Frontiers in Signal Processing, Vol. 3, No. 1, January 2019
https://dx.doi.org/10.22606/fsp.2019.31001

Robust RLS Wiener Signal Estimators for Discrete-Time Stochastic Systems with Uncertain Parameters

Seiichi Nakamori

Department of Technology, Faculty of Education, Kagoshima University, Kagoshima, Japan
Email: k6165161@kadai.jp

Abstract This paper proposes the robust recursive least-squares (RLS) Wiener fixed-point smoother and filter in linear discrete-time stochastic systems with parameter uncertainties. The uncertain parameters exist in the observation matrix and the system matrix. The uncertain parameters cause to generate the degraded signal. In this paper, the degraded signal process is fitted to the finite order autoregressive (AR) model. The robust RLS Wiener estimators use the system matrices and the observation matrices for both the signal and the degraded signal, the variance of the state vector for the degraded signal, the cross-variance of the state vector for the signal with the state vector for the degraded signal, and the variance of the white observation noise. Also, this paper proposes the robust recursive fixed-point smoother and filter, by using the covariance information of the state vector for the degraded signal, the cross-covariance information of the state vector for the signal with the state vector for the degraded signal, the observation matrices for both the degraded signal and the signal besides the variance of the white observation noise. In estimating the signal process expressed by the second order AR model, the proposed robust RLS Wiener filter is superior in estimation accuracy to the robust Kalman filter and the RLS Wiener filter.

Keywords: Discrete-time stochastic systems; robust RLS Wiener filter; robust RLS Wiener fixed-point smoother; robust estimation technique; uncertain parameters.

1 Introduction

The state-space model in macroeconomic and financial problems includes parameter uncertainties [1]. An improved robust Kalman filter is applied to precise point positioning/inertial navigation system (PPP/INS) [2]. In the network communication systems (NCS), the delayed observation and the packet dropout [3] occur during trans-receiving process [4], [5], [6], [7]. In estimating the signal process, the parameter uncertainties in the state-space model might be taken into account. The Kalman filter calculates the estimate recursively by use of the precise dynamic model. The standard Kalman filter or the recursive Wiener filter are not robust against the state-space model including parameter uncertainties. For this reason the robust Kalman filtering problems have been investigated intensively. The case for the uncertain systems without multiplicative noise has been treated in [8], [9], [10], [11] and [12]. In [13] and [14], Wang and Balakrishnan adopt the linear matrix inequality (LMI) method in the stationary robust filtering problem for the uncertain systems with multiplicative noises. In [15], a robust finite-horizon Kalman filter is proposed in discrete time-varying uncertain systems with both additive and multiplicative noises. Note that the system matrix and output matrix contain both deterministic and stochastic parametric uncertainties. Sufficient conditions for the filter to guarantee an optimized upper bound on the state estimation error variance for admissible uncertainties are established in terms of two discrete Riccati difference equations. In [16], a robust Kalman filter is designed for the uncertain time-varying discrete-time stochastic systems with state delay and missing measurement characterized by the Bernoulli random variables. In [17] and [18], the regularized robust filters are proposed in linear discrete-time stochastic systems. In [18], the LMI method is used. In [19], the robust H-infinity filter is proposed in linear discrete-time stochastic systems. In [20], the robust filter is presented based on the regularization approach and the penalty function. In [21], the finite and infinite horizon robust Kalman filtering algorithms are presented in linear discrete-time stochastic systems. The norm-bounded condition is posed on the uncertain parameters in the system and observation matrices. The filtering estimate is calculated recursively by solving the Riccati-type equations. In [22], the robust filter is designed by minimizing the mean square error according to the most feasible model. The risk sensitive filter is also extended using the τ-divergence family.

The robust Kalman filter is designed to estimate the state vector for the state-space model with parameter uncertainties. The norm-bounded condition is posed on the uncertain parameters in the system and observation matrices. The drawback of the robust Kalman filter [21] is that the estimation accuracy is not satisfactory as shown in the numerical simulation example in this paper. The main task of this paper is to design the robust recursive least-squares (RLS) Wiener estimators for signal estimation with better estimation accuracy than the robust Kalman filter [21] in linear discrete-time stochastic systems. The specific character of the proposed robust estimators is that they do not use any information of the uncertain parameters in the state and observation equations. It is assumed that the signal process is expressed by the autoregressive (AR) model. The degraded signal, by the uncertain parameters in the observation and system matrices, is fitted to the finite order AR model. Based on the AR model, the system and observation matrices are obtained in the state and observation equations for the degraded signal process. The fixed-point smoothing estimate is given as a linear transformation of the observed values in terms of the impulse response function. Minimizing the mean-square value (MSV) of the fixed-point smoothing errors, we obtain the Wiener-Hopf equation, which the optimal impulse response function satisfies. From the Wiener-Hopf equation the robust RLS Wiener fixed-point smoothing and filtering algorithms are derived in Theorem 1 by the invariant imbedding method. The robust RLS Wiener estimation algorithms use the system and observation matrices for both the signal and degraded signal processes, the variance of the state vector for the degraded signal, the cross-variance function of the state vector for the signal process with the state vector for the degraded signal, and the variance of the white observation noise. Also, this paper proposes the robust RLS fixed-point smoothing and filtering algorithms in Corollary 1, by using the covariance information of the state vector for the degraded signal, the cross-covariance information of the state vector for the signal with the state vector for the degraded signal, the observation matrices for both the signal and degraded signal processes besides the variance of the white observation noise. In the robust Kalman filter, the state vector is estimated under the norm-bounded condition on the uncertain parameters in the system and observation matrices. It is advantageous that the robust RLS Wiener estimators and the robust RLS estimators do not use the norm-bounded condition at all.

A numerical simulation example for estimating the signal process, expressed by the AR model, is demonstrated to compare the estimation accuracy of the proposed robust RLS Wiener filter with those of the robust Kalman filter [21] and the RLS Wiener filter [23].

2 Least-Squares Fixed-Point Smoothing Problem

Let an m-dimensional observation equation and an n-dimensional state equation be given by

$$\begin{aligned}&\breve{y}(k)=\breve{z}(k)+v(k),\breve{z}(k)=\bar{H}(k)\bar{x}(k),\bar{H}(k)=H+\varDelta H(k),\\&\bar{x}(k+1)=\bar{\varPhi}(k)\bar{x}(k)+\varGamma w(k),\bar{\varPhi}(k)=\varPhi+\varDelta\varPhi(k),\\&E[v(k)v^T(s)]=R\delta_K(k-s),E[w(k)w^T(s)]=Q\delta_K(k-s),\end{aligned}\tag{1}$$

in linear discrete-time stochastic systems with uncertain parameters [21]. It is assumed that $\varDelta H(k)$ and $\varDelta\varPhi(k)$ include uncertain parameters respectively. Here, $v(k)$ is the white observation noise with the variance R. $w(k)$ is the white input noise with the variance Q. Their auto-covariance functions are expressed with the Kronecker delta function $\delta_K(k-s)$. The state equation generating $\bar{x}(k+1)$ includes the uncertain quantity $\varDelta\varPhi(k)$ in the system matrix $\bar{\varPhi}(k)$. In addition, in the observation equation the observation matrix $\bar{H}(k)$ includes the uncertain quantity $\varDelta H(k)$. Hence, $\breve{z}(k)$ has the different value from the nominal signal $z(k)$ of the state-space model (2), not including the uncertain quantities. In (1), as the sum of the degraded signal $\breve{z}(k)$ and the observation noise $v(k)$, $\breve{y}(k)$ is measured. The state-space model without including the uncertain quantities $\varDelta H(k)$ and $\varDelta\varPhi(k)$ in (1) is expressed as

$$\begin{aligned}&y(k)=z(k)+v(k),z(k)=Hx(k),\\&x(k+1)=\varPhi x(k)+\varGamma w(k).\end{aligned}\tag{2}$$

In (2), $z(k)$ represents the signal to be estimated. H is an m by n observation matrix, $x(k)$ is the state vector and $v(k)$ is the white observation noise with the auto-covariance function in (1). Also, the auto-covariance function of the input noise $w(k)$ is given in (1). It is assumed that the signal and the observation noise are mutually independent and have zero means. The purpose of this paper is to design the RLS Wiener fixed-point smoother and filter to obtain the estimates of the signal $z(k)$ with the measurement data $\breve{y}(k)$ without requiring any knowledge of the uncertain quantities $\Delta\Phi(k)$ and $\Delta H(k)$.

Let the sequence of the degraded signal $\breve{z}(k)$ be fitted to the Nth order AR model.

$$\begin{aligned} &\breve{z}(k) = -a_1\breve{z}(k-1) - a_2\breve{z}(k-2)\cdots - a_N\breve{z}(k-N) + \breve{e}(k), \\ &E[\breve{e}(k)\breve{e}^T(s)] = \breve{Q}\delta_K(k-s) \end{aligned} \tag{3}$$

Let $\breve{z}(k)$ be expressed by

$$\begin{aligned} &\breve{z}(k) = \breve{H}\breve{x}(k), \\ &\breve{x}(k) = \begin{bmatrix} \breve{x}_1(k) \\ \breve{x}_2(k) \\ \vdots \\ \breve{x}_{N-1}(k) \\ \breve{x}_N(k) \end{bmatrix} = \begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix}, \\ &\breve{H} = \begin{bmatrix} I_{m\times m}\ 0\ 0 \cdots\ 0\ 0 \end{bmatrix}. \end{aligned} \tag{4}$$

Henceforth, the state equation for the state vector $\breve{x}(k)$ is expressed as

$$\begin{aligned} &\begin{bmatrix} \breve{x}_1(k+1) \\ \breve{x}_2(k+1) \\ \vdots \\ \breve{x}_{N-1}(k+1) \\ \breve{x}_N(k+1) \end{bmatrix} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -\breve{a}_N & -\breve{a}_{N-1} & -\breve{a}_{N-2} & \cdots & -\breve{a}_1 \end{bmatrix} \begin{bmatrix} \breve{x}_1(k) \\ \breve{x}_2(k) \\ \vdots \\ \breve{x}_{N-1}(k) \\ \breve{x}_N(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_{m\times m} \end{bmatrix} \zeta(k), \\ &\zeta(k) = \breve{e}(k+N), E[\zeta(k)\zeta^T(s)] = \breve{Q}\delta_K(k-s). \end{aligned} \tag{5}$$

Let $\breve{K}(k,s) = \breve{K}(k-s)$ represent the auto-covariance function of the state vector $\breve{x}(k)$ in wide-sense stationary stochastic systems [24], and let $\breve{K}(k,s)$ be expressed in the form of

$$\breve{K}(k,s) = \begin{cases} A(k)B^T(s), 0 \le s \le k, \\ B(k)A^T(s), 0 \le k \le s. \end{cases} \tag{6}$$

$A(k) = \breve{\Phi}^k, B^T(s) = \breve{\Phi}^{-s}\breve{K}(s,s)$. Here, $\breve{\Phi}$ is the state transition matrix for the state vector $\breve{x}(k)$. The system matrix $\breve{\Phi}$ in the state equation (5) is given by

$$\breve{\Phi} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -\breve{a}_N & -\breve{a}_{N-1} & -\breve{a}_{N-2} & \cdots & -\breve{a}_1 \end{bmatrix}. \tag{7}$$

Also, by putting $K_{\breve{z}}(k,s) = K_{\breve{z}}(k-s) = E\left[\breve{z}(k)\breve{z}^T(s)\right]$, the auto-variance function $\breve{K}(k,k)$ of the state vector $\breve{x}(k)$ is expressed as

$$\begin{aligned} \breve{K}(k,k) &= E\left[\begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix} \right. \\ &\quad \times \left. \begin{bmatrix} \breve{z}^T(k)\ \breve{z}^T(k+1) \cdots\ \breve{z}^T(k+N-2)\ \breve{z}^T(k+N-1) \end{bmatrix}\right] \\ &= \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(-1) & \cdots & K_{\breve{z}}(-N+2) & K_{\breve{z}}(-N+1) \\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(-N+3) & K_{\breve{z}}(-N+2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(-1) \\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0) \end{bmatrix}. \end{aligned} \tag{8}$$

By using $K_{\breve{z}}(k-s)$, the Yule-Walker equation for the AR parameters is given by

$$\hat{K}(k,k)\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_{N-1}^T \\ a_N^T \end{bmatrix} = -\begin{bmatrix} K_{\breve{z}}^T(1) \\ K_{\breve{z}}^T(2) \\ \vdots \\ K_{\breve{z}}^T(N-1) \\ K_{\breve{z}}^T(N) \end{bmatrix},$$
$$\hat{K}(k,k) = \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1) \\ K_{\breve{z}}^T(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}^T(N-2) & K_{\breve{z}}^T(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1) \\ K_{\breve{z}}^T(N-1) & K_{\breve{z}}^T(N-2) & \cdots & K_{\breve{z}}^T(1) & K_{\breve{z}}(0) \end{bmatrix}. \tag{9}$$

Let $K_{x\breve{x}}(k,s) = K_{x\breve{x}}(k-s) = E[x(k)\breve{x}^T(s)]$ represent the cross-covariance function of the state vector $x(k)$ with $\breve{x}(s)$ in wide-sense stationary stochastic systems. Let $K_{x\breve{x}}(k,s)$ be expressed in the form of

$$K_{x\breve{x}}(k,s) = \alpha(k)\beta^T(s), 0 \le s \le k, \tag{10}$$

$\alpha(k) = \Phi^k, \beta^T(s) = \Phi^{-s}K_{x\breve{x}}(s,s)$. Here, Φ is the state transition matrix for the state vector $x(k)$.

Let the fixed-point smoothing estimate $\hat{x}(k,L)$ of the state vector $x(k)$ at the fixed point k be expressed by

$$\hat{x}(k,L) = \sum_{i=1}^{L} h(k,i,L)\breve{y}(i) \tag{11}$$

in terms of the observed values $\{\breve{y}(i), 1 \le i \le L\}$. In (11), $h(k,i,L)$ is a time-varying impulse response function. Let us consider the estimation problem, which minimizes the MSV

$$J = E[||x(k) - \hat{x}(k,L)||^2] \tag{12}$$

of the fixed-point smoothing error. From an orthogonal projection lemma [24],

$$x(k) - \sum_{i=1}^{L} h(k,i,L)\breve{y}(i) \perp \breve{y}(s), 1 \le s \le L, \tag{13}$$

the impulse response function satisfies the Wiener-Hopf equation

$$E[x(k)\breve{y}^T(s)] = \sum_{i=1}^{L} h(k,i,L)E[\breve{y}(i)\breve{y}^T(s)]. \tag{14}$$

Here '$\perp$' denotes the notation of the orthogonality. Substituting (1) into (14), from (4) and (8), and noting that $E[x(k)\breve{y}^T(s)] = K_{x\breve{x}}(k,s)\breve{H}^T = K_{x\breve{z}}(k,s)$, we obtain

$$h(k,s,L)R = K_{x\breve{x}}(k,s)\breve{H}^T - \sum_{i=1}^{L} h(k,i,L)\breve{H}\breve{K}(i,s)\breve{H}^T. \tag{15}$$

Here, $K_{x\breve{z}}(k,s)$ represent the cross-covariance function of the state vector $x(k)$ with the degraded signal $\breve{z}(s)$ as $E[x(k)\breve{z}^T(s)]$.

3 Robust RLS Wiener Fixed-Point Smoothing and Filtering Algorithms

Under the linear least-squares estimation problem of the signal $z(k)$ in section 2, Theorem 1 presents the robust RLS Wiener fixed-point smoothing and filtering algorithms.

Theorem 1 Let the state equation and the observation equation, including the uncertain quantities $\Delta\Phi$ and ΔH respectively, be given by (1). Let Φ and H represent the system and observation matrices respectively for the signal process of $z(k)$. Let $\breve{\Phi}$ and $\breve{H}$ represent the system and observation matrices respectively for the degraded signal process of $\breve{z}(k)$, fitted to the AR model (3) of the order N. Let the variance $\breve{K}(k,k)$ of the state vector $\breve{x}(k)$ for the degraded signal $\breve{z}(k)$ and the cross-variance function $K_{x\breve{x}}(k,k)$ of the state vector $x(k)$ for the signal $z(k)$ with the state vector $\breve{x}(k)$ for the degraded signal $\breve{z}(k)$ be given. Let the variance of the white observation noise $v(k)$ be R. Then, the robust RLS Wiener algorithms for the fixed-point smoothing estimate $\hat{z}(k,L)$, at the fixed point k, and the filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ consist of (16)-(26) in linear discrete-time stochastic systems.

Fixed-point smoothing estimate of the signal $z(k)$: $\hat{z}(k,L)$

$$\hat{z}(k,L) = H\hat{x}(k,L) \tag{16}$$

Fixed-point smoothing estimate of the state vector $x(k)$: $\hat{x}(k,L)$

$$\begin{aligned}&\hat{x}(k,L) = \hat{x}(k,L-1) + h(k,L,L)(\breve{y}(L) - \breve{H}\breve{\Phi}\hat{\breve{x}}(L-1,L-1)),\\ &\hat{x}(k,L)|_{L=k} = \hat{x}(k,k)\end{aligned} \tag{17}$$

Smoother gain: $h(k,L,L)$

$$\begin{aligned}&h(k,L,L) = [K_{x\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T - q(k,L-1)\breve{\Phi}^T\breve{H}^T]\\ &\times\{R + \breve{H}[\breve{K}(L,L) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1}\end{aligned} \tag{18}$$

$$\begin{aligned}&q(k,L) = q(k,L-1)\breve{\Phi}^T + h(k,L,L)\breve{H}[\breve{K}(L,L) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T],\\ &q(k,k) = S_0(k)\end{aligned} \tag{19}$$

Filtering estimate of the signal $z(k)$: $\hat{z}(k,k)$

$$\hat{z}(k,k) = H\hat{x}(k,k) \tag{20}$$

Filtering estimate of $x(k)$: $\hat{x}(k,k)$

$$\begin{aligned}&\hat{x}(k,k) = \Phi\hat{x}(k-1,k-1) + G(k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)),\\ &\hat{x}(0,0) = 0\end{aligned} \tag{21}$$

Filter gain for $\hat{x}(k,k)$ in the equation (21): $G(k)$

$$\begin{aligned}&G(k) = [K_{x\breve{z}}(k,k) - \Phi S(k-1)\breve{\Phi}^T\breve{H}^T]\\ &\times\{R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1},\\ &K_{x\breve{z}}(k,k) = K_{x\breve{x}}(k,k)\breve{H}^T\end{aligned} \tag{22}$$

Filtering estimate of $\breve{x}(k)$: $\hat{\breve{x}}(k,k)$

$$\begin{aligned}&\hat{\breve{x}}(k,k) = \breve{\Phi}\hat{\breve{x}}(k-1,k-1) + g(k)(\breve{y}(k) - \breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)),\\ &\hat{\breve{x}}(0,0) = 0\end{aligned} \tag{23}$$

Filter gain for $\hat{\breve{x}}(k,k)$ in the equation (23): $g(k)$

$$\begin{aligned}&g(k) = [\breve{K}(k,k)\breve{H}^T - \breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T]\\ &\times\{R + \breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T\}^{-1}\end{aligned} \tag{24}$$

Auto-variance function of $\hat{\breve{x}}(k,k)$: $S_0(k) = E[\hat{\breve{x}}(k,k)\hat{\breve{x}}^T(k,k)]$

$$\begin{aligned}&S_0(k) = \breve{\Phi}S_0(k-1)\breve{\Phi}^T + g(k)\breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T],\\ &S_0(0) = 0\end{aligned} \tag{25}$$

Cross-variance function of $\hat{x}(k,k)$ with $\hat{\breve{x}}(k,k)$: $S(k) = E[\hat{x}(k,k)\hat{\breve{x}}^T(k,k)]$

$$\begin{aligned}&S(k) = \Phi S(k-1)\breve{\Phi}^T + G(k)\breve{H}[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T],\\ &S(0) = 0\end{aligned} \tag{26}$$

Proof

The impulse response function $h(k,s,L)$ satisfies (15). Subtracting $h(k,s,L-1)R$ from $h(k,s,L)R$, we have

$$\begin{aligned}&(h(k,s,L)-h(k,s,L-1))R=-h(k,L,L)\breve{H}\breve{K}(L,s)\breve{H}^T\\&-\sum_{i=1}^{L-1}(h(k,i,L)-h(k,i,L-1))\breve{H}\breve{K}(i,s)\breve{H}^T.\end{aligned} \tag{27}$$

Introducing

$$J_0(s,L)R=\breve{\Phi}^{-s}\breve{K}(s,s)\breve{H}^T-\sum_{i=1}^{L}J_0(i,L)\breve{H}\breve{K}(i,s)\breve{H}^T, \tag{28}$$

we obtain

$$h(k,s,L)-h(k,s,L-1)=-h(k,L,L)\breve{H}\breve{\Phi}^L J_0(s,L-1). \tag{29}$$

Subtracting $J_0(s,L-1)R$ from $J_0(s,L)R$, we have

$$\begin{aligned}&(J_0(s,L)-J_0(s,L-1))R=-J_0(L,L)\breve{H}\breve{K}(L,s)\breve{H}^T\\&-\sum_{i=1}^{L-1}(J_0(i,L)-J_0(i,L-1))\breve{H}\breve{K}(i,s)\breve{H}^T.\end{aligned} \tag{30}$$

From (28) and (30), we obtain

$$J_0(s,L)-J_0(s,L-1)=-J_0(L,L)\breve{H}\breve{\Phi}^L J_0(s,L-1). \tag{31}$$

The filtering estimate is given by

$$\hat{x}(k,k)=\sum_{i=1}^{k}h(k,i,k)\breve{y}(i). \tag{32}$$

From (15), the impulse response function $h(k,s,k)$ satisfies

$$h(k,s,k)R=K_{x\breve{x}}(k,s)\breve{H}^T-\sum_{i=1}^{k}h(k,i,k)\breve{H}\breve{K}(i,s)\breve{H}^T. \tag{33}$$

Introducing

$$J(s,k)R=\Phi^{-s}K_{x\breve{x}}(s,s)\breve{H}^T-\sum_{i=1}^{k}J(i,k)\breve{H}\breve{K}(i,s)\breve{H}^T, \tag{34}$$

we obtain

$$h(k,s,k)=\Phi^k J(s,k). \tag{35}$$

Subtracting $J(s,L-1)R$ from $J(s,L)R$, we have

$$\begin{aligned}&(J(s,k)-J(s,k-1))R=-J(k,k)\breve{H}\breve{K}(k,s)\breve{H}^T\\&-\sum_{i=1}^{k-1}(J(i,k)-J(i,k-1))\breve{H}\breve{K}(i,s)\breve{H}^T.\end{aligned} \tag{36}$$

From (28) and (36), we obtain

$$J(s,k)-J(s,k-1)=-J(k,k)\breve{H}\breve{\Phi}^k J_0(s,k-1). \tag{37}$$

From (34), $J(k,k)$ satisfies

$$J(k,k)R=\Phi^{-k}K_{x\breve{x}}(k,k)\breve{H}^T-\sum_{i=1}^{k}J(i,k)\breve{H}\breve{K}(i,k)\breve{H}^T. \tag{38}$$

Using (6) and introducing

$$r(k) = \sum_{i=1}^{k} J(i,k)\breve{H}B(i), \tag{39}$$

we obtain

$$J(k,k)R = \Phi^{-k}K_{x\breve{x}}(k,k)\breve{H}^T - r(k)A^T(k)\breve{H}^T. \tag{40}$$

Subtracting $r(k-1)$ from $r(k)$, we have

$$r(k) - r(k-1) = J(k,k)\breve{H}B(k) + \sum_{i=1}^{k-1}(J(i,k) - J(i,k-1))\breve{H}B(i). \tag{41}$$

Introducing

$$r_0(k) = \sum_{i=1}^{k} J_0(i,k)\breve{H}B(i), \tag{42}$$

from (37), we obtain

$$\begin{aligned} & r(k) - r(k-1) = J(k,k)\breve{H}B(k) - J(k,k)\breve{H}\breve{\Phi}^k r_0(k-1), \\ & r(0) = 0. \end{aligned} \tag{43}$$

Substituting (43) into (40), we have

$$\begin{aligned} & J(k,k) = [\Phi^{-k}K_{x\breve{x}}(k,k)\breve{H}^T - r(k-1)A^T(k)\breve{H}][R + \breve{H}\breve{K}(k,k)\breve{H}^T \\ & -\breve{H}\breve{\Phi}^k r_0(k-1)(\breve{\Phi}^T)^k\breve{H}^T]^{-1}. \end{aligned} \tag{44}$$

Introducing the function

$$S_0(k) = A(k)r_0(k)A^T(k), A(k) = \breve{\Phi}^k, \tag{45}$$

we rewrite (44) as

$$\begin{aligned} & J(k,k) = [\Phi^{-k}K_{x\breve{x}}(k,k)\breve{H}^T - r(k-1)A^T(k)\breve{H}][R + \breve{H}\breve{K}(k,k)\breve{H}^T \\ & -\breve{H}\breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T]^{-1}. \end{aligned} \tag{46}$$

Subtracting $r_0(k-1)$ from $r_0(k)$ and using (31), we obtain

$$\begin{aligned} & r_0(k) - r_0(k-1) = J_0(k,k)\breve{H}B(k) + \sum_{i=1}^{k-1}(J_0(i,k) - J_0(i,k-1))\breve{H}B(i) \\ & = J_0(k,k)\breve{H}(B(k) - A(k)r_0(k-1)), \\ & r_0(0) = 0. \end{aligned} \tag{47}$$

From (28), $J_0(k,k)$ satisfies

$$J_0(k,k)R = \breve{\Phi}^{-k}\breve{K}(k,k)\breve{H}^T - \sum_{i=1}^{k} J_0(i,k)\breve{H}\breve{K}(i,k)\breve{H}^T.$$

From (6) and (42), it follows that

$$J_0(k,k)R = \breve{\Phi}^{-k}\breve{K}(k,k)\breve{H}^T - r_0(k)A^T(k)\breve{H}^T. \tag{48}$$

Substituting (47) into (48), we obtain an expression for $J_0(k,k)$ as

$$\begin{aligned} & J_0(k,k) = [B^T(k)\breve{H}^T - r_0(k-1)A^T(k)\breve{H}^T][R + \breve{H}\breve{K}(k,k)\breve{H}^T \\ & -\breve{H}\breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T]^{-1}. \end{aligned} \tag{49}$$

From (45) and (47), it follows that

$$\begin{aligned}&S_0(k)=A(k)[r_0(k-1)+J_0(k,k)\breve{H}(B(k)-A(k)r_0(k-1))]A^T(k)\\&=\breve{\Phi}S_0(k-1)\breve{\Phi}^T+A(k)J_0(k,k)\breve{H}(\breve{K}(k,k)-\breve{\Phi}S_0(k-1)\breve{\Phi}^T),\\&S(0)=0.\end{aligned} \tag{50}$$

Let us introduce a function

$$g(k)=A(k)J_0(k,k). \tag{51}$$

From (49) and (51), it follows that

$$\begin{aligned}&g(k)=[\breve{K}(k,k)\breve{H}^T-\breve{\Phi}S_0(k-1)\breve{\Phi}^T(k)\breve{H}^T][R+\breve{H}\breve{K}(k,k)\breve{H}^T\\&-\breve{H}\breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T]^{-1}.\end{aligned} \tag{52}$$

Now, from (32) and (35), the filtering estimate $\hat{x}(k,k)$ of $x(k)$ is given by

$$\hat{x}(k,k)=\Phi^k\sum_{i=1}^{k}J(i,k)\breve{y}(i). \tag{53}$$

Introducing a function

$$e(k)=\sum_{i=1}^{k}J(i,k)\breve{y}(i), \tag{54}$$

the filtering estimate is expressed as

$$\hat{x}(k,k)=\Phi^k e(k). \tag{55}$$

Subtracting $e(k-1)$ from $e(k)$, using (31) and introducing a function

$$e_0(k)=\sum_{i=1}^{k}J_0(i,k)\breve{y}(i), \tag{56}$$

we obtain

$$\begin{aligned}&e(k)-e(k-1)=J(k,k)(\breve{y}(k)-\breve{H}\breve{\Phi}^k\sum_{i=1}^{k-1}J_0(i,k-1)\breve{y}(i))\\&=J(k,k)(y(k)-\breve{H}\breve{\Phi}^k e_0(k-1),\\&e_0(0)=0.\end{aligned} \tag{57}$$

Let us introduce a function

$$\hat{\breve{x}}(k,k)=\breve{\Phi}^k e_0(k), \tag{58}$$

which represents the filtering estimate of $\breve{x}(k)$. Subtracting $e_0(k-1)$ from $e_0(k)$ and using (31), we obtain

$$\begin{aligned}&e_0(k)-e_0(k-1)=J_0(k,k)(\breve{y}(k)-\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)),\\&e_0(0)=0.\end{aligned} \tag{59}$$

Substituting (57) into (55), we have

$$\begin{aligned}&\hat{x}(k,k)=\Phi\hat{x}(k-1,k-1)+\Phi^kJ(k,k)(\breve{y}(k)-\breve{H}\breve{\Phi}^k e_0(k-1))\\&=\Phi\hat{x}(k-1,k-1)+G(k)(\breve{y}(k)-\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)),\\&G(k)=\Phi^kJ(k,k),\\&\hat{x}(0,0)=0.\end{aligned} \tag{60}$$

From (44), and by introducing a function

$$S(k)=\Phi^k r(k)(\breve{\Phi}^T)^k, \tag{61}$$

$G(k)$ is expressed as

$$\begin{aligned}&G(k)=[K_{x\breve{x}}(k,k)\breve{H}^T-\Phi S(k-1)\breve{\Phi}^T\breve{H}][R+\breve{H}\breve{K}(k,k)\breve{H}^T\\&-\breve{H}\breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T]^{-1}.\end{aligned} \tag{62}$$

From (58) and (59), it follows that

$$\begin{aligned}&\hat{\breve{x}}(k,k)=\breve{\Phi}^k e_0(k-1)+\breve{\Phi}^k J_0(k,k)(\breve{y}(k)-\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1))\\&=\breve{\Phi}\hat{\breve{x}}(k-1,k-1)+g(k)(\breve{y}(k)-\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)),\\&\hat{\breve{x}}(0,0)=0.\end{aligned} \tag{63}$$

From (43) and (61), it follows that

$$\begin{aligned}&S(k)=\Phi^k r(k-1)(\breve{\Phi}^T)^k+G(k)(\breve{H}B(k)-\breve{H}\breve{\Phi}^k r_0(k-1))(\breve{\Phi}^T)^k\\&=\Phi S(k-1)\breve{\Phi}^T+G(k)\breve{H}(\breve{K}(k,k)-\breve{\Phi}S_0(k-1)\breve{\Phi}^T),\\&S(0)=0.\end{aligned} \tag{64}$$

From (15), $h(k,L,L)$ satisfies

$$\begin{aligned}&h(k,L,L)R=K_{x\breve{x}}(k,L)\breve{H}^T-\sum_{i=1}^{L}h(k,i,L)\breve{H}\breve{K}(i,L)\breve{H}^T\\&=K_{x\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T-\sum_{i=1}^{L}h(k,i,L)\breve{H}B(i)A^T(L)\breve{H}^T.\end{aligned} \tag{65}$$

Introducing a function

$$P(k,L)=\sum_{i=1}^{L}h(k,i,L)\breve{H}B(i), \tag{66}$$

we have an expression for $h(k,L,L)R$ as

$$h(k,L,L)R=K_{x\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T-P(k,L)(\breve{\Phi}^T)^L\breve{H}^T. \tag{67}$$

Subtracting $P(k,L-1)$ from $P(k,L)$, from (29) and (42), we have

$$\begin{aligned}&P(k,L)-P(k,L-1)=h(k,L,L)\breve{H}B(L)+\sum_{i=1}^{L-1}(h(k,i,L)-h(k,i,L-1))\breve{H}B(i)\\&=h(k,L,L)\breve{H}B(L)-h(k,L,L)\breve{H}\breve{\Phi}^L\sum_{i=1}^{L-1}J_0(i,L)\breve{H}B(i)\\&=h(k,L,L)\breve{H}B(L)-h(k,L,L)\breve{H}\breve{\Phi}^L r_0(L-1).\end{aligned} \tag{68}$$

Let us introduce a function

$$q(k,L)=P(k,L)(\breve{\Phi}^T)^L. \tag{69}$$

From (45), (68) and (69), it follows that

$$q(k,L)=q(k,L-1)\breve{\Phi}^T+h(k,L,L)(\breve{H}\breve{K}(L,L)-\breve{H}\breve{\Phi}S_0(L-1)\breve{\Phi}^T). \tag{70}$$

From (67), it is clear that

$$h(k,L,L)R=K_{x\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T-q(k,L)\breve{H}^T. \tag{71}$$

Substituting (70) into (71), we have

$$\begin{aligned}&h(k,L,L)=[K_{x\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T-q(k,L-1)\breve{\Phi}^T]\\&\times[R+\breve{H}(\breve{K}(L,L)-\breve{\Phi}S_0(L-1)\breve{\Phi}^T)\breve{H}^T]^{-1}.\end{aligned} \tag{72}$$

From (35), (39), (61), (66) and (69), the initial condition $q(k,k)$ on (70) for $q(k,L)$ at $L=k$ is given by

$$\begin{aligned}&q(k,k)=P(k,k)(\breve{\Phi}^T)^k\\&=\sum_{i=1}^{k}h(k,i,k)\breve{H}B(i)(\breve{\Phi}^T)^k\\&=\Phi^k\sum_{i=1}^{k}J(i,k)\breve{H}B(i)(\breve{\Phi}^T)^k\\&=\Phi^k r(k)(\breve{\Phi}^T)^k\\&=S(k).\end{aligned} \tag{73}$$

The fixed-point smoothing estimate is given by (11). Subtracting $\hat{x}(k,L-1)$ from $\hat{x}(k,L)$ and using (29), (56) and (58), we have

$$\begin{aligned}
&\hat{x}(k,L)=\hat{x}(k,L-1)+h(k,L,L)(\breve{y}(L)-h(k,L,L)\breve{H}\breve{\Phi}^L\sum_{i=1}^{L-1}J_0(s,L-1)\breve{y}(i))\\
&=\hat{x}(k,L-1)+h(k,L,L)(\breve{y}(L)-\breve{H}\breve{\Phi}^L e_0(L-1))\\
&=\hat{x}(k,L-1)+h(k,L,L)(\breve{y}(L)-\breve{H}\breve{\Phi}\hat{\breve{x}}(L-1,L-1)),\\
&\hat{x}(k,L)|_{L=k}=\hat{x}(k,k).
\end{aligned} \quad (74)$$

(Q.E.D.)

For the stability of the filtering and fixed-point smoothing algorithms, the following conditions are required.

1. All the real parts in the eigenvalues of the matrix $\breve{\Phi}-g(k)\breve{H}\breve{\Phi}$ are negative.
2. $R+\breve{H}[\breve{K}(k,k)-\breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T>0$

Also, the robust RLS fixed-point smoothing and filtering algorithms, using the covariance information, are summarized in Corollary 1.

Corollary 1 Let the auto-covariance function $\breve{K}(k,s)$ of the state vector $\breve{x}(k)$ be given by (6). Let the cross-covariance function of the state vector $x(k)$ with $\breve{x}(s)$ be given by (10). Let the observation matrix H for the signal $z(k)$ and the observation matrix $\breve{H}$ for the degraded signal $\breve{z}(k)$ be given in (2) and (4) respectively. Let the state equation and the observation equation, including the uncertain quantities $\Delta\Phi$ and ΔH be given in (1). Let the variance of white observation noise be R. Then, by using the covariance information, the robust RLS algorithms for the fixed-point smoothing estimate $\hat{z}(k,L)$, at the fixed point k, and the filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ consist of (75)-(87) in linear discrete-time stochastic systems.

Fixed-point smoothing estimate of the signal $z(k)$: $\hat{z}(k,L)$

$$\hat{z}(k,L)=H\hat{x}(k,L) \quad (75)$$

Fixed-point smoothing estimate of the state vector $x(k)$: $\hat{x}(k,L)$

$$\hat{x}(k,L)=\hat{x}(k,L-1)+h(k,L,L)(\breve{y}(L)-\breve{H}A(L)e_0(L-1)) \quad (76)$$

Smoother gain: $h(k,L,L)$

$$\begin{aligned}
&h(k,L,L)=[\alpha(k)\beta^T(k)(A^T)^{L-k}\breve{H}^T-P(k,L-1)A^T(L)\breve{H}^T]\\
&\times\{R+\breve{H}[B(L)-A(L)r_0(L-1)]A^T(L)\breve{H}^T\}^{-1}
\end{aligned} \quad (77)$$

$$\begin{aligned}
&P(k,L)=P(k,L-1)+h(k,L,L)\breve{H}[B(L)-A(L)r_0(L-1)],\\
&P(k,k)=A(k)r(k)
\end{aligned} \quad (78)$$

Filtering estimate of $z(k)$: $\hat{z}(k,k)$

$$\hat{z}(k,k)=H\hat{x}(k,k) \quad (79)$$

Filtering estimate of $x(k)$: $\hat{x}(k,k)$

$$\hat{x}(k,k)=\alpha(k)e(k) \quad (80)$$

$$\begin{aligned}
&e(k)=e(k-1)+J(k,k)(\breve{y}(k)-\breve{H}A(k)e_0(k-1)),\\
&e(0)=0
\end{aligned} \quad (81)$$

$$\begin{aligned}
&J(k,k)=[\beta^T(k)\breve{H}^T-r(k-1)A^T(k)\breve{H}]\\
&\times\{R+\breve{H}[B(k)-A(k)r_0(k-1)]A^T(k)\breve{H}^T\}^{-1}
\end{aligned} \quad (82)$$

$$\begin{aligned}
&r(k)=r(k-1)+J(k,k)\breve{H}(B(k)-A(k)r_0(k-1)),\\
&r(0)=0.
\end{aligned} \quad (83)$$

Filtering estimate of $\breve{x}(k)$: $\hat{\breve{x}}(k,k)$

$$\hat{\breve{x}}(k,k) = A(k)e_0(k) \tag{84}$$

$$\begin{aligned} &e_0(k) = e_0(k-1) + J_0(k,k)(\breve{y}(k) - \breve{H}A(k)e_0(k-1)), \\ &e_0(0) = 0 \end{aligned} \tag{85}$$

$$\begin{aligned} &J_0(k,k) = [B^T(k)\breve{H}^T - r_0(k-1)A^T(k)\breve{H}^T] \\ &\times\{R + \breve{H}[B(k) - A^k r_0(k-1)]A^T(k)\breve{H}^T\}^{-1} \end{aligned} \tag{86}$$

$$\begin{aligned} &r_0(k) = r_0(k-1) + J_0(k,k)\breve{H}(B(k) - A(k)r_0(k-1)), \\ &r_0(0) = 0 \end{aligned} \tag{87}$$

Proof

From (6), (56) and (74), (76) is obtained. From (6), (10), (42), (67) and (68), (77) is obtained. From (6) and (68), (78) is obtained. From (35), (39) and (66), $P(k,k)$ is obtained. From (6) and (55), (80) is obtained. From (6) and (57), (81) is obtained. Initial condition $e(0) = 0$ is given from (54). From (6), (10) and (44), (82) is obtained. From (6), and (43), (83) is obtained. Initial condition $r(0) = 0$ is given from (39). From (6) and (53) and (54), (84) is given. From (6), (58) and (59), (85) is obtained. From (6), (45) and (49), (86) is obtained. (87) is equivalent to (47). Initial condition $r_0(0) = 0$ is given from (42).

(Q.E.D.)

4 Filtering Error Variance Function of Signal

In this section the filtering error variance function $\tilde{P}_z(k)$ for the signal $z(k)$ is shown. Let the auto-covariance function $K(k,s)$ of the state vector $x(k)$ be expressed by

$$\begin{aligned} &K(k,s) = \begin{cases} A_x(k)B_x^T(s), 0 \le s \le k, \\ B_x(k)A_x^T(s), 0 \le k \le s, \end{cases} \\ &A_x(k) = \alpha(k) = \Phi^k, B_x^T(s) = \Phi^{-s}K(s,s). \end{aligned} \tag{88}$$

The filtering error variance function is given by

$$\begin{aligned} &\tilde{P}_z(k) = H[K(k,k) - E[\hat{x}(k,k)\hat{x}^T(k,k)]H^T \\ &= H[K(k,k) - E[x(k)\hat{x}^T(k,k)]H^T. \end{aligned} \tag{89}$$

By using (10) and (53), and introducing a function,

$$r_s(k) = \sum_{i=1}^{k} J(i,k)\beta(i), \tag{90}$$

(89) is written as

$$\tilde{P}_z(k) = H(K(k,k) - \alpha(k)r_s^T(k)(\alpha^T)^k)H^T. \tag{91}$$

Subtracting $r_s(k-1)$ from $r_s(k)$ and introducing a function

$$\overline{r}_0(k) = \sum_{i=1}^{k} J_0(i,k)\beta(i), \tag{92}$$

we have

$$\begin{aligned} &r_s(k) = r_s(k-1) + J(k,k)(\beta(k) - \breve{H}\breve{\Phi}^k\overline{r}_0(k-1)), \\ &r_s(0) = 0. \end{aligned} \tag{93}$$

Subtracting $\overline{r}_0(k-1)$ from $\overline{r}_0(k)$, we have

$$\begin{aligned} &\overline{r}_0(k) = \overline{r}_0(k-1) + J_0(k,k)(\beta(k) - \breve{H}\breve{\Phi}^k\overline{r}_0(k-1)), \\ &r_s(0) = 0. \end{aligned} \tag{94}$$

Hence, the filtering error variance function $\tilde{P}_z(k)$ is calculated by (91) with (82), (83), (86), (87), (93) and (94) recursively.

Since $\tilde{P}_z(k)$ is the semi-definite function, the filtering variance function $HE[\hat{x}(k,k)\hat{x}^T(k,k)]H^T = H\Phi^k r_s^T(k)(\Phi^T)^k H^T$ is upper bounded by $HK(k,k)H^T$ and lower bounded by the zero matrix as

$$0 \le HE[\hat{x}(k,k)\hat{x}^T(k,k)]H^T \le HK(k,k)H^T. \tag{95}$$

This shows the existence of the robust filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$.

5 A Numerical Simulation Example

Let a scalar observation equation and the state equation for $x(k)$ be given by

$$\begin{aligned}
&y(k) = z(k) + v(k), z(k) = Hx(k), H = \begin{bmatrix} 1 & 0 \end{bmatrix}, x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},\\
&x(k+1) = \Phi x(k) + \Gamma w(k), \Phi = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, a_1 = -0.1, a_2 = -0.8, \Gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\\
&E[v(k)v(s)] = R\delta_K(k-s), E[w(k)w(s)] = Q\delta_K(k-s), Q = 0.5^2.
\end{aligned} \tag{96}$$

In (96) the signal process of $z(k)$ is expressed by the second order AR model. Let the state-space model containing the uncertain quantities $\Delta H(k)$ and $\Delta\Phi(k)$ be given by

$$\begin{aligned}
&\breve{y}(k) = \breve{z}(k) + v(k), \breve{z}(k) = \bar{H}(k)\bar{x}(k), \bar{x}(k) = \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \end{bmatrix},\\
&\bar{H}(k) = H + \Delta H(k) = \begin{bmatrix} 1 + \Delta_3(k) & 0 \end{bmatrix}, \Delta H(k) = \begin{bmatrix} \Delta_3(k) & 0 \end{bmatrix}, \Delta_3(k) = 0.05\zeta(k),\\
&\bar{x}(k+1) = \bar{\Phi}(k)\bar{x}(k) + \Gamma w(k), \bar{\Phi}(k) = \Phi + \Delta\Phi(k), \Delta\Phi(k) = \begin{bmatrix} 0 & 0 \\ \Delta_2(k) & \Delta_1(k) \end{bmatrix},\\
&\Delta_1(k) = 0.01\zeta(k), \Delta_2(k) = -0.1\zeta(k)
\end{aligned} \tag{97}$$

in linear discrete-time stochastic systems. It should be noted that the uncertain quantities $\Delta H(k)$ and $\Delta\Phi(k)$ are not known. It is a task to estimate the signal $z(k)$ recursively in terms of the observed value $\breve{y}(k)$, which is given as the sum of the degraded signal $\breve{z}(k)$ and the observation noise $v(k)$. $\zeta(k)$ in (97) represents the uniform random variables between 0 and 1 generated by the MATLAB command "*rand*". $\Delta_1(k)$, $\Delta_2(k)$ and $\Delta_3(k)$ consist of the deterministic mean values and the stochastic variables with the mean zero. Let $\breve{z}(k)$ be fitted to the N-th order AR model of

$$\begin{aligned}
&\breve{z}(k) = -\breve{a}_1\breve{z}(k-1) - \breve{a}_2\breve{z}(k-2) - \cdots - \breve{a}_N\breve{z}(k-N) + \breve{e}(k),\\
&E[\breve{e}(k)\breve{e}(s)] = \breve{Q}\delta_K(k-s).
\end{aligned} \tag{98}$$

From (4), for the scalar observation equation (96), the degraded signal $\breve{z}(k)$ is given by

$$\breve{z}(k) = \breve{H}\breve{x}(k), \breve{H} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{99}$$

In this example, the state equation for $\breve{x}(k)$, given by (5), corresponds to the case of $m = 1$. $\breve{K}(k,s) = \breve{K}(k-s)$ represents the auto-covariance function of the state vector $\breve{x}(k)$ in wide-sense stationary stochastic systems. $\breve{K}(k,s)$ is expressed in the form of the semi-degenerate function (6). In (6), $\breve{\Phi}$ represents the state transition matrix for the state vector $\breve{x}(k)$. $\breve{\Phi}$ is given by (7). Also, from $K_{\breve{z}}(k-s) = K_{\breve{z}}(s-k) = E[\breve{z}(k)\breve{z}(s)]$ for the scalar degraded signal $\breve{z}(k)$, the auto-variance function $\breve{K}(k,k)$ of the state vector

$\breve{x}(k)$ is expressed as

$$\breve{K}(k,k) = E\left[\begin{bmatrix} \breve{z}(k) \\ \breve{z}(k+1) \\ \vdots \\ \breve{z}(k+N-2) \\ \breve{z}(k+N-1) \end{bmatrix} \right. \\ \left. \times \begin{bmatrix} \breve{z}(k) & \breve{z}(k+1) & \cdots & \breve{z}(k+N-2) & \breve{z}(k+N-1) \end{bmatrix} \right] \\ = \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1) \\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1) \\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0) \end{bmatrix}. \tag{100}$$

Let $K_{z\breve{z}}(k,s) = E[z(k)\breve{z}(s)]$ represent the cross-covariance function between the signal $z(k)$ and the degraded signal $\breve{z}(s)$. From (4) and (96), the cross-covariance function $K_{x\breve{x}}(k,s)$ is expressed as

$$K_{x\breve{x}}(k,s) = \Phi^{k-s} K_{x\breve{x}}(s,s), 0 \le s \le k, \\ K_{x\breve{x}}(k,k) = E\left[\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \begin{bmatrix} \breve{z}(k) & \breve{z}(k+1) & \cdots & \breve{z}(k+N-2) & \breve{z}(k+N-1) \end{bmatrix}\right] \\ = \begin{bmatrix} E[x_1(k)\breve{z}(k)] & E[x_1(k)\breve{z}(k+1) & \cdots & E[x_1(k)\breve{z}(k+N-2)] & E[x_1(k)\breve{z}(k+N-1)] \\ E[x_2(k)\breve{z}(k)] & E[x_2(k)\breve{z}(k+1) & \cdots & E[x_2(k)\breve{z}(k+N-2)] & E[x_2(k)\breve{z}(k+N-1)] \end{bmatrix} \\ = \left[\begin{matrix} E[z(k)\breve{z}(k)] & E[z(k)\breve{z}(k+1) \\ E[z(k+1)\breve{z}(k)] & E[z(k+1)\breve{z}(k+1) \end{matrix}\right. \\ \left.\begin{matrix} \cdots & E[z(k)\breve{z}(k+N-2)] & E[z(k)\breve{z}(k+N-1)] \\ \cdots & E[z(k+1)\breve{z}(k+N-2)] & E[z(k+1)\breve{z}(k+N-1)] \end{matrix}\right] \\ = \begin{bmatrix} K_{z\breve{z}}(k,k) & K_{z\breve{z}}(k,k+1) & \cdots & K_{z\breve{z}}(k,k+N-2) & K_{z\breve{z}}(k,k+N-1) \\ K_{z\breve{z}}(k+1,k) & K_{z\breve{z}}(k+1,k+1) & \cdots & K_{z\breve{z}}(k+1,k+N-2) & K_{z\breve{z}}(k+1,k+N-1) \end{bmatrix}. \tag{101}$$

The AR parameters $\breve{a}_1, \breve{a}_2, \cdots, \breve{a}_{N-1}, \breve{a}_N$ in (98) are calculated by

$$\begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1) \\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1) \\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0) \end{bmatrix} \begin{bmatrix} \breve{a}_1 \\ \breve{a}_2 \\ \vdots \\ \breve{a}_{N-1} \\ \breve{a}_N \end{bmatrix} = \begin{bmatrix} -K_{\breve{z}}(1) \\ -K_{\breve{z}}(2) \\ \vdots \\ -K_{\breve{z}}(N-1) \\ -K_{\breve{z}}(N) \end{bmatrix}.$$

Substituting H, $\breve{H}$, Φ, $\breve{\Phi}$, $K_{x\breve{x}}(k,k)$, $\breve{K}(k,k) = \breve{K}(L,L)$ and R into the robust RLS Wiener estimation algorithms of Theorem 1, the fixed-point smoothing and filtering estimates are calculated recursively. In evaluating $\breve{\Phi}$ in (7) for $m = 1$, $\breve{K}(k,k)$ in (100) and $K_{x\breve{x}}(k,k)$ in (101), 2,000 number of the signal and degraded signal data are used. Fig.1 illustrates the signal $z(k)$, the fixed-point smoothing estimate $\hat{z}(k, k+2)$ and the filtering estimate $\hat{z}(k,k)$ of the signal vs. k for the white Gaussian observation noise $N(0, 0.3^2)$ in the case of the AR model order $N = 5$. Fig.2 illustrates the signal process $z(k)$ and its degraded signal $\breve{z}(k)$ by the uncertain parameters for the white Gaussian observation noise $N(0, 0.3^2)$. Fig.2 shows, in comparison with the signal process, that the degraded signal is influenced by the uncertain parameters in the observation vector and the state transition matrix in (97). Fig.3 illustrates the mean-square values (MSVs) of the filtering errors $z(k) - \hat{z}(k,k)$ and the fixed-point smoothing errors $z(k) - \hat{z}(k, k+Lag)$ vs. Lag, $0 \le Lag \le 5$, for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$ in the case of the AR model order $N = 5$. For $Lag = 0$, the MSV of the filtering errors $z(k) - \hat{z}(k,k), 1 \le k \le 2000$, is shown. Similarly, Fig.4 illustrates the MSVs of the filtering and the fixed-point smoothing errors vs. Lag for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$ in the case of the AR model order $N = 10$. From Fig.3 and Fig.4. for the white Gaussian observation noises $N(0, 0.1^2)$ and $N(0, 0.3^2)$, it is seen that, as Lag increases, the MSVs increase and the estimation accuracy of the robust fixed-point smoother is degraded in comparison with the robust filter.

In Fig.3 and Fig.4, among the filter and the fixed-point smoother, for the white Gaussian observation noise $N(0, 0.5^2)$, the estimation accuracy of the fixed-point smoother at $Lag = 2$ is most feasible. Also, the MSVs of the filtering and fixed-point smoothing errors for the AR model order $N = 5$ is smaller than those for the AR model order $N = 10$. Fig.5 shows the MSVs of the filtering errors by the RLS Wiener filter [23] for the AR model orders $N = 5$, the robust Kalman filter [21], and the robust RLS Wiener filter, proposed in this paper for the AR model orders $N = 5$ and $N = 10$, vs. the standard deviation of the observation noise. Here, by using the parameter notations in [21], the quantities used in the simulation of the robust Kalman filter are as follows. $F_k = \begin{bmatrix} 0.1\zeta(k) & -0.1\zeta(k) \\ 0 & 0.1\zeta(k) \end{bmatrix}$, $E = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.01 \end{bmatrix}$, $H_1 = \begin{bmatrix} 0 & 0 \\ -10 & 0 \end{bmatrix}$, $H_2 = \begin{bmatrix} 10 & 10 \end{bmatrix}$, $\varepsilon_k = 8$. From Fig.5 the MSV by the robust RLS Wiener filter for the AR model order $N = 5$ is smaller slightly than that for the AR model order $N = 10$. From Fig.3, Fig.4 and Fig.5, it is seen that the proposed robust RLS Wiener filter and fixed-point smoother for $1 \leq lag \leq 5$ for are preferable in estimation accuracy to the RLS Wiener filter [23] and the robust Kalman filter [21]. Here, the MSVs of the fixed-point smoothing and filtering errors are evaluated by $\sum_{i=1}^{2000} (z(k) - \hat{z}(k, k + Lag))^2/2000$ and $\sum_{i=1}^{2000} (z(k) - \hat{z}(k, k))^2/2000$ respectively.

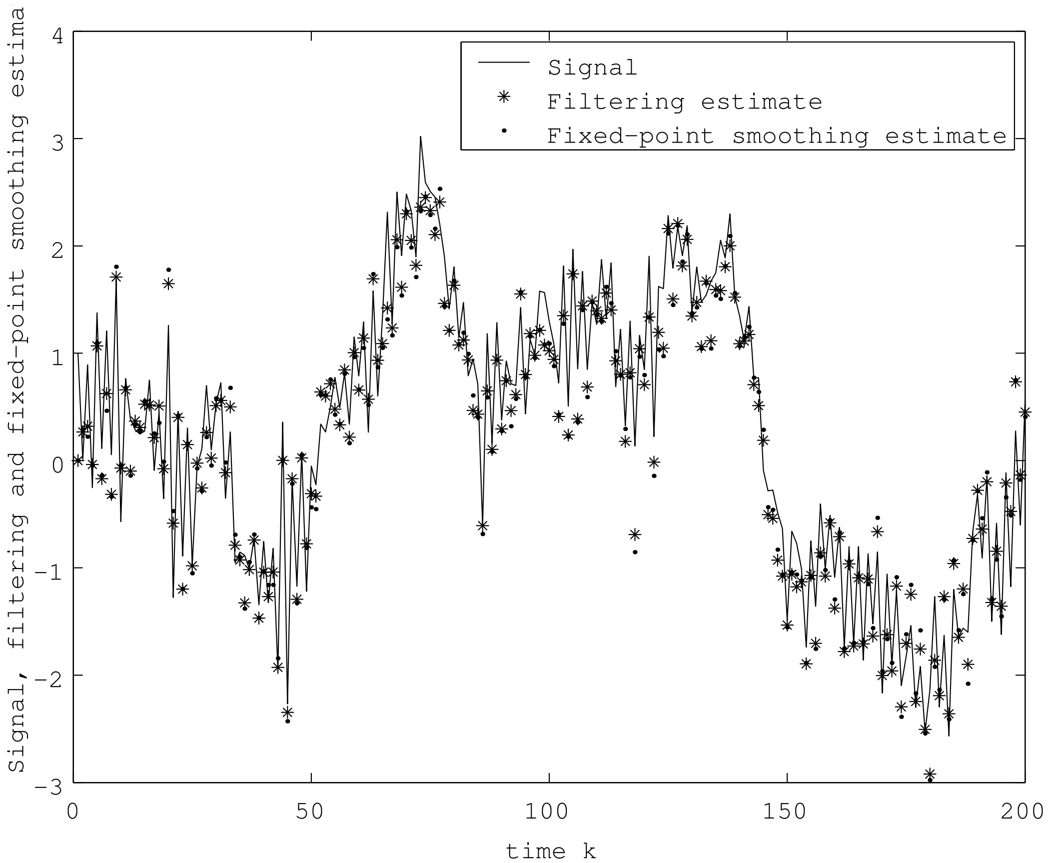

Figure 1. Signal $z(k)$, fixed-point smoothing estimate $\hat{z}(k, k + 2)$ and filtering estimate $\hat{z}(k, k)$ of the signal vs. k for the white Gaussian observation noise $N(0, 0.3^2)$ in the case of the AR model order $N = 5$.

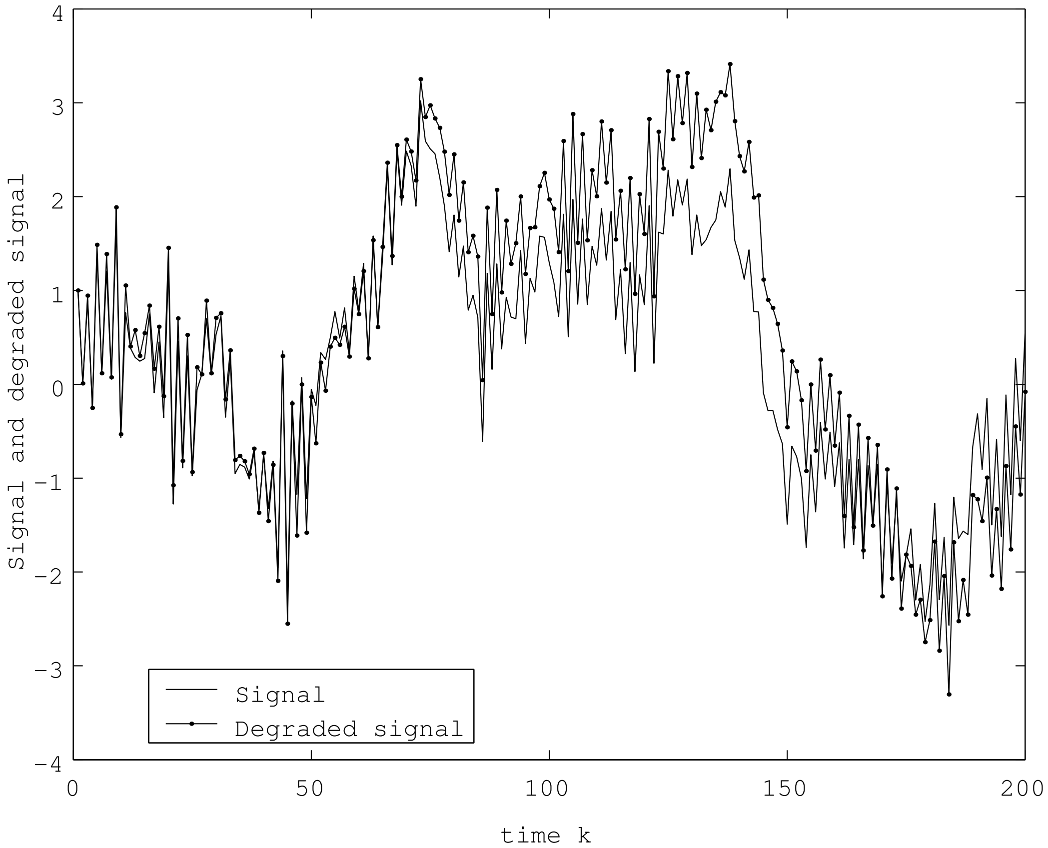

Figure 2. Signal $z(k)$ and degraded signal $\breve{z}(k)$ by the uncertain parameters for the white Gaussian observation noise $N(0, 0.3^2)$.

6 Conclusions

As a purpose of estimating the signal process, this paper has proposed the robust RLS Wiener fixed-point smoother and filter in linear discrete-time stochastic systems with stochastic parameter uncertainties. The uncertain parameters are contained in the observation matrix and the system matrix. The uncertain parameters generate the degraded signal. In this paper, in Theorem 1, by fitting the degraded signal process to the finite order AR model, the robust RLS Wiener estimators are designed. This paper, in Corollary 1, proposes the robust recursive fixed-point smoother and filter, by using the covariance information of the state vector for the degraded signal, the cross-covariance information of the state vector for the signal with the state vector for the degraded signal, the observation matrices for the degraded signal and the signal besides the variance of the white observation noise.

A numerical simulation example, in estimating the signal process expressed by the second order AR model, has shown that the robust RLS Wiener filter and fixed-point smoother for $1 \leq lag \leq 5$ are superior in estimation accuracy to the robust Kalman filter [21] and the RLS Wiener filter [23].

References

1. Y. Luo, J. Nie and E. R. Young, “Model Uncertainty, State Uncertainty, and State-space Models,” *The Federal Research Bank of Kansas City*, *RWP* 12-02, pp. 1-23, 2012.
2. Z. Li, Y. Yao, J. Wang and J. Gao, “Application of improved robust Kalman filter in data fusion for PPP/INS tightly coupled positioning system,” *Metrology and Measurement Systems*, vol. 24, no.2, pp. 289-301, 2017.
3. S. Nakamori, “RLS Wiener filter and fixed-point smoother with randomly delayed or uncertain observations in linear discrete-time stochastic systems,” *Computer Reviews Journal*, vol. 1, no.1, pp. 115-135, 2018.

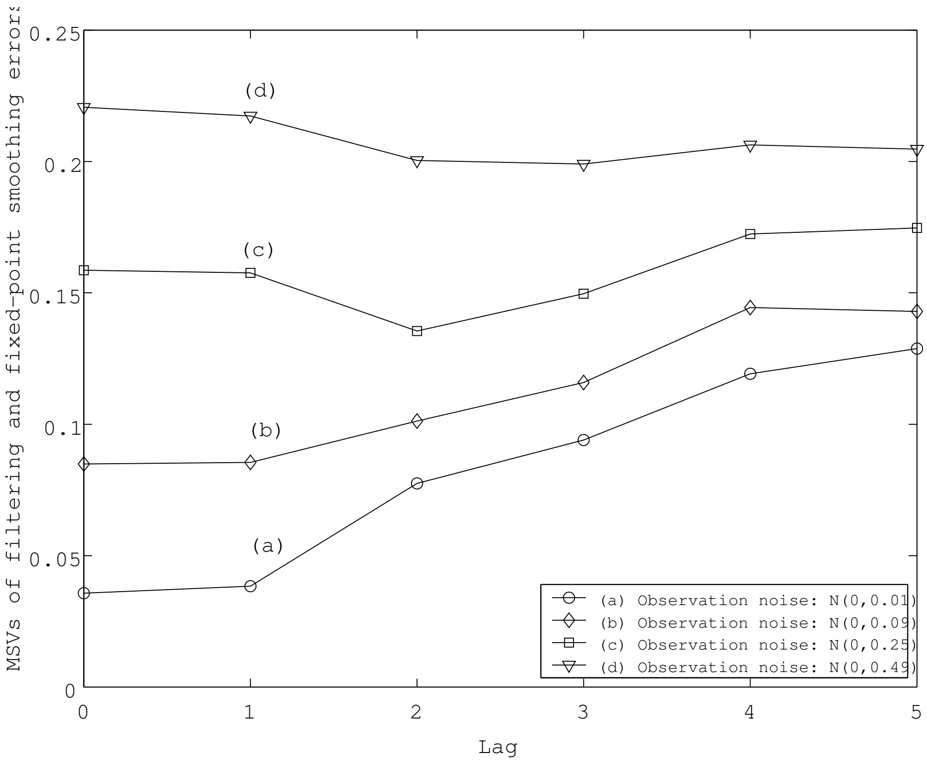

Figure 3. MSVs of the filtering errors $z(k) - \hat{z}(k, k)$ and the fixed-point smoothing errors $z(k) - \hat{z}(k, k + Lag)$ vs. Lag, $0 \leq Lag \leq 5$, for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$ in the case of the AR model order $N = 5$.

4. N. S. Tripathy, I. N. Kar and K. Paul, "Stabilization of uncertain discrete-time linear system with limited communication," *IEEE Trans. on Automatic Control*, vol. 62, no. 9, pp. 4727-4733, 2017.
5. E. Garcia and P. J. Antsaklis, "Model-based event-triggered control for systems with quantization and time-varying network delays," *IEEE Trans. Automatic Control*, vol. 58, no. 2, pp. 422-434, 2013.
6. E. M. Xia, V. Gupta and P. J. Antsaklis. "Networked state estimation over a shared communication medium," *American Control Conference*, *Washinton*, pp. 4128-4133, 2013.
7. W. Wu, S. Reimann, D. Gorges, and S. Liu, "Suboptimal event-triggered control for time-delayed linear systems", *IEEE Trans. Automatic Control*, vol. 60, no. 5, pp. 1386-1391, 2015.
8. M. Fu, C. E. de Souza, and Z. Luo, "Finite horizon robust Kalman filter design," *Proc. IEEE Conf. Decision Control, Phoenix*, *AZ*, pp. 4555–4560, 1999.
9. I. Petersen and D. McFarlane, "Optimal guaranteed cost filtering for uncertain discrete-time linear systems," *Int. J. Robust Nonlinear Control*, vol. 6, no. 4, pp. 267–280, 1996.
10. U. Shaked and C. E. de Souza, "Robust minimum variance filtering," *IEEE Trans. Signal Processing*, vol. 43, no. 11, pp. 2474–2483, 1995.
11. Y. Theodor and U. Shaked, "Robust discrete-time minimum-variance filtering," *IEEE Trans. Signal Processing*, vol. 44, no. 2, pp. 181–189, 1996.
12. L. Xie, Y. C. Soh, and C. E. de Souza, "Robust Kalman filtering for uncertain discrete-time systems," *IEEE Trans. Automat. Control*, vol. 39, no. 6, pp. 1310–1314, 1994.
13. F. Wang and V. Balakrishnan, "Robust adaptive Kalman filters for linear time-varying systems with stochastic parametric uncertainties," in *Proc. Amer. Control Conf.*, *San Diego*, *CA*, pp. 440–444, 1999.
14. F. Wang and V. Balakrishnan, "Robust steady-state filtering for systems with deterministic and stochastic uncertainties," *IEEE Trans. Signal Processing*, vol. 51, no. 10, pp. 2550–2558, 2003.
15. F. Yang, Z. Wang, and Y. S. Hung, "Robust Kalman filtering for discrete time-varying uncertain systems with multiplicative noises," *IEEE Trans. Automatic Control*, vol. 47, no. 7, pp. 1179–1183, 2002.
16. J. H. Zheng and J. F. Liu, "A robust finite-horizon Kalman filter for uncertain discrete time varying systems with state-delay and missing measurements," *Int. J. of Grid and Distributed Computing*, vol. 9, no. 3, pp. 229–242, 2016.

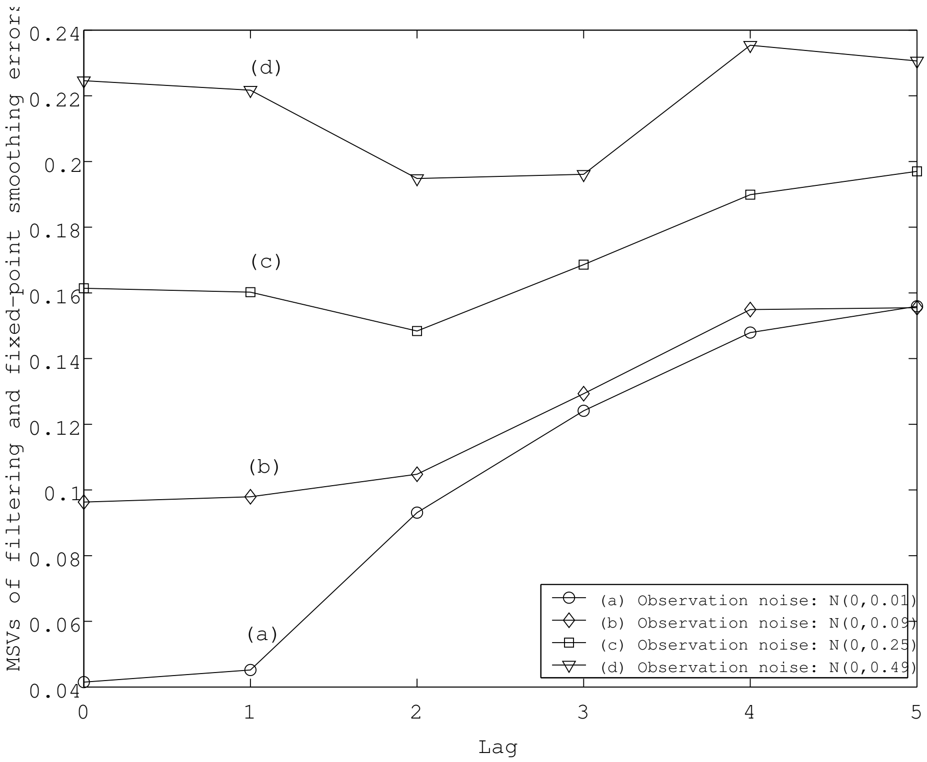

Figure 4. MSVs of the filtering errors $z(k) - \hat{z}(k,k)$ and the fixed-point smoothing errors $z(k) - \hat{z}(k,k+Lag)$ vs. Lag, $0 \le Lag \le 5$, for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$ and $N(0,0.7^2)$ in the case of the AR model order $N = 10$.

17. A. H. Sayed, "A framework for state-space estimation with uncertain models," I*EEE Trans. Automatic Control*, vol. 46, no. 7, pp. 998–1013, 2001.
18. A. Subramanian and A. H. Sayed, "Regularized robust filters for time-varying uncertain discrete-time systems," *IEEE Trans. Automatic Control*, vol. 49, no. 6, pp. 970–976, 2004.
19. F. Yang and Y. S. Hung, "Robust H_∞ filtering with error variance constraints for uncertain discrete-time systems," *Proceedings of the 2000 IEEE Int. Conference on Control Applications*, *Anchorage*, *Alaska*, pp. 635–640, 2000.
20. J. Y. Ishihara, M. H. Terra and J. P.Cerri, "Optimal robust filtering for systems subject to uncertainties," *Automatica*, vol. 52, no. 2, pp. 111–117, 2015.
21. X. Zhu, Y. C. Soh and L. Xie, "Design and analysis of discrete-time robust Kalman filters," *Automatica*, vol. 38, no. 6, pp. 1069–1077, 2002.
22. M. Zorzi, "Robust Kalman filtering under model perturbations," *IEEE Trans. Automatic Control*, vol. 62, no.6, pp. 2902-2907, 2017.
23. S. Nakamori, "Recursive estimation technique of signal from output measurement data in linear discrete-time systems," *IEICE Trans. Fundamentals of Electronics, Communication and Computer Sciences*, vol. E78-A, no.5, pp. 600–607, 1995.
24. A. P. Sage and J. L. Melsa, *Estimation Theory with Applications to Communications and Control*, New York: McGraw-Hill, 1971.

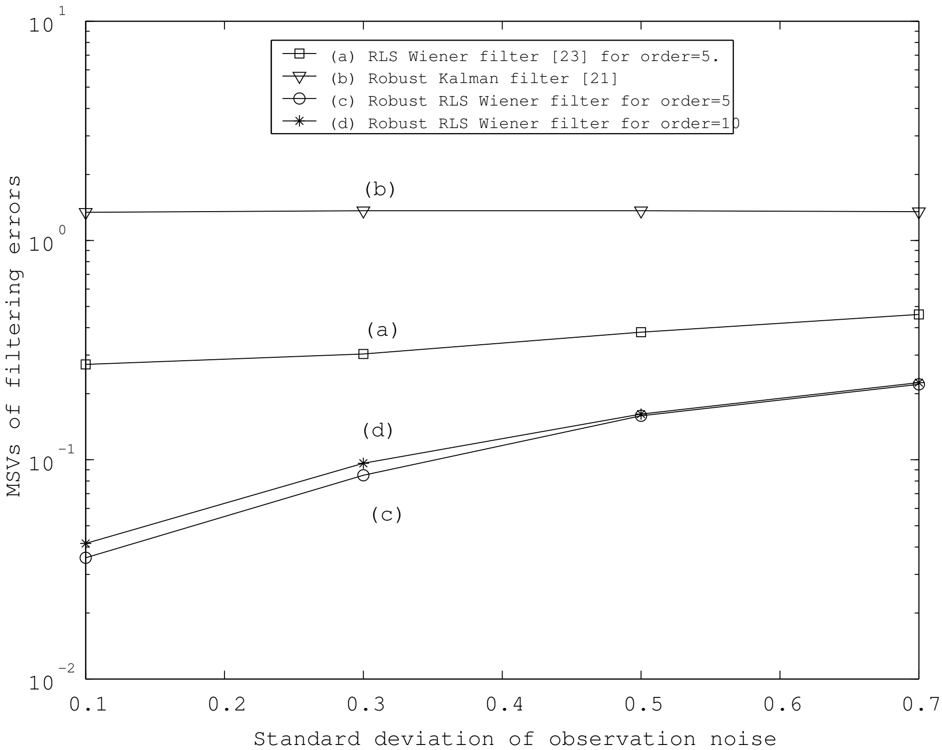

Figure 5. MSVs of the filtering errors $z(k) - \hat{z}(k,k)$ by the RLS Wiener filter [23] for the AR model orders $N = 5$, the robust Kalman filter [21] and the robust RLS Wiener filter, proposed in this paper, for the AR model orders $N = 5$ and $N = 10$ vs. the standard deviation of the observation noise.

