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Robust Stability of Networked Control Systems with Time-varying Network-induced Delays

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Abstract— In this paper, the stability of a Networked Control System (NCS) with time-varying delays is analyzed. A discrete-time state-space model is used to analyze the dynamics of the NCS. The delay is introduced by the network itself and is assumed to be upperbounded by a fraction of the sample-time. A typical motion control example is presented in which the time-variation of the delay results in an unstable system, although for each fixed delay the system is stable. Conditions in terms of LMIs are presented guaranteeing the robust asymptotic stability of the discrete-time system, given bounds on the uncertain time-varying delay. Moreover, it is shown that the robust stability conditions also guarantee asymptotic stability of the intersample behavior. Additionally, LMIs are presented to synthesize a feedback controller that stabilizes the system for the uncertain time-varying delay. The results are illustrated on an example concerning a mechanical model of a motor driving a roller in a printer.

I. INTRODUCTION

A networked control system (NCS) is a control system in which (part of) the control loop is closed over a real-time network. Examples can be found in e.g. DC motors, robots, and automobiles as described in [1]. The use of a communication network that is shared between different devices complicates the analysis and design of an NCS. Standard analysis tools are not applicable due to the non-ideal behavior of the network. Three effects [2], i.e. time-delay, data packet dropouts ([3], [4]) and multiple packets, occur in an NCS. In this paper, we assume that all data arrives and is transmitted in one packet. Hence, our focus is on the effect of delays. The time-delay, consisting of the previously described network delay and the computation time, is assumed to be time-varying, uncertain, and upperbounded by a fraction of the constant sample-time. For this class of NCSs, we investigate the influence of such delays on the stability.

In literature, many modeling approaches for NCSs with delays are given, as well as different methods to assess the stability. One of the first contributions is by Halevi and Ray [5], where a discrete-time representation of an NCS is derived, resulting in a finite-dimensional, time-varying discrete-time model. The model is based on a system with a time-driven controller and sensor and an event-driven actuator. The stability is analyzed for systems with constant and periodic time-delays. A comparable NCS model is given

in [6] and [2], although the assumptions are slightly different, because an event-driven controller is used. The stability analysis is applicable for systems with constant delays only. An extension is presented in [7], where random delays are described. Optimal controllers and state estimators are designed, dependent of the covariance of the delayed signals. Stability results are obtained based on stochastic analysis.

A different modeling approach is used in [3] and [4], where a continuous-time description, with a zero-order-hold controller, is proposed. In [4], they show that their stability conditions and controller design are less conservative than those in [3]. Still, their result is conservative, because it is based on Lyapunov-Krasovskii functionals that give conservative results, in general. Here, our focus is on modeling and analysis of NCSs in discrete-time exploiting stability analysis tools for discrete-time switched systems.

It is well known that a constant delay decreases the performance of a system and can even result in instability [1]. Examples showing the effect of *time-variations* in the delay on the stability are rare. In this paper we show that an NCS, based on a typical motion system, may become unstable for time-varying delays, varying in a bounded set; even when the NCS with any constant delay taken from this set is asymptotically stable. The fact that the time-varying nature may induce instability was also shown in [8]. The stability of NCSs for *time-varying* delays is only investigated recently in literature. In [9], Frequency-domain stability criteria, based on the small gain theorem, are proposed to investigate the stability of single-input-single-output control systems with time-varying delays. Note that, in the current work, we consider state-feedback designs and propose a synthesis approach based on alternative stability criteria. In [10], one studies the stability and stabilization problem of a NCS via approximating the discrete-time NCS description depending on the time-varying delay via a Taylor series. That leads to an uncertain system with polytopic uncertainties and they obtain LMIs for both the analysis and synthesis of a controller for the *approximated* system. The procedure is iterative in the sense that the order of the Taylor series approximation is increased until - if ever - a feasible controller is found for the approximated system. An additional LMI test has to be performed to evaluate if the constructed controller is also stabilizing for the original plant (i.e. including the approximation error). In our approach, we propose a direct convex embedding of the discrete-time NCS description in an uncertain system that leads to an LMI condition without the need for an iterative procedure. Another difference is that, in [10], another controller structure is applied than in this paper.

In this paper no assumptions on the occurrence of the

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delays are made and no online routine to determine the previous delays is implemented. The limitation in the time-delay is that it is upperbounded by a fraction of the sample-time. We adopt the discrete-time NCS model of [6]. For this class of systems, LMI conditions for the robust stability of the NCS are proposed, where the robustness refers to robustness with respect to uncertain time-varying delays taken from a bounded set. Moreover, it is shown that these LMI conditions also ensure stability of the intersample behavior. Based on the same method, LMIs are proposed, with which a robust feedback controller for given bounds on the time-varying delay can be constructed.

The outline of the paper is as follows. In Section II, the NCS model is derived. Section III describes an example from the document printing domain to motivate the stability analysis of NCSs with time-varying delays. In Section IV, a method to guarantee robust asymptotic stability of discrete-time models of NCSs, with time-varying bounded delays, is described. The proposed stability conditions are formulated in terms of LMIs and guarantee stability at the sampling instants. In Section V, we present a model for the intersample behavior and show that the proposed LMIs for robust stability also guarantee the asymptotic stability of the intersample behavior. In Section VI, LMIs are proposed, solving the synthesis problem of a robust feedback controller. In Section VII, the results of Section IV are applied to the printer example. Conclusions and directions for future research are presented in Section VIII.

II. A NCS MODEL

In this paper, a discrete-time description of an NCS, based on [6], is used. The NCS is schematically depicted in Figure 1. It consists of a continuous-time plant and a discrete-time controller, which receives information from the plant only at the sampling instants $t_k = kh$ (with h the sample-time). Additionally, in the model, the computation time and the networked induced delays, i.e. the sensor-to-controller delay τ_{sc} and the controller-to-actuator delay τ_{ca} have to be taken into account. Similar to [6] the sensor acts in a time-driven fashion, and the controller and actuator (including the zero-order-hold (ZOH) in Figure 1) act in an event-driven fashion. Under these assumptions, in combination with a controller that is independent of the time-delays, and the assumption that vacant sampling does not occur ($\tau_{sc} < h$) all delays can be represented by a single delay τ_k, which is taken into account in the discrete-time control signal u_k [7], [8]. The sampling instants t_k are determined by the time-driven sensor output. Moreover, we assume that the total delay τ_k is smaller than the constant sample-time h: $\tau_k < h$. The continuous-time model of the NCS can then be given by:

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu^*(t) \\
u^*(t) &= u_k, \quad \text{for } t \in [kh + \tau_k, (k+1)h + \tau_{k+1})
\end{aligned} \quad (1)
$$

with A and B the continuous-time system and input matrices, $x(t) \in R^f$ the state, $t \in R$ the time, τ_k the delay at sampling moment k, and $u_k \in R$ the delayed discrete-time input. For the sake of simplicity, we assume that we measure the entire state, i.e. $y_k = x_k$, at the sampling instants.
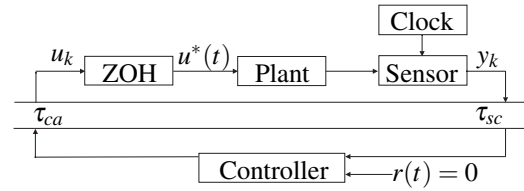

Fig. 1. Schematic overview of the networked control system.

The discretization of (1) on the sampling instants $t_k = kh$ (the sampling moments) gives the NCS model, which is the basis of our analysis:

$$
x_{k+1} = e^{Ah}x_k + \int_0^{h-\tau_k} e^{As}ds Bu_k + \int_{h-\tau_k}^h e^{As}ds Bu_{k-1}. \quad (2)
$$

This equation is only valid at the sampling instants t_k, where the state is given by $x_k := x(t_k)$ and the input by u_k. In this work, we adopt a linear state feedback law and the reference input ($r(t) = 0$ in Figure 1) of the feedback controller is assumed to be zero, which results in the control law $u_k = -Kx_k$. The closed-loop NCS model is then given by:

$$
x_{k+1} = e^{Ah}x_k - \int_0^{h-\tau_k} e^{As}ds BKx_k - \int_{h-\tau_k}^h e^{As}ds BKx_{k-1}. \quad (3)
$$

Now, by defining the state of the closed-loop NCS model by $\xi_k = \begin{pmatrix} x_k^T & x_{k-1}^T \end{pmatrix}^T$, we obtain the following state-space model, given the maximum delay $\tau_{max} \in [0, h]$:

$$
\xi_{k+1} = \tilde{A}(\tau_k)\xi_k, \ \tau_k \in [0, \tau_{max}], \quad (4)
$$

with $\tilde{A}(\tau_k) = \begin{pmatrix} e^{Ah} - \int_0^{h-\tau_k} e^{As}ds BK & -\int_{h-\tau_k}^h e^{As}ds BK \\ I & 0 \end{pmatrix}$, and $\xi_k \in R^n$, with $n = 2f$. Note that in (4) arbitrary time-varying delays, upperbounded by τ_{max}, are accounted for.

Before analyzing the stability of NCSs, we present a motivating example showing the potentially destabilising effect of time-varying delays.

III. A MOTIVATING EXAMPLE

The example is from the document printing domain [11]. In general, a paperpath, consisting of pinches (rollers), driven by motors, is used to move a paper through the printer. Here, the motor controllers share the CPU-time of one processor, which is connected to the motors and sensors via a network resulting in unpredictable time-varying delays in the control loop.

We limit ourselves to one single motor driving one pinch, as depicted in Figure 2. Still, the controller is connected to the motor via the network. In the motor-pinch model, the motor is assumed to behave ideally and slip between the paper and the pinch is neglected, which gives:

$$
\ddot{x}_s = \frac{nr_P}{J_M + n^2 J_P} u, \quad (5)
$$

with $J_M = 1.95 \cdot 10^{-5} \text{kg/m}^2$ the inertia of the motor, $J_P = 6.5 \cdot 10^{-5} \text{ kg/m}^2$ the inertia of the pinch, $r_P = 14$ mm the radius of the pinch, $n = 0.2$ the transmission ratio between motor and pinch, x_s the sheet position and u the motor torque.
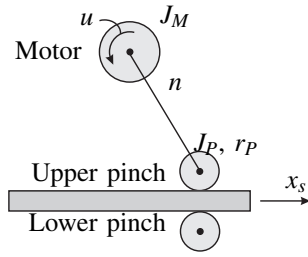
Fig. 2. Schematic overview of the motor-pinch example

The continuous-time state-space representation of (5), where the delays are accounted for in the discrete-time input u_k is given by (1), with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \frac{n r_P}{J_M + n^2 J_P} \end{pmatrix}$, and $x(t) = \begin{pmatrix} x_s(t) & \dot{x}_s(t) \end{pmatrix}^T$. Adopting a feedback controller of the form $u_k = -K x_k$, with $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$, the integrals in $\tilde{A}(\tau_k)$ of (4) can be computed, which yields:

$$\tilde{A}(\tau_k) = \begin{pmatrix} 1 - \frac{1}{2}\alpha^2 K_1 b & h - \frac{1}{2}\alpha^2 K_2 b & \tau_k \beta K_1 b & \tau_k \beta K_2 b \\ -\alpha K_1 b & 1 - \alpha K_2 b & -\tau_k K_1 b & -\tau_k K_2 b \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
(6)

with $b = \frac{n r_P}{J_M + n^2 J_P}$, $\alpha = h - \tau_k$, and $\beta = \frac{1}{2}\tau_k - h$.

If the delay τ_k is constant, the stability of (6) can be determined by checking the eigenvalues of $\tilde{A}(\tau_k)$. Consider this system with a sample-time $h = 1$ ms and two possible constant delays: $\tau^a = 0.2$ ms, and $\tau^b = 0.6$ ms. A linear feedback gain $K = \begin{pmatrix} 50 & 11.8 \end{pmatrix}$ results in a stable system (4), (6) for any constant delay τ in the interval $[0, \tau^b]$, see Figure 3. The eigenvalues of the matrix $\tilde{A}(\tau^a)$ are $\lambda_1 = 0.996$, $\lambda_{2,3} = -0.097 \pm 0.539i$, and $\lambda_4 = 0$. The eigenvalues of $\tilde{A}(\tau^b)$ are $\lambda_1 = 0.996$, $\lambda_{2,3} = 0.203 \pm 0.927i$, and $\lambda_4 = 0$.

However, the system becomes unstable if the delays occur in an alternating sequence ($\tau^a, \tau^b, \tau^a, \tau^b, ...$), as is shown in the lower plot in Figure 3. The stability of this periodic system can be obtained from the eigenvalues of the matrix $\tilde{A}(\tau^b)\tilde{A}(\tau^a)$ [5], which are: $\lambda_1 = 0.992$, $\lambda_2 = -1.012$, $\lambda_3 = 0$, and $\lambda_4 = -0.267$.
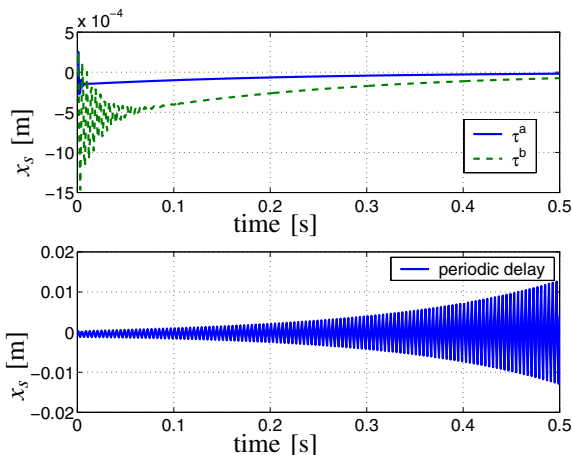

Fig. 3. Response of system (4), (6) for: (upper figure) constant $\tau^a = 2 \cdot 10^{-4}$ s, $\tau^b = 6 \cdot 10^{-4}$ s, and (lower figure) the alternating sequence τ^a, τ^b.

In many practical situations this periodic stability test is too limited. The use of the network results in variations in the time-delay, which are in general not periodic (see e.g. [7]). Therefore, in the next section we propose a stability condition for uncertain time-varying delays.

IV. ROBUST STABILITY OF THE NCS FOR TIME-VARYING UNCERTAIN DELAYS

Consider again system (4). The time-delay τ_k is time-varying, but upperbounded by $\tau_{max} \in [0, h]$. This results in a discrete-time switching system, on the sample instants t_k, due to the dependence of the matrix $\tilde{A}(\tau_k)$ on τ_k. A sufficient condition for stability is that for all matrices $\tilde{A}(\tau_k)$, $\tau_k \in [0, \tau_{max}]$ a common quadratic Lyapunov function exists, i.e. that the following LMIs are feasible:

$$\begin{aligned} P = P^T &> 0 \\ \tilde{A}^T(\tau_k)P\tilde{A}(\tau_k) - P &< 0, \ \forall \tau_k \in [0, \tau_{max}]. \end{aligned}$$
(7)

According to (7), an infinite number of LMIs needs to be checked, because τ_k can take infinitely many distinct values in the interval $[0, \tau_{max}]$. In Theorem 4.1, we will propose a result that uses a finite number of LMIs to guarantee robust asymptotic stability for time-varying delays.

Theorem 4.1: Consider system (4), with the delay-dependent matrix $\tilde{A}(\tau_k)$, $\tau_k \in [0, \tau_{max}]$, and $\tau_{max} \in [0, h]$. Define the set of matrices $\bar{\mathscr{A}}$ by:

$$\bar{\mathscr{A}} \triangleq \{\bar{A} \in R^{n \times n} : \bar{a}_{ij} = q_{ij} \text{ or } \bar{a}_{ij} = r_{ij}, \ i, j = 1, 2, ..., n\}, \ (8)$$

with $\bar{a}_{ij}$ the $(i,j)^{th}$ element of $\bar{A}$, $q_{ij} = \min_{\tau \in [0, \tau_{max}]} \tilde{a}_{ij}(\tau)$, and $r_{ij} = \max_{\tau \in [0, \tau_{max}]} \tilde{a}_{ij}(\tau)$ the minimum and maximum value of the $(i,j)^{th}$ element $\tilde{a}_{ij}(\tau)$ of $\tilde{A}(\tau)$, respectively.

If there exists a solution to the discrete-time Lyapunov matrix inequalities:

$$\begin{aligned} P = P^T &> 0 \\ \bar{A}^T P \bar{A} - P &< 0, \ \forall \bar{A} \in \bar{\mathscr{A}}, \end{aligned}$$
(9)

then system (4) is robustly globally asymptotically stable for any sequence of delays τ_k taking values in $[0, \tau_{max}]$.

Proof: Note that the set $\tilde{\mathscr{A}} := \{\tilde{A}(\tau) | \tau \in [0, \tau_{max}]\}$ satisfies $\tilde{\mathscr{A}} \subset \mathscr{A}$ with

$$\mathscr{A} := \bar{\text{co}}(\bar{\mathscr{A}}) = \{\hat{A} = (\hat{a}_{ij}) : q_{ij} \leq \hat{a}_{ij} \leq r_{ij}; \ i, j = 1, 2, ..., n\},$$
(10)

with "$\bar{\text{co}}$" denoting the convex hull. Therefore, the set of interval matrices $\mathscr{A}$ is a convex overestimation of the set of matrices $\tilde{\mathscr{A}}$. Due to the fact that $\mathscr{A} = \bar{\text{co}}(\bar{\mathscr{A}})$, we can rewrite every matrix $\hat{A} \in \mathscr{A}$ as a function of the matrices $\bar{A} \in \bar{\mathscr{A}}$:

$$\hat{A} = \sum_{i=1}^{L} \delta_i \bar{A}_i, \text{ with } \delta_i \geq 0, \sum_{i=1}^{L} \delta_i = 1,$$
(11)

with L the size of the set $\bar{\mathscr{A}}$.

We will prove that, under condition (9), $V(\xi) = \xi^T P \xi$ is a common quadratic Lyapunov function for the system

$$\xi_{k+1} = \hat{A}_k \xi_k, \text{ for } \hat{A}_k \in \bar{\text{co}}(\bar{\mathscr{A}}),$$
(12)

which implies GAS of (4) since $\tilde{\mathscr{A}} \subset \bar{\text{co}}(\bar{\mathscr{A}})$. $V(\xi)$ is a common quadratic Lyapunov function for system (12) if the following LMIs are feasible:

$$\begin{aligned} P = P^T &> 0 \\ (\sum_{i=1}^{L} (\delta_i \bar{A}_i)^T) P (\sum_{i=1}^{L} \delta_i \bar{A}_i) - P &< 0. \end{aligned}$$

Applying Schur's complement gives:

$$\begin{pmatrix} P & \sum_{i=1}^{L}(\delta_i \bar{A}_i)^T P \\ \sum_{i=1}^{L}(P\delta_i \bar{A}_i) & P \end{pmatrix} = \sum_{i=1}^{L} \delta_i \begin{pmatrix} P & \bar{A}_i^T P \\ P\bar{A}_i & P \end{pmatrix} > 0. \tag{13}$$

By applying Schur's complement to the inequality in (13), combined with the fact that every $\delta_i > 0$, $i = 1, \ldots, L$, we can show that (9) implies (13). ∎

The number of LMIs we need to test for our stability condition in (9), depends on the number of matrices in $\bar{\mathscr{A}}$, as defined in (8). In general, this number of matrices L is equal to 2^m, with $m = n^2$, and n the dimension of $\tilde{A}(\tau_k)$. Due to the specific form of $\tilde{A}(\tau_k)$ in (4), where the lower half of the matrix $\tilde{A}$ is independent of τ_k, the size of $\bar{\mathscr{A}}$ is equal to $2^{\frac{m}{2}}$.

Note that the use of the overestimation $\mathscr{A} = \bar{\text{co}}\{\bar{\mathscr{A}}\}$ for $\bar{\mathscr{A}}$ results in a conservative stability criterion for (4), because specific knowledge on the dependence of the different matrix entries of $\tilde{A}(\tau_k)$ on τ_k is lost. A tighter approximation of $\tilde{A}(\tau_k)$, to derive less conservative stability conditions, is a topic of future research.

V. INTERSAMPLE BEHAVIOR

In Theorem 4.1, we have provided sufficient conditions for asymptotic stability at the sampling instances kh, $k \in \{1, 2, \ldots\}$, but the behavior of the continuous-time system (1) between the sample-times remains unknown. In this section, we will show that the intersample behavior is asymptotically stable as well.

Consider the continuous-time system (1). To study the intersample behavior an additional variable $\tilde{t} = t - kh$, $t \in [kh, kh + h]$, is introduced for which holds $\tilde{t} \in [0, h]$. To determine the time-evolution of the state of the continuous-time system for $t \in [kh, kh + h]$, the well-known convolution integral has to be solved. Two different cases can be distinguished, due to the uncertainty of the value of the delay $\tau_k \in [0, h]$, namely $\tau_k > \tilde{t}$ and $\tau_k \leq \tilde{t}$. For $\tau_k > \tilde{t}$ and $\tau_k \leq \tilde{t}$ the time-evolutions of the state are, respectively, given by:

$$x(kh + \tilde{t}) = e^{A\tilde{t}}x(kh) - \int_0^{\tilde{t}} e^{As} ds BK x(kh - h) \tag{14}$$

and

$$x(kh + \tilde{t}) = \left(e^{A\tilde{t}} - \int_0^{\tilde{t} - \tau_k} e^{As} ds BK \right) x(kh) - \int_{\tilde{t} - \tau_k}^{\tilde{t}} e^{As} ds BK x(kh - h). \tag{15}$$

For both cases, an upper bound for $\|x(kh + \tilde{t})\|$ can be derived, as is stated in the following lemma.

Lemma 5.1: Consider the continuous-time system (1) and the continuous-time state evolutions (14) and (15), and the discrete-time system (3). Then the norms of the states of the continuous-time system (1) are linearly related to the norms of the states of the discrete-time system (3), according to the following relations: if $\lambda_{max} \neq 0$:

$$\|x(kh + \tilde{t})\| \leq \max\left\{ e^{\lambda_{max}h}, 1 \right\} \|x(kh)\| + \frac{1}{\lambda_{max}} \left(e^{\lambda_{max}h} - 1 \right) \|BK\| \left(\|x(kh)\| + \|x(kh - h)\| \right), \tag{16}$$

and if $\lambda_{max} = 0$:

$$\|x(kh + \tilde{t})\| \leq (1 + h\|BK\|) \|x(kh)\| + h\|BK\| \|x(kh - h)\|, \tag{17}$$

for all $\tilde{t} \in [0, h]$ and $\lambda_{max} = \frac{1}{2} \max(\text{eig}(A + A^T))$.

The proof of this Lemma is given in the Appendix. Next, Theorem 5.2 shows that the conditions in Theorem 4.1 under which the discrete-time system is asymptotically stable, imply asymptotic stability of the intersample behavior.

Theorem 5.2: If system (4) satisfies the LMI conditions in (9), then the continuous-time system (1) is asymptotically stable.

Proof: Lemma 5.1 shows that the intersample behavior is bounded, given boundedness of the states of the discrete-time system (4). The Lyapunov-based stability argument in Theorem 4.1 implies such boundedness of the states of system (4). Moreover, if Theorem 4.1 is satisfied, then the states of the discrete-time system converge to zero as $k \to \infty$, which, according to Lemma 5.1, results in convergence to zero of the evolution of the states of the continuous-time system.

Finally, condition (9) in Theorem 4.1 guarantees that $\|x_{k+1}\|_P \leq \gamma \|x_k\|_P$, with $\gamma < 1$. Lemma 5.1 can be rewritten in terms of the P-norms of the states. The fact that the discrete-time P-norms of the states at the sampling instants are decreasing, combined with the adapted form of Lemma 5.1 implies that the intersample behavior is asymptotically stable. ∎

VI. ROBUST CONTROLLER SYNTHESIS FOR THE NCS WITH TIME-VARYING UNCERTAIN DELAYS

In section IV, we analyzed the robust stability of the NCS system (4), given the feedback gain K. In this section we will, based on this analysis, obtain results for the synthesis problem of a robust feedback controller. To do so, we have to rewrite $\tilde{A}(\tau)$ in (4) in a suitable form for controller synthesis:

$$\tilde{A}(\tau) = A_x + \tilde{B}(\tau) \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} = A_x + \tilde{B}(\tau) K_{tot}, \tag{18}$$

with $A_x = \begin{pmatrix} e^{Ah} & 0 \\ I & 0 \end{pmatrix}$, and $\tilde{B}(\tau) = \begin{pmatrix} -\int_0^{h-\tau} e^{As} B ds & -\int_{h-\tau}^{h} e^{As} B ds \\ 0 & 0 \end{pmatrix}$. We define, analogously to $\bar{\mathscr{A}}$ in (8), the set of matrices $\bar{\mathscr{B}} = \{\bar{B}_1, \ldots, \bar{B}_G\}$ such that:

$$\{\tilde{B}(\tau) | \tau \in [0, \tau_{max}]\} \subset \mathscr{B} = \bar{\text{co}}\{\bar{\mathscr{B}}\}, \tag{19}$$

with $G = 2^n$ the number of matrices in $\bar{\mathscr{B}}$, due to the form of $\tilde{B}(\tau)$ in (18) and $\mathscr{B}$ defined analogue to (10). The set of matrices $\bar{\mathscr{B}}$ is thus defined as:

$$\bar{\mathscr{B}} \triangleq \{\bar{B} \in R^{n \times 2} : \bar{b}_{ij} = s_{ij} \text{ or } \bar{b}_{ij} = t_{ij}, i = 1, 2, \ldots, n, \ j = 1, 2\}, \tag{20}$$

with $\bar{b}_{ij}$ the $(i, j)^{th}$ element of $\bar{\mathscr{B}}$ and $s_{ij} = \min_{\tau \in [0, \tau_{max}]} \tilde{b}_{ij}(\tau)$ and $t_{ij} = \max_{\tau \in [0, \tau_{max}]} \tilde{b}_{ij}(\tau)$ the minimum and maximum value of the $(i, j)^{th}$ element $\tilde{b}_{ij}(\tau)$ of $\tilde{B}(\tau)$ for values of $\tau \in [0, \tau_{max}]$, respectively.

Theorem 6.1: If there exists $Z = \begin{pmatrix} \bar{Z} & 0 \\ 0 & \bar{Z} \end{pmatrix}$, and $Y = \begin{pmatrix} \bar{Y} & 0 \\ 0 & \bar{Y} \end{pmatrix}$, with $Y = Y^T > 0$ such that it holds that:

$$\begin{pmatrix} Y & YA_x^T + Z^T \bar{B}_i^T \\ A_x Y + \bar{B}_i Z & Y \end{pmatrix} > 0, \forall \bar{B}_i \in \bar{\mathscr{B}}, \tag{21}$$

then $K = \bar{Z}\bar{Y}^{-1}$ and $P = Y^{-1}$ give the feedback gain K ($K_{tot} = ZY^{-1}$) and the Lyapunov function $V(\xi) = \xi^T P\xi$, and the discrete-time system (4), with (18), and the continuous-time system (1) both are globally asymptotically stable.

Proof: We will show that LMI (21) is a sufficient condition for the GAS of system (4), (18). Since (21) holds for all $\bar{B}_i \in \mathcal{B}$, we have that for all $\mu_1, ...\mu_G \geq 0$, with $\sum_{i=1}^{G} \mu_i = 1$:

$$
\begin{aligned}
0 &< \sum_{i=1}^{G} \mu_i \begin{pmatrix} Y & YA_x^T + Z^T \bar{B}_i^T \\ A_xY + \bar{B}_i Z & Y \end{pmatrix} \\
&= \begin{pmatrix} Y & YA_x^T + Z^T (\sum_{i=1}^{G} \mu_i \bar{B}_i)^T \\ A_xY + (\sum_{i=1}^{G} \mu_i \bar{B}_i)Z & Y \end{pmatrix}.
\end{aligned}
$$

Hence, analogously to (11), we obtain:

$$
\begin{pmatrix} Y & YA_x^T + Z^T \hat{B}^T \\ A_xY + \hat{B}Z & Y \end{pmatrix} > 0, \forall \hat{B} \in \mathcal{B}. \quad (22)
$$

Pre- and postmultiplying (22) by $\begin{pmatrix} Y^{-1} & 0 \\ 0 & Y^{-1} \end{pmatrix}$ gives:

$$
\begin{pmatrix} Y^{-1} & A_x^T Y^{-1} + Y^{-1}Z^T \hat{B}^T Y^{-1} \\ Y^{-1}A_x + Y^{-1}\hat{B}ZY^{-1} & Y^{-1} \end{pmatrix} > 0,
$$

for all $\hat{B} \in \mathcal{B}$. Substituting $Y^{-1} = P$, and $ZY^{-1} = K_{tot}$ yields:

$$
\begin{pmatrix} P & A_x^T P + K_{tot}^T \hat{B}^T P \\ PA_x + P\hat{B}K_{tot} & P \end{pmatrix} > 0, \forall \hat{B} \in \mathcal{B}.
$$

Hence, using Schur complements, we obtain:

$$
\begin{aligned}
P = P^T &> 0 \\
P - (A_x + \hat{B}K_{tot})^T P (A_x + \hat{B}K_{tot}) &> 0, \forall \hat{B} \in \mathcal{B}.
\end{aligned} \quad (23)
$$

Using (19), it is obvious that (23) is a sufficient condition for:

$$
\begin{aligned}
P = P^T &> 0 \\
P - (A_x + \tilde{B}(\tau)K_{tot})^T P (A_x + \tilde{B}(\tau)K_{tot}) &> 0, \forall \tau \in [0, \tau_{max}].
\end{aligned}
$$

Hence, $V(\xi) = \xi^T P\xi$ is a common quadratic Lyapunov function for (4), (18) and proves GAS on the sample instants kh. Using Theorem 5.2, we can also prove asymptotic stability of system (1), including the intersample behavior. ∎

Remark As already mentioned, the "structured state feedback synthesis" problem is known to be notoriously difficult. To obtain a convex problem we introduced some additional conservatism. However, if instead of state feedback $u_k = -Kx_k$, we would use a different control structure of the form $u_k = K_1 x_k + K_2 u_{k-1}$ as proposed in [10], the "stability" LMIs as in (9) could be transformed into "synthesis" LMIs without introducing additional conservatism. The reason for this fact is that the system description (4) would be based on the state variable $(x_k^T, u_{k-1}^T)^T$ and $K_{tot} = [K_1 \ K_2]$ in (18) would be unstructured.

VII. EXAMPLE

In this section, Theorem 4.1 is applied to system (4) with (6). For $u_k = \begin{pmatrix} 50 & K_2 \end{pmatrix} x_k$ and a given τ_{max}, we will determine the values of K_2 for which robust global asymptotic stability is guaranteed. Under the assumption that $\tau_k \in [0, \tau_{max}]$, with $\tau_{max} \in [0, h]$, the upper and lower bounds of the elements of
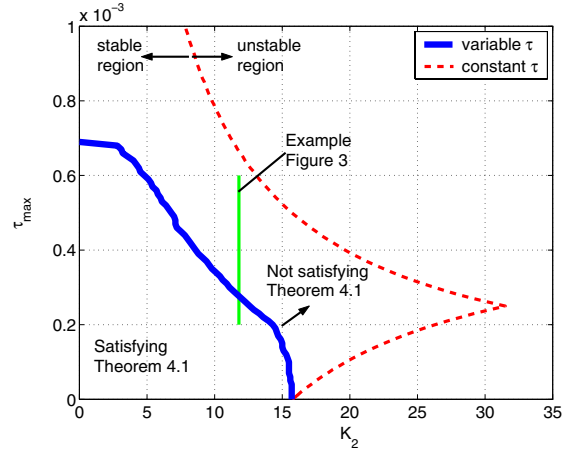

Fig. 4. Robust stability bound for time-varying delays, with $K_1 = 50$.

$\tilde{A}(\tau_k)$ in (6) can be determined. This leads to the following matrices $Q = (q_{ij})$ and $R = (r_{ij})$, as defined in (8):

$$
Q = \begin{pmatrix} 1 - \frac{1}{2}h^2 K_1 b & h(1 - \frac{1}{2}h)K_2 b & \tau_{max}\hat{\beta}K_1 b & \tau_{max}\hat{\beta}K_2 b \\ -hK_1 b & 1 - hK_2 b & -\tau_{max}K_1 b & -\tau_{max}K_2 b \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
$$

$$
R = \begin{pmatrix} 1 - \frac{1}{2}\hat{\alpha}^2 K_1 b & h - \frac{1}{2}\hat{\alpha}^2 K_2 b & 0 & 0 \\ -\hat{\alpha}K_1 b & 1 - \hat{\alpha}K_2 b & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
$$

with $b = \frac{nr_P}{J_M + n^2 J_P}$, $\hat{\alpha} = h - \tau_{max}$, and $\hat{\beta} = \frac{1}{2}\tau_{max} - h$. The set of matrices $\mathcal{A}$ in (8) consists of 2^8 matrices for our example.

The maximum values of K_2 that still guarantee robust asymptotic stability for time-delay variations $\tau_k \in [0, \tau_{max}]$, $\tau_{max} \in [0, h]$ are given in Figure 4. For maximum delays larger than 0.68 ms (68 % of the sample-time) no values of K_2 satisfying Theorem 4.1 could be found. For such large τ_{max} (and $K_1 = 50$), robust stability can not be guaranteed anymore, on the basis of Theorem 4.1.

Additionally, a dashed line is given in Figure 4, representing the maximum allowable value of K_2 that ensures stability for the constant delay τ_{max}. As expected, the controller, used in Section III, is located in the area not satisfying (9), but also in the stable region for constant delays. In the no-delay case ($\tau_{max} = 0$), Theorem 4.1 gives the same result as the constant delay case. For delays for which $\tau_{max} \leq 0.68h$, the stability conditions in (9) do not seem overly conservative. For larger delays we can not give a precise answer on the conservativeness of our results. Reducing conservatism is one of the topics for future research.

VIII. CONCLUSIONS AND FUTURE WORK

In this paper, we studied the robust stability of networked control systems (NCS) with uncertain, time-varying, bounded time-delays. A typical motion control example is presented where time-variations in the delay, varying in a bounded set, result in instability, although the controller is chosen such that it stabilizes the system for all constant

values of the delay in this bounded set. Sufficient conditions for the robust global asymptotic stability (GAS) of the discrete-time NCS, with uncertain bounded time-varying delays, are proposed in terms of LMIs. Herein, the robustness refers to the fact that the conditions guarantee GAS of the NCS for any time-varying uncertain delay satisfying these bounds. Moreover, it is shown that the proposed LMIs also guarantee the stability of the intersample behavior. Based on the analysis for robust stability, the synthesis problem of robust controller design is solved. LMIs are presented that characterise controllers that induce stability for a given interval of the time-varying delay.

Future research deals with the reduction of the number of LMIs, in combination with a reduction of the overapproximation, based on knowledge of the dependence between the different matrix entries of $\tilde{A}$ in (4). Additionally, to reduce the conservatism caused by the use of a *common* quadratic Lyapunov function, methods, such as in [13], where LMI conditions for stability of convex bounded polytopes based on a parameter-dependent Lyapunov function are presented, can be used.

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APPENDIX: PROOF OF LEMMA 5.1

First the case $\tau_k \leq \tilde{t}$ is considered. Then, the norm of (15) satisfies:

$$\|x(kh+\tilde{t})\| \leq \|e^{A\tilde{t}}x(kh)\| + \|\int_0^{\tilde{t}-\tau_k} e^{As}ds BKx(kh)\| + \|\int_{\tilde{t}-\tau_k}^{\tilde{t}} e^{As}ds BKx(kh-h)\|. \tag{24}$$

The first term in the right-hand side can be rewritten using Wazewski's inequalities [14], and [15]: $\|e^{A\tilde{t}}x(kh)\| \leq \|x(kh)\|e^{\lambda_{max}\tilde{t}}$, with $\lambda_{max} = \frac{1}{2}\max\left(\text{eig}(A+A^T)\right)$. Under the assumption that $\tilde{t} \in [0,h]$ and $\tau_k \in [0,\tilde{t}]$, it holds that:

$$\|e^{A\tilde{t}}x(kh)\| \leq \|x(kh)\|\max\left\{e^{\lambda_{max}h},1\right\}. \tag{25}$$

To rewrite the second term of the right-hand side of (24), the integrals are rewritten: $\|\int_0^{\tilde{t}-\tau_k}\left(e^{As}\right)BKx(kh)ds\| \leq \|BK\|\|x(kh)\|\int_0^{\tilde{t}-\tau_k}\left(e^{\lambda_{max}s}\right)ds$. This integral can be solved exactly, because λ_{max} is a real number, yielding:

$$\|\int_0^{\tilde{t}-\tau_k}\left(e^{As}\right)BKx(kh)ds\| \leq$$
$$\begin{cases} \|BK\|\|x(kh)\|\frac{e^{\lambda_{max}h}-1}{\lambda_{max}} & \text{if } \lambda_{max} \neq 0 \\ \|BK\|\|x(kh)\|h & \text{if } \lambda_{max} = 0. \end{cases} \tag{26}$$

In a similar fashion, it holds that:

$$\|\int_{\tilde{t}-\tau_k}^{\tilde{t}}\left(e^{As}BKx(kh-h)\right)ds\| \leq$$
$$\begin{cases} \|BK\|\|x(kh-h)\|\frac{e^{\lambda_{max}h}-1}{\lambda_{max}} & \text{if } \lambda_{max} \neq 0 \\ \|BK\|\|x(kh-h)\|h & \text{if } \lambda_{max} = 0. \end{cases} \tag{27}$$

Substituting (25), (26), and (27) in (24), leads to the result presented by (16) and (17) in Lemma 5.1 for $\tau_k \leq \tilde{t}$.

Next the second case ($\tau_k > \tilde{t}$) has to be studied. The norm of $\|x(kh+\tilde{t})\|$ in (14) is bounded as:

$$\|x(kh+\tilde{t})\| \leq \|e^{A\tilde{t}}x(kh)\| + \|\int_0^{\tilde{t}}\left(e^{As}\right)BKx(kh-h)ds\|. \tag{28}$$

The first part of the right-hand side of (28) is exactly equal to the first part of (24). The corresponding upper bound is given by (25). The upper bound of the second part of the right-hand side of (28) can be derived analogously to (26):

$$\|\int_0^{\tilde{t}}\left(e^{As}\right)BKx(kh-h)ds\| \leq$$
$$\begin{cases} \|BK\|\|x(kh-h)\|\frac{e^{\lambda_{max}h}-1}{\lambda_{max}} & \text{if } \lambda_{max} \neq 0 \\ \|BK\|\|x(kh-h)\|h & \text{if } \lambda_{max} = 0. \end{cases} \tag{29}$$

For the assumption $\tau_k > \tilde{t}$ two equations can hold, dependent of the value of λ_{max}: if $\lambda_{max} \neq 0$:

$$\|x(kh+\tilde{t})\| \leq \|x(kh)\|\max\left\{e^{\lambda_{max}h},1\right\} + \|BK\|\|x(kh-h)\|\frac{e^{\lambda_{max}h}-1}{\lambda_{max}}, \tag{30}$$

if $\lambda_{max} = 0$:

$$\|x(kh+\tilde{t})\| \leq \|x(kh)\|\max\left\{e^{\lambda_{max}h},1\right\} + \|BK\|\|x(kh-h)\|h. \tag{31}$$

Since the right-hand side of (30) and (31) are upperbounded by the right-hand side of (16) and (17), respectively, the result follows.