Technical Notes and Correspondence

Robust Stability of Systems with Integral Control

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Abstract—**A number of necessary and sufficient conditions are derived, which must be satisfied by the plant d.c. gain matrix of a linear time invariant system in order for an integral controller to exist for which the closed loop system is stable. Based on these results, the robustness of integral control systems is analyzed, i.e., the family of plants is defined which are stable when controlled with the same integral controller. Conditions for actuator/sensor failure tolerance of systems with integral control are also given. Finally, parallels are drawn between the results of this paper and the bifurcation theory of nonlinear systems.**

INTRODUCTION

Process control, and in particular, chemical process control, is characterized by open-loop stable and sluggish processes, severe modeling problems, and the overriding need for reliability, robustness, and good steady-state performance of the control system, i.e., negligible offset. In order to reduce the system sensitivity at $\omega = 0$ to a small value, controllers with integral action are typically employed in all important situations. Therefore, the modeling requirements for the design of controllers with integral action, their robustness in the event of plant changes, and their tolerance to actuator and/or sensor failure are of significant practical interest.

Unless stated otherwise, we will assume throughout the paper that the plant is an open loop stable, linear, time-invariant system. Let $G(s)$ denote the plant transfer matrix. We will assume that the plant is functionally controllable [1], i.e., that the right inverse of $G(s)$ exists, because only then it is possible to install controllers with integral action on all the outputs. For simplicity in notation but without loss of generality, we will restrict $G(s)$ to be a square matrix relating n inputs to n outputs.

We will use the following notation: C^+ is the open right half and C^- the open left half complex plane; A^{ij} is the matrix A with the ith row and the jth column removed, and $\lambda_j(A)$ and det (A) are the jth eigenvalue and the determinant of the matrix A, respectively.

INTEGRAL CONTROLLABILITY

The basic control system configuration is shown in Fig. 1. Here $G(s)$ and $C(s)$ are the transfer matrices of the plant and the dynamic compensator, respectively, both of which are assumed to be strictly stable. Throughout the paper $K = \text{diag}(k_1 \cdots k_n)$. For this section $K = kI$ where I is the identity matrix and k is a positive constant. We define $H(s) = G(s)C(s)$. Note that $H(s)$ can be improper. For realizability of the controller, only $C(s)/s$ has to be proper. In this paper we would like to address the following questions. What are the requirements on $H(s)$, or equivalently, how does the compensator $C(s)$ have to be designed, for a positive k to exist for which the closed loop system is stable? How tolerant is a control system of this type to plant changes and actuator and/or sensor failures? A necessary condition for a positive k to exist is provided by the following theorem.

Theorem 1: Assume that $H(s)$ is a proper rational transfer matrix. There exists a $k > 0$ such that the closed loop system in Fig. 1 with $K = kI$ is stable only if det $(H(0)) > 0$.

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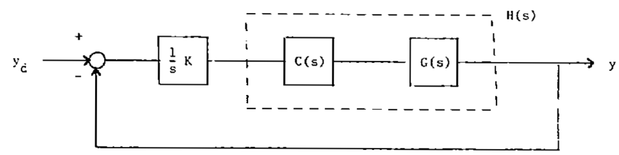

Fig. 1.

Proof: The characteristic equation for the closed loop system of Fig. 1 is given by

$$s\phi(s) \cdot \det\left(I + H(s)\frac{k}{s}\right) = 0 \quad (1)$$

where $\phi(s)$ is the open loop characteristic polynomial of $H(s)$. Express $H(s)$ as $H(s) = N(s)d^{-1}(s)$ where $d(s)$ is the common denominator of the elements of $H(s)$ and $N(s)$ is a polynomial matrix. Equation (1) can then be expressed as

$$\frac{\phi(s)}{d(s)} \cdot \det\left(sd(s)I + kN(s)\right) = 0 \quad (2)$$

where $\phi(s)$ and $d(s)$ are stable polynomials with all coefficients positive.

Upon expansion of the determinant, this expression becomes

$$\frac{\phi(s)}{d(s)} \cdot (s^n d^n(s) + \cdots + k^n \det (N(0))) = 0. \quad (3)$$

If $H(s)$ is proper, the coefficient of the highest power of s in (2) will be the coefficient of the highest power of S in $d(s)$ which is positive. The closed loop system will be stable only if all the coefficients in det $(sd(s)I + kN(s))$ are positive. The constant coefficient is det $(kN(0))$ and therefore, for closed loop stability, it is required that det $(N(0)) > 0$ and det $(H(0)) > 0$. Q.E.D.

Note that systems where $H(s)$ is improper can be stable for some $k > 0$ even when det $(H(0)) < 0$. For SISO systems, the condition in Theorem 1 becomes necessary and sufficient.

Theorem 2: Assume that $h(s)$ is a proper rational transfer function. There exists a $k > 0$ such that the closed loop system in Fig. 1 is stable if and only if $h(0) > 0$.

Theorems 1 and 2 strengthen a result by Sandell and Athans [7]. It is our objective to derive sufficient stability conditions for other than just 2 × 2 systems. It is also practically useful to restrict the range of k somewhat. For this purpose we will introduce the following definition.

Definition 1: The open-loop stable system $H(s)$ is called *integral controllable* if there exists a $k^* > 0$ such that the closed loop system shown Fig. 1 with $K = kI$ is stable for all k satisfying $0 < k \leq k^*$ and exhibits zero tracking error for all asymptotically constant inputs.

It is important to note that we exclude conditionally stable systems in this definition. There could be a $k = k' > 0$ for which the system in Fig. 1 is stable. But unless k' can be made arbitrarily small, the system is not integral controllable according to our definition. Conditionally stable systems which are only stable for high gains k are undesirable from a practical point of view. We will discuss this issue in more detail later.

The following theorem is the main result of this paper and forms the basis of some of the subsequent theorems on robustness and failure tolerance.

Theorem 3: The system $H(s)$ is integral controllable if all the eigenvalues of $H(0)$ lie in C^+. The system is not integral controllable if any of the eigenvalues lie in C^-.

Proof: Let the Nyquist D-contour be indented at the origin to the right to exclude the pole of $1/sH(s)$ at the origin. The system will be closed loop stable if none of the characteristic loci (CL) [2] encircles the point $(-1/k, 0)$. For integral controllability it is sufficient that the CL intersect the negative real axis only at finite values. An intersection at $(-\infty, 0)$ could only occur because of the pole of $H(s)/s$ at the origin. Along the indentation, the small semicircle with radius ϵ around the origin, the CL can be described by

$$\lambda_j(H(0)) \cdot \frac{1}{\epsilon} e^{-i\phi} \qquad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}; \; j=1, n \tag{4}$$

for small ϵ. Let $\lambda_j(H(0)) = r_j e^{i\theta_j}$; then the expression can be rewritten as

$$\frac{r_j}{\epsilon} e^{i(\theta_j - \phi)} \qquad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$$

$$j=1, n. \tag{5}$$

The CL do not cross the negative real axis if $-\pi < \theta_j - \phi < \pi$ or $-\pi/2 < \theta_j < \pi/2$ which means $\lambda_j(H(0)) \epsilon C^+$, $j = 1, n$.

Similar arguments show that there will be an intersection at $(-\infty, 0)$ if $\lambda_j(H(0)) \epsilon C^-$ for any j. Q.E.D.

Theorem 3 says nothing about systems for which the eigenvalues of $H(0)$ lie in the closed right half-plane (RHP) and include eigenvalues on the imaginary axis (not at the origin).[1]

A comparison of Theorem 3 with Theorems 1 and 2 shows that conditional stability without integral controllability is only possible if an even number of eigenvalues of $H(0)$ is in C^-. If the number of eigenvalues of $H(0)$ in C^- is odd, the closed loop system is unstable for all positive gains k. In particular, if the steady-state gain $h(0)$ of a single-input-single-output system is negative, it is unstable for all positive gains.

It would be useful to know how to design the compensator $C(s)$ such that the system is integral controllable. A possibility is to choose $C(s)$ such that $C(0) = \alpha G(0)^T$; $(\alpha > 0)$ or $C(0) = G(0)^{-1}$, that is, to completely "decouple" the system at the steady state. In practice, we often like to reduce the complexity of $C(s)$ and to restrict its structure, for example, to the following form: $C(0) = PD$, where D is a diagonal matrix of constants and P a permutation matrix. This form of $C(0)$ would imply that a set of single-input-single-output controllers can be used. Guadarbassi *et al.* [5] prove that such a compensator always exists when $\det(H(0)) \neq 0$. As we will show in the next section, such systems are often not "failure tolerant."

Failure Tolerance

Obviously, for any actuator or sensor failure, the system shown in Fig. 1 is unstable because of the integral mode. The problem can be remedied by placing the controller in the failure loop on "manual." In Fig. 1 where $K = \text{diag}(k_1, \cdots, k_n)$, this corresponds to removing one integrator and setting one of the elements of K to zero. In such a situation it is desirable that, without readjustment to the other parts of the control system, stability is preserved.

If an actuator fails and only $(n - 1)$ actuators are operating, only $(n - 1)$ variables can be controlled in an offset-free manner. Thus, any actuator failure requires that one controlled variable be left uncontrolled. For simplicity in notation, we will assume that output y_j is left uncontrolled when the actuator of u_j fails. The following theorems, derived directly from Theorem 1, state necessary conditions for stability in the event of actuator or sensor failure.

Theorem 4: Assume that $H(s)$ is rational and proper, and that there exists a $k > 0$ such that the closed loop system in Fig. 1 is stable for $K = kI$ $(\det(H(0)) > 0)$. If sensor j fails and loop j is taken off line $(k_j = 0)$, then the system will be stable only if $\det(H(0)^{jj}) > 0$.

Theorem 5: Assume that $H(s)$ is rational and proper, and that there exists a $k > 0$ such that the closed loop system in Fig. 1 is stable for $K = kI$ $(\det(H(0)) > 0)$. If actuator j fails, variable y_j and loop j are taken off line $(k_j = 0)$, then the system will be stable only if $\det(G(0)^{jj}C(0)^{jj}) > 0$.

Summarizing, we can say that if, upon removal of an actuator and/or sensor, the sign of the determinant of the d.c. gain matrix changes sign, the whole control system has to be redesigned to maintain stability. Every

[1] We are grateful to Dr. N. Schiavoni for pointing out this fact to us.

effort should be made to design the compensator $C(s)$ such that these problems are avoided. Instability in the event of sensor failure can be easily prevented by a steady-state decoupler $C = G(0)^{-1}$. Then $\det(H(0)) = 1$ and $\det(H(0))^{jj}) = 1$. No such simple scheme exists to avoid stability problems associated with actuator failure. Of special interest is the case when the structure of the compensator $C(s)$ is "decentralized," that is, one input-output pair is controlled separately from the rest. It turns out that the relative gain array (RGA) [3] provides some information in this respect.

Let the elements of $G(0)$ be denoted by g_{ij} and the elements of $G(0)^{-1}$ by $\hat{g}_{ij}$. Define the matrix M with the elements

$$m_{ij} = g_{ij}\hat{g}_{ji}. \tag{6}$$

M is called the RGA and enjoys widespread use in process control as an interaction measure, despite its empirical derivation. M can be easily shown to be invariant under input and output scaling of G and to satisfy

$$\sum_{i=1}^{n} m_{ij} = 1 \qquad j=1, n$$

$$\sum_{j=1}^{n} m_{ij} = 1 \qquad i=1, n. \tag{7}$$

Theorem 6: If $m_{jj} < 0$, then for any compensator $C(s)$ with the properties

a) $G(s)C(s)$ is proper
b) $c_{jl} = c_{lj} = 0 \quad \forall l \neq j$
(y_j affects u_j only, u_j is affected by y_j only)

and any $k > 0$, the closed loop system shown in Fig. 1 with $K = kI$ has at least one of the following properties:

a) the closed loop system is unstable
b) loop j is unstable by itself, i.e., with all the other loops opened
c) the closed loop system is unstable as loop j is removed.

Proof: Because m_{ij} is invariant under input and output scaling, we have for any diagonal compensator $C(0)$

$$m_{ij} = (-1)^{i+j} g_{ij} \frac{\det(G(0)^{ji})}{\det(G(0))} \tag{8}$$

$$= (-1)^{i+j} h_{ij} \frac{\det(H(0)^{ji})}{\det(H(0))}. \tag{9}$$

If $m_{jj} < 0$, then one or three of the terms in (9) is negative. For property a), $\det(H(0)) < 0$; for property b), $h_{jj} < 0$; for property c), $\det(H^{jj}(0)) < 0$. Q.E.D.

This theorem can be interpreted in two ways. Let us assume first that loop j is to be designed independently of the others. Then Theorem 6 implies that if loop j by itself is stable and if all the other loops with the loop j removed are stable [b) and c) are not met], then the closed loop system *must* be unstable. Thus, it is *impossible* to design loop j independently of the others.

On the other hand, let us assume that for a particular $C(s)$ there exists a $k > 0$ such that the closed loop system is stable. Then either loop j is unstable by itself or the system becomes unstable when loop j fails, or both. Thus, the system is extremely failure sensitive.

There are two ways around this problem. One could sacrifice the single loop structure of loop j, e.g., introduce a steady-state decoupler. This will avoid the problems of sensor failure, as was argued previously. The other possibility is to look for an alternate pairing of manipulated and controlled variables. Trivially, because of the properties of the RGA, for 2×2 systems there is always a pairing such that $m_{11} = m_{22} > 0$. However, examples show that for 3×3 and larger systems, there might be *no* pairing for which all the m_{jj}'s are positive, that is, there does not exist a fault tolerant decentralized single-loop control structure.

Finally, it is worth emphasizing again that $m_{jj} < 0$ are sufficient but not necessary for the properties of Theorem 6. For 3×3 and larger systems, all properties of Theorem 6 might hold even when $m_{jj} > 0$.

So far, only sufficient conditions for instability have been derived. Using the newly introduced idea of integral controllability (Definition 1), sufficient conditions for stability can be stated. Let us first rigorously define failure tolerance.

Definition 2: The system shown in Fig. 1 is j-sensor failure tolerant (j-SFT) if both the complete system and the reduced system with the jth sensor removed ($k_j = 0$) are integral controllable.

Again we have to assume that the sensor failure has been recognized and that the faulty sensor has been removed from service. j-SFT is a very rich system property. The controller of a j-SFT system can always be tuned such that the closed loop system will remain stable when sensor j fails. After failure, all the inputs are used to control the remaining outputs and the control quality might very well degrade, but without any controller adjustments, stability will be preserved.

Just as in the previous discussion, we will assume that output y_j is left uncontrolled when the actuator of u_j fails.

Definition 3: The system shown in Fig. 1 is j-actuator failure tolerant (j-AFT) if both the complete system and the reduced system with the jth actuator and the jth sensor removed are integral controllable.

The following theorems, which follow directly from Theorem 3, specify the conditions for sensor and actuator failure tolerance.

Theorem 7: The system shown in Fig. 1 with $H(s)$ rational is j-SFT if all the eigenvalues of $H(0)$ and $H(0)^{jj}$ are in C^+. It is not j-SFT if any of the eigenvalues of $H(0)$ or $H(0)^{jj}$ are in C^-.

Theorem 8: The system shown in Fig. 1 with $H(s)$ rational is j-AFT if all the eigenvalues of $H(0)$ and $G(0)^{jj}C(0)^{jj}$ are in C^+. It is not j-AFT if any of the eigenvalues of $H(0)$ or $G(0)^{jj}C(0)^{jj}$ are in C^-.

Except for 2×2 systems, the RGA gives no information on SFT and AFT.

Theorem 9: Let $G(s)$ be a 2×2 system. If $m_{jj}(G) > 0$, then there exists a diagonal compensator $C(s)$ such that $H(s)$ is 1-SFT/AFT and 2-SFT/AFT. Moreover, any 2×2 system can always be brought into a form such that $m_{jj} > 0$ by a permutation of the inputs.

Proof: The necessity follows from Theorem 6. The sufficiency can be proved as follows:

$$m_{11} = m_{22} = \frac{h_{11}h_{22}}{\det (H(0))}.$$

There always exists a diagonal compensator C such that $h_{11} > 0$ (2-SFT/AFT) and $h_{22} > 0$ (1 − SFT/AFT). Therefore, $m_{11} > 0$ implies $\det (H(0)) > 0$. The eigenvalues of $H(0)$ are the roots of

$$\lambda^2 - (h_{11} + h_{22})\lambda + \det (H(0)) = 0.$$

For this second-order polynomial, $\det (H(0)) > 0$ and $h_{11} + h_{22} > 0$ implies that all the eigenvalues of $H(0)$ are in the RHP. $H(s)$ is therefore integral controllable. Moreover, when $m_{11} < 0$, define

$$\mathbf{G}' = \mathbf{G} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

that is, exchange the system inputs. Then

$$m_{11}(\mathbf{G}') = m_{12}(\mathbf{G}) = 1 - m_{11}(\mathbf{G}) > 0. \qquad \text{Q.E.D.}$$

Robustness

The model of a plant is never perfect, and therefore, it is important that the control system is not only stable for the nominal plant but also for a family of plants in some "neighborhood" of the nominal plant. We would like to investigate the *robust stability* of plants with integral control. This property is enjoyed by a family of plants $\mathcal{P}$ if there exists a single compensator C which makes all the members of the family integral controllable. Let the transfer matrix of any member of the family be denoted by $G(s)$ and the transfer matrix of the nominal plant by $G_0(s)$. We may define for each plant in the family the matrix function

$$\Pi(s) = G(s)G_0^{-1}(s). \tag{10}$$

The function $\Pi(s)$ can be interpreted as a multiplicative perturbation of the nominal plant. Then we have the following result.

Theorem 10: Suppose that the family $\mathcal{P}$ of plants satisfies the following assumptions.

a) Each plant in $\mathcal{P}$ is open-loop stable.

b) There exists a fixed square matrix N such that for each plant the matrix $\Pi(0)N$ has all its eigenvalues in a bounded region in C^+.

Then there exists a single compensator C and a single $k > 0$ such that each plant in $\mathcal{P}$ is stable with the control figuration shown in Fig. 1.

Proof: Theorem 10 is an immediate consequence of Theorem 3 when $C = G_0(0)^{-1}N$. Note that when the nominal plant $G_0(s)$ is also a member of the family, as is usually the case, then all the eigenvalues of N also have to be in the open right half-plane.

Theorem 11: Assume that each plant in $\mathcal{P}$ is open loop stable. Then there exists a single compensator C and a single K such that each plant in $\mathcal{P}$ is stable with the control configuration shown in Fig. 1 only if $\det (G(0))$ has the same sign for all plants in $\mathcal{P}$.

This result is disturbing, because quite frequently systems are ill-conditioned, and small uncertainties in the parameters can change the sign of the determinant.

It would almost not be worthwhile to state Theorem 10 because it is so similar to Theorem 3, were it not for a striking resemblance with a result obtained by Kwakernaak.[2]

Theorem 12 [4]: Suppose that the family $\mathcal{P}$ of plants satisfies the following assumptions.

a) Each plant in $\mathcal{P}$ is finite dimensional, is a nonsingular perturbation of the nominal plant such that $\Pi_\infty = \lim_{|s| \to \infty} \Pi(s)$ exists, has the same number of transmission zeros as the nominal plant, and is stabilizable and detectable.

b) The transmission zeros of each plant all lie in a bounded region in C^-.

c) There exists a fixed square matrix N with all its eigenvalues in C^+ such that for each plant, the matrix $\Pi_\infty N$ has all its eigenvalues in a bounded region in C^+.

Then there exists a single controller that stabilizes the control system for each plant in the family $\mathcal{P}$.

Let us analyze the similarities and differences between Theorems 10 and 12. Theorem 12 puts no restriction on the pole location of the different plants (the plants can be unstable), but requires the zeros to be in C^-. Theorem 10 puts no restriction on the zero location of the different plants (the plants can be nonminimum phase), but requires the poles to be in C^-.

Theorem 12 puts a restriction on the asymptotic behavior of the characteristic loci for $\omega \to \infty$; the CL of all the plants in $\mathcal{P}$ together with the compensator have to approach the origin from the same half plane. Theorem 10 puts a restriction on the asymptotic behavior of the CL as $\omega \to 0$; it is required that the CL of all the plants in $\mathcal{P}$ together with the compensator do not cross the negative real axis in the limit.

Let the plant be a single-input-single-output system

$$G(s) = k\chi(s)/\phi_p(s) \tag{11}$$

where ϕ_p is the plant characteristic polynomial, χ a monic polynomial, and k a scalar constant. Then Theorem 12 requires that for each plant in $\mathcal{P}$ the constant k, i.e., the gain at very high frequencies, has the same sign. For single-input-single-output systems, Theorem 10 requires the gain at very low frequencies (d.c. gain) of all the plants in $\mathcal{P}$ to have the same sign.

Thus, Theorems 10 and 12 complement each other in an interesting manner and can be regarded as dual to each other.

Conclusion

A variety of results relating to the stability and robustness of linear and nonlinear systems with integral controllers has been derived. It is most significant that the conditions which have to be satisfied for the controller design to be feasible can all be obtained from steady-state information about the plant. Several issues remain unresolved. Of particular impor-

[2] I am indebted to Prof. Kwakernaak for pointing out this resemblance.

tance is the question of how many restrictions can be placed on the structure of the compensator, which makes a system integral controllable and robust. Any restrictions imply a simplified control structure and are, therefore, practically significant.

Morari [6] has discussed the implications of the results derived here for nonlinear systems. He has shown that for systems with input multiplicities, det ($\tilde{G}$) of the linearized system changes sign. The resulting robustness problems cannot be removed with linear compensators.

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Improved Measures of Stability Robustness for Linear State Space Models

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Abstract—**In this paper, the aspect of "stability robustness" of linear systems is analyzed in the time domain. A bound on the structured perturbation of an asymptotically stable linear system is obtained to maintain stability using a Lyapunov matrix equation solution. The resulting bound is shown to be an improved bound over the ones recently reported in the literature. Also, special cases of the nominal system matrix are considered, for which the bound is given in terms of the nominal matrix, thereby, avoiding the solution of the Lyapunov matrix equation. Examples given include comparison of the proposed approach with the recently reported results.**

NOMENCLATURE

R^α	=	Real vector space of dimension α
$\rho[\cdot]$	=	Spectral radius of the matrix $[\cdot]$
	=	The largest of the modulus of the eigenvalues of $[\cdot]$
$\sigma[\cdot]$	=	Singular values of the matrix $[\cdot]$
$\lambda[\cdot]$	=	Eigenvalues of the matrix $[\cdot]$
$[\cdot]_s$	=	Symmetric part of a matrix $[\cdot]$
$\|[\cdot]\|$	=	Modulus matrix = Matrix with modulus entries
$\forall i$	=	For all i

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I. INTRODUCTION

The problem of maintaining the stability of a nominally stable system subjected to perturbations has been of considerable interest to researchers for quite some time [1]–[5]. The recent published literature on this "stability robustness" analysis can be viewed from two perspectives, namely i) frequently domain analysis and ii) time domain analysis. The analysis in the *frequency domain* is carried out using the singular value decomposition [6]–[8], where the nonsingularity of a matrix is the criterion in developing the robustness conditions. Barrett [8] presents a useful summary and comparison of the different robustness tests available, with respect to their conservatism. Bounds are obtained by Kantor and Andres [9] in the frequency domain using eigenvalue and M matrix analysis. On the other hand, the *time domain* stability robustness analysis is presented using Lyapunov stability analysis starting from Barnett and Storey [2], Bellman [1], Davison [10] (in the context of robust controller design), and Desoer *et al.* [11], among others. Despite the availability of considerable analysis in the time domain stability conditions in the above references, *explicit* bounds on the perturbation of a linear system to maintain stability have been reported only recently by Patel, Toda, and Sridhar [12], Patel and Toda [13], and Lee [14]. In [13], bounds are given for "highly structured perturbations" as well as for "weakly structured perturbations" (according to the classification given by Barrett [8]), while Lee's condition [14] treats "weakly structured perturbations." Highly structured perturbations are those for which only a magnitude bound on individual matrix elements is known for a given model structure. Weakly structured perturbations are those for which only a spectral norm bound for the error is known.

In this paper, we consider the time domain analysis. A new mathematical result is presented [15], which when extended to the result of Patel and Toda [13], provides an improved upper bound for highly structured perturbation. Then some special cases of the nominally stable matrix are considered, for which the bound is given in terms of the nominal matrix, thereby avoiding the solution of the Lyapunov equation. Examples presented include a comparison with the approaches of Patel and Toda [13].

II. STABILITY ROBUSTNESS MEASURES IN THE TIME DOMAIN FOR LINEAR STATE SPACE MODELS

Robustness Measures Due to Patel and Toda

In [13], Patel and Toda consider the following state space description of a dynamic system:

$$\dot{x}(t) = Ax(t) + Ex(t) = (A+E)x(t) \tag{1}$$

where x is the n-dimensional state vector (R^n), A is an $n \times n$ time invariant, asymptotically stable matrix, and E is an $n \times n$ "error" matrix. However, in a practical situation, one does not exactly know the matrix E. One may only have knowledge of the magnitude of the maximum deviation that can be expected in the entries of A. In this case of highly structured perturbation, the entries of E are such that

$$|E_{ij}| \leq \epsilon \tag{2}$$

where ϵ is the magnitude of the maximum deviation.

For this situation, it is shown in [13] that the system of (1) is stable if

$$\epsilon < \frac{1}{n\sigma_{\max}[P]} \equiv \mu_{\epsilon P} \tag{3}$$