Robust Synchronization of Uncertain Linear Multi-Agent Systems

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Abstract—This paper deals with robust synchronization of uncertain multi-agent networks. Given a network with for each of the agents identical nominal linear dynamics, we allow uncertainty in the form of additive perturbations of the transfer matrices of the nominal dynamics. The perturbations are assumed to be stable and bounded in $\mathcal{H}_\infty$-norm by some a priori given desired tolerance. We derive state space formulas for observer based dynamic protocols that achieve synchronization for all perturbations bounded by this desired tolerance. It is shown that a protocol achieves robust synchronization if and only if each controller from a related finite set of feedback controllers robustly stabilizes a given, single linear system. Our protocols are expressed in terms of real symmetric solutions of certain algebraic Riccati equations and inequalities, and also involve weighting factors that depend on the eigenvalues of the graph Laplacian. For undirected network graphs we show that within the class of such dynamic protocols, a guaranteed achievable tolerance can be obtained that is proportional to the quotient of the second smallest and the largest eigenvalue of the Laplacian. We also extend our results to additive nonlinear perturbations with $\mathcal{L}_2$-gain bounded by a given tolerance.

I. INTRODUCTION

In recent years, a major research effort has been put into the study of networks of systems, in particular the distributed control of networked multi-agent systems. A networked multi-agent system is a dynamical system composed of a group of input-output systems that interact by exchanging information with their neighbours. These input-output systems are called the *agents* of the network. Interaction between the agents is represented by a graph, called the *network graph*, describing which agents on the network are neighbours of a given one. The vertices of the network graph represent the agents, while the edges of the graph represent the interconnection topology of the network. Depending on the context, the network graph may be undirected or directed. A crucial object in networked multi-agent systems is the so-called *Laplacian matrix* of the network graph. Many properties of networked systems can be expressed in terms of the spectrum of the Laplacian, see [30], [13].

Each agent on the network exchanges information with each of its neighbours. Once the precise form of this information exchange is fixed, the dynamics of the individual agents together

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with the interaction with their neighbours will result in the overall dynamics of the network. The form of the information exchange is often called a *protocol*. A protocol acts as a feedback controller on the network, with the important feature that it acts locally, with the feedback processer for each of the agents acting on the information from its neighbours. An important issue in the theory of networked multi-agent systems is the design of protocols to achieve a desired overall behavior of the network.

Several related problem formulations involving interconnection of dynamical systems in various application areas can be cast in the framework described in the previous paragraphs. Among these problem formulations perhaps the most well-known is the *consensus problem*, see [17], [18], [21], [15], [20] and pioneering work in [28]. We also mention more recent work in [3], [10], [32], [26], [16] and [12]. In the consensus set-up, the agents may for example represent sensor devices that exchange information only with their neighbours. The aim of the information exchange is to reach agreement on the values of certain quantities of interest that depends on the states of all agents. A protocol that achieves this aim is said to achieve consensus. A strongly related problem is the *synchronization problem*, see for example [8], [14], [24], [22], [31], in which the agents may be identical physical systems, modeled for example as oscillators, and where the problem is to find conditions on the protocol under which the states of a typically large number of these coupled systems converge to a common trajectory. If this is the case then the network is said to be synchronized. The problem of *distributed formation control* deals with cooperation among a collection of vehicles (e.g. satellites, airplanes, mobile robots, cars) that communicate in order to coordinate their actions, see [4],[6]. In this case, the vehicles are the agents, and their communication topology is represented by the network graph. The problem is to have the vehicle formation evolve as much as possible along a certain desired trajectory, and the question is to find protocols that achieve this goal. An excellent overview of the literature can be found in [19].

Whereas most of the initial literature on synchronization and consensus has been dealing with simple systems of scalar, single or double integrators, recently interest has shifted to networked systems in which the dynamics of the agents is a general finite dimensional linear input-output system, see [4], [22], [29], [3], [12], [32]. Here, the problem is to design protocols that use relative state or output measurements of the neighboring agents to obtain synchronization. These protocols are in general static or sometimes *observer based*, in which case they consist of a dynamic part that acts as an observer

for the relative states, combined with a static part that feeds back the estimated relative state to the agents.

In the present paper, we will extend the theory developed on consensus and synchronization until so far to the problem of *robust* consensus and synchronization of linear multi-agent systems. We will deal with the situation that all agents on the network have identical *nominal* dynamics, but that every agent is *uncertain*, in the sense that its transfer matrix can be any transfer matrix obtained as an *additive perturbation* of the common nominal one. The only assumption on the additive perturbation is that it is stable, and its $\mathcal{H}_\infty$-norm is bounded by some a priori given tolerance. Thus, in effect, the network is allowed to be *heterogeneous*, in the sense that the actual agent dynamics can vary from agent to agent, but is contained in a ball of fixed radius around the common nominal dynamics. The aim is then to design, for a given tolerance, a dynamic protocol that synchronizes the network for *all* such additive perturbations. We will show how to obtain, for a given tolerance, such dynamic protocols. These protocols will depend both on the nominal agent dynamics as well as on the Laplacian eigenvalues of the underlying graph. Of course, one would like to *maximize the permitted tolerance*, i.e. the radius of the balls of uncertainty. Among other things, in this paper we will show that, for undirected network graphs, within the class of observer based dynamic protocols a guaranteed radius can be obtained that is proportional to the quotient λ_2/λ_p of the second smallest and largest eigenvalue of the Laplacian. It will also be shown that our protocols achieve robustness against nonlinear additive perturbations with finite $\mathcal{L}_2$ gain.

To the authors' best knowledge, this paper is the first work that addresses the problem of robust consensusability and synchronizability with uncertainty in the agent dynamics for agents given by general linear input-output systems. For work on robustness in the context of consensus with agents given scalar systems we refer to [33], [25] and [2]. The recent paper [5] deals with robust stability *analysis* of multi-agent systems. On the problem of achieving consensus or synchronization in networks with heterogenity using a somewhat different perspective, we mention [9], and we also refer to [31] and [7]. Problems of designing protocols that provide robustness under perturbations of the coupling strengths in the network graph have been studied in [26]. Robustness against communication delays in the network was studied in [16]. The paper [32] deals with consensus protocols that remain to achieve consensus under quantization of the relative state information, thus providing a robustness result under information quantization.

The outline of this paper is as follows. In Section II we introduce the basic material on graph theory needed in this paper, and formulate a version of the bounded real lemma that will be intrumental in proving our main results. In Section III we set the scene by reviewing the 'plain' synchronization problem for homogeneous networks. In Section IV we formulate the problem of robust synchronization and show that for the undirected graph case this problem is equivalent to solving a simultaneous robust stabilization problem, in the sense that a single linear system should be robustly stabilized by each controller from a given set of feedback controllers. A similar result will hold for directed graphs. Then, in Section

V we will formulate our main results, describing how to compute the required protocols in terms of solutions of Riccati equations and inequalities associated with the nominal agent dynamics and the spectrum of the Laplacian. Subsection V-A deals with undirected graphs, and Subsection V-B deals with directed graphs. In Section VI, for undirected graphs we establish a guaranteed uncertainty radius proportional to the quotient of the second smallest and largest eigenvalue of the Laplacian. Section VII briefly explains how our results extend to nonlinear additive perturbations. Finally, Section VIII gives some conclusions.

II. PRELIMINARIES

In this paper we consider multi-agent systems whose interconnection structures are described by directed or undirected unweighted graphs. In general a directed graph is a pair $(\mathcal{V}, \mathcal{E})$, where the elements of $\mathcal{V} = \{1, 2, \ldots, p\}$ are called *vertices*, and where the elements of $\mathcal{E}$ are pairs (i, j), called *edges*. The pair $(i, j) \in \mathcal{E}$ with $i, j \in \mathcal{V}$, $i \neq j$, represents an edge from vertex i to vertex j. If for every $(i, j) \in \mathcal{E}$ also $(j, i) \in \mathcal{E}$, then the graph is called undirected. For a given vertex, say i, its *neighboring set* $\mathcal{N}_i$ is defined by $\mathcal{N}_i := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. For a given graph, its adjacency matrix A is defined by $A = (a_{ij})$ where $a_{ii} = 0$, $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The *Laplacian matrix* of the graph is defined as $L = (l_{ij})$, where $l_{ii} = \sum_{j \neq i} a_{ij}$, $l_{ij} = -a_{ij}$, $i \neq j$. If the graph is undirected, then L is a positive semi-definite real symmetric matrix, so all eigenvalues of L are non-negative real. If the graph is directed, L need no longer be symmetric, so its eigenvalues need not be real. However, the eigenvalues can still be shown to have nonnegative real part. Both for the directed and undirected case, zero is always an eigenvalue of the Laplacian, so it has rank at most $p - 1$.

An undirected graph is called *connected* if for every pair of distinct vertices i and j there exists a path from i to j, i.e. a finite set of edges (i_k, i_{k+1}) $k = 1, 2, \ldots, r-1$ such that $i_1 = i$ and $i_r = j$. An undirected graph is connected if and only if its Laplacian has rank $p - 1$. In that case the zero eigenvalue has multiplicity one, and all other eigenvalues are positive. The remaining $p - 1$ eigenvalues are ordered in increasing order as $0 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_p$.

A directed graph is said to contain a *spanning tree* if it contains a node i such that there exists a path from this node to every other node j. A directed graph contains a spanning tree if and only if its Laplacian has rank $p - 1$. In that case the zero eigenvalue has multiplicity one, and all other eigenvalues have positive real part. The remaining $p - 1$ eigenvalues are numbered $\lambda_2, \lambda_3, \ldots, \lambda_p$ in arbitrary order.

In this paper, we will denote by $R\mathcal{H}_\infty$ the set of all proper and stable rational transfer matrices. If $G \in R\mathcal{H}_\infty$, then $\|G\|_\infty$ will denote its usual infinity norm, $\|G\|_\infty = \sup_{\mathrm{Re}(\lambda) \geq 0} \|G(\lambda)\|$. A square matrix is called *Hurwitz* if all its eigenvalues λ satisfy $\mathrm{Re}(\lambda) < 0$. For a given real or complex matrix C with n columns, we denote by $\ker(C)$ the nullspace of C, i.e. all $x \in \mathbb{R}^n$ ($x \in \mathbb{C}^n$) such that $Cx = 0$.

For future use we state and prove the following version of the bounded real lemma, tailored for our purposes:

Lemma 2.1: Consider the system $\dot{x} = Ax + Bu$, $y = Cx$, with A, B and C real matrices. Let $G(s) = C(sI - A)^{-1}B$ be its transfer matrix. Assume that A is Hurwitz. Let D be a real matrix with n columns such that $\ker(D) \subset \ker(C)$. Let $\rho > 0$. If there exists $\epsilon > 0$ such that the Riccati inequality

$$A^\top P + PA + \frac{1}{\rho^2}PBB^\top P + C^\top C \leq -\epsilon D^\top D$$

has a real symmetric solution P, then $\|G\|_\infty < \rho$.

Proof: First note that there exists a matrix M such that $C = MD$. Let $\delta > 0$ be sufficiently small so that $\delta M^\top M - \epsilon I \leq 0$. Obviously, such δ exists. It is then easily verified that

$$A^\top P + PA + \frac{1}{\rho^2}PBB^\top P + (1+\delta)C^\top C \leq$$
$$D^\top(\delta M^\top M - \epsilon I)D \leq 0.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}x^\top Px =$$
$$x^\top(A^\top P + PA)x + u^\top B^\top Px + x^\top PBu \leq$$
$$-\frac{1}{\rho^2}x^\top PBB^\top Px + u^\top B^\top Px + x^\top PBu -$$
$$(1+\delta)x^\top C^\top Cx =$$
$$-\|\rho u - \frac{1}{\rho}B^\top Px\|^2 + \rho^2\|u\|^2 - (1+\delta)\|y\|^2 \leq$$
$$\rho^2\|u\|^2 - (1+\delta)\|y\|^2.$$

Taking $x(0) = 0$ and $u \in \mathcal{L}_2(\mathbb{R}^+)$, by integrating from 0 to ∞ this yields $0 \leq \rho^2\|u\|_2^2 - (1+\delta)\|y\|_2^2$. Thus $\|y\|_2^2 \leq \frac{\rho^2}{1+\delta}\|u\|_2^2$ for all $u \in \mathcal{L}_2(\mathbb{R}^+)$. This implies that the induced norm $\|G\|_\infty$ of the operator from u to y satisfies $\|G\|_\infty \leq \frac{\rho}{\sqrt{1+\delta}} < \rho$. $\square$

Remark 2.2: In this paper we will also use the complex version of the above lemma, where A, B, C and D are matrices with complex coefficients. In the Riccati inequality, then, transpose should be replaced by conjugate transpose, and the inequality should have a Hermitian solution. The proof is easily adapted to the complex case.

III. SYNCHRONIZATION

In this paper, we consider multi-agent networks with p agents, where the underlying network graph is a directed or undirected graph whose Laplacian is denoted by L. The dynamics of agent i is given by the nominal finite-dimensional linear time-invariant system

$$\dot{x}_i = Ax_i + Bu_i, \ y_i = Cx_i. \qquad (1)$$

Thus, the nominal dynamics of each agent is represented by one and the same linear input-output system. Throughout this paper, we assume that the pair (A, B) is stabilizable, and the pair (C, A) is detectable. Each state x_i takes its values in $\mathbb{R}^n$, the input u_i and output y_i take their values in $\mathbb{R}^m$ and $\mathbb{R}^q$ respectively.

The synchronization problem is the problem of finding a protocol that makes the network synchronized. Following [3],

[22], we consider *dynamic* protocols of the form

$$\dot{w}_i = Aw_i + BF\sum_{j\in\mathcal{N}_i}(w_i - w_j) + G(\sum_{j\in\mathcal{N}_i}(y_i - y_j) - Cw_i),$$
$$u_i = Fw_i. \qquad (2)$$

To understand the structure of this protocol, note that agent i receives information $\sum_{j\in\mathcal{N}_i}(y_i - y_j)$, i.e. the sum of the relative outputs with respect to its neighbours. The first equation of (2) has the structure of an *observer* for the sum of the relative *states*, i.e. $\sum_{j\in\mathcal{N}_i}(x_i - x_j)$, with w_i the estimated value. Indeed, it is easily seen that the error $e_i := w_i - \sum_{j\in\mathcal{N}_i}(x_i - x_j)$ satisfies the dynamics $\dot{e}_i = (A - GC)e_i$. The second equation in (2) is a static gain, feeding back the estimate to agent i. By interconnecting the agents using this protocol, we obtain the closed loop dynamics of the overall network. Denote the aggregate state vector by $\mathbf{x} = \mathrm{col}(x_1, x_2, \ldots, x_p)$ and likewise define $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{y}$. Then we obtain

$$\dot{\mathbf{x}} = (I \otimes A)\mathbf{x} + (I \otimes B)\mathbf{u}, \ \mathbf{y} = (I \otimes C)\mathbf{x} \qquad (3)$$

and

$$\dot{\mathbf{w}} = [(I\otimes(A-GC)+(L\otimes BF)]\mathbf{w}+(L\otimes G)\mathbf{y}, \ \mathbf{u} = (I\otimes F)\mathbf{w}. \qquad (4)$$

This leads to the network dynamics

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes BF \\ L \otimes GC & I \otimes (A - GC) + (L \otimes BF) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix}. \qquad (5)$$

Definition 3.1: The network is said to be synchronized by the protocol if for all $i, j = 1, 2\ldots, p$ we have $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ as $t \to \infty$.

In this section we first consider the case that the network graph is *undirected*. In that case the Laplacian is a real symmetric matrix, so there exists an orthogonal $p \times p$ matrix U that brings L to diagonal form $U^\top LU = \Lambda := \mathrm{diag}(0, \lambda_2, \lambda_3, \ldots, \lambda_p)$. In addition we assume that the graph is connected, equivalently $\lambda_2 > 0$. Then, by applying the state transformation

$$\begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} U^\top \otimes I & 0 \\ 0 & U^\top \otimes I \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix}, \qquad (6)$$

the network equation becomes

$$\begin{pmatrix} \dot{\tilde{\mathbf{x}}} \\ \dot{\tilde{\mathbf{w}}} \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes BF \\ \Lambda \otimes GC & I \otimes (A - GC) + (\Lambda \otimes BF) \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix}. \qquad (7)$$

This brings us to the following well-known fact (see also [4],[3]) that we record for future use:

Lemma 3.2: Consider the network with agent dynamics (1). Assume the network graph is undirected and connected. Then the protocol (2) synchronizes the network if and only if for $i = 2, 3, \ldots p$ the systems

$$\begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{w}}_i \end{pmatrix} = \begin{pmatrix} A & BF \\ \lambda_i GC & A - GC + \lambda_i BF \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} \qquad (8)$$

are stable.

Proof: Note that $\ker(L) = \mathrm{im}(1_p)$, where 1_p denotes the vector $1_p = (1, 1, \ldots, 1)^\top$ in $\mathbb{R}^p$. Let U be an orthogonal matrix as above such that $LU = \Lambda U$ with $\Lambda = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_p)$.

Clearly, $x_i(t) - x_j(t) \to 0$ for all i, j if and only if $\mathbf{x}(t) \to \mathrm{im}(1_p \otimes I) = \ker(L \otimes I)$. This holds if and only if $(L \otimes I)\mathbf{x}(t) \to 0$. Since $\mathbf{x} = (U \otimes I)\tilde{\mathbf{x}}$ the latter holds if and only if $(LU \otimes I)\tilde{\mathbf{x}} \to 0$. Since $LU = U\Lambda$ and U is nonsingular, this holds if and only if $(\Lambda \otimes I)\tilde{\mathbf{x}} \to 0$, equivalently $\tilde{x}_i(t) \to 0$ for $i = 2, 3, \ldots, p$. The same argument applies to the variables w_i and $\tilde{w}_i$. $\qquad\square$

After having completed the proof of the previous lemma, we apply one more state transformation to (8). By defining $\overline{x}_i = \tilde{x}_i$, $\overline{w}_i = \frac{1}{\lambda_i}\tilde{w}_i$ we see that the network is synchronized if and only if for $i = 2, 3, \ldots, p$ the systems

$$\begin{pmatrix} \dot{\overline{x}}_i \\ \dot{\overline{w}}_i \end{pmatrix} = \begin{pmatrix} A & \lambda_i BF \\ GC & A - GC + \lambda_i BF \end{pmatrix} \begin{pmatrix} \overline{x}_i \\ \overline{w}_i \end{pmatrix} \qquad (9)$$

are stable. The latter closed loop system can be interpreted as the feedback interconnection of the system $\dot{\overline{x}}_i = A\overline{x}_i + B\overline{u}_i$, $\overline{y}_i = C\overline{x}_i$ with the controller $\dot{\overline{w}}_i = A\overline{w}_i + B\overline{u}_i + G(\overline{y}_i - C\overline{w}_i)$, $\overline{u}_i = \lambda_i F\overline{w}_i$. Since the set of eigenvalues of the system matrix in (9) is the union of those of $A - GC$ and $A + \lambda_i BF$, we can make the following useful observation:

Lemma 3.3: Consider the network with agent dynamics given by (1). Assume the network graph is undirected and connected. Then the protocol (2) synchronizes the network if and only if the linear system

$$\dot{x} = Ax + Bu, \ y = Cx, \qquad (10)$$

is stabilized by all $p - 1$ controllers

$$\dot{w} = Aw + Bu + G(y - Cw), \ u = \lambda_i Fw, \ i = 2, 3, \ldots, p. \qquad (11)$$

This holds if and only if $A - GC$ and $A + \lambda_i BF$ ($i = 2, 3, \ldots, p$) are Hurwitz.

We now briefly discuss the *directed graph* case. Assume that the graph contains a spanning tree, equivalently $\lambda_i \neq 0$ ($i = 2, \ldots, p$). In this case, the Laplacian need no longer be symmetric. It is however easily seen that it can be brought to *upper triangular form* by means of a unitary transformation, i.e. there exists a complex unitary $p \times p$ matrix U such that $U^* LU = \Lambda_u$, where Λ_u is a complex upper triangular matrix with $0, \lambda_2, \lambda_3, \ldots, \lambda_p$ on the diagonal. Repeating the argument in the proof of Lemma 3.2 it is then straightforward to check that *both Lemma 3.2 and Lemma 3.3 hold through unchanged for directed graphs that contain a spanning tree*. Note however that in the directed graph case, due to the fact that the λ_i's are no longer real, the controllers (11) will in general be complex. The gain matrices F and G are of course still required to be real.

To summarize, the above results show that both for the directed and undirected graph case, the dynamic protocol (2) synchronizes the network if and only if the gain matrices F and G are chosen such that all $p - 1$ controllers (11) stabilize the single system (10). A similar result will turn out to hold for *robust* synchronization in the next section. It can be proven that such F and G exist if and only if (C, A) is detectable and (A, B) is stabilizable. The detectability condition is of course obvious. The fact that stabilizability is sufficient for the existence of a single F such that $A + \lambda_i BF$ is Hurwitz for

$i = 2, 3, \ldots, p$ is less obvious and was e.g. proven in [3], see also [12]. Conditions in the discrete-time case were obtained in [32]. The observation that the design of synchronizing protocols amounts to simultaneous stabilization was made before in [4], [3], [12] and [32].

IV. ROBUST SYNCHRONIZATION

The main topic of this paper is *robust* synchronization. Again consider a multi-agent network with dynamics of agent i given by the nominal system (1). The idea of robust synchronization is that the dynamics of each agent is *uncertain*, accounting for heterogenity, and that the dynamics of any of the agents can be given by any system in a ball around a nominal system. In this paper we will quantify this by *additive perturbations* of the agent transfer matrices. In particular, as $G(s) = C(sI - A)^{-1}B$ represents the nominal system for agent i, we will consider perturbations $G(s) + \Delta_i(s)$, where $\Delta_i \in R\mathcal{H}_\infty$. If we realize $\Delta_i(s) = C_{\Delta_i}(sI - A_{\Delta_i})^{-1}B_{\Delta_i} + D_{\Delta_i}$ this means that the dynamics of agent i is perturbed to the system obtained by interconnecting

$$\dot{x}_i = Ax_i + Bu_i, \ y_i = Cx_i + d_i, \ z_i = u_i \qquad (12)$$

with

$$\dot{\xi}_i = A_{\Delta_i}\xi_i + B_{\Delta_i}z_i, \ d_i = C_{\Delta_i}\xi_i + D_{\Delta_i}z_i. \qquad (13)$$

We allow all such perturbations with transfer matrix $\Delta_i \in R\mathcal{H}_\infty$ with $\|\Delta_i\|_\infty \leq \eta$, where $\eta > 0$ is a given uncertainty radius. Thus, the system describing the dynamics of agent i is any system with transfer matrix of the form $G + \Delta_i$ with $\|\Delta_i\|_\infty \leq \eta$. Instead of explicitly writing out equations of the form (13) for the perturbation, in the sequel we often simply write: $d_i = \Delta_i z_i$.

Definition 4.1: Given a desired tolerance $\eta > 0$, the problem of *robust synchronization* is to find a dynamic protocol such that for all i and for all $\Delta_i \in R\mathcal{H}_\infty$ with $\|\Delta_i\|_\infty \leq \eta$ the network (5) is synchronized, i.e. for all $i, j = 1, 2 \ldots, p$ we have $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ as $t \to \infty$. The tolerance η will be called the *synchronization radius* of the network.

For the purpose of robust synchronization we slightly modify the earlier protocol (2) to include a *weighting factor* on the Laplacian L. Thus, in the sequel we consider protocols of the form

$$\dot{w}_i = Aw_i + BF \sum_{j \in \mathcal{N}_i} \frac{1}{N}(w_i - w_j) +$$
$$G\left(\sum_{j \in \mathcal{N}_i} \frac{1}{N}(y_i - y_j) - Cw_i\right),$$
$$u_i = Fw_i. \qquad (14)$$

Here N is a positive real number that, next to F and G, needs to be determined. In this section we will derive conditions under which, for a given desired radius η, there exists such robustly synchronizing protocol. Note that we only require that the state components of the nominal agent dynamics and of the protocol are synchronized, and not the state components of the systems that represent the perturbations.

We now derive the equations of the network with uncertain agents. The aggregate dynamics of the extended systems (12) is of course represented by

$$\dot{\mathbf{x}} = (I \otimes A)\mathbf{x} + (I \otimes B)\mathbf{u}, \ \mathbf{y} = (I \otimes C)\mathbf{x} + (I \otimes I)\mathbf{d}, \ \mathbf{z} = \mathbf{u}. \tag{15}$$

Combining this with (14) leads to the dynamics of the perturbed network:

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes BF \\ \frac{1}{N} L \otimes GC & I \otimes (A - GC) + (\frac{1}{N} L \otimes BF) \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \\ + \begin{pmatrix} 0 \\ \frac{1}{N} L \otimes G \end{pmatrix} \mathbf{d}, \tag{16}$$

$$\mathbf{z} = \begin{pmatrix} 0 & I \otimes F \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix}, \tag{17}$$

$$\mathbf{d} = \begin{pmatrix} \Delta_1 & 0 & \cdots & 0 \\ 0 & \Delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_p \end{pmatrix} \mathbf{z}. \tag{18}$$

We now first consider the case that the network graph is an undirected, connected graph. As before, we apply the state transformation (6), this time together with the transformations $\tilde{\mathbf{d}} = (U^\top \otimes I)\mathbf{d}$, $\tilde{\mathbf{z}} = (U^\top \otimes I)\mathbf{z}$, to obtain the transformed equations

$$\begin{pmatrix} \dot{\tilde{\mathbf{x}}} \\ \dot{\tilde{\mathbf{w}}} \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes BF \\ \frac{1}{N} \Lambda \otimes GC & I \otimes (A - GC) + (\frac{1}{N} \Lambda \otimes BF) \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} \\ + \begin{pmatrix} 0 \\ \frac{1}{N} \Lambda \otimes G \end{pmatrix} \tilde{\mathbf{d}}, \tag{19}$$

$$\tilde{\mathbf{z}} = \begin{pmatrix} 0 & I \otimes F \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix}, \tag{20}$$

$$\tilde{\mathbf{d}} = (U^\top \otimes I) \begin{pmatrix} \Delta_1 & 0 & \cdots & 0 \\ 0 & \Delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_p \end{pmatrix} (U \otimes I) \, \tilde{\mathbf{z}}. \tag{21}$$

The following theorem gives *necessary and sufficient* conditions on the gain matrices F and G such that the dynamic protocol (14) robustly synchronizes the uncertain network:

Theorem 4.2: Consider the network with agent dynamics given by (1). Assume the network graph is undirected and connected. Let $\eta > 0$. The following two statements are equivalent:

1) The dynamic protocol (14) synchronizes the network with perturbed agent dynamics

$$\dot{x}_i = Ax_i + Bu_i, \ y_i = Cx_i + d_i, \ z_i = u_i, \\ d_i = \Delta_i z_i, \quad i = 1, 2, \ldots, p \tag{22}$$

for all $\Delta_i \in R\mathcal{H}_\infty$ with $\|\Delta_i\|_\infty \leq \eta$,
2) the perturbed linear system

$$\dot{x} = Ax + Bu, \ y = Cx + d, \ z = u, \ d = \Delta z \tag{23}$$

is internally stabilized for all $\Delta \in R\mathcal{H}_\infty$ such that $\|\Delta\|_\infty \leq \eta$ by all $p - 1$ controllers

$$\dot{w} = Aw + Bu + G(y - Cw), \ u = \frac{1}{N} \lambda_i Fw, \\ i = 2, 3, \ldots, p. \tag{24}$$

Proof: Referring to the proof of Lemma 3.2 we see that $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ for all i, j if and only if $\tilde{x}_i(t) \to 0$ and $\tilde{w}_i(t) \to 0$ for $i = 2, 3, \ldots, p$.

(only if) Assume now that the protocol (14) synchronizes the network for all perturbations Δ_i with $\|\Delta_i\|_\infty \leq \eta$. Consider the system (23) and take an arbitrary $\Delta \in R\mathcal{H}_\infty$ with $\|\Delta\|_\infty \leq \eta$. Let $\dot{\xi} = A_\Delta \xi + B_\Delta z$, $d = C_\Delta \xi + D_\Delta z$ be a realization of $\Delta(s)$ with A_Δ Hurwitz. We want to show that for $i = 2, 3, \ldots, p$ the closed loop system obtained by interconnecting (23) and (24), i.e.

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} A & \frac{1}{N} \lambda_i BF \\ GC & A - GC + \frac{1}{N} \lambda_i BF \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ G \end{pmatrix} d, \tag{25}$$

$$z = \frac{1}{N} \lambda_i Fw, \\ \dot{\xi} = A_\Delta \xi + B_\Delta z, \tag{26} \\ d = C_\Delta \xi + D_\Delta z,$$

is internally stable. In order to show this, in the network perturb each agent i with the given perturbation Δ, i.e. $\Delta_i = \Delta$ for all i. Then in (21) we obtain $\tilde{\mathbf{d}} = (U^\top \otimes I)(I \otimes \Delta)(U \otimes I)\tilde{\mathbf{z}} = (I \otimes \Delta)\tilde{\mathbf{z}}$. The network with this perturbation is synchronized by our protocol, so in (19) we have $\tilde{x}_i(t) \to 0$ and $\tilde{w}_i(t) \to 0$ for $i = 2, 3, \ldots, p$. This however implies that for each $i = 2, 3, \ldots, p$ in the system

$$\begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{w}}_i \end{pmatrix} = \begin{pmatrix} A & BF \\ \frac{1}{N} \lambda_i GC & A - GC + \frac{1}{N} \lambda_i BF \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} + \\ \begin{pmatrix} 0 \\ \frac{1}{N} \lambda_i G \end{pmatrix} \tilde{d}_i, \\ \tilde{z}_i = F \tilde{w}_i, \\ \dot{\xi}_i = A_\Delta \xi_i + B_\Delta \tilde{z}_i, \\ \tilde{d}_i = C_\Delta \xi_i + D_\Delta \tilde{z}_i,$$

we have $\tilde{x}_i(t) \to 0$ and $\tilde{w}_i(t) \to 0$. Since $\tilde{w}_i(t) \to 0$, also $\tilde{z}_i(t) \to 0$ and therefore, since A_Δ is Hurwitz, $\xi_i(t) \to 0$ as $t \to \infty$. By the simple transformation $\tilde{w}_i = \frac{1}{N} \lambda_i \overline{w}_i$, this results in a copy of the system given by (25) and (26), which is therefore internally stable.

(if) We now prove the converse. Assume the $p - 1$ controllers (24) all internally stabilize the system (23) for all $\Delta \in R\mathcal{H}_\infty$ with $\|\Delta\|_\infty \leq \eta$. By the small gain theorem then, for $i = 2, 3, \ldots, p$ the closed loop systems (25) are internally stable and their transfer matrices G_i from d to z satisfy $\|G_i\|_\infty < \frac{1}{\eta}$. We now show that the protocol (14) synchronizes the perturbed network for all agent perturbations Δ_i with $\|\Delta_i\|_\infty \leq \eta$. Thus, take arbitrary perturbations Δ_i with $\|\Delta_i\|_\infty \leq \eta$. We need to show that for $i = 2, 3, \ldots, p$ we have $\tilde{x}_i(t) \to 0$ and $\tilde{w}_i(t) \to$

0, where $\tilde{x}_i$ and $\tilde{w}_i$ satisfy (19), (20) and (21). Denote

$$
\begin{pmatrix} \Delta_{11} & \cdots & \Delta_{1p} \\ \vdots & \ddots & \vdots \\ \Delta_{p1} & \cdots & \Delta_{pp} \end{pmatrix} :=
$$

$$
(U^\top \otimes I) \begin{pmatrix} \Delta_1 & 0 & \cdots & 0 \\ 0 & \Delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_p \end{pmatrix} (U \otimes I). \quad (27)
$$

Since U is orthogonal, the $\mathcal{H}_\infty$-norm of the left hand side is less than η. Now we want to write out the dynamics of $\tilde{x}_2, \tilde{x}_2, \ldots, \tilde{x}_p$, and $\tilde{w}_2, \tilde{w}_2, \ldots, \tilde{w}_p$. First note from (19) that $\tilde{w}_1$ is governed by the equation $\dot{\tilde{w}}_1 = (A - GC)\tilde{w}_1$. Note that $A - GC$ is Hurwitz. Now denote $\bar{\mathbf{x}} = \mathrm{col}(\tilde{x}_2, \tilde{x}_2, \ldots, \tilde{x}_p)$, $\bar{\mathbf{w}} = \mathrm{col}(\tilde{w}_2, \tilde{w}_2, \ldots, \tilde{w}_p)$, $\bar{\mathbf{z}} = \mathrm{col}(\tilde{z}_2, \tilde{z}_2, \ldots, \tilde{z}_p)$ and $\bar{\mathbf{d}} = \mathrm{col}(\tilde{d}_2, \tilde{d}_2, \ldots, \tilde{d}_p)$. Then from (19) we obtain

$$
\begin{pmatrix} \dot{\bar{\mathbf{x}}} \\ \dot{\bar{\mathbf{w}}} \end{pmatrix} = \begin{pmatrix} I_{p-1} \otimes A & I_{p-1} \otimes BF \\ \frac{1}{N}\Lambda_1 \otimes GC & I_{p-1} \otimes (A - GC) + (\frac{1}{N}\Lambda_1 \otimes BF) \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix}
$$
$$
+ \begin{pmatrix} 0 \\ \frac{1}{N}\Lambda_1 \otimes G \end{pmatrix} \bar{\mathbf{d}}, \quad (28)
$$

$$
\bar{\mathbf{z}} = \begin{pmatrix} 0 & I_{p-1} \otimes F \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix}, \quad (29)
$$

$$
\bar{\mathbf{d}} = \begin{pmatrix} \Delta_{22} & \cdots & \Delta_{2p} \\ \vdots & \ddots & \vdots \\ \Delta_{p2} & \cdots & \Delta_{pp} \end{pmatrix} \bar{\mathbf{z}} + \begin{pmatrix} \Delta_{21} \\ \vdots \\ \Delta_{p1} \end{pmatrix} \tilde{z}_1. \quad (30)
$$

Here $\Lambda_1 := \mathrm{diag}(\lambda_2, \ldots, \lambda_p)$. In this system the transfer matrix from $\bar{\mathbf{d}}$ to $\bar{\mathbf{z}}$ is equal to $G := \mathrm{blockdiag}(G_2, \ldots G_p)$ so $\|G\|_\infty < \frac{1}{\eta}$. We also have

$$
\| \begin{pmatrix} \Delta_{22} & \cdots & \Delta_{2p} \\ \vdots & \ddots & \vdots \\ \Delta_{p2} & \cdots & \Delta_{pp} \end{pmatrix} \|_\infty \le \eta,
$$

Finally, since $\tilde{z}_1 = F\tilde{w}_1$ with $\dot{\tilde{w}}_1 = (A - GC)\tilde{w}_1$ stable, we conclude that $\bar{\mathbf{x}}(t) \to 0$ and $\bar{\mathbf{w}}(t) \to 0$. This completes the proof of the theorem. $\quad\square$

We now turn to the case that the network graph is directed and contains a spanning tree. It turns out that the results for the undirected graph case basically carry over to this case, in the sense that robust stabilization by $p-1$ controllers is equivalent with robust synchronization where for each agent i the perturbation Δ_i is equal to one and the same Δ. In other words, the agents are assumed to be perturbed identically. A proof of this can be given by suitably adapting the corresponding proof of Theorem 4.2. As in the previous section, in the directed graph case the role of the orthogonal transformation U is taken over by a complex unitary transformation U that brings the Laplacian L to *upper diagonal form*: $U^*LU = \Lambda_u$, with Λ_u complex upper triangular with $0, \lambda_2, \lambda_3, \ldots, \lambda_p$ on the diagonal. A key ingredient in the proof is that if $\Delta_i = \Delta$ for all $i = 1, 2, \ldots, p$, then the left hand side of (27) will remain block diagonal, so that in (30) the second term vanishes and the

small gain argument continues to hold. The precise statement is as follows:

Proposition 4.3: Consider the network with agent dynamics given by (1). Assume the network graph is directed and contains a spanning tree. Let $\eta > 0$. Then the following two statements are equivalent:
  1) the dynamic protocol (14) synchronizes the network with perturbed agent dynamics (22) where for each $i = 1, 2, \ldots, p$ we have $\Delta_i = \Delta$ with $\Delta \in R\mathcal{H}_\infty$ and $\|\Delta\|_\infty \le \eta$,
  2) the perturbed linear system (23) is internally stabilized for all $\Delta \in R\mathcal{H}_\infty$ with $\|\Delta\|_\infty \le \eta$ by all $p-1$ controllers (24).

Remark 4.4: Proposition 4.3 brings about a striking difference between the undirected and directed graph case. Whereas in the undirected graph case the N, F and G appearing in the set of $p-1$ controllers (24) yield a protocol that robustly synchronizes the perturbed network for all perturbations Δ_i with $\|\Delta_i\|_\infty \le \eta$, possibly different for different agents, in the directed graph case the protocol only robustly synchronizes the network against perturbations that are identical for each i, i.e., $\Delta_i = \Delta$ with $\|\Delta\|_\infty \le \eta$, and will in general not robustly synchronize against 'heterogeneous' additive perturbations.

Remark 4.5: By Theorem 4.2 and Proposition 4.3, both in the directed and undirected graph case, in order to obtain a protocol that robustly synchronizes the network with synchronization radius $\eta > 0$, it suffices to find a positive real number N, and gain matrices F and G such that all $p-1$ controllers (24) robustly internally stabilize the (single) system (23) with stability radius η. Obviously, by the small gain theorem (see e.g. [27]), this requires that any of the controllers (24) solves the $\mathcal{H}_\infty$-control problem for the system $\dot{x} = Ax + Bu$, $y = Cx + d$, $z = u$ in the sense that the closed loop system is internally stable and $\|G_i\|_\infty < \frac{1}{\eta}$, where G_i is the closed loop transfer matrix from d to z. In the sequel, we will explain how to obtain such N, F and G.

V. ROBUSTLY SYNCHRONIZING PROTOCOLS

In this section we will, for given desired synchronization radius, establish conditions for the existence of robustly synchronizing dynamic protocols that achieve this radius, and algorithms to compute such protocols.

The idea that we will use is the following. It follows from Theorem 4.2 and Proposition 4.3 that the protocol (14) robustly synchronizes the network if the agent dynamics is robustly internally stabilized by every controller in the collection of $p-1$ controllers given by (24). In the sequel, we will propose methods to compute a positive real number N, gain matrices F and G and a tolerance η such that all controllers (24) robustly stabilize the system (23) with respect to this tolerance. We will first do this in detail for the undirected graph case, and subsequently treat the more intricate case that the network graph is directed.

A. Undirected graph case

For simplicity, we first consider the case that the matrix A does not have eigenvalues on the imaginary axis. Associated

with (A, B, C) we consider the following algebraic Riccati equation

$$A^\top P + PA - \gamma PBB^\top P = 0, \qquad (31)$$

together with the strict Riccati inequality

$$AQ + QA^\top - QC^\top CQ < 0. \qquad (32)$$

In (31), γ is a positive real number that will be specified later. Let $P(\gamma)$ be the maximal real symmetric solution of (31). Then $P(\gamma) \geq 0$. Also, $A - \gamma BB^\top P(\gamma)$ is Hurwitz (this uses the assumption that A has no eigenvalues on the imaginary axis). Let $Q > 0$ be any real symmetric positive definite solution to (32). It is easily seen that such Q exists.

Assume now that our network graph is undirected and connected. Recall that λ_2 and λ_p are the second smallest and largest eigenvalue of the Laplacian L, and that $\lambda_2 > 0$. The following theorem yields a robustly synchronizing dynamic protocol for the perturbed multi-agent network. The synchronization radius that we obtain depends on the spectral radius $\rho(P(\gamma)Q)$ of the product of $P(\gamma)$ and Q as given by (31) and (32):

Theorem 5.1: Consider the network with p agents, where the network graph is undirected and connected. Let perturbed agent i be given by

$$\dot{x}_i = Ax_i + Bu_i, \ y_i = Cx_i + d_i, \ z_i = u_i, \ d_i = \Delta_i z_i$$

Assume that A has no eigenvalues on the imaginary axis. Choose N any positive real number such that

$$N > \frac{\lambda_p^2}{\lambda_2}, \qquad (33)$$

equivalently $(\frac{\lambda_p}{N})^2 < \frac{\lambda_2}{N}$. Next, choose γ such that

$$(\frac{\lambda_p}{N})^2 < \gamma < \frac{\lambda_2}{N}. \qquad (34)$$

Then, let $P(\gamma)$ be the maximal real symmetric solution of (31) and let $Q > 0$ be any solution of (32). Let η be any positive real number such that

$$\eta < \frac{1}{\sqrt{\rho(P(\gamma)Q)}}. \qquad (35)$$

Define

$$F := -B^\top P(\gamma) \qquad (36)$$

$$G := (I - \eta^2 QP(\gamma))^{-1} QC^\top \qquad (37)$$

Then the dynamic protocol (14) synchronizes the network for all perturbations $\Delta_i \in R\mathcal{H}_\infty$ $(i = 1, 2 \ldots, p)$ such that $\|\Delta_i\|_\infty \leq \eta$.

Proof: According to Theorem 4.2, we should prove that any of the controllers (24), with N, F and G chosen as in the theorem statement, solves the $\mathcal{H}_\infty$-control problem for the system $\dot{x} = Ax + Bu$, $y = Cx + d$, $z = u$ in the sense that the closed loop system is internally stable and $\|G_i\|_\infty < \frac{1}{\eta}$, where G_i is the closed loop transfer matrix from d to z $(i = 2, 3, \ldots, p)$. Recall that these closed loop systems are given by (25). In order to show that they are internally stable and that $\|G_i\|_\infty < \frac{1}{\eta}$, we first apply a state transformation

$$\begin{pmatrix} \overline{x} \\ \overline{w} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

to these systems. This yields

$$\begin{pmatrix} \dot{\overline{x}} \\ \dot{\overline{w}} \end{pmatrix} = \begin{pmatrix} A + \frac{1}{N}\lambda_i BF & \frac{1}{N}\lambda_i BF \\ 0 & A - GC \end{pmatrix} \begin{pmatrix} \overline{x} \\ \overline{w} \end{pmatrix} + \begin{pmatrix} 0 \\ G \end{pmatrix} d, \quad (38)$$

$$z = (\frac{1}{N}\lambda_i F \quad \frac{1}{N}\lambda_i F) \begin{pmatrix} \overline{x} \\ \overline{w} \end{pmatrix}. \qquad (39)$$

Next, we apply Lemma 2.1 to the systems (38). In fact, we will show that for each $i = 2, 3, \ldots, p$, the relevant Riccati inequality associated with (38) has a positive semidefinite real symmetric solution. In the following, for notational convenience we denote $P := P(\gamma)$ and

$$\mu_i := \frac{\lambda_i}{N} \quad (i = 2, 3, \ldots, p).$$

First note that, since $(\eta^2 Q)^{-1} - P > 0$ (which follows from (35)), for all $i = 2, 3, \ldots, p$ we have

$$\begin{pmatrix} \mu_i P & 0 \\ 0 & (\eta^2 Q)^{-1} - P \end{pmatrix} \geq 0. \qquad (40)$$

Recall that F and G are given by (36) and (37). By straightforward calculation, for all i we have

$$\begin{pmatrix} A + \mu_i BF & \mu_i BF \\ 0 & A - GC \end{pmatrix}^\top \begin{pmatrix} \mu_i P & 0 \\ 0 & (\eta^2 Q)^{-1} - P \end{pmatrix} +$$
$$\begin{pmatrix} \mu_i P & 0 \\ 0 & (\eta^2 Q)^{-1} - P \end{pmatrix} \begin{pmatrix} A + \mu_i BF & \mu_i BF \\ 0 & A - GC \end{pmatrix} +$$
$$\eta^2 \begin{pmatrix} \mu_i P & 0 \\ 0 & (\eta^2 Q)^{-1} - P \end{pmatrix} \begin{pmatrix} 0 \\ G \end{pmatrix}$$
$$\times \begin{pmatrix} 0 \\ G \end{pmatrix}^\top \begin{pmatrix} \mu_i P & 0 \\ 0 & (\eta^2 Q)^{-1} - P \end{pmatrix} +$$
$$(\mu_i F \quad \mu_i F)^\top (\mu_i F \quad \mu_i F) \leq \qquad (41)$$
$$\begin{pmatrix} \mu_i(\gamma - \mu_i) & 0 \\ 0 & -\gamma + \mu_i^2 \end{pmatrix} \otimes F^\top F.$$

By (34) we have $-\gamma + \mu_i^2 < 0$ and $\gamma - \mu_i < 0$ for all $i = 2, 3, \ldots, p$, and hence the 2×2 matrix on the right hand side of the inequality (41) is negative definite. Now, for fixed i, let $\epsilon > 0$ be sufficiently small such that

$$\begin{pmatrix} \mu_i(\gamma - \mu_i) & 0 \\ 0 & -\gamma + \mu_i^2 \end{pmatrix} \leq -\epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the right hand side of the inequality (41) is bounded from above by

$$-\epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes F^\top F = -\epsilon \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}^\top \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}.$$

Since, obviously

$$\ker \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \subset \ker (\mu_i F \quad \mu_i F),$$

we are now in a position to apply Lemma 2.1, provided that

$$\begin{pmatrix} A + \mu_i BF & \mu_i BF \\ 0 & A - GC \end{pmatrix}$$

is Hurwitz. This will be proven next.

We first prove that $A - \mu_i BB^\top P$ is Hurwitz. Using (31) we obtain

$$(A - \mu_i BB^\top P)^\top P + P(A - \mu_i BB^\top P) = (\gamma - 2\mu_i)PBB^\top P.$$

Let ρ be an eigenvalue of $A - \mu_i BB^\top P$ with eigenvector, say v. Then we obtain $2\,\mathrm{Re}(\rho)v^*Pv = (\gamma - 2\mu_i)\|B^\top Pv\|^2$. First consider the case $v^*Pv = 0$. Since $\gamma - 2\mu_i < 0$ we must have $B^\top Pv = 0$. This however yields $(A - \gamma BB^\top P)v = \rho v$, so $\mathrm{Re}(\rho) < 0$. The case $v^*Pv > 0$ in as similar way yields to $\mathrm{Re}(\rho) < 0$. A proof that $A - GC$ is Hurwitz can be given along the same lines using the Riccati inequality (41) together with detectability of (C, A).

Using Lemma 2.1 we finally conclude that for each i the transfer matrix G_i of (38) satisfies $\|G_i\|_\infty < \frac{1}{\eta}$. $\qquad\square$

Remark 5.2: We note that here, and also elsewhere in this paper, if $\rho(P(\gamma)Q) = 0$ then $\frac{1}{\sqrt{\rho(P(\gamma)Q)}}$ should simply be interpreted as $+\infty$. In that case the inequality (35) does not give an upper bound on η, and the synchronization radius is $+\infty$. This occurs if A is Hurwitz and consequently $P(\gamma) = 0$.

Remark 5.3: Thus, for the case that the network graph is undirected and connected, under the assumption that A has no imaginary axis eigenvalues, computation of a robustly synchronizing protocol can be performed as follows.

1) Choose $N > \frac{\lambda_p^2}{\lambda_2}$,
2) Choose γ in the thus established non-empty interval $(\frac{\lambda_p^2}{N^2}, \frac{\lambda_2}{N})$,
3) Compute the maximal real symmetric solution $P(\gamma)$ of the ARE (31) and a positive definite real symmetric solution Q of the strict Riccati inequality (32)
4) Choose a value of the synchronization radius $\eta < \frac{1}{\sqrt{\rho(P(\gamma)Q)}}$,
5) Compute the gains F and G given by (36) and (37).

Note that $\rho(P(\gamma)Q)$ decreases with decreasing Q. Thus, the achievable radius can be increased by decreasing the solution $Q > 0$ of the inequality (32). In effect, this can be done by taking for Q the maximal solution $Q(\tau)$ of the ARE $AQ + QA^\top - QC^\top CQ + \tau I = 0$ with $\tau > 0$. It is easily seen that if τ decreases, then $Q(\tau)$ decreases.

We now deal with the general case that A can have imaginary axis eigenvalues, which is of course required in order to be able to deal with, for example, periodic agent dynamics. The computation of robustly synchronizing protocols is then only slightly more complicated.

Again, first choose N such that $N > \frac{\lambda_p^2}{\lambda_2}$, and choose γ such that $(\frac{\lambda_p}{N})^2 < \gamma < \frac{\lambda_2}{N}$. Then, consider the pair of algebraic Riccati equations

$$A^\top P + PA - \gamma PBB^\top P = 0, \qquad (42)$$
$$AQ + QA^\top - QC^\top CQ = 0. \qquad (43)$$

Let $P(\gamma)$ and Q be the maximal real symmetric solutions. We claim that for each $\eta > 0$ satisfying

$$\eta < \frac{1}{\sqrt{\rho(P(\gamma)Q)}} \qquad (44)$$

robust synchronization can be achieved. The construction of a protocol goes along the following lines. Consider the pair of parametrized Riccati equations

$$A^\top P + PA - \gamma PBB^\top P + \sigma I = 0, \qquad (45)$$
$$AQ + QA^\top - QC^\top CQ + \tau I = 0, \qquad (46)$$

with parameters $\sigma, \tau > 0$. Denote the maximal real symmetric solutions by $P(\gamma, \sigma)$ and $Q(\tau)$, respectively. It is easily seen that for each γ, $P(\gamma, \sigma) \downarrow P(\gamma)$ as $\sigma \downarrow 0$, and $Q(\tau) \downarrow Q$ as $\tau \downarrow 0$. Thus, $\rho(P(\gamma, \sigma)Q(\tau)) \downarrow \rho(P(\gamma)Q)$ as $\sigma \downarrow 0$ and $\tau \downarrow 0$. As a consequence, (44) implies that for σ and τ sufficiently small, say for $\sigma, \tau \in (0, \epsilon)$, we have

$$\eta < \frac{1}{\sqrt{\rho(P(\gamma, \sigma)Q(\tau))}}. \qquad (47)$$

Now take $\tau \in (0, \epsilon)$ and compute $Q(\tau)$. Take $\sigma \in (0, \epsilon)$ such that the inequality

$$\sigma I \leq \frac{\tau}{\eta^2} Q(\tau)^{-2} \qquad (48)$$

is satisfied. Define now

$$F := -B^\top P(\gamma, \sigma), \qquad (49)$$
$$G := (I - \eta^2 Q(\tau)P(\gamma, \sigma))^{-1}Q(\tau)C^\top. \qquad (50)$$

Theorem 5.4: Assume the network graph is undirected and connected. Then the dynamic protocol with $N > \frac{\lambda_p^2}{\lambda_2}$, γ chosen such that $(\frac{\lambda_p}{N})^2 < \gamma < \frac{\lambda_2}{N}$, and F and G defined by (49) and (50), with $\sigma >$ and $\tau > 0$ sufficiently small, achieves synchronization for all $\Delta_i \in R\mathcal{H}_\infty$ such that $\|\Delta_i\|_\infty \leq \eta$.

Proof: As in the proof of Theorem 5.1 we apply Lemma 2.1 to the systems (38). We will again show that for each $i = 2, 3, \ldots, p$ the Riccati inequality associated with (38) has a positive semi-definite real symmetric solution (this time even positive definite). Recall that $\mu_i := \frac{\lambda_i}{N}$ ($i = 2, 3, \ldots, p$).

For F and G given by (49) and (50) and with $P := P(\gamma, \sigma)$ and $Q := Q(\tau)$, we obtain that the left hand side of the inequality (41) this time is bounded from above by

$$\begin{pmatrix} \mu_i(\gamma - \mu_i) & 0 \\ 0 & -\gamma + \mu_i^2 \end{pmatrix} \otimes F^\top F + \\ \begin{pmatrix} -\mu_i \sigma I & 0 \\ 0 & -\frac{\tau}{\eta^2}(Q^{-1})^2 + \sigma I \end{pmatrix}.$$

Clearly the inequality (48) is equivalent with $-\frac{\tau}{\eta^2}(Q^{-1})^2 + \sigma I \leq 0$. We can now repeat the argument in the proof of Theorem 5.1. The fact that for each i the system matrix is Hurwitz is proven along the same lines as the corresponding proof in Theorem 5.1. $\qquad\square$

B. Directed graph case

In this subsection we deal with the case that the network graph is directed. This case requires a more intricate analysis, but also here explicit closed form results analogous to the undirected graph case can be obtained. As before, assume the network graph contains a spanning tree. Within the set of nonzero eigenvalues $\{\lambda_2, \lambda_3, \ldots, \lambda_p\}$ of L, let λ_m have minimal real part, λ_M have maximal modulus, and λ_ℓ have maximal argument, i.e.

$$\mathrm{Re}(\lambda_m) = \min_{i=2,\ldots,p} \mathrm{Re}(\lambda_i), \qquad (51)$$
$$|\lambda_M| = \max_{i=2,\ldots,p} |\lambda_i|, \qquad (52)$$
$$\mathrm{Arg}(\lambda_\ell) = \max_{i=2,\ldots,p} \mathrm{Arg}(\lambda_i). \qquad (53)$$

Here, 'Arg' denotes the principal value of the argument. Note that $-\pi/2 < \text{Arg}(\lambda_i) < \pi/2$. For $N > 0$, define

$$f_N := \frac{\text{Re}(\lambda_m)}{|\lambda_M|^2} N. \tag{54}$$

The following theorem states how to compute robustly synchronizing dynamic protocols for the given network. The protocols have the same structure as in the undirected case, and are determined by a real weighting coefficient N and gain matrices F and G. The main difference lies in the choice of the weighting coefficient N and the parameter γ in the ARE (31).

Theorem 5.5: Consider the network with p agents, where the network graph is directed and contains a spanning tree. Let perturbed agent i be given by

$$\dot{x}_i = A x_i + B u_i, \ y_i = C x_i + d_i, \ z_i = u_i, \ d_i = \Delta z_i$$

Assume that A has no eigenvalues on the imaginary axis. Choose N any positive real number such that

$$f_N > 1 \quad (\text{ equivalently, } N > \frac{|\lambda_M|^2}{\text{Re}(\lambda_m)}) \tag{55}$$

and

$$f_N + \frac{1}{f_N} > 2 + 4\tan^2(\text{Arg}(\lambda_\ell)) \tag{56}$$

(such N always exists). Next, choose γ as

$$\gamma = \frac{1}{2}\left(\frac{|\lambda_M|^2}{N^2} + \frac{\text{Re}(\lambda_m)}{N} \right) \tag{57}$$

Then, let $P(\gamma)$ be the maximal real symmetric solution of (31) and let $Q > 0$ be any solution of (32). Let η be any positive real number such that (35) holds. Define F by (36) and G by (37). Then the dynamic protocol (14) synchronizes the network for all agent perturbations $\Delta \in R\mathcal{H}_\infty$ with $\|\Delta\|_\infty \leq \eta$.

Proof: According to Proposition 4.3 it suffices to choose real N and gain matrices F and G such that each of the $p-1$ (complex) controllers (24) robustly stabilizes the single system (23). Again, denote $\mu_i = \lambda_i/N$. A first idea is to mimic the proof of the undirected graph case, and check under what conditions the complex versions of the quadratic inequalities (41) have complex Hermitian positive semi-definite solutions, see also Remark 2.2. Note that the 'old' solutions

$$\begin{pmatrix} \mu_i P & 0 \\ 0 & (\eta^2 Q)^{-1} - P \end{pmatrix} \tag{58}$$

($i = 2, 3, \ldots, p$) will not be Hermitian if μ_i is not real, and therefore do no longer qualify as solutions. Instead, as candidate solutions we replace (58) by the following:

$$\begin{pmatrix} k_i P & 0 \\ 0 & (\eta^2 Q)^{-1} - P \end{pmatrix} \tag{59}$$

where the k_i are real and nonnegative, and are to be determined. Substituting (59) into the complex version of (41)

yields

$$\begin{pmatrix} A + \mu_i B F & \mu_i B F \\ 0 & A - GC \end{pmatrix}^* \begin{pmatrix} k_i P & 0 \\ 0 & (\eta^2 Q)^{-1} - P \end{pmatrix} +$$

$$\begin{pmatrix} k_i P & 0 \\ 0 & (\eta^2 Q)^{-1} - P \end{pmatrix} \begin{pmatrix} A + \mu_i B F & \mu_i B F \\ 0 & A - GC \end{pmatrix} +$$

$$\eta^2 \begin{pmatrix} k_i P & 0 \\ 0 & (\eta^2 Q)^{-1} - P \end{pmatrix} \begin{pmatrix} 0 \\ G \end{pmatrix}$$

$$\times \begin{pmatrix} 0 \\ G \end{pmatrix}^* \begin{pmatrix} k_i P & 0 \\ 0 & (\eta^2 Q)^{-1} - P \end{pmatrix} +$$

$$\begin{pmatrix} \mu_i F & \mu_i F \end{pmatrix}^* \begin{pmatrix} \mu_i F & \mu_i F \end{pmatrix} \leq$$

$$\begin{pmatrix} k_i(\gamma - 2\text{Re}(\mu_i)) + |\mu_i|^2 & -k_i \mu_i + |\mu_i|^2 \\ -k_i \overline{\mu_i} + |\mu_i|^2 & -\gamma + |\mu_i|^2 \end{pmatrix} \otimes F^\top F.$$

Thus, we need to find real N, γ and $k_i \geq 0$ such that

$$\begin{pmatrix} k_i(\gamma - 2\text{Re}(\mu_i)) + |\mu_i|^2 & -k_i \mu_i + |\mu_i|^2 \\ -k_i \overline{\mu_i} + |\mu_i|^2 & -\gamma + |\mu_i|^2 \end{pmatrix} < 0 \tag{60}$$

holds for each $i = 2, 3, \ldots, p$. In the sequel we show that this is always possible. Indeed, define

$$k_i := \frac{|\mu_i|^2}{\text{Re}(\mu_i)}.$$

Then, the inequality (60) becomes

$$\begin{pmatrix} \frac{|\mu_i|^2}{\text{Re}(\mu_i)}(\gamma - 2\text{Re}(\mu_i)) + |\mu_i|^2 & |\mu_i|^2(1 - \frac{\mu_i}{\text{Re}(\mu_i)}) \\ |\mu_i|^2(1 - \frac{\overline{\mu_i}}{\text{Re}(\mu_i)}) & -\gamma + |\mu_i|^2 \end{pmatrix} < 0. \tag{61}$$

Clearly, (61) holds if and only if the two diagonal elements are negative and the determinant is positive, equivalently

$$|\mu_i|^2 < \gamma < \text{Re}(\mu_i) \tag{62}$$

and

$$(\frac{|\mu_i|^2}{\text{Re}(\mu_i)}(\gamma - 2\text{Re}(\mu_i)) + |\mu_i|^2)(-\gamma + |\mu_i|^2) -$$
$$|\mu_i|^4 |1 - \frac{\mu_i}{\text{Re}(\mu_i)}|^2 > 0 \tag{63}$$

It is easily verified that (63) is equivalent to

$$(1 - \frac{\gamma}{\text{Re}(\mu_i)})(\frac{\gamma}{|\mu_i|^2} - 1) - \tan^2(\text{Arg}(\mu_i)) > 0. \tag{64}$$

Now, referring to (51), (52) and (53), note that μ_m minimizes $\text{Re}(\mu_i)$ and that μ_M and μ_ℓ maximize μ_i and $\text{Arg}(\mu_i)$, respectively. It is then easy to observe that the inequalities (62) and (64) are satisfied for all $i = 2, 3, \ldots, p$ if both

$$|\mu_M|^2 < \gamma < \text{Re}(\mu_m) \tag{65}$$

and

$$(1 - \frac{\gamma}{\text{Re}(\mu_m)})(\frac{\gamma}{|\mu_M|^2} - 1)) > \tan^2(\text{Arg}(\mu_\ell)) \tag{66}$$

hold. Note that (65) is equivalent to the condition that (62) holds for all i, and (66) provides a sufficient condition for (64) to hold for all i. Also note that there exists γ such that (65) holds if and only if (55) holds.

Observe now that the right hand side of (66) is independent of N and γ. It can be verified that, as a polynomial function of γ, the left hand side is maximized by choosing γ as

$$\gamma = \frac{|\mu_M|^2 + \mathrm{Re}(\mu_m)}{2}, \qquad (67)$$

which, clearly, satisfies (65). Note that this expression for γ coincides with (57). Then, the inequality (66) simplifies to

$$(1 - \frac{|\mu_M|^2}{\mathrm{Re}(\mu_m)})(\frac{\mathrm{Re}(\mu_m)}{|\mu_M|^2} - 1) > 4\tan^2(\mathrm{Arg}(\mu_\ell)).$$

This can be rewritten as

$$(1 - \frac{1}{f_N})(f_N - 1) > 4\tan^2(\mathrm{Arg}(\mu_\ell)) \qquad (68)$$

with f_N given by (54). Clearly, (68) is satisfied if (56) holds. We conclude that the inequality (61), with γ chosen as (57), is satisfied if both (55) and (56) are satisfied. The remainder of the proof is analogous to the proof of Theorem 5.1, with 'transpose' replaced by 'conjugate transpose', and using the fact that $\gamma < \mathrm{Re}(\mu_i)$ for all i. □

Remark 5.6: In Section 6, we will show that the upper bound (44) on the tolerance η increases if γ increases, equivalently, N decreases. Note that the size of N depends on how the nonzero Laplacian eigenvalues are distributed over the open right half plane. It follows from (55) and (56) that N tends to be large if the maximal modulus $|\lambda_M|$ is large, the minimal real part $\mathrm{Re}(\lambda_m)$ is small and if the maximal argument $\mathrm{Arg}(\lambda_\ell)$ is close to $\pi/2$, i.e. the eigenvalue λ_ℓ is close to the imaginary axis. It was shown in [1] that for a graph with p nodes actually $-\frac{\pi}{2} + \frac{\pi}{p} \le \mathrm{Arg}(\lambda_i) \le \frac{\pi}{2} - \frac{\pi}{p}$ ($i = 2, 3, \ldots, p$), which indicates that smaller values of p tend to require smaller values of N.

Remark 5.7: Note that, as expected, Theorem 5.5 also captures the undirected case. Indeed, if the Laplacian eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_p$ are real, then with the usual ordering $\lambda_2 \le \lambda_3 \le \ldots \le \lambda_p$ we have $\mathrm{Re}(\lambda_m) = \lambda_2$, $|\lambda_M| = \lambda_p$ and the argument of all eigenvalues is equal to 0, so $\mathrm{Arg}(\lambda_\ell) = 0$. Thus $f_N = \frac{\lambda_2}{\lambda_p^2} N$ and the condition $f_N > 1$ is equivalent to $N > \frac{\lambda_p^2}{\lambda_2}$, i.e. condition (33). Since $\tan(\mathrm{Arg}(\lambda_\ell)) = 0$, condition (56) becomes $f_N + \frac{1}{f_N} > 2$, which is satisfied automatically for any positive f_N. Finally, the choice $\gamma = \frac{1}{2}\left(\frac{\lambda_p^2}{N^2} + \frac{\lambda_2}{N}\right)$ obviously satisfies $(\frac{\lambda_p}{N})^2 < \gamma < \frac{\lambda_2}{N}$, i.e. condition (34) (in fact it lies exactly in the middle of this interval).

We will now give a simple example to illustrate the above method for directed graphs.

Example 5.8: Consider the network with agent dynamics as usual given by A, B and C and network graph given by the three-node directed circle graph with Laplacian

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

The nonzero eigenvalues are $\lambda_2 = \frac{3}{2} + j\frac{1}{2}\sqrt{3}$, $\lambda_3 = \frac{3}{2} - j\frac{1}{2}\sqrt{3}$. Clearly, $\mathrm{Re}(\lambda_m) = \frac{3}{2}$, $|\lambda_M|^2 = 3$ and $\tan^2(\mathrm{Arg}(\lambda_\ell)) = \frac{1}{3}$. Thus $f_N = \frac{N}{2}$ so conditions (55) and (56) hold if and only if $\frac{N}{2} > 1$ and $\frac{N}{2} + \frac{2}{N} > \frac{10}{3}$, equivalently $N > 6$. As an example

take $N = 10$. Then take $\gamma = \frac{1}{2}\left(\frac{3}{N^2} + \frac{3}{2N}\right) = 0.09$, and solve the ARE (31) to obtain $P(0.09)$ and the inequality (32) to obtain Q. Next take η such that (34) is satisfied and compute F and G to obtain a protocol that achieves synchronization radius η.

We conclude this subsection by noting that the limiting argument used in the undirected graph case for the situation that we allow the matrix A to have imaginary axis eigenvalues carries over unchanged to the directed graph case. Thus, the analogue of Theorem 5.4 for directed graphs can be formulated, using the choices of N and γ as in Theorem 5.5. We omit the details.

VI. GUARANTEED ROBUST SYNCHRONIZATION RADIUS

In this section we will study the problem of obtaining, for a given multi-agent network, a *guaranteed* robust synchronization radius, i.e. the supremum over all values of $\eta > 0$ such that a suitable dynamic protocol of the form (2) achieves synchronization for all Δ_i with $\|\Delta_i\|_\infty \le \eta$. For given $\gamma > 0$, consider the algebraic Riccati equation (42). Again, denote by $P(\gamma)$ the maximal real symmetric solution. In addition consider the equation

$$A^\top P + PA - PB^\top BP = 0. \qquad (69)$$

Denote its maximal real symmetric solution by $\overline{P}$. It is easily seen that for all $\gamma > 0$

$$P(\gamma) = \frac{1}{\gamma}\overline{P}. \qquad (70)$$

Consider also the equation

$$AQ + QA^\top - QC^\top CQ = 0, \qquad (71)$$

and let $\overline{Q}$ be its maximal real symmetric solution. By Theorem 5.4, for each $\eta < \frac{1}{\sqrt{\rho(P(\gamma)\overline{Q})}}$ synchronization with uncertainty radius η is achieved by a suitable protocol. By (70), $\eta < \frac{1}{\sqrt{\rho(P(\gamma)\overline{Q})}}$ if and only if $\eta < \frac{\sqrt{\gamma}}{\sqrt{\rho(\overline{P}\,\overline{Q})}}$. We see that the upper bound improves by taking γ as large as possible.

We will now restrict ourselves to the undirected graph case. It will be shown that, for a given network, a guaranteed radius can be found that is proportional to the quotient λ_2/λ_p of the second smallest and the largest eigenvalue of the Laplacian. In this case, recall the restrictions $(\frac{\lambda_p}{N})^2 < \gamma < \frac{\lambda_2}{N}$ and $N > \frac{\lambda_p^2}{\lambda_2}$. We see that the upper bound on γ increases with decreasing N. Of course, the "best" (but not permitted) choice is $N = \frac{\lambda_p^2}{\lambda_2}$ and $\gamma = \frac{\lambda_2}{N}$, which would lead to $\gamma = (\frac{\lambda_2}{\lambda_p})^2$. This provides the intuition for the following theorem:

Theorem 6.1: Consider the network with p agents, where the network graph is undirected and connected. Let $\overline{P}$ and $\overline{Q}$ be the maximal real symmetric solutions of the Riccati equations (69) and (71). Then for each positive real number η that satisfies

$$\eta < \frac{\lambda_2}{\lambda_p}\frac{1}{\sqrt{\rho(\overline{P}\,\overline{Q})}} \qquad (72)$$

there exists a dynamic protocol achieving synchronization for all perturbations $\Delta_i \in R\mathcal{H}_\infty$ with $\|\Delta_i\|_\infty \le \eta$.

Proof: Let η satisfy (72). Define $\gamma := \frac{\lambda_2^2}{\lambda_p^2 + 2\delta}$ with $\delta > 0$ chosen sufficiently small so that

$$\eta < \frac{\sqrt{\gamma}}{\sqrt{\rho(\overline{P}\,\overline{Q})}}.$$

Let $P(\gamma)$ be the maximal solution of (42) corresponding to γ. Then by (70) we have

$$\eta < \frac{1}{\sqrt{\rho(P(\gamma)\overline{Q})}}.$$

Choose $N = \frac{\lambda_p^2 + \delta}{\lambda_2}$. Then obviously $N > \frac{\lambda_p^2}{\lambda_2}$. It can also be verified that γ satisfies $(\frac{\lambda_p}{N})^2 < \gamma < \frac{\lambda_2}{N}$. Now, let $P(\gamma, \sigma)$ and $Q(\tau)$ be the maximal solutions of (45) and (46). Then by Theorem 5.4, for σ and τ sufficiently small while satisfying (48), the protocol defined by N as specified above, with gain matrices (49) and (50), achieves synchronization with radius η. This completes the proof. $\qquad\square$

The above theorem establishes the intuitively appealing result that, for undirected network graphs, the guaranteed synchronization radius is proportional to the quotient λ_2/λ_p of the second smallest and the largest eigenvalue of the Laplacian. Obviously, this quotient is maximal if $\lambda_2 = \lambda_p = p$, which occurs in complete graphs. The quotient λ_2/λ_p also plays an important role in [32] where it was called *the eigenratio* of the undirected graph. In [11], page 290, it was shown that, in fact, $\lambda_2/\lambda_p \leq \frac{\min_{i=1,2\ldots,p} \deg_i}{\max_{i=1,2\ldots,p} \deg_i}$, where $\deg_i$ denotes the degree of node i.

To conclude this section, we discuss the guaranteed radius for a number of important classes of undirected graphs (see [30], [13]).

Complete graphs
For complete graphs $\lambda_2 = \lambda_p = p$. We should take $N > p$, and subsequently $\frac{p^2}{N^2} < \gamma < \frac{p}{N}$. We have $\lambda_2/\lambda_p = 1$, which is maximal.

Star graphs
For star graphs $\lambda_2 = 1$ and $\lambda_p = p$. We should take $N > p^2$ and $\frac{p^2}{N^2} < \gamma < \frac{1}{N}$. We have $\lambda_2/\lambda_p = \frac{1}{p}$, which obviously decreases with increasing number of agents.

Line graphs
For line graphs we have $\lambda_2 = 2(1 - \cos\frac{\pi}{p})$ and $\lambda_p = 2(1 + \cos\frac{\pi}{p})$. Thus for large number of agents p we have $\lambda_2 \approx 0$ and $\lambda_p \approx 4$. N will then be very large, while γ will be very small. The guaranteed radius λ_2/λ_p will be small for large p.

Cycle graphs
For cycle graphs $\lambda_2 = 2(1 - \cos\frac{2\pi}{p})$ and

$$\lambda_p = \begin{cases} 4 & p \text{ even} \\ 2(1 + \cos\frac{\pi}{p}) & p \text{ odd} \end{cases}$$

Thus, for large p we have $\lambda_2 \approx 0$ and $\lambda_p \approx 4$. Also here, the guaranteed radius λ_2/λ_p will be small for large p.

VII. EXTENSION TO NONLINEAR ADDITIVE PERTURBATIONS

In this paper we have focused on *linear* additive perturbations. In the present section we briefly outline how to extend our theory to *nonlinear* additive perturbations. Given the nominal linear agent dynamics (12) we consider perturbations given by nonlinear systems Δ_i represented by

$$\dot{\xi}_i = f_i(\xi_i, z_i), \; d_i = h_i(\xi_i, z_i), \tag{73}$$

where f_i and h_i are sufficiently smooth, and are such that for all initial conditions $\xi_i(0) = \xi_{i0}$ the system Δ_i defines an input-output map $\Delta_{i,\xi_{i0}} : L_{2,e}^m \to L_{2,e}^q$, $d_i = \Delta_{i,\xi_{i0}}(z_i)$, in the obvious way. Here $L_{2,e}$ denotes the space of all measurable functions from $\mathbb{R}^+$ to $\mathbb{R}$ that are square integrable on each finite interval $[0,T]$. We assume that the systems Δ_i have finite $\mathcal{L}_2$-gain, and the $\mathcal{L}_2$-gain of Δ_i is denoted by $g(\Delta_i)$ (see [23]). For robust synchronization we again consider weighted dynamic protocols of the form (14). Interconnecting the nominal agents (12), the nonlinear perturbations (73) and the protocol (14) yields the overall network equations in the form of a system of nonlinear differential equations of the form

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{w}} \\ \dot{\xi} \end{pmatrix} = H\left(\begin{pmatrix} \mathbf{x} \\ \mathbf{w} \\ \xi \end{pmatrix} \right) \tag{74}$$

for a given nonlinear function H. Here, as before, $\mathbf{x}, \mathbf{w}$ and ξ denote the aggregate state vectors. To avoid technicalities, we assume that, for a given protocol, all functions f_i and h_i that represent the perturbation Δ_i have the property that (74) has a unique solution for each initial state $(\mathbf{x}(0), \mathbf{w}(0), \xi(0))$. Then, allowing nonlinear perturbation with finite $\mathcal{L}_2$-gain, the problem of robust synchronization is formulated as follows:

Definition 7.1: Given a desired tolerance $\eta > 0$, find a dynamic protocol such that for all i and for all systems Δ_i of the form (73) with finite $\mathcal{L}_2$-gain $g(\Delta_i) \leq \eta$, for all $i,j = 1,2\ldots,p$ we have $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ as $t \to \infty$.

As expected, the dynamic protocols that we have constructed for robustness against linear perturbations also work for nonlinear perturbations. This follows immediately from the following theorem:

Theorem 7.2: Consider the network with agent dynamics given by (12). Assume the network graph is undirected and connected. Let $\eta > 0$. Then the dynamic protocol (14) robustly synchronizes the network with tolerance η for all nonlinear perturbations Δ_i of the form (73) with finite $\mathcal{L}_2$-gain $g(\Delta_i) \leq \eta$ if and only if the perturbed linear system (23) is internally stabilized for all $\Delta \in R\mathcal{H}_\infty$ with $\|\Delta\|_\infty \leq \eta$ by all $p - 1$ controllers (24).

Proof: The proof is along the lines of the proof of Theorem 4.2. Using the nonlinear version of the small gain theorem ([23], Theorem 2.11), it can be proven that in the interconnection of (28), (29) and the nonlinear version of (30), for all initial conditions on $\bar{\mathbf{x}}$ and $\bar{\mathbf{w}}$ and state ξ of the perturbation, the signal $\bar{\mathbf{d}}$ is in $\mathcal{L}_2(\mathbb{R}^+)$. Then, since (28) is internally stable, $\bar{\mathbf{x}}$ and $\bar{\mathbf{w}}$ must be in $\mathcal{L}_2(\mathbb{R}^+)$. This implies that also their derivatives $\dot{\bar{\mathbf{x}}}$ and $\dot{\bar{\mathbf{w}}}$ are in $\mathcal{L}_2(\mathbb{R}^+)$, which then implies that

$\bar{\mathbf{x}}(t) \to 0$ and $\bar{\mathbf{w}}(t) \to 0$ as $t \to \infty$, proving synchronization. $\square$

The analogous result holds for directed graphs containing a spanning tree and where the perturbations of the agents are assumed to be identical nonlinear systems Δ with finite $\mathcal{L}_2$-gain. Note that this implies that both for the undirected as well as the directed graph case the protocols that we have constructed in Subsections 5.1 and 5.2 also yield robust synchronization against nonlinear perturbations. Indeed, for given tolerance η, the N, F and G defining the protocol have been constructed so that the controllers (24) solve the $\mathcal{H}_\infty$-control problem for the system $\dot{x} = Ax + Bu$, $y = Cx + d$, $z = u$ in the sense that the closed loop system is internally stable and $\|G_i\|_\infty < \frac{1}{\eta}$, where G_i is the closed loop transfer matrix from d to z. As noted before, by the small gain theorem each of the controllers (24) then robustly stabilizes this single system against linear perturbations with transfer matrices $\Delta \in R\mathcal{H}_\infty$ satisfying $\|\Delta\|_\infty \leq \eta$, i.e. statement (2) of Theorem 4.2 holds.

VIII. CONCLUSIONS

In this paper we have studied the problem of robust synchronization of multi-agent networks. Given such a network with identical nominal linear dynamics for each of the agents, we allow additive perturbations of the transfer matrices of the nominal dynamics. The perturbations are assumed to be stable and bounded in $\mathcal{H}_\infty$-norm by some a priori given tolerance. Both for the case that the network graph is undirected as well as for the directed graph case we have given explicit methods to compute, for a given tolerance, observer based dynamic protocols that achieve synchronization for all tolerated perturbations. These methods require the computation of maximal real symmetric solutions of certain algebraic Riccati equations and inequalities, and also involve weighting factors that depend on the spectrum of the network graph. In the undirected graph case these factors are determined by the second smallest and the largest eigenvalue of the Laplacian. In the case of directed network graphs, the factors depend on the spectrum of the Laplacian in a more intricate way, and are determined by the minimal real part, the maximal modulus, and the maximal argument over all nonzero eigenvalues of the Laplacian. For the undirected graph case, we have shown that within the class of such dynamic protocols, a guaranteed achievable synchronization radius can be obtained that is proportional to the quotient of the second smallest and the largest eigenvalue of the Laplacian. Finally, we have shown that the protocols that we have designed also achieve robust synchronization against nonlinear perturbations with bounded $\mathcal{L}_2$-gain.

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