Now, we choose

$$\Phi = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix} \quad \Psi = \begin{bmatrix} 19, & -5 \\ -5, & 7 \end{bmatrix}.$$

The corresponding Riccati equation (13) admits the maximal solution

$$\Pi = \begin{bmatrix} 1, & 0 \\ 0, & 7 \end{bmatrix}.$$

By Theorem 2, all optimal controls are given as follows:

$$\bar{u}(t) = \begin{bmatrix} 0, & 0 \\ 0, & 7 \end{bmatrix} \bar{x}(t) + v(t)$$

with $v(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$. Moreover, one feedback law is

$$\bar{u}(t) = \begin{bmatrix} 0, & 0 \\ 0, & 7 \end{bmatrix} \bar{x}(t).$$

Next, we would like to see how the choice of Φ and Ψ might affect the form of the optimal controls. Take the following matrices:

$$\Phi_\varepsilon = \begin{bmatrix} 1+\varepsilon, & 0 \\ 0, & 1+\varepsilon \end{bmatrix} \quad \Psi_k = \begin{bmatrix} 19k, & -5k \\ -5k, & 7k \end{bmatrix}$$

parameterized by ε and k with $|\varepsilon| < 1/2$ and $0 < k < 18$. Both Φ_ε and Ψ_k are positive–definite and the corresponding Riccati equation (13) admits the maximal solution

$$\Pi_{\varepsilon,k} = \begin{bmatrix} k, & 0 \\ 0, & (1+\varepsilon)\left(3 + \sqrt{9 + \frac{7}{1+\varepsilon}}\right) \end{bmatrix}.$$

In this case, it follows from Theorem 2 that all optimal controls of the original LQ problem can also be given as follows:

$$\bar{u}(t) = \begin{bmatrix} 0, & 0 \\ 0, & 3 + \sqrt{9 + \frac{7}{1+\varepsilon}} \end{bmatrix} \bar{x}(t) + v(t)$$

with $v(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$. Hence, a different optimal feedback law is

$$\bar{u}(t) = \begin{bmatrix} 0, & 0 \\ 0, & 3 + \sqrt{9 + \frac{7}{1+\varepsilon}} \end{bmatrix} \bar{x}(t).$$

It is interesting to note that the aforementioned optimal controls do not depend on the parameter k.

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Robust Stability and Stabilization for Singular Systems With State Delay and Parameter Uncertainty

Shengyuan Xu, Paul Van Dooren, Radu Ştefan, and James Lam

***Abstract*—This note considers the problems of robust stability and stabilization for uncertain continuous singular systems with state delay. The parametric uncertainty is assumed to be norm bounded. The purpose of the robust stability problem is to give conditions such that the uncertain singular system is regular, impulse free, and stable for all admissible uncertainties, while the purpose of robust stabilization is to design a state feedback control law such that the resulting closed-loop system is robustly stable. These problems are solved via the notions of generalized quadratic stability and generalized quadratic stabilization, respectively. Necessary and sufficient conditions for generalized quadratic stability and generalized quadratic stabilization are derived. A strict linear matrix inequality (LMI) design approach is developed. An explicit expression for the desired robust state feedback control law is also given. Finally, a numerical example is provided to demonstrate the application of the proposed method.**

***Index Terms*—Continuous singular systems, delay systems, linear matrix inequality (LMI), robust stability, robust stabilization.**

Notation

Throughout this note, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive–semidefinite (respectively, positive–definite). I is the identity matrix with appropriate dimension, the superscript "T" represents the transpose, $\|x\|$ is the Euclidean norm of the vector x, while $\rho(M)$ denotes the spectral radius of the matrix M.

I. Introduction

Control of delay systems has been a topic of recurring interest over the past decades since time delays are often the main causes for instability and poor performance of systems and encountered in various engineering systems such as chemical processes, long transmission lines in pneumatic systems, and so on [8]. Recently, the problems of robust stability analysis and robust stabilization for uncertain delay systems have been studied. Like in the case of uncertain systems without delay, the method based on the concepts of *quadratic stability* and *quadratic stabilizability* has been shown to be effective in dealing with these problems in both continuous and discrete contexts [12], [18].

On the other hand, control of singular systems has been extensively studied in the past years due to the fact that singular systems better describe physical systems than regular ones. Singular systems are also referred to as descriptor systems, implicit systems, generalized state-space systems, differential-algebraic systems, or semistate systems [4], [11]. A great number of results based on the theory of regular systems (or state-space systems) have been extended to the area of singular systems [4], [11]. Recently, robust stability and robust stabiliza-

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tion for uncertain singular systems have been considered. The notions of *quadratic stability* and *quadratic stabilizability* for regular systems have been extended [20], [22]. It should be pointed out that the robust stability problem for singular systems is much more complicated than that for regular systems because it requires to consider not only stability robustness, but also regularity and absence of impulses (for continuous singular systems) and causality (for discrete-singular systems) at the same time [6], [7], and the latter two need not be considered in regular systems. Very recently, much attention has been paid to singular systems with time delay. For the discrete-time case, when structured uncertainty appears, some results on robust stability were given in [19] by using properties of modulus matrix. When unstructured uncertainty appears, the results on robust stability and robust stabilization were reported in [17], where a linear matrix inequality (LMI) design method was developed. For the continuous-time case, numerical methods for such systems were discussed in [1] and [3], while [23] studied the stability problem by analyzing the system's characteristic equation and some frequency domain conditions for stability were given. It is worth pointing out that no parameter uncertainty was considered in [23]. To the best of our knowledge, when parameter uncertainty appears, there are no results on the problems of robust stability and stabilization for continuous singular delay systems in the literature.

In this note, we address the problems of robust stability and stabilization for uncertain continuous singular systems with state delay. The parameter uncertainties are time invariant and unknown, but norm bounded. The purpose of the robust stability problem is to develop conditions such that the uncertain singular system is regular, impulse free and stable for all admissible uncertainties. Following the same idea as in dealing with the robust stability problem for uncertain singular systems without delay [20], [22], we introduce the concept of *generalized quadratic stability*. It is shown that *generalized quadratic stability* implies robust stability. A necessary and sufficient condition for *generalized quadratic stability* is obtained in terms of a strict LMI. Similarly, the concept of *generalized quadratic stabilization* is proposed when dealing with the robust stabilization problem, the purpose of which is the design of memoryless state feedback control laws such that the resultant closed-loop system is regular, impulse free and stable for all admissible uncertainties. A strict LMI design approach is proposed and an explicit expression for the desired robust state feedback control law is given. It is worth pointing out that most LMI-type conditions for singular systems in the literature contain equality constraints [13], [21], [22], which will result in numerical problems when checking such nonstrict LMI conditions since equality constraints are fragile and usually not met perfectly [15]. Therefore, the strict LMI design approach proposed in this note is much more reliable in numerical computation.

II. Preliminaries and Problem Formulation

Consider a linear singular system with state delay and parameter uncertainties described by

$$(\Sigma):\quad \begin{aligned} E\dot{x}(t) =&(A+\Delta A)x(t)\\ &+(A_d+\Delta A_d)x(t-\tau)\\ &+(B+\Delta B)u(t) \end{aligned} \tag{1}$$

$$x(t) = \phi(t),\quad t\in[-\tau,0] \tag{2}$$

where $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control input. The matrix $E\in\mathbb{R}^{n\times n}$ may be singular, we shall assume that rank $E=r\leq n$. A, A_d and B are known real constant matrices with appropriate dimensions. $\tau>0$ is a constant time delay, $\phi(t)$ is a compatible vector valued continuous function. ΔA, ΔA_d and ΔB are time-invariant matrices representing norm-bounded parameter uncertainties, and are assumed to be of the following form:

$$[\,\Delta A\quad \Delta A_d\quad \Delta B\,] = MF(\sigma)[\,N_A\quad N_d\quad N_B\,] \tag{3}$$

where M, N_A, N_d and N_B are known real constant matrices with appropriate dimensions. The uncertain matrix $F(\sigma)$ satisfies

$$F(\sigma)F(\sigma)^T\leq I \tag{4}$$

and $\sigma\in\Theta$, where Θ is a compact set in $\mathbb{R}$. Furthermore, it is assumed that given any matrix $F: FF^T\leq I$, there exists a $\sigma\in\Theta$ such that $F=F(\sigma)$. ΔA, ΔA_d and ΔB are said to be admissible if both (3) and (4) hold.

Remark 1: It should be pointed out that the structure of the uncertainty with the form (3) and (4) has been used in other papers dealing with the problem of robust stabilization for regular and singular uncertain systems in both continuous and discrete time contexts; see, e.g., [16] and [22].

The nominal unforced singular delay system of (1) can be written as

$$E\dot{x}(t) = Ax(t) + A_d x(t-\tau). \tag{5}$$

Definition 1: [4], [11]:

1) The pair (E,A) is said to be regular if $\det(sE-A)$ is not identically zero.
2) The pair (E,A) is said to be impulse free if $\deg(\det(sE-A)) = \operatorname{rank} E$.

The singular delay system (5) may have an impulsive solution, however, the regularity and the absence of impulses of the pair (E,A) ensure the existence and uniqueness of an impulse free solution to this system, which is shown in the following lemma.

Lemma 1: Suppose the pair (E,A) is regular and impulse free, then the solution to (5) exists and is impulse free and unique on $[0,\infty)$.

Proof: Noting the regularity and the absence of impulses of the pair (E,A) and using the decomposition as in [4], the desired result follows immediately. □

In view of this, we introduce the following definition for singular delay system (5).

Definition 2:

1) The singular delay system (5) is said to be regular and impulse free if the pair (E,A) is regular and impulse free.
2) The singular delay system (5) is said to be stable if for any $\varepsilon>0$ there exists a scalar $\delta(\varepsilon)>0$ such that, for any compatible initial conditions $\phi(t)$ satisfying $\sup_{-\tau\leq t\leq 0}\|\phi(t)\|\leq\delta(\varepsilon)$, the solution $x(t)$ of system (5) satisfies $\|x(t)\|\leq\varepsilon$ for $t\geq 0$. Furthermore
$$x(t)\to 0\quad t\to\infty.$$

Throughout this note, we shall use the following notion of robust stability and robust stabilization for uncertain singular delay system (Σ).

Definition 3: The uncertain singular delay system (Σ) is said to be robustly stable if the system (Σ) with $u(t)\equiv 0$ is regular, impulse free and stable for all admissible uncertainties ΔA, and ΔA_d.

Definition 4: The uncertain singular delay system (Σ) is said to be robustly stabilizable if there exists a linear state feedback control law $u(t)=Kx(t)$, $K\in\mathbb{R}^{m\times n}$ such that the resultant closed-loop system is robustly stable in the sense of Definition 3. In this case, $u(t)=Kx(t)$ is said to be a robust state feedback control law for system (Σ).

The problem to be addressed in this note is the development of conditions for robust stability and robust stabilizability for the uncertain singular delay system (Σ) given in (1) and (2).

III. Main Results

In this section, we give a solution to the robust stability analysis and robust stabilization problems formulated previously, by using a strict LMI approach. First, we present the following result for singular delay system (5), which will play a key role in solving the aforementioned problems.

Theorem 1: The singular delay system (5) is regular, impulse free and stable if there exist a matrix $Q > 0$ and a matrix P such that

$$EP^T = PE^T \geq 0 \tag{6}$$

$$AP^T + PA^T + A_d P^T Q^{-1} P A_d^T + Q < 0. \tag{7}$$

For the proof of Theorem 1, we need the following results.

Lemma 2 [13]: The singular system

$$E\dot{x}(t) = Ax(t) \tag{8}$$

is regular, impulse free and stable if and only if there exists a matrix P such that

$$EP^T = PE^T \geq 0$$
$$AP^T + PA^T < 0.$$

Lemma 3: Consider the function $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}$. If $\dot{\varphi}$ is bounded on $[0, \infty)$, that is, there exists a scalar $\alpha > 0$ such that $|\dot{\varphi}(t)| \leq \alpha$ for all $t \in [0, \infty)$, then φ is uniformly continuous on $[0, \infty)$.

Lemma 4 (Barbalat's Lemma) [9]: Consider the function $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}$. If φ is uniformly continuous and $\int_0^\infty \varphi(s)ds < \infty$, then

$$\lim_{t\rightarrow\infty} \varphi(t) = 0.$$

Proof of Theorem 1: Suppose both (6) and (7) hold for $Q > 0$, then from (7) it is easy to see that

$$AP^T + PA^T < 0. \tag{9}$$

By Lemma 2, it follows from (6) and (9) that the pair (E, A) is regular and impulse free. Next, we shall show the stability of the singular delay system (5). To this end, we note that the regularity and the absence of impulses of the pair (E, A) implies that there exist two invertible matrices G and $H \in \mathbb{R}^{n\times n}$ such that [4]

$$\bar{E} := GEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{A} := GAH = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \tag{10}$$

where $I_r \in \mathbb{R}^{r\times r}$ and $I_{n-r} \in \mathbb{R}^{(n-r)\times(n-r)}$ are identity matrices, $A_1 \in \mathbb{R}^{r\times r}$. According to (10), let

$$\begin{aligned} \bar{A}_d &:= GA_dH = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}, \\ \bar{P} &:= GPH^{-T} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix}, \\ \bar{Q} &:= GQG^T = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{12}^T & \bar{Q}_{22} \end{bmatrix}. \end{aligned} \tag{11}$$

Then, from (6) and (7), we have

$$\bar{E}\bar{P}^T = \bar{P}\bar{E}^T \geq 0 \tag{12}$$

$$\bar{A}\bar{P}^T + \bar{P}\bar{A}^T + \bar{A}_d\bar{P}^T\bar{Q}^{-1}\bar{P}\bar{A}_d^T + \bar{Q} < 0. \tag{13}$$

By using a Schur complement argument, it follows from (13) that

$$\begin{bmatrix} \bar{A}\bar{P}^T + \bar{P}\bar{A}^T + \bar{Q} & \bar{A}_d\bar{P}^T \\ \bar{P}\bar{A}_d^T & -\bar{Q} \end{bmatrix} < 0. \tag{14}$$

Noting the expression of $\bar{E}$ in (10) and using (12), we can deduce that $\bar{P}_{11} = \bar{P}_{11}^T \geq 0$ and $\bar{P}_{21} = 0$, therefore $\bar{P}$ reduces to

$$\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ 0 & \bar{P}_{22} \end{bmatrix}. \tag{15}$$

Substituting (10), (11) and (15) into (14), one eventually gets (16), as shown at the bottom of the page. Thus

$$\begin{bmatrix} \bar{P}_{22} + \bar{P}_{22}^T + \bar{Q}_{22} & A_{d22}\bar{P}_{22}^T \\ \bar{P}_{22}A_{d22}^T & -\bar{Q}_{22} \end{bmatrix} < 0. \tag{17}$$

Since $\bar{Q}_{22} > 0$ and the inequality (17) holds, we have that $\bar{P}_{22}$ is invertible. Therefore, it follows from (17) that

$$\begin{bmatrix} -\tilde{Q}_{22} & A_{d22}^T\bar{P}_{22}^{-1} \\ \bar{P}_{22}^{-T}A_{d22} & \bar{P}_{22}^{-1} + \bar{P}_{22}^{-T} + \tilde{Q}_{22} \end{bmatrix} < 0$$

where

$$\tilde{Q}_{22} = \bar{P}_{22}^{-1}\bar{Q}_{22}\bar{P}_{22}^{-T} > 0. \tag{18}$$

By [5, Th. 1], we have that (18) implies

$$A_{d22}^T\tilde{Q}_{22}A_{d22} - \tilde{Q}_{22} < 0. \tag{19}$$

Therefore

$$\rho(A_{d22}) < 1. \tag{20}$$

Now, let

$$\zeta(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix} = H^{-1}x(t)$$

where $\zeta_1(t) \in \mathbb{R}^r$, $\zeta_2(t) \in \mathbb{R}^{n-r}$. Using the expressions in (10) and (11), the singular delay system (5) can be decomposed as

$$\begin{aligned} (\Sigma_D): \quad \dot{\zeta}_1(t) &= A_1\zeta_1(t) + A_{d11}\zeta_1(t-\tau) \\ &\quad + A_{d12}\zeta_2(t-\tau) && (21) \\ 0 &= \zeta_2(t) + A_{d21}\zeta_1(t-\tau) \\ &\quad + A_{d22}\zeta_2(t-\tau). && (22) \end{aligned}$$

It is easy to see that the stability of the singular delay system (5) is equivalent to that of the system (Σ_D). In view of this, next we shall prove that the system (Σ_D) is stable. Since $\bar{P}_{11} = \bar{P}_{11}^T \geq 0$ and

$$\bar{P}_{11}A_1^T + A_1\bar{P}_{11} + \bar{Q}_{11} < 0$$

as (16) shows, it follows that $\bar{P}_{11} > 0$. Define

$$V(\zeta_t) = \zeta_1(t)^T\bar{P}_{11}^{-1}\zeta_1(t) + \int_{t-\tau}^{t} \zeta(s)^T\bar{P}^{-1}\bar{Q}\bar{P}^{-T}\zeta(s)ds$$

where

$$\zeta_t = \zeta(t+\beta), \quad \beta \in [-\tau, 0].$$

Recall that for any matrices K_1, K_2 and K_3 of appropriate dimensions with $K_2 > 0$

$$K_1^TK_3 + K_3^TK_1 \leq K_1^TK_2K_1 + K_3^TK_2^{-1}K_3.$$

Then, the time-derivative of $V(\zeta_t)$ along the solution of (21) and (22) is given by

$$\begin{aligned} \dot{V}(\zeta_t) &= \frac{d}{dt}(\zeta(t)^T\bar{P}^{-1}\bar{E}\zeta(t)) + \zeta(t)^T\bar{P}^{-1}\bar{Q}\bar{P}^{-T}\zeta(t) \\ &\quad - \zeta(t-\tau)^T\bar{P}^{-1}\bar{Q}\bar{P}^{-T}\zeta(t-\tau) \\ &= \zeta(t)^T\bar{P}^{-1}\bar{E}\dot{\zeta}(t) + \dot{\zeta}(t)^T\bar{E}^T\bar{P}^{-1}\zeta(t) \\ &\quad + \zeta(t)^T\bar{P}^{-1}\bar{Q}\bar{P}^{-T}\zeta(t) - \zeta(t-\tau)^T\bar{P}^{-1}\bar{Q}\bar{P}^{-T}\zeta(t-\tau) \\ &= 2\zeta(t)^T\bar{P}^{-1}\bar{A}\zeta(t) + \zeta(t)^T\bar{P}^{-1}\bar{Q}\bar{P}^{-T}\zeta(t) \\ &\quad + 2\zeta(t)^T\bar{P}^{-1}\bar{A}_d\zeta(t-\tau) - \zeta(t-\tau)^T\bar{P}^{-1}\bar{Q}\bar{P}^{-T}\zeta(t-\tau) \\ &\leq \zeta(t)^T\bar{P}^{-1}\left(\bar{A}\bar{P}^T + \bar{P}\bar{A}^T + \bar{A}_d\bar{P}^T\bar{Q}^{-1}\bar{P}\bar{A}_d^T + \bar{Q}\right) \\ &\quad \times \bar{P}^{-T}\zeta(t). \end{aligned}$$

$$\begin{bmatrix} \bar{P}_{11}A_1^T + A_1\bar{P}_{11} + \bar{Q}_{11} & \bar{P}_{12} + \bar{Q}_{12} & A_{d11}\bar{P}_{11} + A_{d12}\bar{P}_{12}^T & A_{d12}\bar{P}_{22}^T \\ \bar{P}_{12}^T + \bar{Q}_{12}^T & \bar{P}_{22} + \bar{P}_{22}^T + \bar{Q}_{22} & A_{d21}\bar{P}_{11} + A_{d22}\bar{P}_{12}^T & A_{d22}\bar{P}_{22}^T \\ \bar{P}_{11}A_{d11}^T + \bar{P}_{12}A_{d12}^T & \bar{P}_{11}A_{d21}^T + \bar{P}_{12}A_{d22}^T & -\bar{Q}_{11} & -\bar{Q}_{12} \\ \bar{P}_{22}A_{d12}^T & \bar{P}_{22}A_{d22}^T & -\bar{Q}_{12}^T & -\bar{Q}_{22} \end{bmatrix} < 0. \tag{16}$$

It follows from this inequality and (13) that $\dot{V}(\zeta_t) < 0$ and

$$\begin{aligned}\lambda_1\|\zeta_1(t)\|^2 - V(\zeta_0) \leq& \zeta_1(t)^T \bar{P}_{11}^{-1}\zeta_1(t) - V(\zeta_0)\\ \leq& \zeta_1(t)^T \bar{P}_{11}^{-1}\zeta_1(t)\\ &+ \int_{t-\tau}^{t} \zeta(s)^T \bar{P}^{-1}\bar{Q}\bar{P}^{-T}\zeta(s)ds\\ &- V(\zeta_0)\\ =& \int_0^t \dot{V}(\zeta_s)ds\\ \leq& -\lambda_2\int_0^t \|\zeta(s)\|^2 ds\\ \leq& -\lambda_2\int_0^t \|\zeta_1(s)\|^2 ds < 0 \end{aligned} \tag{23}$$

where

$$\begin{aligned}\lambda_1 =& \lambda_{\min}\left(\bar{P}_{11}^{-1}\right) > 0,\\ \lambda_2 =& -\lambda_{\max}[\bar{P}^{-1}(\bar{A}\bar{P}^T + \bar{P}\bar{A}^T + \bar{A}_d\bar{P}^T\bar{Q}^{-1}\bar{P}\bar{A}_d^T + \bar{Q})\\ &\times \bar{P}^{-T}] > 0.\end{aligned}$$

Taking into account (23), we can deduce that

$$\lambda_1\|\zeta_1(t)\|^2 + \lambda_2\int_0^t \|\zeta_1(s)\|^2 ds \leq V(\zeta_0).$$

Therefore

$$\|\zeta_1(t)\|^2 \leq m_1 \tag{24}$$

and

$$\int_0^t \|\zeta_1(s)\|^2 ds \leq m_2 \tag{25}$$

where

$$m_1 = \frac{1}{\lambda_1}V(\zeta_0) > 0, \quad m_2 = \frac{1}{\lambda_2}V(\zeta_0) > 0.$$

Thus, $\|\zeta_1(t)\|$ is bounded. Considering this and (20), it can be deduced from (22) that $\|\zeta_2(t)\|$ is bounded and, hence, it follows from (21) that $\|\dot{\zeta}_1(t)\|$ is bounded. Therefore, $(d/dt)\|\zeta_1(t)\|^2$ is bounded too. By Lemma 3, we have that $\|\zeta_1(t)\|^2$ is uniformly continuous. Therefore, noting (25) and using Lemma 4, we obtain

$$\lim_{t\to\infty}\|\zeta_1(t)\| = 0. \tag{26}$$

Now, noting that for any $t > 0$, there exists a positive integer k such that $k\tau - \tau \leq t < k\tau$, and considering (22) we have

$$\zeta_2(t) = (-A_{d22})^k\zeta_2(t-k\tau) - \sum_{i=1}^{k}(-A_{d22})^{i-1}A_{d21}\zeta_1(t-i\tau).$$

This, together with (20) and (26), implies that

$$\lim_{t\to\infty}\|\zeta_2(t)\| = 0.$$

Thus, (Σ_D) is stable. This completes the proof. □

Remark 2: Theorem 1 provides a sufficient condition for the singular delay system (Σ) to be regular, impulse free and stable. When $E = I$, the singular delay system (Σ) reduces to a state-space delay system and it is easy to show that Theorem 1 coincides with [10, Lemma 1]. Therefore, Theorem 1 can be viewed as an extension of existing results on state-space delay systems to singular delay systems. Furthermore, by comparing Theorem 1 with [13, Lemma 2], we can regard Theorem 1 as an extension of existing results on singular systems without delay to singular delay systems.

Following the same philosophy as in dealing with the problems of robust stability and robust stabilization for uncertain singular systems without delay [20], [22], and taking into account Theorem 1, we introduce the following definitions.

Definition 5: The uncertain singular delay system (Σ) is said to be generalized quadratically stable if there exist matrices $Q > 0$ and P such that

$$EP^T = PE^T \geq 0 \tag{27}$$

$$\begin{aligned}&(A+\Delta A)P^T + P(A+\Delta A)^T\\ &\quad + (A_d+\Delta A_d)P^TQ^{-1}P(A_d+\Delta A_d)^T + Q < 0\end{aligned} \tag{28}$$

for all admissible uncertainties ΔA and ΔA_d.

Definition 6: The uncertain singular delay system (Σ) is said to be generalized quadratically stabilizable if there exists a linear state feedback control law $u(t) = Kx(t)$, $K \in \mathbb{R}^{m\times n}$, matrices $Q > 0$ and P such that

$$EP^T = PE^T \geq 0 \tag{29}$$

$$\begin{aligned}&(A_K+\Delta A_K)P^T + P(A_K+\Delta A_K)^T\\ &\quad + (A_d+\Delta A_d)P^TQ^{-1}P(A_d+\Delta A_d)^T + Q < 0\end{aligned} \tag{30}$$

for all admissible uncertainties ΔA, ΔA_d and ΔB, where

$$A_K = A + BK, \quad \Delta A_K = \Delta A + \Delta BK. \tag{31}$$

The following lemma shows that generalized quadratic stability and generalized quadratic stabilization imply robust stability and robust stabilization, respectively.

Lemma 5: Consider the uncertain singular delay system (Σ). If it is generalized quadratically stable, then it is robustly stable. If it is generalized quadratically stabilizable, then it is robustly stabilizable.

Proof: From Theorem 1, the desired results follow immediately. □

In view of this, necessary and sufficient conditions for generalized quadratic stability and generalized quadratic stabilizability for the uncertain singular delay system (Σ) are derived. In order to obtain these results, the following lemma is needed.

Lemma 6 [14]: Given matrices Ω, Γ and Ξ of appropriate dimensions and with Ω symmetrical, then

$$\Omega + \Gamma F(\sigma)\Xi + (\Gamma F(\sigma)\Xi)^T < 0$$

for all $F(\sigma)$ satisfying $F(\sigma)F(\sigma)^T \leq I$, if and only if there exists a scalar $\epsilon > 0$ such that

$$\Omega + \epsilon\Gamma\Gamma^T + \epsilon^{-1}\Xi^T\Xi < 0.$$

For simplicity we introduce the matrix $\Phi \in \mathbb{R}^{n\times(n-r)}$ satisfying $E\Phi = 0$ and $\operatorname{rank}\Phi = n - r$. Now, we are in a position to give the quadratic stability result.

Theorem 2: The uncertain singular delay system (Σ) is generalized quadratically stable if and only if there exist a scalar $\epsilon > 0$, matrices $X > 0$, $Q > 0$ and Y such that the LMI (32) holds, as shown at the bottom of the page.

Proof:

(*Sufficiency*) Assume that there exist a scalar $\epsilon > 0$, matrices $X > 0$, $Q > 0$ and Y satisfying (32). By setting $P = EX + Y\Phi^T$, it is easy to see that

$$EP^T = PE^T \geq 0. \tag{33}$$

$$\begin{bmatrix} A(EX+Y\Phi^T)^T + (EX+Y\Phi^T)A^T + \epsilon MM^T + Q & A_d(EX+Y\Phi^T)^T & (EX+Y\Phi^T)N_A^T \\ (EX+Y\Phi^T)A_d^T & -Q & (EX+Y\Phi^T)N_d^T \\ N_A(EX+Y\Phi^T)^T & N_d(EX+Y\Phi^T)^T & -\epsilon I \end{bmatrix} < 0. \tag{32}$$

Observe that for any $F(\sigma)$ satisfying (4) and any scalar $\epsilon > 0$

$$\begin{aligned}\begin{bmatrix} \Delta A P^T + P\Delta A^T & \Delta A_d P^T \\ P\Delta A_d^T & 0 \end{bmatrix} &= \begin{bmatrix} M \\ 0 \end{bmatrix} F(\sigma) \\ &\quad \times [\, N_A P^T \quad N_d P^T \,] \\ &\quad + \begin{bmatrix} PN_A^T \\ PN_d^T \end{bmatrix} F(\sigma)^T \\ &\quad \times [\, M^T \quad 0 \,] \\ &\leq \begin{bmatrix} \epsilon M M^T & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad + \epsilon^{-1} \begin{bmatrix} PN_A^T \\ PN_d^T \end{bmatrix} \\ &\quad \times [\, N_A P^T \quad N_d P^T \,].\end{aligned}$$

Therefore

$$\begin{aligned}&\begin{bmatrix} (A+\Delta A)P^T + P(A+\Delta A)^T + Q & (A_d + \Delta A_d)P^T \\ P(A_d+\Delta A_d)^T & -Q \end{bmatrix} \\ &\quad \leq \begin{bmatrix} AP^T + PA^T + \epsilon M M^T + Q & A_d P^T \\ PA_d^T & -Q \end{bmatrix} \\ &\qquad + \epsilon^{-1} \begin{bmatrix} PN_A^T \\ PN_d^T \end{bmatrix} [\, N_A P^T \quad N_d P^T \,].\end{aligned}$$

By using a Schur complement argument, it follows from this inequality and (32) that:

$$\begin{bmatrix} (A+\Delta A)P^T + P(A+\Delta A)^T + Q & (A_d + \Delta A_d)P^T \\ P(A_d+\Delta A_d)^T & -Q \end{bmatrix} < 0$$

or, equivalently

$$\begin{aligned}(A+\Delta A)P^T &+ P(A+\Delta A)^T \\ &+ (A_d + \Delta A_d)P^T Q^{-1} P(A_d + \Delta A_d)^T \\ &+ Q < 0.\end{aligned}$$

This inequality and (33) are precisely (27) and (28) in Definition 5. Hence, the uncertain singular delay system (Σ) is generalized quadratically stable.

(*Necessity*) Assume that the uncertain singular delay system (Σ) is generalized quadratically stable. It follows from Definition 5 that there exist matrices $Q > 0$ and P such that (27) and (28) hold. Thus, for all $F(\sigma)$ satisfying (3) and (4), the following inequality holds:

$$\begin{bmatrix} (A+\Delta A)P^T + P(A+\Delta A)^T + Q & (A_d + \Delta A_d)P^T \\ P(A_d+\Delta A_d)^T & -Q \end{bmatrix} < 0$$

which can be rewritten as

$$\begin{aligned}&\begin{bmatrix} AP^T + PA^T + Q & A_d P^T \\ PA_d^T & -Q \end{bmatrix} \\ &\quad + \begin{bmatrix} M \\ 0 \end{bmatrix} F(\sigma) [\, N_A P^T \quad N_d P^T \,] \\ &\quad + \begin{bmatrix} PN_A^T \\ PN_d^T \end{bmatrix} F(\sigma)^T [\, M^T \quad 0 \,] < 0.\end{aligned}$$

By Lemma 6, it follows that there exists a scalar $\epsilon > 0$ such that:

$$\begin{aligned}&\begin{bmatrix} AP^T + PA^T + Q & A_d P^T \\ PA_d^T & -Q \end{bmatrix} + \epsilon \begin{bmatrix} MM^T & 0 \\ 0 & 0 \end{bmatrix} \\ &\qquad + \epsilon^{-1} \begin{bmatrix} PN_A^T \\ PN_d^T \end{bmatrix} [\, N_A P^T \quad N_d P^T \,] < 0.\end{aligned}$$

Invoking again a Schur complement argument, one obtains

$$\begin{bmatrix} AP^T + PA^T + \epsilon MM^T + Q & A_d P^T & PN_A^T \\ PA_d^T & -Q & PN_d^T \\ N_A P^T & N_d P^T & -\epsilon I \end{bmatrix} < 0. \quad (34)$$

From Lemma 2, it can be shown that (34) implies that the pair (E, A) is regular and impulse free. Therefore, it follows from [4] that there exist two invertible matrices U and $V \in \mathbb{R}^{n\times n}$ such that:

$$\bar{E} := UEV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{A} := UAV = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \quad (35)$$

where $I_r \in \mathbb{R}^{r\times r}$ and $I_{n-r} \in \mathbb{R}^{(n-r)\times(n-r)}$ are identity matrices, $A_1 \in \mathbb{R}^{r\times r}$. Let $\bar{P} := UPV^{-T}$, then from the proof of Theorem 1, we have that $\bar{P}$ takes the form

$$\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ 0 & \bar{P}_{22} \end{bmatrix} \quad (36)$$

where $\bar{P}_{11} > 0$, $\bar{P}_{12} \in \mathbb{R}^{r\times(n-r)}$ and $\bar{P}_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$.

On the other hand, from $E\Phi = 0$ and $\operatorname{rank}\Phi = n - r$, we can show that there exists an invertible matrix $\Lambda \in \mathbb{R}^{(n-r)\times(n-r)}$ such that

$$\Phi = V \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} \Lambda. \quad (37)$$

Hence

$$\begin{aligned}P =& U^{-1} \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ 0 & \bar{P}_{22} \end{bmatrix} V^T \\ =& \left(U^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} V^{-1} \right) \left(V \begin{bmatrix} \bar{P}_{11} & 0 \\ 0 & I_{n-r} \end{bmatrix} V^T \right) \\ &+ \left(U^{-1} \begin{bmatrix} \bar{P}_{12} \\ \bar{P}_{22} \end{bmatrix} \Lambda^{-T} \right) \left(\Lambda^T [\, 0 \quad I_{n-r} \,] V^T \right) \\ =& EX + Y\Phi^T\end{aligned}$$

where

$$X = V \begin{bmatrix} \bar{P}_{11} & 0 \\ 0 & I_{n-r} \end{bmatrix} V^T, \quad Y = U^{-1} \begin{bmatrix} \bar{P}_{12} \\ \bar{P}_{22} \end{bmatrix} \Lambda^{-T}.$$

Finally, since $X > 0$ and by replacing P into (34), the desired result follows immediately. □

The generalized quadratic stabilizability result is presented in the following theorem.

Theorem 3: The uncertain singular delay system (Σ) is generalized quadratically stabilizable if and only if there exist a scalar $\epsilon > 0$, matrices $X > 0, Q > 0, Y$ and Z such that the LMI (38) holds, as shown at the bottom of the page, where

$$\begin{aligned}W =& A\Upsilon(X,Y)^T + \Upsilon(X,Y)A^T + BZ \\ &+ Z^T B^T + \epsilon MM^T + Q \\ \Upsilon(X,Y) =& EX + Y\Phi^T\end{aligned}$$

with $\Upsilon(X,Y)$ invertible. In this case, a robustly stabilizing state feedback control law is given by

$$u(k) = Z\Upsilon(X,Y)^{-T} x(t). \quad (39)$$

$$\begin{bmatrix} W & A_d\Upsilon(X,Y)^T & \Upsilon(X,Y)N_A^T + Z^T N_B^T \\ \Upsilon(X,Y)A_d^T & -Q & \Upsilon(X,Y)N_d^T \\ N_A\Upsilon(X,Y)^T + N_B Z & N_d\Upsilon(X,Y)^T & -\epsilon I \end{bmatrix} < 0 \quad (38)$$

Proof: According to Definition 6, the system (Σ) is generalized quadratically stabilizable with respect to the uncertainty structure (3) if and only if there exists $K \in \mathbb{R}^{m\times n}$ such that the resultant closed-loop system

$$E\dot{x}(t) = (A_c + \Delta A_c)x(t) + (A_d + \Delta A_d)x(t-\tau) \qquad (40)$$

with

$$A_c = A + BK, \quad \Delta A_c = \Delta A + \Delta BK$$

is quadratically stable with respect to the uncertainty structure

$$[\,\Delta A_c \quad \Delta A_d\,] = MF(\sigma)\,[\,N_A + N_B K \quad N_d\,].$$

By invoking now Theorem 2 for the closed-loop system (40), one deduces that (Σ) is generalized quadratically stabilizable if and only if there exists $K \in \mathbb{R}^{m\times n}$ and a scalar $\epsilon > 0$, matrices $X > 0$, $Q > 0$ and Y such that the LMI holds, as shown in (41) at the bottom of the page, with

$$\Theta = (A+BK)(EX + Y\Phi^T)^T + (EX+Y\Phi^T)(A+BK)^T + \epsilon MM^T + Q.$$

Define

$$Z = K(EX + Y\Phi^T)^T$$

and observe that the LMI (41) is precisely inequality (38) in the statement of Theorem 3. Hence, necessity is proved.

Now, without loss of generality, we can assume that $\Upsilon(X,Y) = EX + Y\Phi^T$ is invertible, otherwise we can choose a sufficiently small scalar $\theta > 0$ such that $\hat{\Upsilon}(X,Y) = \Upsilon(X,Y) + \theta I$ also satisfies (38) with $\hat{\Upsilon}(X,Y)$ invertible. If (38) holds, then (41) is satisfied for $K = Z\Upsilon(X,Y)^{-T}$. Taking into account the aforementioned considerations, it follows that (Σ) is generalized quadratically stabilizable. This proves sufficiency. □

Remark 3: Theorem 3 presents a necessary and sufficient condition for generalized quadratic stabilizability. The desired robustly stabilizing state feedback for uncertain singular system (Σ) can be obtained by solving the strict LMI (38), which can be solved numerically very efficiently by using interior-point algorithm, and no tuning of parameters is involved [2]. It is worth pointing out that strict LMI (38) is expressed by using the system matrices of (Σ). The design procedure involves no decomposing of the system, which can get around certain numerical problems arising from decomposition of matrices and thus makes the design procedure relatively simple and reliable.

IV. Numerical Example

In this section, we give an example to demonstrate the effectiveness of the proposed method.

Consider the linear uncertain singular delay system (Σ) with parameters as follows:

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1.5 & 0.5 & 1 \\ -1 & 0 & 1 \\ 0.5 & 0 & 1 \end{bmatrix}$$
$$A_d = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0.5 \\ 0.3 & 0.5 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$M = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.1 \end{bmatrix} \quad N_A = [\,0.2 \quad 0.1 \quad 0.3\,]$$
$$N_d = [\,0.1 \quad 0.2 \quad 0.5\,] \quad N_B = [\,0.1 \quad 0.1\,].$$

In this example, we assume that the time delay $\tau = 1.5$ and the uncertain matrix $F(\sigma) = \sin(\sigma)$. The purpose is the design of a state feedback control law such that, for all admissible uncertainties, the resultant closed-loop system is regular, impulse free and stable. To this end, we choose

$$\Phi = [\,-1 \quad 1 \quad 2\,]^T.$$

Using Matlab LMI Control Toolbox to solve the LMI (38), we obtain the solution as follows:

$$X = \begin{bmatrix} 0.2682 & -0.1067 & -0.3102 \\ -0.1067 & 0.2976 & 0.3568 \\ -0.3102 & 0.3568 & 0.6443 \end{bmatrix}$$
$$Q = \begin{bmatrix} 0.9575 & 0.0475 & 0.0475 \\ 0.0475 & 0.9538 & -0.0503 \\ 0.0475 & -0.0503 & 0.9538 \end{bmatrix}$$
$$Y = \begin{bmatrix} 0.2467 \\ -0.1484 \\ -0.2103 \end{bmatrix}$$
$$Z = \begin{bmatrix} -0.9452 & -0.4160 & 0.5859 \\ -0.7454 & 0.5912 & -0.7552 \end{bmatrix} \quad \epsilon = 1.0021.$$

Therefore, by Theorem 3, a robustly stabilizing state feedback control law can be obtained as

$$u(t) = \begin{bmatrix} -13.5354 & 19.4496 & -19.6474 \\ 6.7469 & -15.0227 & 11.8584 \end{bmatrix} x(t).$$

V. Conclusion

The problems of robust stability and stabilization for uncertain continuous singular systems with state delay and parameter uncertainty have been studied. Based on the notions of generalized quadratic stability and generalized quadratic stabilization, these problems have been solved. Necessary and sufficient conditions for generalized quadratic stability and generalized quadratic stabilization are presented in terms of a strict LMI, respectively. The proposed state feedback control law guarantees that the resultant closed-loop system is regular, impulse free as well as stable for all admissible uncertainties.

$$\begin{bmatrix} \Theta & A_d(EX+Y\Phi^T)^T & (EX+Y\Phi^T)(N_A+N_BK)^T \\ (EX+Y\Phi^T)A_d^T & -Q & (EX+Y\Phi^T)N_d^T \\ (N_A+N_BK)(EX+Y\Phi^T)^T & N_d(EX+Y\Phi^T)^T & -\epsilon I \end{bmatrix} < 0 \qquad (41)$$

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Singular LQ Problem for Nonregular Descriptor Systems

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***Abstract*—In this note, a singular linear quadratic (LQ) problem for nonregular descriptor systems is investigated. Under some general conditions, the optimal control and the optimal state of the LQ problem are given. The optimal control is synthesized as state feedback. All the finite eigenvalues of the closed-loop system are located on the left-half complex plane. The state of the closed-loop system has the least free entries.**

***Index Terms*—Nonregular descriptor systems, singular linear quadratic (LQ) problem, state feedback.**

I. INTRODUCTION

Descriptor systems have comprehensive practical background, such as power systems [11], social economic systems [18], circuit systems [21], and so on. Great progress [1], [3], [8] has been made in the theory and its applications since 1970s. Linear quadratic (LQ) optimal control problem (LQ problem) is important in control theory and has been used in practice widely. There have been a lot of excellent results [2], [5], [7], [16], [23] about LQ problem for descriptor systems. In the case of the weighting matrix R in the linear quadratic cost being positive–definite, the theory has matured. Cobb [7] considered the problem with geometric method. Bender and Laub [2] reduced the problem to solving a generalized Riccati equation. Cheng *et al.* [5] transformed the nonsingular LQ problem for descriptor systems into a nonsingular LQ problem for standard state space systems. They gave sufficient conditions for the solvability of the nonsingular LQ problem. However, with R being semidefinite–positive, there was not much work until now. Chen *et al.* [4] discussed the problem for a special kind of descriptor systems based on Weierstrass canonical form and the assumption that the control is sufficiently smooth. Zhu *et al.* [24], who transformed the singular LQ problem for descriptor systems into a nonsingular LQ problem for standard systems, gave a new method of dealing with the problem. In recent years, nonregular descriptor systems were discussed [9], [12], [13], with many open problems unsolved. In [9], Geerts discussed the LQ problem via linear matrix inequalities (LMIs).

In this note, we extend the method used in [24] to nonregular descriptor systems and obtain some new results. Using elementary linear algebra and the equivalence principle for optimal control problem, we derive the relationship between the singular LQ problem for nonregular descriptor systems and the singular or nonsingular LQ problem for standard state space systems. Under some general conditions, the optimal control-state pair is derived. The optimal control is expressed as state feedback. The state of the closed-loop system has the least free entries and all the finite eigenvalues are located on the left half complex plane. The restriction imposed on systems in this note is weaker than that in [2], [5], [7], and [24].

This note is organized as follows. Section II is a statement and transformation of the problem. Section III considers the solution of the problem. Section IV is a brief conclusion.

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