Int. J. Appl. Math. Comput. Sci., 2010, Vol. 20, No. 4, 665–670
DOI: 10.2478/v10006-010-0049-8

ROBUST STABILITY OF POSITIVE CONTINUOUS–TIME LINEAR SYSTEMS WITH DELAYS

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The paper is devoted to the problem of robust stability of positive continuous-time linear systems with delays with structured perturbations of state matrices. Simple necessary and sufficient conditions for robust stability in the general case and in the case of systems with a linear uncertainty structure in two sub-cases: (i) a unity rank uncertainty structure and (ii) non-negative perturbation matrices are established. The problems are illustrated with numerical examples.

Keywords: positive continuous-time linear system, delays, robust stability, linear uncertainty, interval system.

1. Introduction

A dynamic system is called positive if any trajectory of the system starting from non-negative initial states remains forever non-negative for non-negative controls. An overview of the state of the art in positive systems theory is given by Farina and Rinaldi (2000) as well as Kaczorek (2002).

The stability and robust stability problems of standard (i.e., non-positive) time-delay systems have been dealt with in many papers and books (Górecki *et al.,* 1993; Busłowicz, 2000; Niculescu, 2001; Gu *et al.,* 2003; Gu and Niculescu, 2006, Wu *et al.,* 2004).

Simple necessary and sufficient conditions for asymptotic stability of positive discrete-time linear systems with delays were given by Hmamed *et al*. (2007) for systems with one delay and by Busłowicz (2008a) for systems with multiple delays. Conditions for robust stability of positive discrete-time systems with one delay with structured perturbations of state matrices were given in the work of Busłowicz (2008b).

Recently, it has been shown that the checking of asymptotic stability of positive continuous-time linear systems with one delay (Rami *et al*., 2007) and with multiple non-commensurate delays (Kaczorek, 2009) can be reduced to checking the asymptotic stability of the corresponding positive systems without delays, similarly as in the case of positive discrete-time systems.

The main purpose of the paper is to formulate new simple necessary and sufficient conditions for robust stability of positive continuous-time linear systems with delays with structured perturbations of state matrices in a general case and in the case of a linear uncertainty structure in two sub-cases: (i) a unity rank uncertainty structure and (ii) non-negative perturbation matrices. To the best of the author's knowledge, the problem of the robust stability of positive continuous-time linear systems with delays has not been studied yet.

The following notation will be used: M_n, the set of $n\times n$ real Metzler matrices; $[A^-,\ A^+]$, an interval matrix; $\mathbb{R}_+^{n\times m}$, the set of real $n\times m$ matrices with non-negative entries and $\mathbb{R}_+^n=\mathbb{R}_+^{n\times 1}$; I_n, the $n\times n$ identity matrix. A real $n\times n$ matrix $A=[a_{ij}]$ is called a Metzler matrix if $a_{ij}\geq 0$ for $i\neq j$.

2. Problem formulation

Consider the positive continuous-time linear system with delays

$$\dot{x}(t)=A_0(q)x(t)+\sum_{k=1}^{p}A_k(q)x(t-h_k)+Bu(t),\qquad q\in Q,\quad (1)$$

where $x(t)\in\mathbb{R}^n$ and $u(t)\in\mathbb{R}^{n_u}$ are respectively the state and input vectors, $A_k(q)$, $q\in Q$ $(k=0,1,\dots,p)$ and B are real matrices of appropriate dimensions, $h_k>0$ $(k=1,2,\dots,p)$ is a delay, $q=[q_1,q_2,\dots,q_m]^{\mathrm{T}}$ is the vector of uncertain physical parameters and

$$Q = \{q : q_r \in [q_r^-,\ q_r^+],\ \ r = 1, 2, \dots, m\} \qquad (2)$$

with $q_r^- \le 0$, $q_r^+ \ge 0$ $(r = 1, 2, \dots, m)$ is a value set of uncertain parameters.

The initial condition for the system (1) has the form $x(\tau) = \varphi(\tau)$ for $\tau \in [-h,\ 0]$, $h = \max_k h_k$, where $\varphi(\tau)$ is a given vector function.

By the generalization of the positivity condition of a continuous-time linear system with delays without uncertain parameters (Kaczorek, 2009) to the case with uncertain parameters, we obtain the following definition and theorem.

Definition 1. The system (1) will be called (internally) *positive* if for any $q \in Q$ the following condition holds: $x(t) \in \mathbb{R}_+^n$ $(t > 0)$ for any $x(\tau) = \varphi(\tau) \in \mathbb{R}_+^n$ $(\tau \in [-h,\ 0])$ and for all inputs $u(t) \in \mathbb{R}_+^{n_u}$, $t \ge 0$.

Theorem 1. *The system (1) is positive if and only if* $B \in \mathbb{R}_+^{n\times n_u}$ *and*

$$A_0(q) \in M_n, \quad A_k(q) \in \mathbb{R}_+^{n\times n}, \quad \forall q \in Q, \quad k = 1, 2, \dots, p. \qquad (3)$$

We will assume that all the entries of matrices $A_k(q)$, $k = 0, 1, \dots, p$ are continuous functions of uncertain parameters, non-linear or linear.

In the case of the positive system (1) with a linear uncertainty structure, we may write

$$A_k(q) = A_{k0} + \sum_{r=1}^{m} q_r E_{kr}, \quad k = 0, 1, \dots, p, \qquad (4)$$

where $A_{00} \in M_n$, $A_{k0} \in \mathbb{R}_+^{n\times n}$ $(k = 1, 2, \dots, p)$ and $E_{kr} \in \mathbb{R}^{n\times n}$ $(k = 0, 1, \dots, p$ and $r = 1, 2, \dots, m)$ are the nominal and the perturbation matrices, respectively, such that the condition (3) holds.

Definition 2. The system (1) will be called a *system with a linear unity rank uncertainty structure* if

$$\operatorname{rank} \sum_{k=0}^{p} E_{kr} = 1, \quad r = 1, 2, \dots, m. \qquad (5)$$

The system (1) has a linear uncertainty structure with non-negative perturbation matrices if

$$E_{kr} \in \mathbb{R}_+^{n\times n}, \quad k = 0, 1, \dots, p, \quad r = 1, 2, \dots, m. \qquad (6)$$

The positive system (1) is robustly stable if and only if all the roots $s_i(q)$ $(i = 1, 2, \dots)$ of the characteristic transcendental equation

$$\det\left[sI_n - A_0(q) - \sum_{k=1}^{p} A_k(q) e^{-sh_k}\right] = 0 \qquad (7)$$

satisfy the condition

$$\operatorname{Re} s_i(q) < 0, \quad \forall q \in Q \quad (i = 1, 2, \dots).$$

The aim of this paper is to provide simple necessary and sufficient conditions for the robust stability of linear positive continuous-time systems with delays (1) in the general case and in the case of systems with a linear uncertainty structure in two sub-cases: (i) a unity rank uncertainty structure (the condition (5) holds) and (ii) non-negative perturbation matrices (the condition (6) holds, satisfying (5) is not necessary).

First, we show that the robust stability of the continuous-time positive system (1) is equivalent to that of the corresponding continuous-time positive system without delays of the same order as (1). Next, we give simple conditions for robust stability.

3. Robust stability of linear positive continuous-time systems with delays

The following theorems and lemma have been proved by Kaczorek (2009).

Theorem 2. *The positive continuous-time linear system with delays*

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{p} A_k x(t - h_k), \qquad (8)$$

in which $A_0 \in M_n$ *and* $A_k \in \mathbb{R}_+^{n\times n}$ $(k = 1, 2, \dots, p)$*, is asymptotically stable (independent of delays) if and only if the positive system without delays*

$$\dot{x}(t) = Ax(t), \quad A = \sum_{k=0}^{p} A_k \in M_n, \qquad (9)$$

is asymptotically stable.

Theorem 3. *The positive system (8) with delays is asymptotically stable (independent of delays) if and only if one of the following equivalent conditions holds:*

1. *the eigenvalues* $s_1, s_2, \dots, s_n$ *of the matrix* A *defined in (9) have negative real parts,*
2. *all the leading principal minors of the matrix* $-A$ *are positive,*
3. *all the coefficients of the characteristic polynomial of the matrix* A *are positive.*

Lemma 1. *The positive continuous-time system (8) with delays is unstable if at least one diagonal entry of the matrix* A_0 *is positive.*

By comparing the asymptotic stability conditions of standard (i.e., non-positive) continuous-time linear systems with delays and those given in Theorems 2 and 3 for positive systems, we obtain the following important remark.

Remark 1. The positive continuous-time linear system with delays (8) (standard or positive) is asymptotically stable if and only if

$$\det(sI_n - A_0 - \sum_{k=1}^{p} A_k e^{-sh_k}) \neq 0, \quad \forall \operatorname{Re} s \geq 0.$$

In the case of the positive system (1) (with $A_0 \in M_n$ and $A_k \in \mathbb{R}_+^{n\times n}$, $k = 1, 2, \ldots, p$), the above condition is equivalent to

$$\det(sI_n - A) \neq 0, \quad \forall \operatorname{Re} s \geq 0,$$

where the matrix A is defined in (9).

3.1. Robust stability in the general case.

By generalizing Theorem 2 to the case of the system (1) with uncertain parameters, we obtain the following theorem.

Theorem 4. *The positive continuous-time linear system with delays (1) is robustly stable (independent of delays) if and only if the positive continuous-time system without delays*

$$\dot{x}(t) = A(q)x(t), \quad q \in Q, \tag{10}$$

is robustly stable, where

$$A(q) = \sum_{k=0}^{p} A_k(q) \in M_n, \quad \forall q \in Q. \tag{11}$$

By generalizing Theorem 3 and Lemma 1 to the system (1) with uncertain parameters, we obtain the following theorem and lemma.

Theorem 5. *The positive continuous-time system with delays (1) is robustly stable if and only if the following equivalent conditions hold:*

1. *all the leading principal minors $\Delta_i(q)$ $(i = 1, 2, \ldots, n)$ of the matrix $-A(q)$ are positive for all $q \in Q$, i.e.,*

$$\min_{q\in Q} \Delta_i(q) > 0, \quad i = 1, 2, \ldots, n, \tag{12}$$

2. *all the coefficients of the characteristic polynomial of the matrix $A(q)$, of the form*

$$w(s, q) = \det(sI_n - A(q)) = s^n + \sum_{i=0}^{n-1} a_i(q)s^i, \tag{13}$$

are positive for all $q \in Q$, i.e.,

$$\min_{q\in Q} a_i(q) > 0, \quad i = 0, 1, \ldots, n-1. \tag{14}$$

Lemma 2. *The positive system (1) with delays is not robustly stable if there is a $q \in Q$ such that at least one diagonal entry of $A_0(q)$ is positive.*

The conditions (12) and (14) can be checked by using computer programmes for minimization of real multivariable functions subject to constraints.

Example 1. Consider the system (1) for $n = 2$, $p = 1$, $m = 2$ with the matrices

$$\begin{aligned} A_0(q) &= \begin{bmatrix} -0.9 + q_1^2 & 0.3 - q_2^2 \\ 0 & -0.6 - q_1 \end{bmatrix}, \\ A_1(q) &= \begin{bmatrix} 0.4 + q_2 & 0 \\ 0.25 + q_1 - q_2 & 0.2 - q_2^2 \end{bmatrix}, \end{aligned} \tag{15}$$

where $q = [q_1,\ q_2]^{\mathrm{T}} \in Q = \{q : q_r \in [-0.1,\ 0.1],\ r = 1,\ 2\}$.

It is easy to ascertain that the condition (3) holds and the system is positive.

For the system, the matrix $A(q) = A_0(q) + A_1(q)$ of the form

$$A(q) = \begin{bmatrix} -0.5 + q_1^2 + q_2 & 0.3 - q_2^2 \\ 0.25 + q_1 - q_2 & -0.4 - q_1 - q_2^2 \end{bmatrix}$$

is a Metzler matrix for all $q \in Q$.

Computing the leading principal minors of $-A(q)$, we obtain

$$\Delta_1(q) = 0.5 - q_1^2 - q_2,$$
$$\Delta_2(q) = \det(-A(q))$$

and

$$\min_{q\in Q} \Delta_1(q) = 0.39 > 0,$$
$$\min_{q\in Q} \Delta_2(q) = 0.102 > 0.$$

Hence, the condition (12) holds and the system is robustly stable, according to the first condition of Theorem 5. ♦

3.2. Robust stability of systems with a linear unity rank uncertainty structure.

In the case of a linear uncertainty structure, using (11) and (4), we can write

$$A(q) = \bar{A}_0 + E(q), \tag{16}$$

where

$$\bar{A}_0 = \sum_{k=0}^{p} A_{k0}, \quad E(q) = \sum_{r=1}^{m} q_r \sum_{k=0}^{p} E_{kr}. \tag{17}$$

The asymptotic stability of the positive nominal system $\dot{x}(t) = \bar{A}_0 x(t)$ is necessary for the robust stability of the positive system (1) with a linear uncertainty structure. For stability analysis of this system, Theorem 3 can be applied for $A = \bar{A}_0$.

Now we consider the system (1) with a linear unity rank uncertainty structure (the condition (5) holds) and denote by $\bar{q}_1,\ \bar{q}_2,\ \dots,\ \bar{q}_K$ the vertices of the hiperrectangle (2) ($K = 2^m$), where $\bar{q}_k = [\hat{q}_1, \hat{q}_2, \dots, \hat{q}_m]$ with $\hat{q}_r = q_r^-$ or $\hat{q}_r = q_r^+$, $r = 1, 2, \dots, m$. Moreover, by $V_k = A(\bar{q}_k)$, $k = 1, 2, \dots, K$, denote the vertex matrices of the family of non-negative matrices $\{A(q) : q \in Q\}$, where $A(q)$ has the form (16). These matrices correspond to the vertices of the set (2).

Theorem 6. *The positive system (1) with a linear unit rank uncertainty structure is robustly stable if and only if all the positive vertex systems without delays*

$$\dot{x}(t) = V_k x(t), \quad k = 1, 2, \dots, K, \tag{18}$$

are asymptotically stable, i.e., the conditions of Theorem 3 are satisfied for $A = V_k$, $k = 1, 2, \dots, K$.

Proof. Necessity is obvious because the systems (18) belong to the family (10) of positive systems.

The proof of sufficiency is based on the following observation: if the system (1) has a linear unit rank uncertainty structure, then the coefficients $a_i(q)$, $i = 0, 1, \dots, n-1$, of (13) are real multilinear functions of uncertain parameters and

$$\min_{q \in Q} a_i(q) = \min_k a_i(\bar{q}_k). \tag{19}$$

From the third condition of Theorem 3 it follows that if the family (18) of positive systems is asymptotically stable, then all coefficients of the characteristic polynomials of matrices V_k, $k = 1, 2, \dots, K$, are positive, i.e., $a_i(\bar{q}_k) > 0$, $i = 0, 1, \dots, n-1$ and $k = 1, 2, \dots, K$. Hence,

$$\min_k a_i(\bar{q}_k) > 0, \quad i = 0, 1, \dots, n-1,$$

and by (19),

$$\min_{q \in Q} a_i(q) > 0, \quad i = 0, 1, \dots, n-1.$$

This means that all the coefficients of the polynomial (13) are positive for all $q \in Q$, and, by the second condition of Theorem 5, the positive system (1) with a linear unity rank uncertainty structure is robustly stable. ■

For the analysis of the asymptotic stability of the positive continuous-time systems (18), we can apply Theorem 3 assuming $V_k = A(\bar{q}_k)$ instead of the matrix A.

Example 2. Check the robust stability of the system (1) with $p = 2$, $m = 2$ and matrices $A_k(q)$, $k = 0, 1, 2$, of the form (4) with

$$\begin{aligned} A_{00} &= \begin{bmatrix} -1 & 0.2 \\ 0.1 & -1 \end{bmatrix}, & E_{01} &= \begin{bmatrix} -2.5 & 0.5 \\ 0 & 0 \end{bmatrix}, \\ E_{02} &= \begin{bmatrix} 0.6 & 0 \\ 1 & 0 \end{bmatrix}, & A_{10} &= \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \\ E_{11} &= \begin{bmatrix} 1 & -0.8 \\ 0 & 0 \end{bmatrix}, & E_{12} &= \begin{bmatrix} 0 & 0 \\ -0.4 & 0 \end{bmatrix}, \\ A_{20} &= \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0 \end{bmatrix}, & E_{21} &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \\ E_{22} &= \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \end{aligned} \tag{20}$$

and

$$Q = \{q = [q_1,\ q_2]^{\mathrm{T}} : q_r \in [-0.1,\ 0.1],\ \ r = 1,\ 2\}.$$

It is easy to check that $A_0(q) \in M_2$ and $A_k(q) \in \mathbb{R}_+^{2\times 2}$ ($k = 1,\ 2$) for all $q \in Q$. Hence, the condition (3) holds and the system (1) with the matrices (4), (20) is positive. Moreover, it is easy to see that this system has a linear unit rank uncertainty structure (the condition (5) holds) and the nominal system is asymptotically stable because all the leading principal minors of the matrix

$$-\bar{A}_0 = -\sum_{k=0}^{2} A_{k0} = \begin{bmatrix} 0.7 & -0.5 \\ -0.4 & 0.8 \end{bmatrix}$$

are positive. We apply Theorem 6 for robust stability analysis.

The set Q of $m = 2$ uncertain parameters has $K = 2^m = 4$ vertices. Hence there are $K = 4$ vertex systems (18). Computing the vertices of the set Q, the vertex matrices $V_k = A(\bar{q}_k)$ and the matrices $-V_k$, $k = 1, 2, \dots, 4$, we obtain

$$\bar{q}_1 = \begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix}, \qquad \bar{q}_2 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix},$$
$$\bar{q}_3 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \qquad \bar{q}_4 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix},$$

$$-V_1 = \begin{bmatrix} 0.71 & -0.43 \\ -0.44 & 0.8 \end{bmatrix}, \quad -V_2 = \begin{bmatrix} 0.59 & -0.43 \\ -0.36 & 0.8 \end{bmatrix},$$
$$-V_3 = \begin{bmatrix} 0.69 & -0.57 \\ -0.36 & 0.8 \end{bmatrix}, \quad -V_4 = \begin{bmatrix} 0.81 & -0.57 \\ -0.44 & 0.8 \end{bmatrix}.$$

It is easy to check that all the leading principal minors of the above matrices are positive. This means that all positive vertex systems (18) are asymptotically stable, according to the second condition of Theorem 3. Hence, from Theorem 6, it follows that the system is robustly stable.

The same result is obtained from Theorem 5, because all the leading principal minors of the matrix

$$-A(q) = \begin{bmatrix} 0.7 - 0.5q_1 - 0.6q_2 & -0.5 - 0.7q_1 \\ -0.4 - 0.4q_2 & 0.8 \end{bmatrix}$$

are positive for all $q \in Q$. ♦

3.3. Robust stability of systems with a linear uncertainty structure and non-negative perturbation matrices.

Recall that a real $n \times n$ interval matrix $[A^-,\ A^+]$ is a set of real $n \times n$ matrices $A = [a_{ij}]$ such that $a_{ij}^- \leq a_{ij} \leq a_{ij}^+$, $i, j = 1, 2, \dots, n$, where $A^- = [a_{ij}^-]$, $A^+ = [a_{ij}^+]$.

Consider the positive continuous-time system (1) with state matrices of form (4) satisfying the conditions (6) and (3). In this case, $q_r E_{kr} \in [q_r^- E_{kr},\ \ q_r^+ E_{kr}]$ for all fixed $q_r \in [q_r^-,\ \ q_r^+]$. This means that $A_0(q) \in [A_0^-,\ A_0^+] \subset M_n$ and $A_k(q) \in [A_k^-,\ A_k^+] \subset \mathbb{R}_+^{n\times n}$ for all $q \in Q$ and $k = 1, 2, \dots, p$, where

$$\begin{aligned} A_k^- &= A_{k0} + \sum_{r=1}^{m} q_r^- E_{kr}, \\ A_k^+ &= A_{k0} + \sum_{r=1}^{m} q_r^+ E_{kr}. \end{aligned} \tag{21}$$

Moreover,

$$\{A_0(q) :\ q \in Q\} \subseteq [A_0^-,\ A_0^+]$$

and

$$\{A_k(q) :\ q \in Q\} \subseteq [A_k^-,\ A_k^+], \quad k = 1, 2, \dots, p.$$

From the above, the formulae (16), (17), (21) and the condition (3), we have $A(q) \in [A^-,\ A^+] \subset M_n$ for all $q \in Q$, where

$$A^- = \sum_{k=0}^{p} A_k^-, \quad A^+ = \sum_{k=0}^{p} A_k^+ \tag{22}$$

and $\{A(q) :\ q \in Q\} \subseteq [A^-,\ A^+]$.

It is easy to see that $[A^-,\ A^+] \subset M_n$ if and only if $A^- \in M_n$.

From the above it follows that the robust stability of the positive interval system

$$\dot{x}(t) = A_I x(t), \quad A_I = [A^-,\ A^+] \subset M_n \tag{23}$$

is sufficient for the robust stability of the positive system (10) with a linear uncertainty structure and non-negative perturbation matrices.

Bhattacharyya *et al.* (1995) showed that robust stability of the positive interval system (23) is equivalent to the asymptotic stability of the positive system

$$\dot{x}(t) = A^+ x(t). \tag{24}$$

It is easy to see that $A^+ \in \{A(q) :\ q \in Q\}$. This means that the robust stability of the positive interval system (23) is also necessary for the robust stability of the positive system (10).

From the above and Theorem 4, it follows that the robust stability of the positive continuous-time system (1) with a linear uncertainty structure and non-negative perturbation matrices is equivalent to the asymptotic stability of the positive system (24).

Hence, we have the following theorem and lemma.

Theorem 7. *The positive continuous-time system (1) with a linear uncertainty structure and non-negative perturbation matrices is robustly stable if and only if the positive continuous-time system without delays (24) is asymptotically stable, where*

$$A^+ = \sum_{k=0}^{p} A_{k0} + \sum_{r=1}^{m} q_r^+ \sum_{k=0}^{p} E_{kr}. \tag{25}$$

Lemma 3. *The positive continuous-time system (1) with a linear uncertainty structure (4) and non-negative perturbation matrices is not robustly stable if at least one diagonal entry of the matrix A^+ is positive.*

Example 3. Check the robust stability of the positive continuous-time system (1) with $p = 2$, $m = 2$ and the matrices (4) with A_{00}, A_{10} and A_{20} of the forms given in (20) and

$$E_{01} = \begin{bmatrix} 2.5 & 0.5 \\ 0 & 0 \end{bmatrix}, \qquad E_{11} = \begin{bmatrix} 1 & 0.8 \\ 0 & 0 \end{bmatrix},$$

$$E_{21} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad E_{02} = \begin{bmatrix} 0.6 & 0 \\ 1 & 0 \end{bmatrix},$$

$$E_{12} = \begin{bmatrix} 0 & 0 \\ 0.4 & 0 \end{bmatrix}, \qquad E_{22} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

with

$$Q = \{q = [q_1,\ q_2]^{\mathrm{T}} : q_r \in [-0.1,\ 0.1],\ \ r = 1,\ 2\}.$$

The system is a positive one with a linear uncertainty structure with non-negative perturbation matrices. Therefore, we use Theorem 7 for robust stability analysis.

Computing the matrix A^+ from (25) and $-A^+$, we obtain

$$-A^+ = \begin{bmatrix} 0.19 & -0.73 \\ -0.64 & 0.8 \end{bmatrix}.$$

It is easy to ascertain that the matrix $-A^+$ has a non-positive leading principal minor $\Delta_2 = \det(-A^+)$. This means that the system is not robustly stable, according to Theorem 7. ♦

Consider the positive continuous-time linear interval system with delays

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{p} A_k x(t - h_k) \tag{26}$$

with $A_0 = [A_0^-,\ A_0^+] \subset M_n$ and $A_k = [A_k^-,\ A_k^+] \subset \mathbb{R}_+^{n\times n}$ for $k = 1, 2, \dots, p$.

Theorem 8. *The robust stability of the positive continuous-time interval system (26) with delays is equivalent to asymptotic stability of the positive continuous-time system (24) without delays with* $A^+ = A_0^+ + A_1^+ + \dots + A_p^+$.

4. Concluding remarks

New, simple, necessary and sufficient conditions for robust stability of the positive continuous-time linear system (1) in the general case and in the case of a system with a linear uncertainty structure in two sub-cases: (i) a unit rank uncertainty structure (the condition (5) holds) and (ii) non-negative perturbation matrices (the condition (6) holds, and satisfaction of (5) is not necessary), were given.

It was shown that

(i) The robust stability of the positive continuous-time system (1) with delays is equivalent to the robust stability of the corresponding continuous-time positive system without delays (10) (Theorem 4).

(ii) The positive continuous-time system (1) with delays with a linear unit rank uncertainty structure is robustly stable if and only if the positive vertex systems (18) are asymptotically stable (Theorem 6).

(iii) The positive continuous-time system (1) with delays with a linear uncertainty structure and non-negative perturbation matrices is robustly stable if and only if the positive system (24) is asymptotically stable (Theorem 7).

(iv) The positive continuous-time interval system (26) with delays is robustly stable if and only if the positive system (24) is asymptotically stable (Theorem 8).

Acknowledgment

The work was supported by the Ministry of Science and High Education of Poland under Grant No. N N514 1939 33.

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Received: 12 February 2010
Revised: 31 May 2010