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Robust Schur Stability of a Polytope of Polynomials

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***Abstract*—The main objective of this note is to provide a necessary and sufficient condition for a polytope of polynomials to have all its zeros inside the unit circle. The criterion obtained serves as a discrete-time counterpart for results in [1] and [7] for the continuous case. Also, the results are reduced to operations on $(n - 1) \times (n - 1)$ matrices.**

I. Introduction and Formulation

The motivation for this note is derived from the so-called robust stability problem for a family of polynomials. That is, given a polynomial $P(\cdot)$ whose coefficients are functions of a vector of uncertain parameters q, the problem is to ascertain whether $P(\cdot)$ remains stable for all q within a prescribed bounding set Q. More specifically, we consider the family of polynomials

$$P(s, q) = s^n + \sum_{k=0}^{n-1} a_k(q)s^k; \qquad q \in Q$$

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where $a_k(\cdot): Q \to R$ are prescribed coefficient functions for $k = 0, 1, 2, \cdots, n - 1$. Given this setup, the objective here is to provide a discrete-time analog of the robust stability results given in [1] for a line of polynomials and in [7] for a polytope of polynomials.

The critical assumption in [7] (and also here) involves the coefficient functions. Namely, it is assumed that the set of possible coefficients

$$\mathfrak{A} = \{a(q) \triangleq (a_0(q), a_1(q), \cdots, a_{n-1}(q)) : q \in Q\}$$

is a polytope. This will be the case when the $a_i(q)$ depend (affine) linearly on q and the bounding set Q is obtained by assuming an upper bound and a lower bound for each component q_i of q. As a consequence of this assumption, it is readily verified that the associated family of polynomials

$$\mathbb{P} \triangleq \{P(s, q) : q \in Q\}$$

is also a polytope generated from the extreme points of the operating range Q, i.e., letting q^j denote the jth extreme point of Q, it follows that $\mathbb{P}$ is the convex hull of the finite set of polynomials of the form

$$P_j(s) \triangleq s^n + \sum_{k=0}^{n-1} a_k(q^j)s^k.$$

To complete the discussion of the problem formulation, it should be noted that this polytope framework provides a more general setting than the one considered in Kharitonov's Theorem [2]. In [2], it is assumed that the coefficient variations are independent, whereas the current formulation allows for linear dependencies. Fundamental to the attainment of our main result is the theorem due to Bartlett, Hollot, and Lin [3]. The authors in [3] show that the zeros of a polytope of polynomials $\mathbb{P}$ lie in a simply connected set D if and only if the edges of $\mathbb{P}$ have all their zeros in D. Hence, one need only test for D-stability of all convex combinations of the form

$$\alpha P_i(s) + (1-\alpha)P_j(s); \; \alpha \in [0, 1].$$

This same simplification is exploited in [7].

II. Main Result

To obtain a discrete-time extension of the result in [1], we use a refinement of the Schur–Cohn stability criterion due to Jury and Pavlidis [4]. For a polynomial

$$P(z) = \sum_{k=0}^{n} a_k z^k = a_n \prod_{i=1}^{n} (z - z_i)$$

define the $(n - 1) \times (n - 1)$ matrix

$$S(P) = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & a_3 & a_2 - a_0 \\ 0 & a_n & a_{n-1} & \cdots & a_4 - a_0 & a_3 - a_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & -a_0 & -a_1 & \cdots & a_n - a_{n-4} & a_{n-1} - a_{n-3} \\ -a_0 & -a_1 & -a_2 & & -a_{n-3} & a_n - a_{n-2} \end{bmatrix}.$$

It is shown in [4] that

$$\det S(P) = a_n^{n-1} \times \prod_{\substack{k=1 \\ i<k}}^{n} (1 - z_i z_k).$$

If the a_k vary continuously, it follows that the zeros z_i of the polynomial $P(z)$ vary continuously and $\det S(P) = 0$ if a complex pair of roots crosses the unit circle. There are two other possibilities for crossing a stability boundary: $P(1) = 0$ and $P(-1) = 0$. The above three cases are the critical stability constraints. For simplicity, we consider monic polynomials in the main result below.

Theorem: Consider the polytope of monic polynomials $\mathbb{P}$ having generating points $\{P_i(z)\}_{i=1}^{l}$ whose zeros are inside the unit circle. Then all polynomials $P(\cdot) \in P$ also have their zeros inside the unit circle if and only if the following condition holds. For all $i, j \in \{1, 2, \cdots, l\}$, the matrix $S(P_i)S^{-1}(P_j)$ has no real eigenvalues in $(-\infty, 0)$.

Proof: In accordance with the Edge Theorem in [3], every polynomial in $\mathbb{P}$ has all its zeros inside the unit circle if and only if the exposed edges of $\mathbb{P}$ have all their zeros inside the unit circle. Equivalently, given any $i, j \in \{1, 2, \cdots, l\}$ and any $\alpha \in [0, 1]$, the polynomial $\alpha P_i(z) + (1 - \alpha)P_j(z)$ must be stable. Now, in view of the stability of the generators of P and the critical stability constraints in [4], we arrive at the following point: every polynomial in $\mathbb{P}$ is stable if and only if given any $i, j \in \{1, 2, \cdots, l\}$ and $\alpha \in (0, 1)$, we have

i) $\alpha P_i(1) + (1 - \alpha)P_j(1) \neq 0$
ii) $\alpha P_i(-1) + (1 - \alpha)P_j(-1) \neq 0$
iii) $\det S[(\alpha P_i + (1 - \alpha)P_j)] \neq 0$.

By stability of the generating points, we have $P_i(1) > 0$, $P_j(1) > 0$, and therefore $\alpha P_i(1) + (1 - \alpha)P_j(1) > 0$. A similar argument holds for $P_i(-1)$, $P_j(-1)$. Therefore, conditions i) and ii) are satisfied for all $\alpha \in (0, 1)$.

Now, we observe that condition iii) holds if and only if $S[\alpha P_i + (1 - \alpha)P_j] = \alpha S(P_i) + (1 - \alpha)S(P_j)$ is nonsingular for all $\alpha \in (0, 1)$. Equivalently, $S(P_i) - ((\alpha - 1)/\alpha)S(P_j)$ is nonsingular for all $\alpha \in (0, 1)$. To complete the proof, we can divide by $S(P_j)$ (since $P_j(z)$ is discrete-time stable) and replace $(\alpha - 1)/\alpha$ by a new variable $\lambda \in (-\infty, 0)$. Therefore, all polynomials in $\mathbb{P}$ are discrete-time stable if and only if

$$\det [\lambda I - S(P_i)S^{-1}(P_j)] \neq 0$$

for all $\lambda \in (-\infty, 0)$ and all $i, j \in \{1, 2, \cdots, l\}$. That is, for each pair $i, j \in \{1, 2, \cdots l\}$, $S(P_i)S^{-1}(P_j)$ has no real eigenvalues in $(-\infty, 0)$. □

Remarks:

1) The conditions of the theorem can be checked numerically using standard software for matrix inversion and eigenvalue calculation.

2) If it is desired to find stable ranges for α along the edges, then for all negative real eigenvalues λ_k of $S(P_i)S^{-1}(P_j)$, the corresponding $\alpha_k = 1/(1 - \lambda_k)$ at the stability boundaries are easily calculated.

3) The requirements of the theorem above can be relaxed in two ways. First, for nonmonic polynomials, the theorem remains valid as long as the leading coefficient a_n does not change sign over the given family $\mathbb{P}$. Second, the requirement that one must check $S(P_i)S^{-1}(P_j)$ for all combinations of i and j is stronger than necessary. In fact, one need only deal with those values of i and j corresponding to exposed edges of $\mathbb{P}$; see [3].

4) The dimension of the matrix S is $(n - 1) \times (n - 1)$. Similarly, the result of [1] can be simplified by using the matrix

$$\bar{H} = \begin{bmatrix} a_1 & a_3 & a_5 \cdots & & \\ a_0 & a_2 & a_4 \cdots & & \\ 0 & a_1 & a_3 & a_5 \cdots & \\ 0 & a_0 & a_2 & a_4 \cdots & \\ & & \cdots & & \\ & & & & a_{n-1} \end{bmatrix}$$

instead of the full Hurwitz matrix H. This follows from Orlando's formula [5]

$$\det \bar{H} = (-1)^{n(n-1)/2} \times a_n^{n-1} \times \prod_{\substack{i=1 \\ i<k}}^{n} (s_i + s_k).$$

If a pair of complex conjugate roots s_i, s_k crosses the imaginary axis, then $\det \bar{H} = 0$.

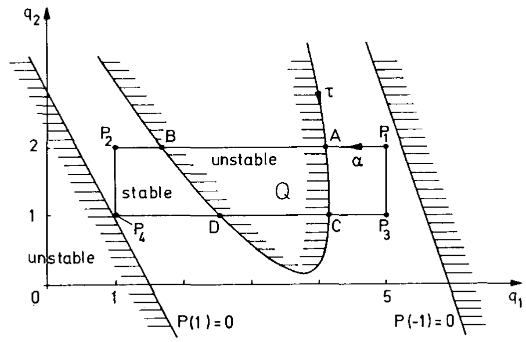

Fig. 1. The polytope of polynomials with vertices P_1, P_2, P_3, P_4 corresponding to the operating range Q is not entirely stable.

III. Example

Consider the polytope of polynomials described by

$$P(z, q_1, q_2) = (-0.825 + 0.225q_1 + 0.1q_2) + (0.895 + 0.025q_1 + 0.09q_2)z + (-2.475 + 0.675q_1 + 0.3q_2)z^2 + z^3$$

with $Q = \{q: q_1 \in [1, 5]; q_2 \in [1, 2]\}$.

The four extreme polynomials are

$$P_1(z) = P(z, 5, 2), \quad P_3(z) = P(z, 5, 1)$$
$$P_2(z) = P(z, 1, 2), \quad P_4(z) = P(z, 1, 1)$$

and it is easily verified that their zeros are inside the unit circle. The four S matrices are

$$S(P_1) = \begin{bmatrix} 1 & 1 \\ -0.5 & -0.2 \end{bmatrix}, \quad S(P_3) = \begin{bmatrix} 1 & 0.8 \\ -0.4 & -0.11 \end{bmatrix}$$

$$S(P_2) = \begin{bmatrix} 1 & -0.8 \\ 0.4 & -0.1 \end{bmatrix}, \quad S(P_4) = \begin{bmatrix} 1 & -1 \\ 0.5 & -0.01 \end{bmatrix}.$$

Only two of them must be inverted to obtain $S(p_i)S^{-1}(P_j)$ for the four edges.

i) Edge $P_1 - P_2$

$$S(P_1)S^{-1}(P_2) = \frac{1}{0.22} \begin{bmatrix} -0.5 & 1.8 \\ 0.13 & -0.6 \end{bmatrix}$$

with eigenvalues $\lambda_1 = -0.289$ and $\lambda_2 = -4.711$.

ii) Edge $P_4 - P_2$

$$S(P_4)S^{-1}(P_2) = \frac{1}{0.22} \begin{bmatrix} 0.3 & -0.2 \\ -0.046 & 0.399 \end{bmatrix}$$

with eigenvalues $\lambda_1 = 2.079$ and $\lambda_2 = 1{,}098$.

iii) Edge $P_1 - P_3$

$$S(P_1)S^{-1}(P_3) = \frac{1}{0.21} \begin{bmatrix} 0.29 & 0.2 \\ -0.025 & 0.2 \end{bmatrix}$$

with eigenvalues $\lambda_{1,2} = 1.167 \pm j0.260$.

iv) Edge $P_4 - P_3$

$$S(P_4)S^{-1}(P_3) = \frac{1}{0.21} \begin{bmatrix} -0.51 & -1.8 \\ -0.059 & -0.41 \end{bmatrix}$$

with eigenvalues $\lambda_1 = -0.621$ and $\lambda_2 = -3.760$.

The edges $P_4 - P_2$ and $P_1 - P_3$ do not yield negative real eigenvalues, i.e., they are discrete-time stable. Edge $P_1 - P_2$, however, has two negative, real eigenvalues and stability boundaries at $q_1 = 4.101$ (A in Fig. 1), and $q_1 = 1.700$ (B in Fig. 1). Edge $P_4 - P_3$ intersects the stability boundary at $q_1 = 2.532$ (D in Fig. 1) and $q_1 = 4.16$ (C in Fig. 1). Thus, the polytope is not entirely discrete-time stable.

For comparison, Fig. 1 also shows the true stability boundaries. They are obtained by the parameter space method [6] in an implicit form as

$$q=\begin{bmatrix}3.592\\0.169\end{bmatrix}+\begin{bmatrix}-5.071\\1.409\end{bmatrix}\tau+\begin{bmatrix}-11.268\\25.352\end{bmatrix}\tau^2;\ \tau\in[-1,1]$$

where τ is the real part of a pair of roots on the unit circle in the z-plane. The intersections with the edges of Q correspond to $\tau_A = -0.298$, $\tau_C = -0.211$, $\tau_D = 0.155$, and $\tau_B = 0.242$.

IV. Conclusions

By the edge result of [3], it suffices to check the exposed edges in order to determine whether a polytope of polynomials has all its zeros in a simply connected region D. The edges could be tested, for example, by plotting a root locus with parameter α and checking whether it is located entirely in the desired D region. This sweep along edges can be avoided for continuous-time systems with D being the left half-plane; see [1]. In the present note this result is extended to the discrete-time case and reduced to operations (inversion, eigenvalue calculation) on $(n-1) \times (n-1)$ matrices.

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A Generalized MFD Criterion for Fixed Modes

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***Abstract*—A generalized MFD criterion for fixed modes with arbitrarily constrained feedback structure is presented. The efficiency of the new criterion in structure analysis is illustrated by a numerical example.**

I. Introduction

Consider the system [1]

$$\dot{x}=Fx+\sum_{i=1}^{m}G_iu_i,\qquad y_i=H_i'x\quad(i=1,\cdots,m).\tag{1.1}$$

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Here, $G_i \in R^{n\times v_i}$, $H_i \in R^{n\times \gamma_i}$. Let $\bar{K}$ be the set of block-diagonal matrices

$$\bar{K}=\{K\,|\,K=\text{block diag}\,(K_1,\cdots,K_m),\ K_i\in R^{v_i\times\gamma_i}\}.\tag{1.2}$$

Then the set of fixed modes of $\{F, G_i, H_i', i = 1, \cdots, m\}$ with respect to $\bar{K}$ is defined as

$$\Lambda(F,G_i,H_i',\bar{K})=\bigcap_{K\in\bar{K}}\sigma\left(F+\sum_{i=1}^{m}G_iK_iH_i'\right)\tag{1.3}$$

where $\sigma(\cdot)$ denotes the set of eigenvalues.

In order to calculate fixed modes, Wang and Davison presented an applicable algorithm in [1]. But the algorithm did not give deep insight into the mechanism and characterization of fixed modes.

Anderson and his co-workers studied the algebraic characterization of fixed modes and achieved some new results. Their main contributions in [2] could be summarized in an algebraic criterion for fixed modes based upon matrix fraction descriptions (MFD). Although Anderson's criterion is well known in the field of decentralized control, it can only be used for block-diagonal feedback structure matrix.

In this note, a new criterion for fixed modes with arbitrarily constrained feedback is developed which generalizes Anderson's criterion.

II. A New Criterion for Fixed Modes

Let $X(\alpha_1, \cdots, \alpha_i/\beta_1, \cdots, \beta_i)$ be the submatrix formed by the rows $\alpha_1, \cdots, \alpha_i$ and the columns $\beta_1, \cdots, \beta_i$ of matrix X, $X_Y(\alpha_1, \cdots, \alpha_i; \beta_1, \cdots, \beta_i)$ the matrix with replacing the columns $\alpha_1, \cdots, \alpha_i$ of X by the columns $\beta_1, \cdots, \beta_i$ of Y, and $G(s) = H'(sI - F)^{-1}G = A^{-1}(s)B(s)$ the irreducible left matrix fraction description of system (H', F, G) [3].

Definition 2.1: For ith order index group $\Omega_i = (\alpha_1, \cdots, \alpha_i; \beta_1, \cdots, \beta_i)$ with $1 \le i \le m$, where both $\{\alpha_1, \cdots, \alpha_i\}$ and $\{\beta_1, \cdots, \beta_i\}$ are strictly increasing subsequences of $\{1, \cdots, m\}$, if there exists a $K \in \bar{K}$ such that $\det K(\alpha_1, \cdots, \alpha_i/\beta_1, \cdots, \beta_i) \not\equiv 0$, then Ω_i is called an ith order effective index group of $\bar{K}$. The total number of the ith order effective index groups is denoted by l_i. We specify that $\Omega_o = \phi$ (empty set) is also effective and $l_o = 1$.

Definition 2.2: If $\Omega_i^j = (\alpha_1^j, \cdots, \alpha_i^j; \beta_1^j, \cdots, \beta_i^j)$ is one of the ith order index groups with $0 \le i \le m$, $1 \le j \le l_i$, where l_i is the number of ith order effective index groups of $\bar{K}$, then

$$f(\Omega_i^j,s)=\det A(s)_{B(s)}(\alpha_1^j,\cdots,\alpha_i^j;\beta_1^j,\cdots,\beta_i^j)$$

is called the adjoint polynomial of Ω_i^j.

Lemma 2.1: There is the following relationship between the closed-loop characteristic polynomial of the system (1.1) with feedback $u_i = K_iy_i$ and all adjoint polynomials

$$\det(sI-F+GKH')=\sum_{i=0}^{m}\sum_{j=1}^{l_i}f(\Omega_i^j,s)K(\Omega_i^j)\tag{2.1}$$

where $K = \text{block diag}\{K_1, \cdots, K_m\}$, $H = \text{block diag}\{H_1, \cdots, H_m\}$ and

$$K(\Omega_i^j)=\begin{cases}1, & \text{for } i=0\\ \det K(\alpha_1^j,\cdots,\alpha_i^j/\beta_1^j,\cdots,\beta_i^j), & \text{for } 1\le i\le m.\end{cases}\qquad\square$$

The proof of Lemma 2.1 is omitted here and the details can be found in [4].

Theorem 2.1: s_o is a fixed mode, with multiplicity r, of system (H', F, G) w.r.t. $\bar{K}$ if and only if

$$\frac{d^{k-1}}{ds^{k-1}}f(\Omega_i^j,s)|_{s=s_0}=0,$$
$$i=0,\cdots,m;\ j=1,\cdots,l_i;\ k=1,\cdots,r.\tag{2.2}$$