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Robust Stability via Polyhedral Lyapunov Functions

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Abstract— **In this paper we study the robustness analysis problem for linear continuous-time systems subject to parametric time-varying uncertainties making use of piecewise linear (polyhedral) Lyapunov functions. A given class of Lyapunov functions is said to be "universal" for the uncertain system under consideration if the search of a Lyapunov function that proves the robust stability of the system can be restricted, without conservatism, to the elements of the class. In the literature it has been shown that the class of polyhedral functions is universal, while, for instance, the class of quadratic Lyapunov functions is not. This fact justifies the effort of developing efficient algorithms for the construction of optimal polyhedral Lyapunov functions. In this context, we provide a novel procedure that enables to construct, in the general n-dimensional case, a polyhedral Lyapunov function to prove the robust stability of a given system. Some numerical examples are included, where we show the effectiveness of the proposed approach comparing it with other approaches proposed in the literature.**

Index Terms— **Linear uncertain systems, robust stability, polyhedral Lyapunov functions.**

I. Introduction

Robust control has been widely investigated by the automatic control community especially during the 1980's and 1990's. Several different approaches have been proposed both for robust stability analysis and for the design of robust control systems (see [11] and [2] for a survey). In this paper we focus on the robust stability analysis problem for linear continuous-time systems subject to parametric time-varying uncertainties. Typically, this problem is tackled by means of quadratic Lyapunov functions (see for instance [13], [3], [8]). As a matter of fact, this approach has been shown to be conservative with respect to approaches using other types of Lyapunov functions [18]; in particular polyhedral Lyapunov functions, namely functions that are positively homogenous and whose level surfaces are the boundary of a polytope, are a strong basis to set up a less conservative robustness analysis tool.

The use of polyhedral Lyapunov functions for robust stability analysis was first proposed in [9], [10]. In [14] it is proved that they are universal for the robustness analysis problem involving linear systems subject to parametric uncertainties, in the sense that the existence of a Lyapunov function which proves robust stability of the given uncertain system implies the existence of a polyhedral Lyapunov function which does the same job. In [6] it is also shown that these functions are universal for the stabilization problem.

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One problem concerning the use of polyhedral functions in the robust stability context, consists in the development of an efficient numerical approach to find, for a given uncertain system, an optimal polyhedral Lyapunov function. In [5], the author proposes a procedure for constructing a polyhedral Lyapunov function for a linear discrete-time system; this procedure is then extended [6] to continuous-time systems via a suitable "Euler approximating system".

Conversely, in this paper we propose a novel procedure to *directly* construct, in the general n-dimensional case, a polyhedral Lyapunov function for the class of linear continuous-time systems subject to parametric uncertainties. A necessary and sufficient condition for the existence of a polyhedral Lyapunov function, which allows to prove the robust stability of the given system, is provided. Such condition requires that a certain optimization problem admits a feasible solution; therefore a workable numerical algorithm is provided to solve the optimization problem.

The paper is organized as follows: in Section II we give some preliminary definitions and results concerning polytopes and the problem we deal with is precisely stated. In Section III the main result of our work is provided. In Section IV some numerical examples, concerning second and third-order linear uncertain systems, illustrate the effectiveness of the proposed approach; in these examples we show that our method performs slightly better than the method proposed in [15] and [16]. Finally, some conclusions are drawn in Section V.

II. Preliminaries

In this paper we deal with the stability of a linear system subject to uncertain parameters

$$\dot{x}(t) = A(p)x(t)\,, \tag{1}$$

where $A(\cdot) : \mathcal{R} \subset \mathbb{R}^q \to \mathbb{R}^{n\times n}$, and $\mathcal{R}$ is a box, i. e.

$$\mathcal{R} := [\underline{p}_1, \overline{p}_1] \times [\underline{p}_2, \overline{p}_2] \times \cdots \times [\underline{p}_q, \overline{p}_q]\,.$$

In the sequel we shall assume the following.

- The vector-valued function
$$p(\cdot) = \begin{pmatrix} p_1(\cdot) & p_2(\cdot) & \cdots & p_q(\cdot) \end{pmatrix}^T\,,$$
is any Lebesgue measurable function $p(\cdot) : [0, +\infty] \to \mathcal{R}$.
- The matrix-valued function $A(\cdot)$ depends multiaffinely on the parameter vector p, that is
$$A(p) = \sum_{\substack{i_1,\ldots,i_q\\=0}}^{1} A_{i_1,\ldots,i_q} p_1^{i_1} \cdots p_q^{i_q}\,. \tag{2}$$

Remark 1: The structure assumed for the system matrices in (2) captures many cases of practical interest and, in particular, the affine dependence on parameters; for more details the interested reader is referred to [2]. $\diamondsuit$

Definition 1 (Robust stability): System (1) is said to be robustly stable if for any Lebesgue measurable vector-valued function $p(\cdot) : [0, +\infty] \to \mathcal{R}$, the resulting linear time-varying system

$$\dot{x}(t) = A\big(p(t)\big)x(t)$$

is exponentially stable. $\diamondsuit$

We focus on the problem of determining some conditions guaranteeing the robust stability of system (1). In order to study this problem, we will make use of the class of (symmetrical) polyhedral Lyapunov functions, which are piecewise linear functions of the following form

$$V(x) = \|Q^T x\|_\infty \,, \tag{3}$$

where $Q \in \mathbb{R}^{n\times m}$ is a full row rank matrix and, given a vector $v \in \mathbb{R}^n$, $\|v\|_\infty := \max_{i=1,\dots,n} |v_i|$ denotes the infinity norm of v.

A. Notions on polytopes

In the following we provide some preliminary definitions and results on linear algebra and polytopes which will be useful to state the main result of the paper.

If we deal with a finite set, say $K = \{x^{(1)}, \dots, x^{(l)}\} \subset \mathbb{R}^n$, the *convex hull* of K turns out to be a *polytope*, whose *dimension* ([19], p. 5), is given by the dimension of the affine hull of K, i. e.

$$\text{rank}\left[\; x^{(2)} - x^{(1)} \quad x^{(3)} - x^{(1)} \quad \dots \quad x^{(l)} - x^{(1)} \;\right].$$

Moreover, as stated in the next lemma, the set of vertices of a given polytope $\mathcal{P}$ is a subset of K.

Lemma 1 ([19]): Given a polytope defined as the convex hull of $K = \{x^{(1)}, \dots, x^{(l)}\} \subset \mathbb{R}^n$, the vertices of the polytope are the points $x^{(i)} \in K$ which satisfy the following property

$$x^{(i)} \notin \text{conv}\left(K - \{x^{(i)}\}\right).$$

$\diamondsuit$

Remark 2: Note that, given a collection of symmetric points $K = \{x^{(1)}, \dots, x^{(2l)}\}$, $x^{(i)} = -x^{(l+i)}, i = 1, \dots, l$, if $x^{(i)}$ is a vertex of $\text{conv}(K)$, then also $x^{(l+i)} = -x^{(i)}$ is a vertex of $\text{conv}(K)$. $\diamondsuit$

In this paper we will focus on polytopes symmetrical with respect to the origin of $\mathbb{R}^n$. To this regard note that, given any symmetrical polytope $\mathcal{P} \subset \mathbb{R}^n$, there always exists a full row rank matrix $Q \in \mathbb{R}^{n\times m}$, $m \geq n$, such that the polytope $\mathcal{P}$ can be alternatively defined as (see [17], p. 6)

$$\mathcal{P} = \wp(Q) := \{x \in \mathbb{R}^n : \|Q^T x\|_\infty \leq 1\}\,. \tag{4}$$

Therefore a given symmetric polytope $\mathcal{P}$ admits two different equivalent descriptions. The Matlab routine *convhulln* allows to find the description matrix Q of a polytope starting from its vertices.

In the following, given a symmetric polytope $\wp(Q)$, we indicate with $x_Q^{(i)}$ with $i = 1 \dots 2l$ the vertices of $\wp(Q)$ and we suppose that $x_Q^{(i)} = -x_Q^{(i+l)}$ for $i = 1 \dots l$. Moreover we indicate with $q_{i,h}$, with $h = 1 \dots s_i$, the s_i columns of Q such that $q_{i,h}^T x_Q^{(i)} = 1$.

B. Quadratic and Polyhedral Stability

Let us consider the following definitions.

Definition 2 (Quadratic stability, [13], [3], [8]): System (1) is said to be quadratically stable if and only if (iff) there exists a quadratic Lyapunov function in the form $x^T Q x$, with Q symmetric positive definite, such that its derivative along the solutions of system (1) is negative definite for all $p \in \mathcal{R}$. $\diamondsuit$

In order to state the definition of polyhedral stability, given a generic system in the form $\dot{x} = f(x)$ and a Lyapunov function $V(x)$, we recall the definition of Dini (upper) derivative [12] of $V(x)$ along the solutions of the system

$$\dot{V}(x) = \lim_{\tau\to 0^+} \sup \left.\frac{V(x+\tau\dot{x}) - V(x)}{\tau}\right|_{\dot{x}=f(x)}.$$

Such definition returns the classical derivative when $V(x)$ is continuously differentiable, but also enables to treat the more general case in which the Lyapunov function is not differentiable everywhere (as it is the case of polyhedral functions).

Definition 3 (Polyhedral stability, [6]): System (1) is said to be polyhedrally stable *iff* there exist a polyhedral Lyapunov function in the form (3) such that its Dini derivative along the solutions of system (1) is negative definite for all $p \in \mathcal{R}$. $\diamondsuit$

Both quadratic stability and polyhedral stability guarantee robust stability of system (1); however polyhedral Lyapunov functions are "better" than quadratic Lyapunov functions.

To clarify this point, recall that, according to [6], a given class of Lyapunov functions is said to be "universal" for system (1) if the existence of a Lyapunov function which proves robust stability of the uncertain system implies the existence of a Lyapunov function belonging to the class which does the same job.

Following this definition, the class of quadratic Lyapunov functions is not universal for systems in the form (1). For example in [7], p. 73, an uncertain system depending on one parameter is shown to be robustly stable, by using a *piecewise quadratic* Lyapunov function, but not quadratically stable (for an interesting discussion on this issue see the seminal paper [9]).

Conversely, as shown in [10], [6], the class of polyhedral Lyapunov functions is universal. This consideration justifies the effort of developing efficient algorithms for the construction of optimal polyhedral Lyapunov functions, which is the topic discussed in the next section.

III. MAIN RESULT

The following result provides a necessary and sufficient condition for polyhedral stability of system (1).

Theorem 1: System (1) is polyhedrally stable *iff* there exists polytope $\wp(Q)$ of dimension n such that the following condition holds for all $i = 1, \ldots, l$, $h = 1, \ldots, s_i$,

$$\max_{p \in \mathrm{vert}(\mathcal{R})} q_{i,h}^T A(p)\, x_Q^{(i)} < 0\,, \tag{5}$$

where $\mathrm{vert}(\mathcal{R})$ denotes the set of all vertices of $\mathcal{R}$.

Proof: According to Definition 3, system (1) is polyhedrally stable *iff* there exists a full row rank matrix $Q \in \mathbb{R}^{n\times m}$ such that the Dini derivative of the polyhedral Lyapunov function $V(x) = \|Q^T x\|_\infty$ along the solutions of system (1) is negative definite for all $p \in \mathcal{R}$.

In the rest of the proof we shall show that such derivative is negative definite *iff* condition (5) holds.

To this end, note that the Dini derivative can be expressed as

$$\dot V(x) = \max_{j\in I(x)} \tilde q_j^T A(p)x\,, \tag{6}$$

where $\tilde Q = \begin{pmatrix} Q & -Q\end{pmatrix}$, $\tilde q_j$ denotes the $j-th$ column of $\tilde Q$ and $I(x)$ is the set of the indexes j such that $V(x) = \tilde q_j^T x$ (see [6]).

Now, the derivative (6) is negative definite for all $p \in \mathcal{R}$ *iff* its maximum on $\mathcal{R}$ is negative. We have

$$\begin{aligned}\max_{p\in\mathcal{R}} \dot V(x) &= \max_{p\in\mathcal{R}} \max_{j\in I(x)} \tilde q_j^T A(p)x\\ &= \max_{j\in I(x)} \max_{p\in\mathcal{R}} \tilde q_j^T A(p)x\\ &= \max_{j\in I(x)} \max_{p\in \mathrm{vert}(\mathcal{R})} \tilde q_j^T A(p)x\\ &= \max_{p\in \mathrm{vert}(\mathcal{R})} \max_{j\in I(x)} \tilde q_j^T A(p)x\,, \end{aligned}\tag{7}$$

where we have used the fact that a multiaffine function defined on a box $\mathcal{R}$ attains its maximum at one of the vertices of $\mathcal{R}$ (see [4]).

Now, it is straightforward to show that the Lyapunov derivative of a polyhedral function enjoys the radial property

$$\dot V(\mu x) = \mu \dot V(x) \qquad \forall \mu \geq 0\,,$$

and therefore its sign behavior can be inferred by the behavior on the boundary of the polytope $\mathcal{P}$ defined in (4). Moreover, notice that, for a given $p \in \mathrm{vert}(\mathcal{R})$, the maximum value of the linear function

$$\tilde q_j^T A(p)x$$

on the j-th face of $\mathcal{P}$ is attained at the vertices of the face itself. Hence, from (7) we have that $\dot V(x)$ is negative definite for all $p \in \mathcal{R}$ *iff*

$$\max_{p\in \mathrm{vert}(\mathcal{R})} \max_{j\in I(x_Q^{(i)})} \tilde q_j^T A(p) x_Q^{(i)} < 0 \tag{8}$$

for all $i = 1, \ldots, 2l$. The symmetry of the polytope implies that (8) is equivalent to

$$\max_{p\in \mathrm{vert}(\mathcal{R})} \max_{j\in I(x_Q^{(i)})} q_j^T A(p) x_Q^{(i)} < 0$$

for all $i = 1, \ldots, l$.

Finally, for a given i, the set $\{q_j\,,\, j \in I(x_Q^{(i)})\}$ is equal to the set $\{q_{i,h}\,,\, h = 1, \ldots, s_i\}$; therefore we can conclude that $\dot V(x)$ is negative definite for all $p \in \mathcal{R}$ *iff*

$$\max_{p\in \mathrm{vert}(\mathcal{R})} q_{i,h}^T A(p)\, x_Q^{(i)} < 0 \tag{9}$$

for all $i = 1, \ldots, l$, and $h = 1, \ldots, s_i$. This concludes the proof. ■

In order to find a polyhedral Lyapunov function satisfying the conditions of Theorem 1, the following procedure can be adopted.

Procedure 1 (Implementation of Theorem 1):

1) Fix an initial number $2l \geq 2n$ of symmetric points $x_Q^{(i)}$ on a hypersphere with radius 1. Let indicate with $K_0 = \{x_Q^{(i)}\}_{i=1,\ldots,2l}$ the set of such points.
2) Find a set of points K solving the problem

$$\min_K \max_{p\in \mathrm{vert}(\mathcal{R})} f(K,p) \tag{10}$$

$$s.t. \quad \mathrm{rank}(Q) = n \tag{11}$$

   with initial condition K_0, where

$$f(K,p) = \max_{i=1,\ldots,l} \max_{h=1,\ldots,s_i} q_{i,h}^T A(p)\, x_Q^{(i)}\,. \tag{12}$$

3) Let $M = \min_K (\max_p f(K,p))$. If $M < 0$ then set

$$K_{opt} = \arg M, \tag{13}$$

   and go to step 4, else choose a new vertex (as explained in Remark 4), set

$$K_0 = K \cup \left\{x_Q^{(l+1)}, -x_Q^{(l+1)}\right\},\ x_Q^{(l+1)} \in \mathbb{R}^n \tag{14a}$$

$$l = l+1 \tag{14b}$$

   and go to step 2.
4) The polyhedral Lyapunov function which proves the polyhedral stability of system (1) is

$$V(x) = \|Q^T x\|_\infty \tag{15}$$

   where Q describes the polytope of vertices K_{opt}.

◊

Remark 3: To solve problem (10), we have made use of the Matlab 7.5.0 and in particular of the Optimization Toolbox routine *fminimax* [1], with variables $x_Q^{(i)}, i = 1, \ldots, l$. ◊

Remark 4: In step 3, the choice of $x_Q^{(l+1)}$ is done putting such point on one of the faces of $\wp(Q)$where the condition of Theorem 1 is not verified. In this way, since at each step the algorithm begins from the solution found in the previous step, the value M decreases (or, at least, does not increase) at each step. ◊

Procedure 1 does not guarantee to find a polyhedral Lyapunov function satisfying the conditions of Theorem 1 even if system (1) is robustly stable. This is due to the fact that (12) is not convex with respect to the optimization variables. Hence even if Theorem 1 provides a necessary and sufficient condition for polyhedral stability (and

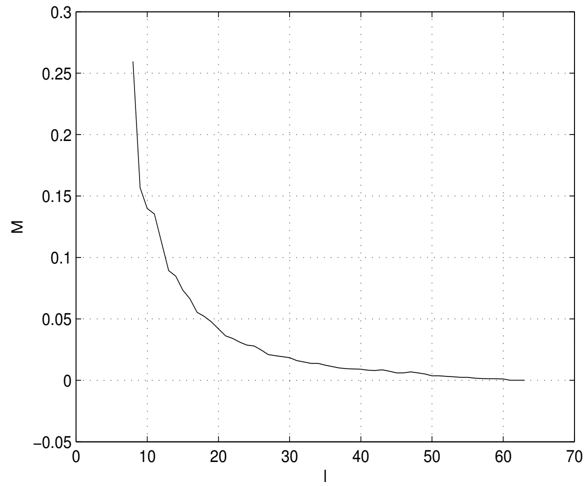

Fig. 1. Trend of the parameter M

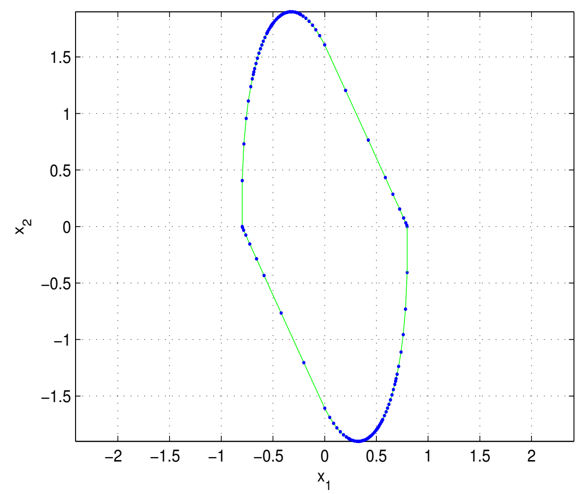

Fig. 2. Polyhedral Lyapunov function for the system in Example 1

therefore for robust stability, since polyhedral functions are universal), applying Procedure 1 we introduce some conservatism.

IV. Numerical Examples

A. Example 1

In this example we will compare our method for constructing a polyhedral Lyapunov function with the one presented in [15]. Let us consider the following linear uncertain system [15]

$$\dot{x} = (p_1 A_1 + p_2 A_2)\, x\,, \quad p_1, p_2 \geq 0 : p_1 + p_2 = 1, \quad (16)$$

with

$$A_1 = \begin{pmatrix} 0 & 1 \\ -0.01 & -2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ -\gamma & -2 \end{pmatrix}.$$

It is easy to prove that system (16) is quadratically stable for $0 < \gamma \leq 4.3$. The author in [15] proves that the system is polyhedrally stable for γ up to 11.3. With our approach, instead, we manage to prove that system (16) is robustly stable for $0 < \gamma \leq 11.45$. We show the result of the Procedure 1 when $\gamma = 11.45$. We started with $l = 5$; Fig. 1 shows the trend of the parameter M defined in step 3 of Procedure 1 depending on the number l of vertices. According to Remark 4, M is a decreasing function of l. Fig. 2 shows the last polytope $\wp(Q)$ of 122 vertices ($l = 61$) which proves the polyhedral (and hence robust) stability of system (16) for $\gamma = 11.45$. $\Diamond$

B. Example 2

Consider the linear uncertain system

$$\dot{x} = (A_1 + A_2 p)\, x\,, \quad p \in [-\gamma, \gamma]\,, \quad (17)$$

with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -6 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

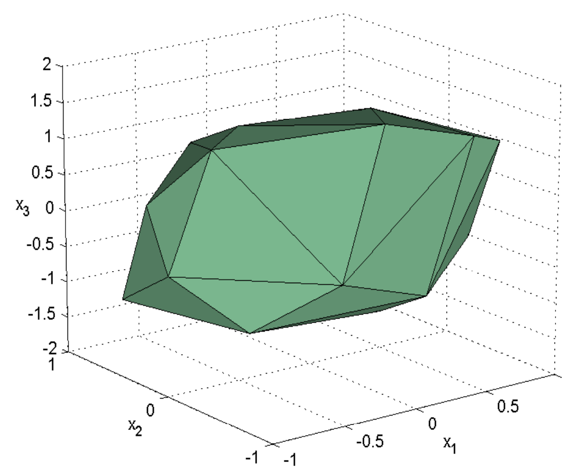

Fig. 3. Polyhedral Lyapunov function for the system in Example 2

System (17) is quadratically stable for $\gamma \leq 0.962$. However using the approach based on polyhedral functions, it is possible to prove the robust stability of (17) up to $\gamma = 0.999$ with the polytope $\wp(Q)$ of 20 vertices ($l = 10$) shown in Fig. 3. $\Diamond$

C. Example 3

Let us consider the feedback loop studied in [16] (see Figure 4) composed by a double integrator with a phase lead element $\frac{1+s}{1+0.1s}$ and a nonlinear time-dependent function $\sigma(e,t)$ which is assumed to satisfy the sector condition

$$\alpha \leq \frac{\sigma(e,t)}{e} \leq \beta\,. \quad (18)$$

The problem is to determine, for a fixed value $\alpha = 0.2$, the bound β such that the feedback system is absolutely stable.

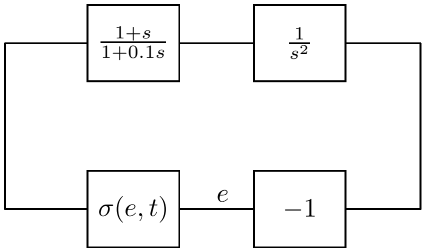

Fig. 4. The feedback system studied in Example 3. The nonlinear time-dependent function $\sigma(e,t)$ is assumed to satisfy the sector condition (18).

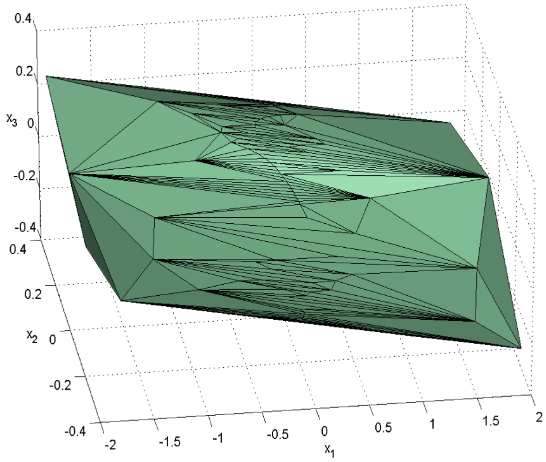

Fig. 5. Polyhedral Lyapunov function for the system in Example 3

In [16] the author shows that the absolute stability of this system is equivalent to the robust stability of the following linear uncertain system

$$\dot{x} = (p_1 A_1 + p_2 A_2)\, x\,, \quad p_1, p_2 \geq 0 : p_1 + p_2 = 1, \quad (19)$$

with

$$A_1 = \begin{pmatrix} -10 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} -10 & -10\beta & -10\beta \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and he verifies the robust stability of (19) for $\beta \leq 1$ (while the maximum value applying the circle criterion is 0.5467). With our approach, instead, we manage to prove that system (19) is robustly stable for $\beta \leq 1.03$ with the polytope $\wp(Q)$ of 168 vertices ($l = 84$) shown in Fig. 5. $\Diamond$

D. Discussion

Theorem 1 is a necessary and sufficient condition for polyhedral stability; therefore it is less conservative than the corresponding necessary and sufficient condition for quadratic stability (since, as said, polyhedral functions are universal, while quadratic are not). However the practical implementation of Theorem 1 through Procedure 1 may introduce a certain degree of conservatism, as clearly explained before.

Moreover, the computational burden increases with the order n of the system, since condition (5) introduces $\sum_{i=1}^{k} s_i$ constraints, where $s_i \geq n$ is related to the number of half-planes associated to the i-th vertex of the polytope and $2k$ is the number of polytope vertices. Therefore as the system order increases, the numbers s_i increases and, in order to keep the computational burden low, we have to limit the number of polytope vertices. Anyway in Examples 1 and 3 above we show that our method performs slightly better than the method proposed in [15] and [16].

Therefore the approach proposed in this paper is not always better than the one based on quadratic Lyapunov functions and should be seen as an alternative to the classical quadratic stability methodology.

V. Conclusions

In this paper we have considered the robustness analysis problem for a linear uncertain system subject to parametric time-varying uncertainties. To tackle this problem we have made use of polyhedral Lyapunov functions, which lead to a condition for robust stability less conservative than the classical quadratic stability test. On the basis of such condition, a novel procedure, which enables to construct a polyhedral Lyapunov function proving robust stability of a given uncertain system, has been provided. Some numerical examples have been included to illustrate both the application and the effectiveness of the proposed approach, in comparison with other methods.

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