Robust Stability Analysis of Constrained l_1-Norm Model Predictive Control

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Sufficient conditions for robust closed-loop stability of a class of dynamic matrix control (DMC) systems are presented. The l_1-norm is used in the objective function of the on-line optimization, thus resulting in a linear programming problem. The ideas of this work, however, are expandable to other DMC-type controllers. The keys to the stability conditions are: to use an end-condition in the moving horizon on-line optimization; to have coefficients of the move suppression term in the objective function of the on-line optimization satisfy certain inequalities; and to express the uncertainty as deviations in the unit pulse response coefficients of the nominal plant. These deviations and disturbances must also satisfy certain inequalities.

An off-line tuning procedure for robust stability and performance of a class of DMC controllers is also included, which determines an optimal moving horizon length and optimal values for coefficients of the move suppression term. The applicability of our approach is elucidated through numerical simulations.

Introduction

In recent years, several multivariable control techniques falling in the general category of model predictive control (MPC) have been studied and successfully implemented to industrial processes. MPC variations include model predictive heuristic control (MPHC) (Richalet et al., 1978), model algorithmic control (MAC) (Mehra et al., 1979), dynamic matrix control (DMC) (Cutler and Ramaker, 1980; Prett and Gillete, 1979), linear dynamic matrix control (LDMC) (Morshedi et al., 1985) and quadratic dynamic matrix control (QDMC) (Garcia and Morshedi, 1986).

The MPHC, MAC, DMC, LDMC and QDMC algorithms share a common characteristic in using a pulse or a step response model, unlike other MPC algorithms that use a parametric model based on physical laws.

Several investigators have tried to develop a theory for analyzing the stability properties of unconstrained and constrained MPC closed-loop systems. Garcia and Morari (1982) were first to study the effects of the tuning parameters (prediction and control horizons, penalty on process input changes) on the closed-loop stability of unconstrained DMC systems in the internal model control (IMC) framework. Their later work

concentrated on utilizing robust linear control theory to determine the robust stability of unconstrained DMC systems (Prett and Garcia, 1988; Morari and Zafiriou, 1989).

Zafiriou and coworkers used the contraction mapping principle proposed by Economou (1985) to develop sufficient conditions for the stability of QDMC closed-loop systems with process input/output constraints (Zafiriou, 1988, 1989, 1990; Zafiriou and Marshal, 1991; Zafiriou and Chiou, 1989). They showed that the tuning rules developed for the unconstrained case may not work well for a constrained system, and in fact may cause instability in the presence of output constraints.

Rawlings and Muske (1993) used a state-space framework to develop sufficient conditions in the absence of modeling error and disturbances for the stability of closed-loop constrained MPC systems with infinite prediction horizons and a finite number of on-line decision variables (process inputs).

No stability results have been derived for the LDMC controller introduced by Morshedi et al. (1985). The lack of stability results may be attributed to the fact that a closed-form expression for an equivalent linear controller does not exist, even in the absence of input/output constraints (Garcia et al., 1989).

In this work, we are deriving sufficient conditions for robust stability of a class of LDMC systems with process input and

output constraints. In particular, we try to answer the following questions:

- How, exactly, does the prediction-horizon length affect the stability characteristics of LDMC with process input and output constraints? What modifications are needed to maintain closed-loop stability with zero offset, if *short* prediction and constraint horizons are used? How do the control and prediction horizon lengths affect the overall performance of the LDMC closed-loop system?

- How should the weights of the input-change terms (move suppression terms) be selected so that robust closed-loop stability is guaranteed?

The key to answering the above questions is the use of an *end condition* in the formulation of the moving-horizon optimization problem. From this point on, we will use the letter E in all acronyms where an end-condition is included in the configuration of an MPC algorithm (for example, ELDMC and EQDMC).

Conditions that the coefficients of the move suppression terms must satisfy to guarantee robust closed-loop stability are provided in the sequel.

The rest of the article is structured as follows. We first state precisely how our LDMC formulation (ELDMC) differs from the original LDMC introduced by Morshedi et al. (1985). We then derive sufficient conditions for the robust stability properties of ELDMC in the presence of process input/output constraints, bounded and converging disturbances, and plant/model mismatch. We subsequently present an ELDMC controller tuning methodology for robust performance. Finally, simulations are used to illustrate the practicality of our approach. The proof of our main theorem is given in Appendices A and B. Sample calculations referring to the simulations are given in Appendix C.

Linear Dynamic Matrix Control with End Condition (ELDMC)

The basic idea behind ELDMC is that the unit pulse response (or step response) coefficients of the plant, assumed to be open-loop stable, are used at the sampling time k to predict future outputs, and then the manipulated variables are calculated by minimizing a cost:

$$\min_{\epsilon_1, \ldots, \epsilon_{nw}, \Delta u(k), \Delta u(k+1), \ldots, \Delta u(k+p)} J(k) \qquad (1)$$

with

$$J(k) = w \sum_{j=1}^{nw} \epsilon_j + \sum_{j=1}^{nh} |\bar{y}(k+j) - y^{sp}| + \sum_{i=0}^{p} r_i |\Delta u(k+i)|$$

subject to

$$\bar{y}(k+i) = \sum_{j=1}^{N} g_j u(k+i-j) + \bar{d}(k)$$

$$\bar{d}(k) = y(k) - \sum_{j=1}^{N} g_j u(k-j)$$

$$\Delta u_{max} \geq \Delta u(k+i) \geq -\Delta u_{max}, \quad i = 0, 1, 2, \ldots, p$$

$$u_{max} \geq u(k+i) \geq u_{min}, \quad i = 0, 1, 2, \ldots, p$$

$$y_{max} + \epsilon_j \geq \bar{y}(k+j) \geq y_{min} - \epsilon_j, \quad j = 1, 2, \ldots, nw$$

$$u(k+p+i) = \frac{y^{sp} - \bar{d}(k)}{G}, \quad i \geq 0 \text{ (end condition)}$$

$$\epsilon_j \geq 0, \quad j = 1, 2, \ldots, nw$$

where $\bar{y}$ is predicted output; y is measured output; u is manipulated input; $\bar{d}$ is an estimate of unmeasured disturbance; y^{sp} is set point; g_j's are unit pulse response model coefficients; G is the model gain; Δu_{max} is the limit for move size; ϵ_j is the constraint relaxation factor; w is the constraint tuning parameter; u_{max}, u_{min}, y_{max}, y_{min} are upper and lower limits for manipulated inputs and predicted outputs respectively; $p+1$ and nh are control and prediction horizon lengths; nw is the output constraint window length; N is the number of terms in the unit pulse response model; and $r_i \geq 0$ is the move suppression parameter.

The above formulation (Eq. 1) can be transformed into a linear program, after a simple substitution of variables as follows:

$$\min_{\epsilon_1, \ldots, \epsilon_{nw}, \Delta u(k), \Delta u(k+1), \ldots, \Delta u(k+p), v_1, \ldots, v_{nh}, \mu_0, \ldots \mu_p} J(k) \qquad (2)$$

with

$$J(k) = w \sum_{j=1}^{nw} \epsilon_j + \sum_{j=1}^{nh} v_j + \sum_{i=0}^{p} \mu_i$$

subject to

$$\bar{y}(k+i) = \sum_{j=1}^{N} g_j u(k+i-j) + \bar{d}(k)$$

$$\bar{d}(k) = y(k) - \sum_{j=1}^{N} g_j u(k-j)$$

$$\Delta u_{max} \geq \Delta u(k+i) \geq -\Delta u_{max}, \quad i = 0, 1, 2, \ldots, p$$

$$u_{max} \geq u(k+i) \geq u_{min}, \quad i = 0, 1, 2, \ldots, p$$

$$y_{max} + \epsilon_j \geq \bar{y}(k+j) \geq y_{min} - \epsilon_j, \quad j = 1, 2, \ldots, nw$$

$$-v_j \leq \bar{y}(k+j) - y^{sp} \leq v_j, \quad j = 1, 2, \ldots, nh$$

$$-\mu_i \leq r_i \Delta u(k+i) \leq \mu_i, \quad i = 0, 1, 2, \ldots, p$$

$$u(k+p+i) = \frac{y^{sp} - \bar{d}(k)}{G}, \quad i \geq 0 \text{ (end condition)}$$

$$v_j \geq 0, \quad j = 1, 2, \ldots, nh$$

$$\mu_i \geq 0, \quad i = 0, 1, 2, \ldots, p$$

$$\epsilon_j \geq 0, \quad j = 1, 2, \ldots, nw$$

Our formulation (ELDMC) differs from the original LDMC introduced by Morshedi et al. (1985) in the following aspects:
- We include an end condition.
- We do not introduce the transpose of the dynamic matrix (Eqs. 14 to 16) in Morshedi et al. (1985) to incorporate the move suppression terms. Hence, the objective function of the new formulation is different than that of the original formulation.

Remarks: feasibility of on-line ELDMC problem

The end condition combined with the constraint on the process input may cause an infeasible solution, as a result of an unexpectedly large disturbance occurring for a short time period. In order not to interrupt the process in such situations, the following end condition adaptation is necessary when implementing ELDMC control.

$$u(k+p+i) = u_\infty, \quad i \geq 0 \quad \text{if} \quad u_{min} \leq u_\infty \leq u_{max}$$

$$u(k+p+i) = u_{min}, \quad i \geq 0 \quad \text{if} \quad u_\infty < u_{min}$$

$$u(k+p+i) = u_{max}, \quad i \geq 0 \quad \text{if} \quad u_\infty > u_{max}$$

where

$$u_\infty = \frac{y^{sp} - \bar{d}(k)}{G}$$

It is clear that p must be large enough to allow $u(k+p)$ to reach any feasible value in (u_{min}, u_{max}). In the worst case where $u(k-1) = u_{min}$ and $u(k+p)$ must reach u_{max}, the inequality

$$u_{max} - u_{min} \leq (p+1)\Delta u_{max}$$

guarantees that u can reach u_{max} in $p+1$ steps.

Even if LDMC and ELDMC have no input/output constraints, the conversion of formulation 1 to 2 introduces some constraints. Hence, it is not possible to utilize linear control techniques to show the stability of LDMC or ELDMC, even in the absence of original input/output constraints.

Stability analysis of unconstrained QDMC in terms of linear control techniques (Prett and Garcia, 1988) may have been one additional reason why QDMC (constrained or unconstrained) has been the preferred DMC version to most industrial applications. However, we are going to show in the sequel that our improved formulation for input/output constrained ELDMC can be used to ensure robust stability with zero offset.

Robust stability of closed-loop ELDMC

Let

$$y(k) = d(k) + \sum_{j=1}^{N} h_j u(k-j) \tag{3}$$

describe the real plant behavior, where only the estimates of h_j's (g_j's) are known:

$$h_j = g_j + e_j, \quad 1 \leq j \leq N$$

$$|e_j| \leq E_j, \quad 1 \leq j \leq N$$

where e_j is a bounded additive error.

Assume that for all k and for a certain $M > 0$

$$d_{min} \leq d(k) \leq d_{max} \tag{4}$$

$$|\Delta d(k)| \leq \Delta d_{max} \tag{5}$$

where

$$\Delta d_{max} > 0, \quad k \leq M; \quad \Delta d_{max} = 0, \quad k > M \ (\Rightarrow \lim_{k \to \infty} d(k) = d_\infty)$$

Theorem: robust stability

For a plant that is described by Eq. 3 and disturbances satisfying inequalities 4 and 5, the ELDMC closed-loop system with controller described by the set of relations 1 is asymptotically stable with zero offset if:

(i) process modeling and disturbance uncertainties satisfy the conditions

$$\left(|G| - \sum_{i=1}^{N} E_i\right) \Delta u_{max} \geq \Delta d_{max} > 0 \tag{6}$$

$$\max\{Gu_{max}, Gu_{min}\} \geq y^{sp} - d_{min} + U \sum_{i=1}^{N} E_i \tag{7}$$

$$\min\{Gu_{max}, Gu_{min}\} \leq y^{sp} - d_{max} - U \sum_{i=1}^{N} E_i \tag{8}$$

where $U = \max\{|u_{max}|, |u_{min}|\}$ and $G = \sum_{i=1}^{N} g_i$;

(ii) the prediction and optimization horizon lengths ($p+1$ and nh) satisfy the inequalities

$$nh - 1 \geq p + 1 \geq \max\left(\frac{u_{max} - u_{min}}{\Delta u_{max}}, 1\right) \tag{9}$$

and the output constraint window length satisfies the inequalities

$$nw - 1 \geq p + 1 \geq \max\left(\frac{u_{max} - u_{min}}{\Delta u_{max}}, 1\right); \tag{10}$$

(iii) the move suppression terms $\{r_j\}_{j=0}^{p}$ are selected according to

$$r_p = \frac{\sum_{j=-N+1}^{p} \delta_j + (b + w\breve{b}) \sum_{i=1}^{N} E_i + \sum_{j=-N+1}^{p} (a_j + w\breve{a}_j)}{1 - \frac{\sum_{i=1}^{N} E_i}{|G|}} \tag{11}$$

$$r_{j-1} = r_j - a_j - w\breve{a}_j - \delta_j, \quad 1 \leq j \leq p \tag{12}$$

where $\delta_j \geq 0$, and

$$a_j = \left|\sum_{i=2+nh-j}^{N+1} g_i\right|, \quad -N+2+nh \leq j \leq p \tag{13}$$

$$a_j = 0, \quad j < -N + 2 + nh \tag{14}$$

$$b = 1 + p + \sum_{i=1+p}^{nh} \left| \frac{\sum_{l=1+i-p}^{N+1} q_l}{G} \right| \tag{15}$$

$$\check{a}_j = \left| \sum_{i=2+nw-j}^{N+1} g_i \right|, \quad -N+2+nw \leq j \leq q \tag{16}$$

$$\check{a}_j = 0, \quad j < -N + 2 + nw, \text{ and } j > q \tag{17}$$

$$\check{b} = 1 + p + \sum_{i=1+p}^{nw} \left| \frac{\sum_{l=1+i-p}^{N+1} g_l}{G} \right| \tag{18}$$

and $q \leq p$ is selected such that the inequality

$$Y \geq \Delta u_{\max} \sum_{j=q+1}^{p} \left| \sum_{l=2+nw-j}^{N+1} g_l \right| \tag{19}$$

is satisfied, or else $q = p$. For proof, see Appendix A and Appendix B.

Remarks

Condition ii of the above theorem implies that a prediction horizon of a minimum length $nh = p + 2$ can be used, with guaranteed closed-loop stability. In particular, if $\Delta u_{\max} \geq u_{\max} - u_{\min}$, then a *short* prediction horizon of length $nh = 2(=nw)$ with one calculated move ($p=0$) guarantees closed-loop stability. Certainly, such a choice may not be optimal, and longer horizons may have to be used for better performance. We address the performance issue in the Corollary of the next section.

• For *long* horizons, that is, such that $nw = nh \geq p + N - 1$, Eqs. 13 and 16 yield $a_j = 0$, $\check{a}_j = 0$. In that case, the move suppression parameters, $\{r_j\}_{j=0}^{p}$, may be selected to have the same value, that is, $r_p = r_{p-1} = \ldots = r_0 > 0$, where r_p is given by Eq. 11, for guaranteed closed-loop stability. In addition, if $E_j = 0$ (no modeling error) then no move suppression terms are necessary to stabilize the system, that is, $r_p = r_{p-1} = \ldots = r_0 = 0$. It should be stressed that the end condition must be included in the on-line optimization for the above remarks to be valid.

• The importance of the end condition in constrained ELDMC can also be realized as follows. In the presence of constraints and in the absence of end condition, it can be shown by a procedure similar to that discussed in Appendix A, that with the proper selection of move suppression parameters $\{\Phi(k)\}_{k=0}^{\infty}$ becomes a nonincreasing and converging sequence, as follows:

$$\Phi(k) \geq \Phi(k+1) + \sum_{j=-N+1}^{p} \delta_j |\Delta \hat{u}(k+j)|$$

where $\delta_j > 0$ (the main difference between the above inequality and Eq. A-41 is the missing term $(|y(k) - y^{sp}|)$.

Hence, $\lim_{k \to \infty} \hat{u}(k) = u_{ss}$. If the input $\hat{u}(k)$ has reached a steady-state value, there are two possibilities:

(i) The controller does not need to drive the system away from the reached steady state, because there is no offset.

(ii) The controller has not driven the system to the desired steady state, but cannot change the attained steady state, because it is saturated (due to operating constraints). Thus, an offset remains.

However, in the case of *unconstrained* ELDMC, the second possibility is ruled out. Hence, the offset will be zero, thus making the end condition dispensable.

Robust performance of closed-loop ELDMC

Once the stability of a control loop with zero offset is guaranteed, then the second most important step is to find the best performing controller out of a set of stabilizing controllers. Hence, a definition of a measure for the performance of an ELDMC closed-loop system is necessary.

Definition

The performance (P) of an ELDMC closed-loop system is measured by the following quantity

$$P = \sum_{k=0}^{\infty} [|y(k) - y^{sp}| + w \max(0, [y(k) - y_{\max}], [y_{\min} - y(k)])]$$

where y is the measured process output.

The following corollary is a direct consequence of the main theorem. It is useful in the selection of proper control and prediction horizons.

Corollary: Robust performance

The ELDMC controller defined by the set of relations (1) achieves a closed-loop performance, P, no worse than the initial on-line calculated optimal objective function value $\Phi(0)$:

$$P \leq \Phi(0) = \min_{\epsilon_1, \ldots, \epsilon_{nw}, \Delta u(0), \Delta u(1), \ldots, \Delta u(p)} J(0)$$

where

$$J(0) = [|y(0) - y^{sp}| + w \max(0, [y(0) - y_{\max}], [y_{\min} y(0)])$$

$$+ \sum_{j=1}^{nh} |\bar{y}(j) - y^{sp}| + w \sum_{j=1}^{nw} \max(0, [\bar{y}(j) - y_{\max}], [y_{\min} - \bar{y}(j)])$$

$$+ \sum_{j=-N+1}^{p} r_j |\Delta u(j)| + f(0)]$$

subject to operating constraints, end condition, and Eqs. B27 to B30.

Proof

Let us choose $\delta_j = 0$ for $-N \leq j \leq p$. Hence, relation B22 becomes

$$\Phi(k) \geq \Phi(k+1) + |y(k) - y^{sp}|$$

$$+ w \max(0, [y(k) - y_{\max}], [y_{\min} - y(k)])$$

Successive substitution of $\Phi(k)$ in the above equation, starting from initial time $k = 0$, yields

$$\Phi(0) \geq \sum_{k=0}^{\infty} [|y(k) - y^{sp}|$$

$$+ w \max(0, [y(k) - y_{\max}], [y_{\min} - y(k)])] = P$$

Remark

In the absence of modeling error and in the presence of constant disturbances, the performance of the closed-loop, as p increases, approaches the *best possible* off-line calculated performance:

$$P = \lim_{p \to \infty} \Phi(0)$$

suggesting to choose p as large as possible and $nw = nh = p + N - 1$.

Realistically, however, we have to consider modeling error. In this case, the value of $\Phi(0)$ increases for large p, since r_p contains p as a multiplier (see Eqs. 11, 12, 15, and 18). Hence, p, nw, and nh should be selected as a result of the following minimization problem:

$$\min_{p, nh, nw} \Phi(0)$$

for the possible set points and disturbances satisfying relations 7 and 8.

Example

We consider a plant described by the equation

$$y(k) = h_1 u(k-1) + h_2 u(k-2) + h_3 u(k-3) + h_4 u(k-4) + d(k)$$

with

$$h_1 = g_1 + e_1 = 0.0 + e_1, \quad \text{where} \quad |e_1| \leq 0.12$$
$$h_2 = g_2 + e_2 = -1.0 + e_2, \quad \text{where} \quad |e_2| \leq 0.10$$
$$h_3 = g_3 + e_3 = 2.0 + e_3, \quad \text{where} \quad |e_3| \leq 0.08$$
$$h_4 = g_4 + e_4 = 0.0 + e_4, \quad \text{where} \quad |e_4| \leq 0.05$$
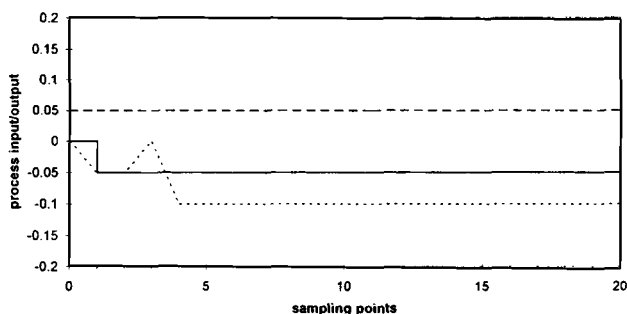

Figure 1. Case 1: no end condition is enforced.

No modeling error, $(e_1, e_2, e_3, e_4) = (0, 0, 0, 0)$; constant disturbance and set point, $[d(k), y^{sp}] = (-0.05, 0.05)$; move suppression coefficients, $r_1 = r_0 < 0.5$; resulting performance, $P = \infty$ (offset).
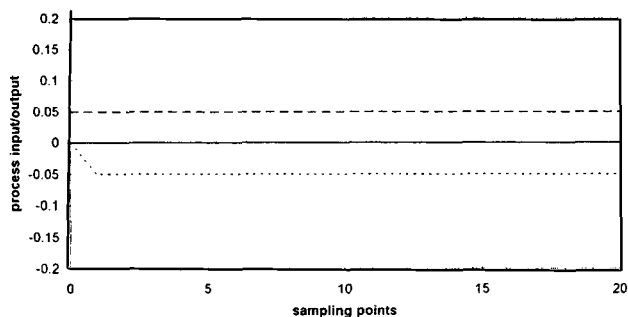

Figure 2. Case 2: no end condition is enforced.

No modeling error, $(e_1, e_2, e_3, e_4) = (0, 0, 0, 0)$; constant disturbance and set point, $[d(k), y^{sp}] = (-0.05, 0.05)$; move suppression coefficients: $r_1 = r_0 \geq 0.5$; resulting performance: $P = \infty$ (offset).

and the ELDMC controller 1, with $p = 1$ (two calculated moves), $nh = 3$, $N = 4$, $\Delta u_{\max} = 0.2$, $u_{\max} = -u_{\min} = 0.2$, with no constraint on the output y. The move suppression parameters are found as $r_1 = r_0 = 2.7$. The details of the selection of the above move suppression values are given in the Appendix C. Five simulation cases are given in Figures 1, 2, 3, 4 and 5 to elucidate our approach. In all figures, u is represented by a continuous line, y^{sp} by a long-dashed line, and y by a short-dashed line.

Simulation cases 1 and 2 clearly show poor closed-loop performance even in the absence of any modeling error when the end condition is not enforced. When the end condition is enforced however, simulations 3, 4 and 5 show robust asymptotic stability of the closed-loop system with no offset.

Discussion and Conclusions

In this article, we have tried to answer the questions raised in the Introduction section. While the results of this article can be implemented to directly design ELDMC controllers, the ideas developed here will also give more insight into understanding robust stability characteristics of general DMC types of controllers.
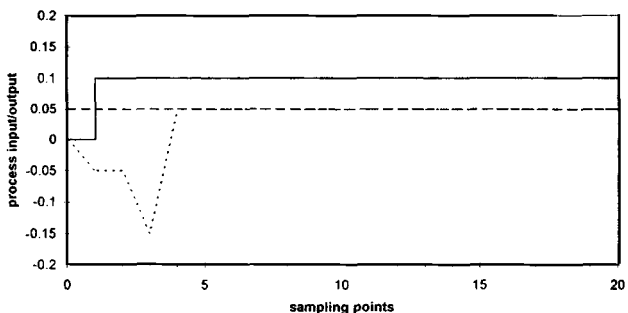
Acknowledgment

Figure 3. Case 3: end condition is enforced.

No modeling error, $(e_1, e_2, e_3, e_4) = (0, 0, 0, 0)$; constant disturbance and set point, $[d(k), y^{sp}] = (-0.05, 0.05)$; move suppression coefficients: $r_1 = r_0 = 2.7$; resulting performance: $P = 0.4 < \Phi(0) = 0.67$ (no offset).
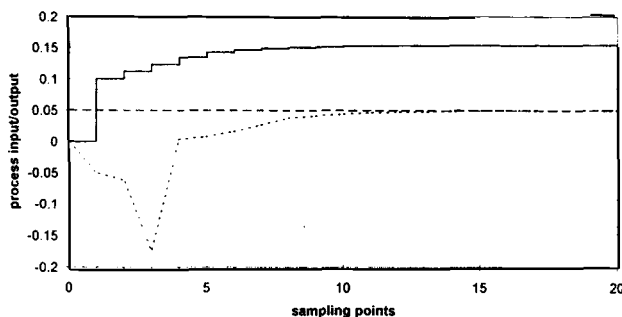
Figure 4. Case 4: end condition is enforced.

Modeling error, $(e_1, e_2, e_3, e_4) = (-0.12, -0.1, -0.08, -0.05)$; constant disturbance and set point, $[d(k), y^{sp}] = (-0.05, 0.05)$; move suppression coefficients, $r_1 = r_0 = 2.7$; resulting performance, $P = 0.6154 < \Phi(0) = 0.67$ (no offset).

are also due to Prof. Jim Rawlings for letting the authors access a manuscript with early results of his research in this field.

Notation

d = output additive disturbance
e = model error
E = maximum absolute model error
$f(\cdot)$ = dummy function bounding the disturbance
g = unit pulse response model coefficient
G = model gain
h = unit pulse response coefficient of real plant
H = real plant gain
$J(\cdot)$ = cost functional
l = coordinate of process output constraint window
M = sampling time which disturbance reach steady state after
nh = ending of prediction-horizon
nw = ending of output constraint window
N = number of coefficients in the unit pulse response model
p = ending of control-horizon
P = performance measure of an LDMC closed-loop system
r = move-suppression parameter
u = process manipulated input
U = maximum absolute input limit
w = soft output constraint tuning parameter
y = process output

Greek letters

δ = positive small number
Δ = difference operator: $\Delta x(k) = x(k) - x(k-1)$
ϵ = soft constraint relaxation factor
$\Phi(\cdot)$ = optimal value of a specific cost function
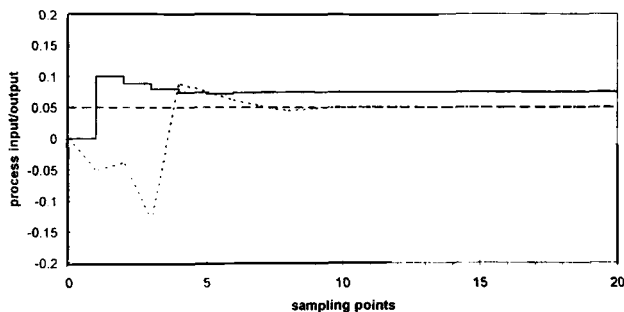

Figure 5. Case 5: end condition is enforced.

Modeling error, $(e_1, e_2, e_3, e_4) = (0.12, 0.1, 0.08, 0.05)$; constant disturbance and set point, $[d(k), y^{sp}] = (-0.05, 0.05)$; move suppression coefficients, $r_1 = r_0 = 2.7$; resulting performance, $P = 0.4531 < \Phi(0) = 0.67$ (no offset).

Subscripts

max = maximum limit
min = minimum limit
ss = steady state

Superscripts

sp = set point
$\tilde{\ }$ = prediction
$\hat{\ }$ = known variable at time k
$\check{\ }$ = known variable at time k for time $k+1$

Acronyms

DMC = dynamic matrix control
ELDMC = linear dynamic matrix control with end condition
IMC = internal model control
LDMC = linear dynamic matrix control
MAC = model algorithmic control
MPC = model predictive control
MPHC = model predictive heuristic control
QDMC = quadratic dynamic matrix control
SISO = single-input-single-output

Literature Cited

Cutler, C. R., and B. L. Ramaker, "Dynamic Matrix Control—A Computer Control Algorithm," *Joint Automatic Control Conf.*, San Francisco (1980).

Economou, C. G., "An Operator Theory Approach to Nonlinear Controller Design," PhD Thesis, Caltech (1985).

Garcia, C. E., and M. Morari, "Internal Robust Control: 1. A Unifying Review and Some New Results," *Ind. Eng. Chem. Process Des. and Dev.*, **21**, 308 (1982).

Garcia, C. E., and A. M. Morshedi, "Quadratic Programming Solution of Dynamic Matrix Control (QDMC)," *Chem. Eng. Commun.*, **46**, 73 (1986).

Garcia, C. E., D. M. Prett, and M. Morari, "Model Predictive Control: Theory and Practice—A Survey," *Automatica*, **25**(3), 335 (1989).

Mehra, R. K., R. Rouhani, A. Rault, and J. G. Reid, "Model Algorithmic Control: Theoretical Results on Robustness," *Proc. JACC*, 387 (1979).

Morari, M., and E. Zafiriou, *Robust Process Control*, Prentice-Hall, Englewood Cliffs, NJ (1989).

Morshedi, A. M., C. R. Cutler, and T. A. Skovanek, "Optimal Solution of Dynamic Matrix Control with Linear Programming Techniques (LDMC)," *Proc. Automatic Control Conf.*, p. 199, Boston (1985).

Prett, D. M., and R. D. Gillette, "Optimization and Constrained Multivariable Control of a Catalytic Cracking Unit," AIChE Meeting, Houston (1979).

Prett, D. M., and C. E. Garcia, *Fundamental Process Control*, Butterworths, Stoneham, MA (1988).

Rawlings, J. B., and K. R. Muske, "The Stability of Constrained Receding Horizon Control," *IEEE Trans. Automat. Contr.*, accepted (1993).

Richalet, J., A. Rault, J. L. Testud, and J. Papon, "Model Predictive Heuristic Control: Applications to Industrial Processes," *Automatica*, **14**, 413 (1978).

Zafiriou, E., "Robust Control of Processes with Hard Constraints," AIChE Meeting, Washington, DC (1988).

Zafiriou, E., "Robustness and Tuning of On-Line Optimizing Control Algorithms," *Model Based Process Control-Proc. IFAC Workshop*, Pergamon Press, Oxford (1989).

Zafiriou, E., "Robust Model Predictive Control of Processes with Hard Constraints," *Comp. Chem. Eng.*, **14**, 359 (1990).

Zafiriou, E., and A. L. Marchal, "Stability of SISO Quadratic Dynamic Matrix Control with Hard Output Constraints," *AIChE J.*, **37**, 1550 (1991).

Zafiriou, E., and H. W. Chiou, "An Operator Control Theory Approach to the Design and Tuning of Constrained Model Predictive Controllers," AIChE Meeting, San Francisco (1989).

Appendix A: Robust Stability Condition for Closed-Loop SISO ELDMC

We will derive here sufficient conditions for robust stability of closed-loop SISO ELDMC systems, for which
- A modeling error is explicitly considered.
- Disturbances are bounded and eventually reach a steady state.
- *No* process output constraints are considered.

The results of Appendix A will be used in Appendix B to derive sufficient conditions for robust stability when process output constraints are present.

We will first formulate the on-line optimization problem corresponding to the ELDMC controller.

Formulation of on-line optimization for ELDMC

The plant and disturbances for all k are described by Eqs. 3, 4 and 5 with manipulated input constraints:

$$u_{min} \le u(k) \le u_{max} \tag{A1}$$

$$|\Delta u(k)| \le \Delta u_{max} \tag{A2}$$

The model used in ELDMC to predict, at time k, future outputs is:

$$\bar{y}(k+i) = \bar{d}(k+i) + \sum_{j=1}^{N} g_j u(k+i-j) \tag{A3}$$

with

$$\bar{d}(k+i) = \bar{d}(k), \quad i \ge 0 \tag{A4}$$

$$\bar{d}(k) = y(k) - \sum_{j=1}^{N} g_j u(k-j) = d(k) + \sum_{j=1}^{N} e_j u(k-j) \tag{A5}$$

Remark

Equation A4 above is a convenient assumption, in the traditional DMC framework, used to include the feedback effect of the measurement $y(k)$. The derivations in the sequel will hold whether or not this assumption is true.

Let the ELDMC on-line solve, at time k, the following minimization problem

$$\Phi(k) = \min_{\Delta u(k), \Delta u(k+1), \ldots, \Delta u(k+p)} J(k) \tag{A6}$$

subject to

$$\Delta u_{max} \ge \Delta u(k+i) \ge -\Delta u_{max}, \quad i = 0, 1, 2, \ldots, p \tag{A7}$$

$$u_{max} \ge u(k+i) \ge u_{min}, \quad i = 0, 1, 2, \ldots, p \tag{A8}$$

$$u(k+p) = \frac{y^{sp} - \bar{d}(k)}{G} \quad \text{(end condition)} \tag{A9}$$

$$\Delta u(k+p+i) = 0, \quad i \ge 1 \tag{A10}$$

where

$$J(k) = |y(k) - y^{sp}| + \sum_{i=1}^{nh} |\bar{y}(k+i) - y^{sp}|$$
$$+ \sum_{i=-N+1}^{p} r_i |\Delta u(k+i)| + f(k) \tag{A11}$$

$\{f(k)\}_{k=0}^{\infty}$ is a non-negative sequence to be determined in the sequel for showing disturbance rejection properties of ELDMC. Thus, Eqs. A3 to A11 constitute the ELDMC controller.

Feasibility of ELDMC on-line problem

The ELDMC on-line problem as defined by the Eqs. A6 to A11 has a solution if

$$(i) \qquad (1+p)\Delta u_{max} \ge u_{max} - u_{min} \tag{A12}$$

$$(ii) \quad u_{max} \ge \frac{y^{sp} - d(k) - \sum_{i=1}^{N} e_i \hat{u}(k-i)}{G}$$

$$\text{(feasibility of end condition)} \ge u_{min} \tag{A13}$$

The conditions (i) and (ii) are satisfied if relations 9, 7 and 8 are true.

Derivation of sufficient conditions for robust stability

We will now derive robust stability conditions for the ELDMC controller defined above by Eqs. A3 to A11. The idea is to develop conditions that ensure the sequence $\{\Phi(k)\}_{k=0}^{\infty}$ converges to 0.

Assuming the conditions 9, 7 and 8 are satisfied, let

$$\{\Delta \hat{u}(k), \Delta \hat{u}(k+1), \ldots, \Delta \hat{u}(k+p)\} \tag{A14}$$

be the solution of Eq. A6 corresponding to the optimal predicted outputs

$$\{\hat{y}(k+1), \ldots, \hat{y}(k+nh)\}.$$

Remark

In the application of ELDMC, $\hat{u}(k)$ is the only calculated variable which is implemented at time k. The actual value of the optimal cost $\Phi(k)$ is not used. Hence, two ELDMC controllers, one minimizing the cost (Eq. A11) and another minimizing

$$J(k) = \sum_{i=1}^{nh} |\bar{y}(k+i) - y^{sp}| + \sum_{i=0}^{p} r_i |\Delta u(k+i)| \tag{A15}$$

are equivalent, both resulting in solutions given by Eq. A14.

Let us also denote all past inputs implemented before the sampling time k by $\hat{u}(k-j)$ where $j \ge 1$.

Hence, at time k, the combination of Eqs. A3 to A11 yields

$$\Phi(k) = |y(k) - y^{sp}| + \sum_{i=1}^{nh} |\hat{y}(k+i) - y^{sp}|$$

$$+ \sum_{i=-N+1}^{p} r_i |\Delta u(k+i)| + f(k) \quad \text{(A16)}$$

For the next sampling time $k+1$

$$\Phi(k+1) = \min_{\Delta u(k+1), \Delta u(k+2), \ldots, \Delta u(k+1+p)} J(k+1) \quad \text{(A17)}$$

subject to

$$\Delta u_{\max} \geq \Delta u(k+1+i) \geq -\Delta u_{\max}, \quad i = 0, 1, 2, \ldots, p \quad \text{(A18)}$$

$$u_{\max} \geq u(k+1+i) \geq u_{\min}, \quad i = 0, 1, 2, \ldots, p \quad \text{(A19)}$$

$$u(k+1+p) = \frac{y^{sp} - \bar{d}(k+1)}{G} \quad \text{(end condition)} \quad \text{(A20)}$$

$$\Delta u(k+1+p+i) = 0, \quad i \geq 1 \quad \text{(A21)}$$

where

$$J(k+1) = |y(k+1) - y^{sp}| + \sum_{i=1}^{nh} |\bar{y}(k+1+i) - y^{sp}|$$

$$+ \sum_{i=-N+1}^{p} r_i |\Delta u(k+1+i)| + f(k+1) \quad \text{(A22)}$$

and

$$\bar{y}(k+1+i) = \bar{d}(k+1+i) + \sum_{j=1}^{N} g_j u(k+i+1-j) \quad \text{(A23)}$$

with

$$\bar{d}(k+1+i) = \bar{d}(k+1), \quad i \geq 0 \quad \text{(A24)}$$

$$\bar{d}(k+1) = y(k+1) - \sum_{j=1}^{N} g_j u(k+1-j)$$

$$= d(k+1) + \sum_{j=1}^{N} e_j u(k+1-j) \quad \text{(A25)}$$

Let us try to create a feasible (but not necessarily optimal) solution to Eq. A17 as

$$\{\Delta \bar{u}(k+1), \Delta \bar{u}(k+2), \ldots, \Delta \bar{u}(k+p), \Delta \bar{u}(k+1+p)\}$$

$$= \{\Delta \hat{u}(k+1), \Delta \hat{u}(k+2), \ldots, \Delta \hat{u}(k+p),$$

$$\Delta \bar{u}(k+1+p)\} \quad \text{(A26)}$$

where $\Delta \bar{u}(k+1+p)$ is to be selected. The corresponding predicted future outputs are denoted by

$$\{\bar{y}(k+2), \ldots, \bar{y}(k+1+nh)\}$$

yielding a suboptimal cost $\Phi^*(k+1)$:

$$\Phi^*(k+1) = |y(k+1) - y^{sp}| + \sum_{i=1}^{nh} |\bar{y}(k+1+i) - y^{sp}|$$

$$+ \sum_{i=-N+1}^{p-1} r_i |\Delta \hat{u}(k+1+i)| + r_p |\Delta \bar{u}(k+1+p)|$$

$$+ f(k+1) \quad \text{(A27)}$$

Obviously,

$$0 \leq \Phi(k+1) \leq \Phi^*(k+1) \quad \text{(A28)}$$

$$\Phi(k) \geq \Phi(k+1) + [\Phi(k) - \Phi^*(k+1)] \quad \text{(A29)}$$

Now let us obtain sufficient conditions for which the set (Eq. A26) is a feasible solution of Eq. A17.

It is clear that all $\hat{u}$'s and $\Delta \hat{u}$'s satisfy the constraints (Eqs. A18 and A19) because they satisfied the constraints (Eqs. A7 and A8) in the previous sampling time k. Hence, $\bar{u}(k+1+p)$ remains to be selected so that it can satisfy the constraint (Eq. A18):

$$\Delta u_{\max} \geq \Delta \bar{u}(k+1+p) \geq -\Delta u_{\max} \quad \text{(A30)}$$

the constraint (Eq. A19), that is,

$$u_{\min} \leq \bar{u}(k+1+p) \leq u_{\max} \quad \text{(A31)}$$

and the following end condition (derived from the combination of Eq. A20 with Eq. A25):

$$\bar{y}(\infty) = y^{sp} = d(k+1) + G\bar{u}(k+1+p)$$

$$+ \sum_{i=1}^{N} e_i \hat{u}(k+1-i) \quad \text{(A32)}$$

Combining Eqs. A31 and A32, we get, equivalently

$$u_{\max} \geq \frac{y^{sp} - d(k+1) - \sum_{i=1}^{N} e_i \hat{u}(k+1-i)}{G} \geq u_{\min}$$

The above condition is satisfied if conditions 7 and 8 are true (starting assumption).

To satisfy the constraint (Eq. A30), we start with the combination of Eqs. A9 and A5, to get

$$\hat{y}(\infty) = y^{sp} = d(k) + G\hat{u}(k+p) + \sum_{i=1}^{N} e_i \hat{u}(k-i) \quad \text{(A33)}$$

Subtracting Eq. A33 from A32 we get

$$\Delta \bar{u}(k+1+p) = \frac{-\Delta d(k+1) - \sum_{i=1}^{N} e_i \Delta \hat{u}(k+1-i)}{G} \quad \text{(A34)}$$

The above $\Delta \bar{u}(k+1+p)$ must satisfy Eq. A30. This is guar-

anteed if inequality 6 is satisfied. From the preceding discussion it is evident that Eqs. 7, 8 and 6 guarantee the feasibility of the set (Eq. A26).

We are now ready to derive closed-loop robust stability conditions, which guarantee that $\{\Phi(k)\}_{k=0}^{\infty}$ is convergent. Inequality A29 yields

$$\Phi(k) \geq \Phi(k+1) + |y(k) - y^{sp}|$$

$$- |\tilde{y}(k+1+nh) - y^{sp}| + (|\hat{y}(k+1) - y^{sp}| - |y(k+1) - y^{sp}|)$$

$$+ \sum_{j=2}^{nh} (|\hat{y}(k+j) - y^{sp}| - |\tilde{y}(k+j) - y^{sp}|)$$

$$- r_p |\Delta\tilde{u}(k+1+p)| + \sum_{j=-N+2}^{p} (r_j - r_{j-1}) |\Delta\hat{u}(k+j)|$$

$$+ (r_{-N+1}) |\Delta\hat{u}(k-N+1)| + f(k) - f(k+1) \quad \text{(A35)}$$

or

$$\Phi(k) \geq \Phi(k+1) + |y(k) - y^{sp}|$$

$$- |\tilde{y}(k+1+nh) - y^{sp}| - |\tilde{y}(k+1) - y(k+1)|$$

$$- \sum_{j=2}^{nh} |\hat{y}(k+j) - \tilde{y}(k+j)| - r_p |\Delta\tilde{u}(k+1+p)|$$

$$+ \sum_{j=-N+1}^{p} (r_j - r_{j-1}) |\Delta\hat{u}(k+j)| + f(k) - f(k+1) \quad \text{(A36)}$$

with $r_{-N} = 0$.

The predicted outputs $\{\hat{y}(k+j),\ j=1,\ldots,nh\}$ and $\{\tilde{y}(k+j), j=2,\ldots,1+nh\}$ can be expressed in terms of the corresponding inputs in Eqs. A14 and A26, by means of Eqs. A3, A23, and A34. The measured outputs $y(k)$ and $y(k+1)$ can be expressed in terms of the corresponding inputs by means of Eq. 3. Therefore, the following equalities are obtained after some manipulations.

$$\tilde{y}(k+1+nh) - y^{sp} = \frac{\sum_{l=1+nh-p}^{N+1} g_l}{G} \Delta d(k+1)$$

$$+ \frac{\sum_{l=1+nh-p}^{N+1} g_l}{G} \sum_{j=-N+1}^{0} e_{1-j}\Delta\hat{u}(k+j)$$

$$- \sum_{j=-N+2+nh}^{p} \left(\sum_{l=2+nh-j}^{N+1} g_l\right)\Delta\hat{u}(k+j) \quad \text{(A37)}$$

$$\hat{y}(k+1) - y(k+1) = -\Delta d(k+1)$$

$$- \sum_{j=-N+1}^{0} e_{1-j}\Delta\hat{u}(k+j) \quad \text{(A38)}$$

$$\hat{y}(k+1+i) - \tilde{y}(k+1+i)$$

$$= -\Delta d(k+1) - \sum_{j=-N+1}^{0} e_{1-j}\Delta\hat{u}(k+j),\ 1 \leq i \leq p \quad \text{(A39)}$$

$$\hat{y}(k+1+i) - \tilde{y}(k+1+i) = \frac{\sum_{l=1+i-p}^{N+1} g_l}{G} \Delta d(k+1)$$

$$+ \frac{\sum_{l=1+i-p}^{N+1} g_l}{G} \sum_{j=-N+1}^{0} e_{1-j}\Delta\hat{u}(k+j),\ 1+p \leq i \leq nh-1 \quad \text{(A40)}$$

Substituting Eqs. A34, A37, A38, A39, and A40 in A36 yields:

$$\Phi(k) \geq \Phi(k+1) + |y(k) - y^{sp}|$$

$$+ \sum_{j=1}^{p} (r_j - r_{j-1} - a_j) |\Delta\hat{u}(k+j)|$$

$$+ \sum_{j=-N+1}^{0} \left(r_j - r_{j-1} - a_j - \left[b + \frac{r_p}{|G|}\right]E_{1-j}\right) |\Delta\hat{u}(k+j)|$$

$$+ \left(f(k) - f(k+1) - \left[b + \frac{r_p}{|G|}\right]\Delta d_{max}\right) \quad \text{(A41)}$$

where a_j and b are given by Eqs. 13, 14 and 15.

We would like to make the quantity added to $\Phi(k+1)$ in the righthand side of inequality (Eq. A41) nonnegative, in order to guarantee that the sequence $\{\Phi(k)\}_{k=0}^{\infty}$ is nonincreasing. This is guaranteed if $\{r_j\}_{j=-N+1}^{p}$ and $\{f(k)\}_{k=0}^{\infty}$ are chosen to satisfy the following equalities, for $\delta_j \geq 0$, and for all disturbances satisfying Eqs. 4, 5, 7, 8, 6:

$$r_j - r_{j-1} - a_j = \delta_j,\ 1 \leq j \leq p \quad \text{(A42)}$$

$$r_j - r_{j-1} - a_j - \left(b + \frac{r_p}{|G|}\right)E_{1-j} = \delta_j,\ -N+1 \leq j \leq 0 \quad \text{(A43)}$$

$$f(k) - f(k+1) - \left(b + \frac{r_p}{|G|}\right)\Delta d_{max} = \delta_{-N},\ 0 \leq k \leq M \quad \text{(A44)}$$

$$f(k) = 0,\ k > M \quad \text{(A45)}$$

The solution of Eqs. A42 to A45 is

$$r_p = \frac{\sum_{j=-N+1}^{p} \delta_j + b\sum_{i=1}^{N} E_i + \sum_{j=-N+1}^{p} a_j}{1 - \frac{\sum_{i=1}^{N} E_i}{|G|}} \quad \text{(A46)}$$

$$r_{j-1} = r_j - a_j - \delta_j,\ 1 \leq j \leq p \quad \text{(A47)}$$

Remark

The following are dummy variables to help the calculation of parameters above, hence their calculated values are not needed but given for completeness.

$$r_{j-1} = r_j - a_j - \left(b + \frac{r_p}{|G|}\right)E_{1-j} - \delta_j,\ -N+1 \leq j \leq 0 \quad \text{(A48)}$$

$$f(k) = \left(\frac{\delta_{-N}}{\Delta d_{max}} + b + \frac{r_p}{|G|} \right) [M + 1 - k] \Delta d_{max},$$

$$k \geq 0, \quad [f(k) = 0, \ k > M] \quad (A49)$$

Hence, if Eqs. A46 to A49 are satisfied, $\{\Phi(k)\}_{k=0}^{\infty}$ is a nonincreasing sequence and bounded below by zero. Therefore, $\{\Phi(k)\}_{k=0}^{\infty}$ converges. This implies that taking the limit of Eq. A41, as $k \to \infty$, yields:

$$0 \leq \lim_{k \to \infty} \left[|y(k) - y^{sp}| + \sum_{j=-N+1}^{p} \delta_j |\Delta \hat{u}(k+j)| \right]$$

$$\Rightarrow \lim_{k \to \infty} y(k) = y^{sp} \quad (A50)$$

Appendix B: Effect of Soft Output Constraints

We will use here the results of Appendix A to derive sufficient conditions for robust stability of closed-loop SISO ELDMC systems, for which
- A modeling error is explicitly considered.
- Disturbances are bounded and eventually reach a steady state.
- Process output constraints *are* considered.

Formulation of ELDMC with process output constraints

Equations A1 to A5 remain the same. Equation A6 becomes

$$\Phi(k) = \min_{\epsilon_1, \dots, \epsilon_{nw}, \Delta u(k), \Delta u(k+1), \dots, \Delta u(k+p)} J(k) \quad (B1)$$

subject to

$$y_{min} - \epsilon_j \leq \bar{y}(k+j) \leq y_{max} + \epsilon_j, \quad j = 1, 2, \dots, nw \quad (B2)$$

$$\epsilon_j \geq 0, \quad 1 \leq j \leq nw \quad (B3)$$

and subject to Eqs. A7, A8, A9 and A10.
The cost functional A11 will change to:

$$J(k) = w\hat{\epsilon}_0 + w \sum_{j=1}^{nw} \epsilon_j + |y(k) - y^{sp}|$$

$$+ \sum_{j=1}^{nh} |\bar{y}(k+j) - y^{sp}| + \sum_{j=-N+1}^{p} r_i |\Delta u(k+j)| + f(k) \quad (B4)$$

where $\hat{\epsilon}_0 = \max([y(k) - y_{max}], [y_{min} - y(k)], 0)$ and is a dummy variable which will be important for showing ELDMC closed-loop performance characteristics.

Derivation of sufficient conditions for robust stability of ELDMC with process output constraints

We will now proceed similarly to Appendix A, namely, we will first find a particular feasible solution at time $k+1$, based on which we will show that $\{\Phi(k)\}_{k=0}^{\infty}$ is a converging sequence, provided that certain sufficient conditions are satisfied.

Let the counterpart of the solution (Eq. A14) be

$$\{\hat{\epsilon}_1, \dots, \hat{\epsilon}_2, \Delta\hat{u}(k), \Delta\hat{u}(k+1), \dots, \Delta\hat{u}(k+p)\} \quad (B5)$$

Hence, at time k,

$$\Phi(k) = w\hat{\epsilon}_0 + w \sum_{j=0}^{nw} \hat{\epsilon}_j + |y(k) - y^{sp}|$$

$$+ \sum_{j=1}^{nh} |\hat{y}(k+j) - y^{sp}| + \sum_{j=-N+1}^{p} r_j |\Delta\hat{u}(k+j)| + f(k) \quad (B6)$$

For the next sampling time $k+1$

$$\Phi(k+1) = \min_{\epsilon_1, \dots, \epsilon_{nw}, \Delta u(k+1), \Delta u(k+2), \dots, \Delta u(k+1+p)} J(k+1) \quad (B7)$$

subject to

$$y_{min} - \epsilon_j \leq \bar{y}(k+1+j) \leq y_{max} + \epsilon_j, \quad j = 1, 2, \dots, nw \quad (B8)$$

$$\epsilon_j \geq 0, \quad 1 \leq j \leq nw \quad (B9)$$

and subject to Eqs. A18, A19, A20 and A21, where

$$J(k+1)$$

$$= w\tilde{\epsilon}_0 + w \sum_{j=1}^{nw} \epsilon_j + |y(k+1) - y^{sp}| + \sum_{j=1}^{nh} |\bar{y}(k+1+j) - y^{sp}|$$

$$+ \sum_{j=-N+1}^{p} r_j |\Delta u(k+1+j)| + f(k+1) \quad (B10)$$

where $\tilde{\epsilon}_0 = \max([y(k+1) - y_{max}], [y_{min} - y(k+1)], 0)$.
Let the counterpart of Eq. A26 be

$$\{\tilde{\epsilon}_1, \dots, \tilde{\epsilon}_{nw}, \Delta\tilde{u}(k+1), \Delta\tilde{u}(k+2),$$

$$\dots, \Delta\tilde{u}(k+p), \Delta\tilde{u}(k+1+p)\} = \{\hat{\epsilon}_2 + |\hat{y}(k+2) - \tilde{y}(k+2)|,$$

$$\dots, \hat{\epsilon}_{nw} + |\hat{y}(k+nw) - \tilde{y}(k+nw)|,$$

$$\max(0, |\tilde{y}(k+1+nw) - y^{sp}| - Y), \Delta\hat{u}(k+1),$$

$$\Delta\hat{u}(k+2), \dots, \Delta\hat{u}(k+p), \Delta\tilde{u}(k+1+p)\} \quad (B11)$$

where $Y = \min(|y_{max} - y^{sp}|, |y^{sp} - y_{min}|)$. We will show that Eq. B11 represents a feasible solution at time $k+1$. The feasibility of all terms but $\tilde{\epsilon}_j$'s was shown in Appendix A. For $\tilde{\epsilon}_j$'s we have

$$y_{min} - \hat{\epsilon}_{j+1} \leq \hat{y}(k+1+j), \quad 1 \leq j \leq nw - 1 \quad (B12)$$

equivalently,

$$y_{min} - \hat{\epsilon}_{j+1} - |\hat{y}(k+1+j) - \tilde{y}(k+1+j)|$$

$$\leq \hat{y}(k+1+j) - |\hat{y}(k+1+j) - \tilde{y}(k+1+j)|$$

$$\leq \tilde{y}(k+1+j), \quad 1 \leq j \leq nw - 1 \quad (B13)$$

and

$$y_{max} + \hat{\epsilon}_{j+1} \geq \hat{y}(k+1+j), \quad 1 \leq j \leq nw - 1 \quad (B14)$$

equivalently,

$$y_{max} + \hat{\epsilon}_{j+1} + |\hat{y}(k+1+j) - \bar{y}(k+1+j)|$$

$$\geq \hat{y}(k+1+j) + |\hat{y}(k+1+j) - \bar{y}(k+1+j)|$$

$$\geq \bar{y}(k+1+j), \quad 1 \leq j \leq nw - 1 \quad (B15)$$

Because

$$y_{min} \leq y^{sp} \leq y_{max} \quad (B16)$$

the following relation can be obtained.

$$\max([\bar{y}(k+1+nw) - y_{max}], [y_{min} - \bar{y}(k+1+nw)], 0)$$

$$\leq \max(0, |\bar{y}(k+1+nw) - y^{sp}| - Y) = \tilde{\epsilon}_{nw} \quad (B17)$$

Therefore, Eq. B11 is indeed feasible.

Based on the above discussion we will now develop conditions for which $\{\Phi(j)\}_{j=k}^{\infty}$ is nonincreasing and convergent.

The counterpart of inequality A36 becomes:

$$\Phi(k) \geq \Phi(k+1) + |y(k) - y^{sp}| + w\hat{\epsilon}_0 - |\bar{y}(k+1+nh) - y^{sp}|$$

$$- |\hat{y}(k+1) - y(k+1)| - w\tilde{\epsilon}_{nw} + w(\hat{\epsilon}_1 - \tilde{\epsilon}_0) + \sum_{j=2}^{nw} w(\hat{\epsilon}_j - \tilde{\epsilon}_{j-1})$$

$$- \sum_{j=2}^{nh} |\hat{y}(k+j) - \bar{y}(k+j)| - r_p |\Delta\bar{u}(k+1+p)|$$

$$+ \sum_{j=-N+1}^{p} (r_j - r_{j-1}) |\Delta\bar{u}(k+j)| + f(k) - f(k+1) \quad (B18)$$

The similar procedure followed from Eqs. B12 to B16 yields:

$$- |\hat{y}(k+1) - y(k+1)| \leq (\hat{\epsilon}_1 - \tilde{\epsilon}_0) \quad (B19)$$

and from Eq. B11 the following set of equalities is obtained:

$$- |\hat{y}(k+j) - \bar{y}(k+j)| = (\hat{\epsilon}_j - \tilde{\epsilon}_{j-1}), \quad \text{for } 2 \leq j \leq nw \quad (B20)$$

Hence, Eq. B18 becomes:

$$\Phi(k) \geq \Phi(k+1) + |y(k) - y^{sp}| + w\hat{\epsilon}_0$$

$$- |\bar{y}(k+1+nh) - y^{sp}| - w\max(0, |\bar{y}(k+1+nw) - y^{sp}| - Y)$$

$$- (1+w)|\hat{y}(k+1) - y(k+1)| - \sum_{j=2}^{nw} w|\hat{y}(k+j) - \bar{y}(k+j)|$$

$$- \sum_{j=2}^{nh} |\hat{y}(k+j) - \bar{y}(k+j)| - r_p |\Delta\bar{u}(k+1+p)|$$

$$+ \sum_{j=-N+1}^{p} (r_j - r_{j-1}) |\Delta\bar{u}(k+j)| + f(k) - f(k+1) \quad (B21)$$

Using Eqs. A37 to A40 (except that nw will substitute nh) and Eq. B21, the counterpart of inequality A41 becomes:

$$\Phi(k) \geq \Phi(k+1) + |y(k) - y^{sp}| + w\hat{\epsilon}_0$$

$$+ \sum_{j=1}^{p} (r_j - r_{j-1} - a_j - w\check{a}_j) |\Delta\hat{u}(k+j)|$$

$$+ \sum_{j=-N+1}^{0} \left(r_j - r_{j-1} - a_j - w\check{a}_j \right.$$

$$- \left[b + w\check{b} + \frac{r_p}{|G|} \right] E_{1-j} \bigg) |\Delta\hat{u}(k+j)|$$

$$+ (f(k) - f(k+1) - \left[b + w\check{b} + \frac{r_p}{|G|} \right] \Delta d_{max} \quad (B22)$$

where a_j, b, $\check{a}_j$, $\check{b}$, and q are given by Eqs. 13, 14, 15, 16, 17, 18 and 19.

We would like to make the quantity added to $\Phi(k+1)$ in the righthand side of inequality B22 nonnegative, in order to guarantee that the sequence $\{\Phi(k)\}_{k=0}^{\infty}$ is nonincreasing. This is guaranteed if $\{r_j\}_{j=-N+1}^{p}$ and $\{f(k)\}_{k=0}^{\infty}$ are chosen to satisfy the following equalities, for $\delta_j \geq 0$, and for all disturbances satisfying Eqs. 4, 5, 7, 8 and 6.

$$r_j - r_{j-1} - a_j - w\check{a}_j = \delta_j, \quad 1 \leq j \leq p \quad (B23)$$

$$r_j - r_{j-1} - a_j - w\check{a}_j - \left(b + w\check{b} + \frac{r_p}{|G|} \right) E_{1-j} = \delta_j,$$

$$-N+1 \leq j \leq 0 \quad (B24)$$

$$f(k) - f(k+1) - \left(b + w\check{b} + \frac{r_p}{|G|} \right) \Delta d_{max} = \delta_{-N},$$

$$0 \leq k \leq M \quad (B25)$$

$$f(k) = 0, \quad k > M \quad (B26)$$

The solution of Eqs. B23 to B26 is:

$$r_p = \frac{\sum_{j=-N+1}^{p} \delta_j + (b + w\check{b}) \sum_{i=1}^{N} E_i + \sum_{j=-N+1}^{p} (a_j + w\check{a}_j)}{1 - \frac{\sum_{i=1}^{N} E_i}{|G|}} \quad (B27)$$

$$r_{j-1} = r_j - a_j - w\check{a}_j - \delta_j, \quad 1 \leq j \leq p \quad (B28)$$

$$r_{j-1} = r_j - a_j - w\check{a}_j - \left(b + w\check{b} + \frac{r_p}{|G|} \right) - \delta_j,$$

$$-N+1 \leq j \leq 0 \quad (B29)$$

$$f(k) = \left(\frac{\delta_{-N}}{\Delta d_{max}} + b + w\check{b} + \frac{r_p}{|G|} \right) [M+1-k] \Delta d_{max},$$

$$k \geq 0, \quad [f(k) = 0, \quad k > M] \quad (B30)$$

The rest of the proof is the same as in Appendix A.

Appendix C: Stability Condition of an Example

Sufficient conditions for the example will be derived according to the theory outlined. Unit pulse response coefficients are:

$$h_1 = g_1 + e_1 = 0.0 + e_1, \quad \text{where} \quad |e_1| \leq 0.12$$
$$h_2 = g_2 + e_2 = -1.0 + e_2, \quad \text{where} \quad |e_2| \leq 0.10$$
$$h_3 = g_3 + e_3 = 2.0 + e_3, \quad \text{where} \quad |e_3| \leq 0.08$$
$$h_4 = g_4 + e_4 = 0.0 + e_4, \quad \text{where} \quad |e_4| \leq 0.05$$

ELDMC parameters are: $p = 1$, $nh = 3$

Settling time is: $N = 4$

Constraints are: $\Delta u_{max} = 0.2$, $u_{max} = 0.2$, $u_{min} = -0.2$

The other calculated parameters are as follows:

$$G = 1, \quad a_j = 0 \quad \text{for} \quad 1 \geq j \geq -3, \quad b = 5$$

Let us choose $\delta_1 = \delta_0 = \delta_{-1} = \delta_{-2} = \delta_{-3} = 0$, then:

$$r_1 = 2.7 \quad \text{and} \quad r_0 = 2.7$$

Hence, ELDMC closed loop will have no offset provided that the disturbance reaches a steady-state value after an arbitrary sampling time M and the following conditions are true.
From Eq. 6

$$\Delta d(k) \leq 0.13 \quad \text{for} \quad k \leq M$$
$$\Delta d(k) = 0 \quad \text{for} \quad k > M$$

From Eqs. 7 and 8

$$|y^{sp} - d(k)| \leq 0.13.$$