal{D}} + \frac{1}{8}(\mathcal{D}^{\max} - \bar{\mathcal{D}})]$ for the electric consumption, and replaced the demand scenarios by scenarios uniformly drawn in $[\bar{\mathcal{D}} - \frac{1}{8}(\bar{\mathcal{D}} - \mathcal{D}^{\min}), \bar{\mathcal{D}} + \frac{1}{8}(\mathcal{D}^{\max} - \bar{\mathcal{D}})]$, the mean cost of $\mathtt{AARC}_2$ would have decreased significantly from $4.7{\times}10^8$ to $4.51{\times}10^8$.

5.1.3. *Comparison of different management methods.* In Table 7, we give the mean cost and the standard deviation of the cost over the 456 scenarios and for different management methods. As in (Guigues et al. 2009), let us denote by `Tree` a scenario tree based optimization method described in (Guigues et al. 2009) and let `Dual.Stab.` be the management method described in (Guigues 2005) based on a stabilization of the dual function. Notice that the scenario tree based optimization method `Tree` was used in (Guigues et al. 2009) for a management horizon of one year but with a daily time step. We aggregated the tree used in (Guigues et al. 2009) and adapted the optimization methods in order to use a 15 day time step. This way, all the methods use the same discretization of the optimization period. Methods `ARC`, $\mathtt{AARC}_2$ and $\mathtt{SAARC}_2$ are defined in the previous paragraph. The methods are ranked according to the value of the sum mean plus standard deviation of the cost.

Method	`ARC`	$\mathtt{AARC}_2$	`Dual.Stab.`	`Tree`	$\mathtt{SAARC}_2$
Mean	$4.54{\times}10^8$	$4.70{\times}10^8$	$4.49\ {\times}10^8$	$4.51{\times}10^8$	$4.75{\times}10^8$
Sd	$1.01{\times}10^6$	$3.57{\times}10^6$	$2.92{\times}10^7$	$3.26{\times}10^7$	$1.2{\times}10^7$
Rank	1	2	3	4	5

TABLE 7. Comparison of the different management methods.

The AARC methods introduced in this article yield a higher average cost and a significantly smaller standard deviation of the cost. The ARC method yields both low mean and standard deviation but is only of theoretical interest.

5.2. **Simulations with real data.** Given the electric consumption in France from 1996 to 2003, we use the first x years of data (for $3 \leq x \leq 7$) to determine a production schedule for year $1996 + x$. We use the same production capacity as before. The inflows for the two reservoirs are fixed (equal to the mean inflows over the 456 scenarios considered in the previous section) and we suppose that all the availability rates are equal to one over the optimization period. We test 3 AARCs and 3 RCs obtained by taking box constrained uncertainty sets.

We define $\mathtt{AARC}_1$ and $\mathtt{RC}_1$ as the AARC and RC obtained taking as uncertainty set on the demands, the prediction interval obtained using the *regression model* introduced in Section 4.3. To explain the electric consumption of year $1996+x$, we thus use the temperature and cloud cover of that year.

We denote by $\mathtt{AARC}_2$ and $\mathtt{RC}_2$ the AARC and RC obtained when the uncertainty set on the consumption is determined using the *SARIMA* model.

We also denote by $\mathtt{AARC}_3$ and $\mathtt{RC}_3$ the AARC and RC obtained when the uncertainty set $[\mathcal{D}^{\min}, \mathcal{D}^{\max}]$ on the consumption is such that $\mathcal{D}_t^{\min}$ (resp. $\mathcal{D}_t^{\max}$) is the smallest (resp. the largest) consumption for time step t over the first x years of historical data.

Finally, let $\mathtt{CS}_1$, $\mathtt{CS}_2$, and $\mathtt{CS}_3$, be the methods consisting of replacing the uncertain consumption by their forecasts using, respectively, the *regression model*, the *SARIMA* model, and the mean consumption over the first x years of historical data. To implement the AARC, we choose, for time step t, $I_t = \{1, \ldots, t-1\}$ and we suppose the adjustable variables are functions of the consumption given by (21). The results are given in Table 8 (the first row of the table gives the optimal costs). The AARC gives the most interesting results. For $\mathtt{AARC}_1$, in particular, the mean cost is only 3.9% above the optimal mean cost.

x	3	4	5	6	7	8	Mean
Opt	3.80	3.86	3.98	4.12	4.27	4.42	4.07
$AARC_1$	3.98	4.13	4.17	4.18	4.44	4.49	4.23
$AARC_2$	-	4.50	4.53	4.64	4.77	4.89	4.67
$AARC_3$	4.05	4.12	4.25	4.39	4.47	-	4.26
RC_1	4.32	4.39	4.52	4.67	4.75	-	4.53
RC_2	-	4.75	4.78	4.83	5.03	5.15	4.91
RC_3	4.25	4.39	4.43	4.55	4.69	4.74	4.51
CS_1	4.69	4.54	4.56	4.62	4.69	4.71	4.63
CS_2	-	4.70	4.79	4.89	5.08	5.19	4.93
CS_3	4.65	4.72	4.84	5.00	5.07	-	4.86

Table 8. Costs (divided by 10^8) of the AARC and RC of problem (7) implemented for years 1998, 1999, 2000, 2001, 2002 and 2003 and using 8 years of electric consumption data in France from 1996 to 2003.

6. Conclusion

The methodology of the AARC applied to the electricity production management problem provides management policies whose mean cost is reasonable (though higher than the mean cost obtained in (Guigues et al. 2009) for instance) and with a low standard deviation of the cost. The superiority of this methodology to a more traditional RC approach is also illustrated in this application.

From the theoretical point of view, an interesting perspective in the area of AARC is to study the AARC of general conic quadratic and SDP problems.

Acknowledgments

The author is grateful to Anatoli Juditski of the "Laboratoire de Modélisation et Calcul" of University Joseph Fourier for helpful discussions. We also thank the referees for their beneficial suggestions for improvement of this paper.

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