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Robust Time-Optimal Control: Frequency Domain Approach

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The design of nonrobust and robust time-optimal controllers for linear systems in the frequency domain is presented. The bang-bang profile is represented as the superposition of time-delayed step inputs or the output of a time-delay filter subject to a step input. A parameter optimization problem is formulated to minimize the final time of the maneuver with the constraint that the time-delay filter cancels all of the poles of the system. The issue of robustness to errors in the model is addressed by placing multiple zeros of the time-delay filter at the estimated locations of the poles of the system. The design technique is illustrated on representative models of large space structures, for rest-to-rest, time-optimal, and robust time-optimal maneuvers. Spin-up maneuvers are shown to be special cases of the general formulation.

I. Introduction

TIME-OPTIMAL control of flexible spacecraft is a topic of current interest.[1] Many computational approaches and analyses have been presented in the recent literature to deal with the effects of flexibility. Most of these works deal with planar (single-axis) rest-to-rest maneuvers under two categories: near-minimum-time control and exact-minimum-time control. The first category of methods is based on smooth approximations to minimum-time control for an equivalent rigid body. This class of methods has been shown to be well suited when applied moments or torques are produced by either throttlable thrusters or reaction wheels.[2,3] Higher modes of the system are not excited due to the smoothness of the control profile. The second category of methods deals with on-off thrusters directly. Rajan[4] formulates the problem including one elastic mode and solves a two-point boundary-value problem. Singh et al.[5] determine the switch times by solving a set of nonlinear algebraic equations. They also show that the control profile is antisymmetric about the midmaneuver time for a rest-to-rest maneuver. Ben-Asher et al.[6] simplify the computational process for linear models by formulating a parameter optimization problem for which the gradients can be computed analytically. They also solve the problem, including nonlinear terms due to the centrifugal stiffening effect, using a shooting technique. Hablani[7] discusses the single-axis slew problem from a geometric viewpoint and gives examples including the effect of damping. It is well known that the time-optimal control is highly sensitive to errors in the system parameters. Liu and Wie[8] present a method to robustify the time-optimal control using input preshaping as proposed by Singer and Seering.[9]

This paper presents the design of the minimum-time profile as the design of a bang-bang control profile/filter that minimizes the total time and whose transfer function cancels all of the poles of the system subject to state boundary conditions and actuator constraints. It is shown that the necessary conditions derived from this point of view are the same as those derived from conventional optimal control theory. We make use of the bang-bang principle for linear controllable systems, which states that "If an optimal control exists, then there is always a bang-bang control that is optimal. Hence, if the optimal control is unique it is bang-bang."[10]

The motivation behind the paper is the fact that a bang-bang input can be viewed as a summation of time-delayed step commands. For example, a one-switch bang-bang input can be written in the frequency domain as $u(s) = U_0[1 - 2e^{-sT_1} + e^{-sT_2}]/s$, where T_1 is the switch time and T_2 is the final time. U_0 is the saturation value of the actuator and can be thought of as a reference input. The transfer function of the time-delay filter is $1 - 2e^{-sT_1} + e^{-sT_2}$. This transfer function has an infinite number of zeros, some of which can be placed to cancel the finite poles of the system to be controlled. We then formulate a parameter optimization problem to determine the time delays to arrive at a filter that produces the time-optimal control profile when it is subject to a step input.

Singh and Vadali[11] have demonstrated the robustness achieved by locating multiple zeros of the time-delay filter at the location of the poles of the system. We use this concept to design robust time-optimal control profiles.

Prior to the development of the technique for the design of time-optimal control profiles for linear systems, it is noted that a function $f(s) = 0$ has a minimum of two roots at $s = s_0$ if

$$f(s_0) = 0 \text{ and } \left.\frac{\mathrm{d}f(s)}{\mathrm{d}s}\right|_{s_0} = 0$$

This fact is utilized to develop constraint equations to design time-delay filters with multiple roots at any given location.

The paper begins with the presentation of the formulation of the problem. A simple technique is presented for the design of a minimum-time control profile for a rest-to-rest maneuver of a flexible spacecraft. Design of a robust minimum-time control profile is presented next. This is followed by the presentation of a general procedure for the design of a bang-bang profile for the control of a system with damped modes. Design of the minimum-time control profile for spin-up maneuvers is presented in the penultimate section of the paper. This is followed by some concluding remarks.

II. Problem Formulation

We consider a general linear model of a flexible system with one rigid body mode and n flexible modes, which can be represented by the vector differential equation

$$M\ddot{x} + C\dot{x} + Kx = bu, \qquad b, x \in R^{n+1} \tag{1}$$

where M is the mass matrix, C the damping, and K the stiffness matrix. The b is the control influence vector, and x and u are the state vector and scalar control input, respectively. Define the modal participation vector

$$(\phi_0 \;\; \phi_1 \dots \phi_n)^T = \Phi^T b \tag{2}$$

where Φ is the matrix of eigenvectors. We can decouple Eq. (1) by a similarity transformation, using the eigenvectors of the system,

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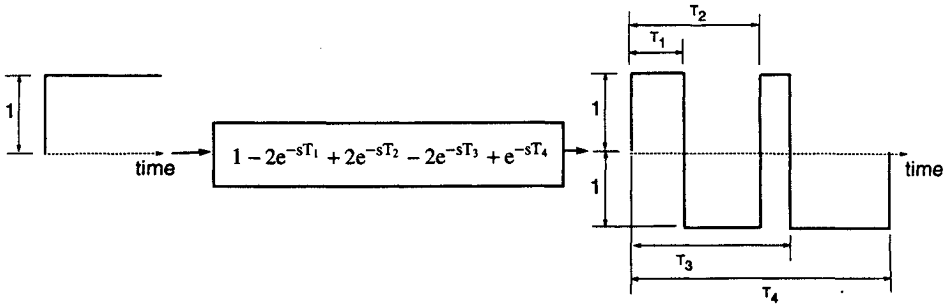

Fig. 1 Time-delay filter.

to the form

$$\ddot{\theta} = \phi_0 u$$
$$\ddot{q}_i + 2\sigma_i\dot{q}_i + \omega_i^2 q = \phi_i u, \qquad i = 1 \text{ to } n \tag{3}$$

where θ is the rigid body coordinate, q_i is the ith modal coordinate, and σ_i and ω_i are the ith damping factor and frequency, respectively. We also assume that the control effort lies in the range

$$-1 \le u \le 1 \tag{4}$$

The objective of the controllers is to cancel all poles of the system, with the constraint that the control is saturated at all times. The design involves selection of time delays of a time-delay filter whose output for a step input is the time-optimal control (Fig. 1).

III. Rest-to-Rest Maneuvers

A. Minimum-Time Control of a Flexible Spacecraft with Undamped Modes

Many researchers have noted the antisymmetric characteristic of the minimum-time control profiles designed for rest-to-rest maneuvers for flexible spacecraft without structural damping.[5,6,8] We first show that a control profile that is antisymmetric about the midmaneuver time leads to a transfer function with two zeros at the origin of the s plane. Figure 2 illustrates the minimum-time control profile for an undamped system with the time delays selected to represent antisymmetry. The transfer function of a time-delay filter containing $2n+2$ time delays ($2n+1$ switches) is

$$1 + 2\sum_{i=1}^{n} (-1)^i e^{-sT_i} + 2(-1)^{n+1} e^{-sT_{n+1}} + 2\sum_{i=1}^{n} (-1)^i e^{-s(2T_{n+1}-T_i)} + e^{-2sT_{n+1}} \tag{5}$$

A zero of the transfer function is located at $s = 0$, as Eq. (5) goes to zero at $s = 0$. The derivative of Eq. (5) with respect to s is

$$-2\sum_{i=1}^{n} (-1)^i T_i e^{-sT_i} - 2(-1)^{n+1} T_{n+1} e^{-sT_{n+1}} - 2\sum_{i=1}^{n} (-1)^i (2T_{n+1}-T_i) e^{-s(2T_{n+1}-T_i)} - 2T_{n+1} e^{-2sT_{n+1}} \tag{6}$$

which has a zero at $s = 0$. Thus we conclude that Eq. (5) has at least two zeros at the origin that automatically cancel the rigid body poles, thus satisfying the velocity boundary condition for a rest-to-rest maneuver.

To arrive at the equations required to cancel the imaginary poles of the system, we substitute

$$s = \sigma + j\omega \tag{7}$$

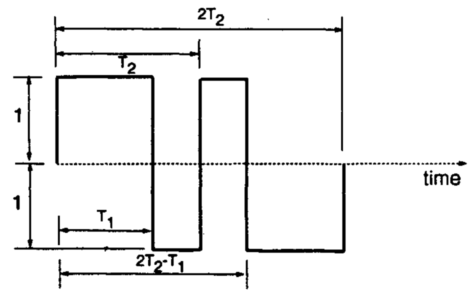

Fig. 2 Antisymmetric control profile for a spring-mass system.

in Eq. (5), and equating the real and imaginary parts, we have, respectively

$$1 + 2\sum_{i=1}^{n} (-1)^i e^{-\sigma T_i} \cos(\omega T_i) + 2(-1)^{n+1} e^{-\sigma T_{n+1}} \cos(\omega T_{n+1}) + 2\sum_{i=1}^{n} (-1)^i e^{-\sigma(2T_{n+1}-T_i)} \cos[\omega(2T_{n+1}-T_i)] + e^{-2\sigma T_{n+1}} \cos(2\omega T_{n+1}) = 0 \tag{8}$$

and

$$2\sum_{i=1}^{n} (-1)^i e^{-\sigma T_i} \sin(\omega T_i) + 2(-1)^{n+1} e^{-\sigma T_{n+1}} \sin(\omega T_{n+1}) + 2\sum_{i=1}^{n} (-1)^i e^{-\sigma(2T_{n+1}-T_i)} \sin[\omega(2T_{n+1}-T_i)] + e^{-2\sigma T_{n+1}} \sin(2\omega T_{n+1}) = 0 \tag{9}$$

As we require the undamped poles of the system to be canceled by the zeros of the time-delay transfer function, we substitute $\sigma = 0$ in Eqs. (8) and (9) and rewrite them, respectively, as

$$\cos(\omega T_{n+1}) \left\{ 2\sum_{i=1}^{n} (-1)^i \cos[\omega(T_{n+1}-T_i)] + (-1)^{n+1} + \cos(\omega T_{n+1}) \right\} = 0 \tag{10}$$

and

$$\sin(\omega T_{n+1}) \left\{ 2\sum_{i=1}^{n} (-1)^i \cos[\omega(T_{n+1}-T_i)] + (-1)^{n+1} + \cos(\omega T_{n+1}) \right\} = 0 \tag{11}$$

To cancel poles at $\omega = \pm j\omega_1, \pm j\omega_2, \ldots, \pm j\omega_n$, we substitute ω_i into the coefficient of $\sin(\omega T_{n+1})$ in Eq. (11), which is the same as the coefficient of $\cos(\omega T_{n+1})$ in Eq. (10), which leads to n equations in $n+1$ unknowns, $T_1, T_2, \ldots, T_{n+1}$.

Another equation is derived from the boundary constraint of the rigid body motion. The response of the rigid body equation to the antisymmetric control profile at $t_f = 2T_{n+1}$ can be represented using the boundary conditions

$$\begin{aligned} \theta(0) &= 0, & \theta(t_f) &\neq 0 \\ \dot{\theta}(0) &= 0, & \dot{\theta}(t_f) &= 0 \end{aligned} \tag{12}$$

as

$$\theta(t_f) = \phi_0\left[\frac{(2T_{n+1})^2}{2} + \sum_{i=1}^{n}(-1)^i(2T_{n+1} - T_i)^2 + (-1)^{n+1}T_{n+1}^2 + \sum_{i=1}^{n}(-1)^i T_i^2\right] \tag{13}$$

Equations (10) and (13) represent $(n+1)$ equations in $(n+1)$ unknowns. These equations allow multiple solutions. Hence we solve for the time delays via an optimization problem that is formulated in the next section.

B. Parameter Optimization

The time delays for the time-optimal filter are solved using a parameter optimization technique. Analytical expressions for the gradients of the cost function and the constraints are used in determining the time delays.

The optimization problem for the undamped system is to minimize the cost function

$$J = (t_f/2)^2 = T_{n+1}^2 \tag{14}$$

subject to the constraints

$$2\sum_{i=1}^{n}(-1)^i\cos[\omega(T_{n+1} - T_i)] + (-1)^{n+1} + \cos(\omega T_{n+1}) = 0, \qquad i = 1, 2, \ldots, n \tag{15}$$

$$\theta(t_f) = \phi_0\left[\frac{(2T_{n+1})^2}{2} + \sum_{i=1}^{n}(-1)^i(2T_{n+1} - T_i)^2 + (-1)^{n+1}T_{n+1}^2 + \sum_{i=1}^{n}(-1)^i T_i^2\right] \tag{16}$$

and

$$0 < T_1 < T_2 < T_3 \ldots < T_{n+1} \tag{17}$$

The antisymmetric structure of the control profile about the mid-maneuver time has been exploited to reduce the number of parameters to be determined for an undamped system. For example, we need to determine $n+1$ parameters for the control of an n-mode system, unlike the previous papers where $2n+2$ parameters had to be determined.[6]

The optimization toolbox of MATLAB has been used to solve the constrained optimization problem.

C. Sufficiency Condition

In this paper we assume that the system model is normal, an assumption that precludes the existence of singular intervals and forces the necessary conditions to also be sufficient. To verify the optimality of the switch times arrived at from the parameter optimization problem, Ben-Asher et al.[6] proposed an elegant technique. Consider the first-order system

$$\dot{x} = Ax + du, \qquad |u| \leq 1, \qquad x \in R^{2n+2} \tag{18}$$

The optimal control is given by $u(t) = -\mathrm{sgn}[d^T\lambda(t)]$, where λ is the costate vector. Furthermore

$$\lambda(t) = \exp(-A^T t)\lambda(0) \tag{19}$$

Hence the switching function is

$$d^T\lambda(t) = d^T\exp(-A^T t)\lambda(0) = P\lambda(0) \tag{20}$$

which is equal to zero at the $(2n+1)$ switch times. Thus $\lambda(0)$ is in the null space of the $(2n+1) \times (2n+2)$ matrix P

$$P = \begin{bmatrix} d^T\exp(-A^T T_1) \\ d^T\exp(-A^T T_2) \\ \cdots \\ d^T\exp[-A^T(2T_{n+1} - T_2)] \\ d^T\exp[-A^T(2T_{n+1} - T_1)] \end{bmatrix} \tag{21}$$

We can determine the null space of P and use that vector as $\lambda(0)$ to determine $\lambda(t)$. Since the parameter optimization permits multiple solutions that satisfy the boundary conditions, the control profile determined from

$$u(t) = -\mathrm{sgn}[\lambda^T(t)d] \tag{22}$$

must switch at the predetermined switch points to be optimal.

D. Numerical Example 1

To illustrate the proposed control design technique, we consider the control of a two-mass-spring problem, which has one flexible and one rigid body mode. The equations of motion are

$$\begin{aligned} m\ddot{x}_1 + k(x_1 - x_2) &= u \\ m\ddot{x}_2 - k(x_1 - x_2) &= 0 \end{aligned} \tag{23}$$

where x_1 and x_2 represent the displacement of the first and second masses with respect to some inertial frame. We intend to control the displacement of the second mass x_2 with a control applied to the first mass. We use $m = 1$ and $k = 1$, following Liu and Wie.[8] The boundary conditions for the rest-to-rest maneuver are

$$\begin{aligned} x_1(0) = x_2(0) &= 0, & x_1(t_f) = x_2(t_f) &= 1 \\ \dot{x}_1(0) = \dot{x}_2(0) &= 0, & \dot{x}_1(t_f) = \dot{x}_2(t_f) &= 0 \end{aligned} \tag{24}$$

The decoupled equations of motion are

$$\begin{aligned} \ddot{\theta} &= 0.7071u \\ \ddot{q} + 2q &= -0.7071u \end{aligned} \tag{25}$$

and the boundary conditions are

$$\begin{aligned} &\theta(0) = q(0) = 0, \qquad \theta(t_f) = 1.4142, \qquad q(t_f) = 0 \\ &\dot{\theta}(0) = \dot{q}(0) = \dot{\theta}(t_f) = \dot{q}(t_f) = 0 \end{aligned} \tag{26}$$

The optimization problem to be solved is

$$\min J = (t_f/2)^2 = T_2^2 \tag{27}$$

subject to the constraints

$$-2\cos[\omega_1(T_2 - T_1)] + 1 + \cos(\omega_1 T_2) = 0 \tag{28}$$

$$0.7071\left[2T_2^2 - (2T_2 - T_1)^2 + T_2^2 - T_1^2\right] - 1.4142 = 0 \tag{29}$$

and

$$T_2 > T_1 > 0 \tag{30}$$

The transfer function of the time-delay filter for the time-optimal solution can be shown to be

$$1 - 2e^{-1.0026s} + 2e^{-2.1089s} - 2e^{-[2(2.1089) - 1.0026]s} + e^{-2(2.1089)s} \tag{31}$$

The optimization toolbox of MATLAB,[12] which uses the sequential quadratic programming method, was used to arrive at the time delays. Figure 3 illustrates the evolution of the states of the system subject to the bang-bang control profile (Fig. 4). The optimality of the solution is corroborated via the technique detailed in Sec. III.C.

E. Robust Minimum-Time Control

It has been shown in Refs. 11 and 13 that multiple zeros of the time-delay transfer function at the location of the poles of the system lead to robustness of the controller with respect to errors in estimated frequency. To add n additional zeros at the location of the poles of the system, we need an additional n time delays. In addition we need to maintain antisymmetry of the control profile, which leads to a total addition of $2n$ time delays. Thus the final robust control profile has $4n + 2$ time delays in the transfer function in addition to the proportional signal. The time-delay transfer function is

$$1 + 2\sum_{i=1}^{2n} (-1)^i e^{-sT_i} - 2e^{-sT_{2n+1}} + 2\sum_{i=1}^{2n} (-1)^i e^{-s(2T_{2n+1} - T_i)} + e^{-2sT_{2n+1}} \tag{32}$$

To cancel the n modes of the plant, we require n zeros of the time-delay transfer function at the location of the n poles of the system, which leads to n equations

$$2\sum_{i=1}^{2n} (-1)^i \cos[\omega(T_{2n+1} - T_i)] - 1 + \cos(\omega T_{2n+1}) = 0, \qquad i = 1, 2, \ldots, n \tag{33}$$

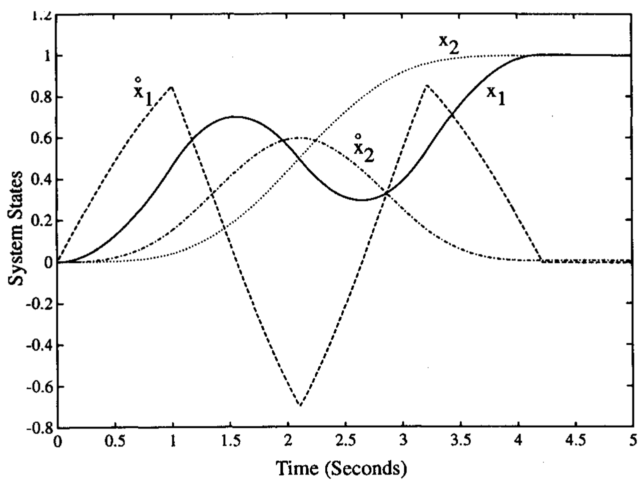

Fig. 3 Evolution of system states subject to time-optimal control.

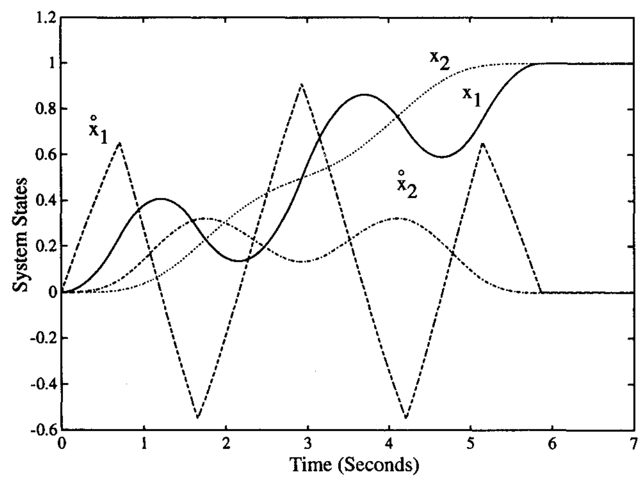

Fig. 5 Evolution of system states subject to robust time-optimal control.

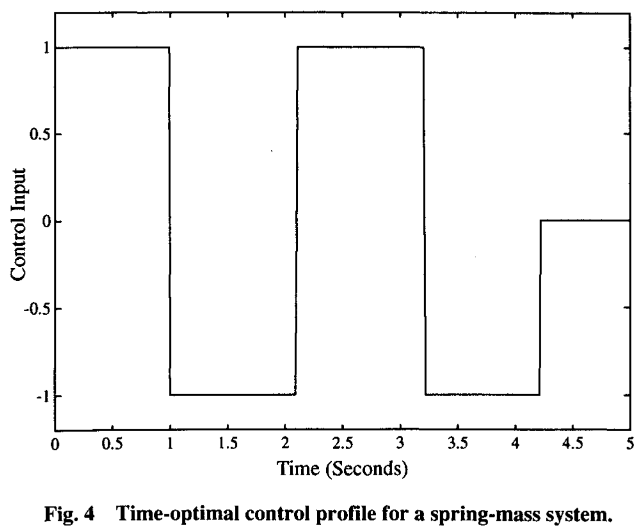

Fig. 4 Time-optimal control profile for a spring-mass system.

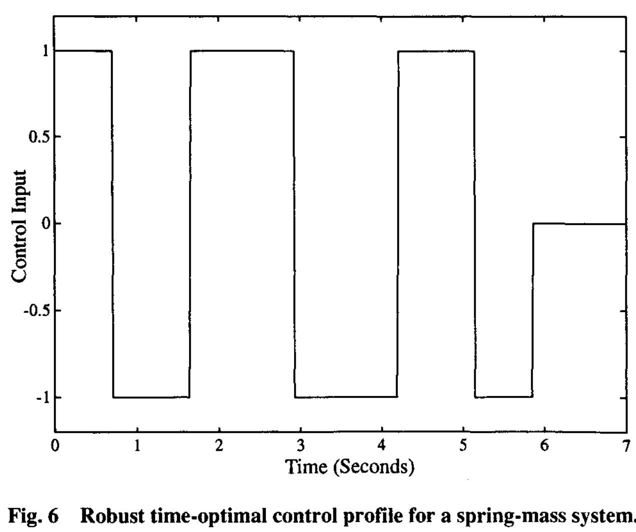

Fig. 6 Robust time-optimal control profile for a spring-mass system.

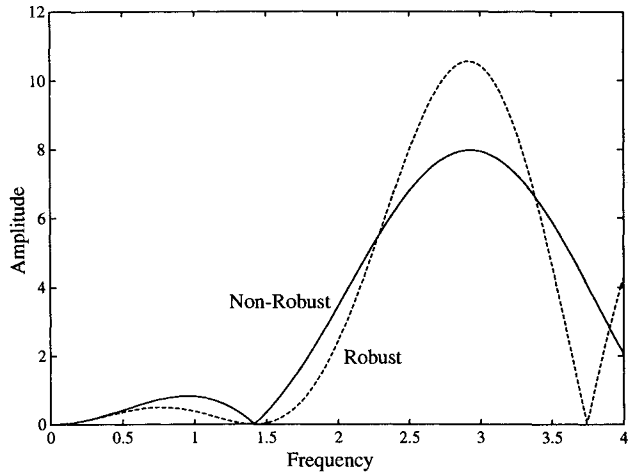

Fig. 7 Magnitude plot of the Bode diagram for the spring-mass system.

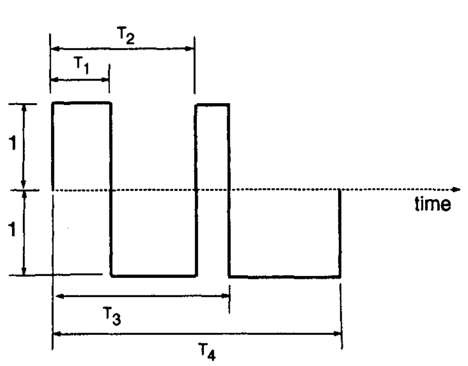

Fig. 8 Control profile for a spring-mass-dashpot system.

We derive the additional n equations by forcing the derivative of Eq. (33) with respect to ω to zero

$$-2\sum_{i=1}^{2n}(-1)^i(T_{2n+1}-T_i)\sin[\omega(T_{2n+1}-T_i)]$$

$$-T_{2n+1}\sin(\omega T_{2n+1}) = 0, \qquad i=1,2,\ldots,n \tag{34}$$

The final equation is derived from the boundary condition of the rigid body motion

$$\theta(t_f)=\phi_0\Bigg[\frac{(2T_{2n+1})^2}{2}+\sum_{i=1}^{2n}(-1)^i(2T_{2n+1}-T_i)^2$$

$$-T_{2n+1}^2+\sum_{i=1}^{2n}(-1)^iT_i^2\Bigg] \tag{35}$$

The time delays are obtained by solving a parameter optimization problem with the constraints given by Eqs. (33–35).

F. Numerical Example 2

We consider the same example as in Sec. III.D. to illustrate the robust time-optimal control methodology. The optimization problem to determine the time delays for robust time-optimal control is

$$\min J=(t_f/2)^2=T_3^2 \tag{36}$$

subject to the constraints

$$-2\cos[\omega_1(T_3-T_1)]+2\cos[\omega_1(T_3-T_2)]-1+\cos(\omega_1T_3)=0 \tag{37}$$

$$-2(T_3-T_1)\sin[\omega_1(T_3-T_1)]+2(T_3-T_2)\sin[\omega_1(T_3-T_2)]$$

$$+T_3\sin(\omega_1T_3)=0 \tag{38}$$

$$0.7071[2T_3^2-(2T_3-T_1)^2+(2T_3-T_2)^2-T_3^2+T_2^2-T_1^2]$$

$$-1.4142=0 \tag{39}$$

and

$$T_3>T_2>T_1>0 \tag{40}$$

The time-delay filter for the time-optimal solution can be shown to be

$$1-2e^{-0.712s}+2e^{-1.6563s}-2e^{-2.9330s}+2e^{-[2(2.9330)-1.6563]s}$$

$$-2e^{-[2(2.9330)-0.712]s}+e^{-2(2.9330)s} \tag{41}$$

Figure 5 illustrates the response of the system subject to the robust bang-bang control (Fig. 6). An increase in the maneuver time is evident. The robustness of the controller can be gauged from the smaller amplitude of the Bode plot (Fig. 7) in the vicinity of the system frequency as compared to the Bode plot for the nonrobust case. This implies that the sensitivity of the overall transfer function is near zero for a small band of frequencies around the natural frequency. The performance of the robust control profile, however, degenerates when the model error is large.

IV. Minimum-Time Control of a Flexible Spacecraft with Damped Modes

Ben-Asher et al.[6] and Hablani[7] have noted the lack of antisymmetry of the minimum-time control profile for systems with damped modes. With that argument in mind, we represent the transfer function of the time-delay filter acting on a unit step input to produce the minimum-time control profile for an n-mode system as (Fig. 8)

$$1+2\sum_{i=1}^{2n+1}(-1)^ie^{-sT_i}+e^{-sT_{2n+2}} \tag{42}$$

This transfer function has one zero at $s = 0$. To force the transfer function to have an additional zero at $s = 0$, we require

$$2\sum_{i=1}^{2n+1}(-1)^iT_ie^{-sT_i}+T_{2n+2}e^{-sT_{2n+2}}=0 \tag{43}$$

which is the derivative of Eq. (42) with respect to s, to have a zero at $s = 0$, which leads to the constraint equation

$$-2T_1+2T_2-\cdots+2T_{2n}-2T_{2n+1}+T_{2n+2}=0 \tag{44}$$

To cancel the damped poles

$$s=\sigma_1\pm j\omega_1,\ \sigma_2\pm j\omega_2,\ldots,\ \sigma_n\pm j\omega_n \tag{45}$$

of the system we require

$$1+2\sum_{i=1}^{2n+1}(-1)^ie^{-\sigma T_i}\cos(\omega T_i)+e^{-\sigma T_{2n+2}}\cos(\omega T_{2n+2})=0 \tag{46}$$

and

$$2\sum_{i=1}^{2n+1}(-1)^i e^{-\sigma T_i}\sin(\omega T_i)+e^{-\sigma T_{2n+2}}\sin(\omega T_{2n+2})=0 \qquad (47)$$

The final constraint is arrived at from the rigid mode boundary condition

$$\theta(t_f)=\phi_0\left[\frac{(T_{2n+2})^2}{2}-(T_{2n+2}-T_1)^2+(T_{2n+2}-T_2)^2-\ldots+0\right] \qquad (48)$$

where $t_f=T_{2n+2}$.

Minimizing T_{2n+2} subject to the constraint equations (44) and (46–48), we arrive at the $2n+2$ time delays.

To address the robustness issue, we require additional constraints derived by forcing the variation of Eqs. (46) and (47) with respect to ω to zero. We will arrive at the same constraint equations if we forced the variation of Eqs. (46) and (47) with respect to σ to zero. We would require an additional $2n$ time delays to satisfy the additional constraints leading to a $(4n+2)$ time-delay filter for an n-mode system.

Numerical example 3. We consider a two-mass-spring-dashpot system to illustrate the technique to design time-optimal controllers for damped systems. The equations of motion of the system are

$$m\ddot{x}_1+c(\dot{x}_1-\dot{x}_2)+k(x_1-x_2)=u$$

$$m\ddot{x}_2-c(\dot{x}_1-\dot{x}_2)-k(x_1-x_2)=0 \qquad (49)$$

The parameters chosen for this numerical example are $m=1$, $c=1$, and $k=1$. The boundary conditions are the same as Eq. (24). The decoupled equations of motion are

$$\ddot{\theta}=0.7071u$$

$$\ddot{q}+2\dot{q}+2q=-0.7071u \qquad (50)$$

and the associated boundary conditions are

$$\theta(0)=q(0)=0, \qquad \theta(t_f)=1.4142, \qquad q(t_f)=0$$

$$\dot{\theta}(0)=\dot{q}(0)=\dot{\theta}(t_f)=\dot{q}(t_f)=0 \qquad (51)$$

The constrained optimization problem to be solved is

$$\min J=t_f^2=T_4^2 \qquad (52)$$

subject to the constraints

$$1-2e^{-\sigma_1 T_1}\cos(\omega_1 T_1)+2e^{-\sigma_1 T_2}\cos(\omega_1 T_2)-2e^{-\sigma_1 T_3}\cos(\omega_1 T_3)$$
$$+e^{-\sigma_1 T_4}\cos(\omega_1 T_4)=0 \qquad (53)$$

$$-2e^{-\sigma_1 T_1}\sin(\omega_1 T_1)+2e^{-\sigma_1 T_2}\sin(\omega_1 T_2)-2e^{-\sigma_1 T_3}\sin(\omega_1 T_3)$$
$$+e^{-\sigma_1 T_4}\sin(\omega_1 T_4)=0 \qquad (54)$$

$$-2T_1+2T_2-2T_3+T_4=0 \qquad (55)$$

$$0.7071[(T_4^2/2)]-(T_4-T_1)^2+(T_4-T_2)^2-(T_4-T_3)^2]-1.4142=0 \qquad (56)$$

and

$$T_4>T_3>T_2>T_1>0 \qquad (57)$$

The transfer function of the time-delay filter for the time-optimal solution can be shown to be

$$1-2e^{-1.4404s}+2e^{-3.368s}-2e^{-4.2586s}+e^{-4.662s} \qquad (58)$$

It can also be shown that the transfer function of the robust time-delay filter for the same problem is

$$1-2e^{-1.4465s}+2e^{-3.5005s}-2e^{-4.7310s}+2e^{-5.7641s}-2e^{-6.413s}$$
$$+e^{-6.6518s} \qquad (59)$$

Figures 9 and 10 illustrate the evolution of the system states for the nonrobust and the robust bang-bang control cases, respectively. Figures 11 and 12 are the control profiles for the nonrobust and the robust cases, respectively.

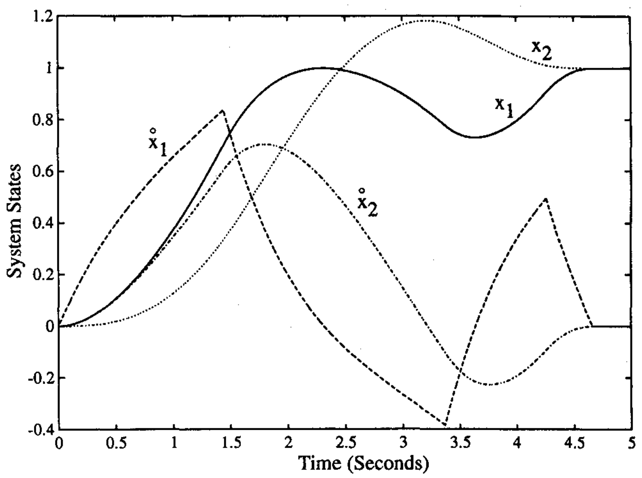

Fig. 9 Response of a spring-mass-dashpot to a time-optimal control profile.

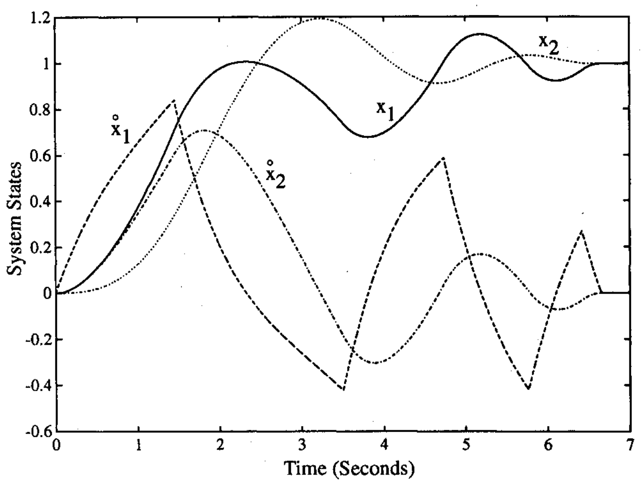

Fig. 10 Response of a spring-mass-dashpot to a robust time-optimal control profile.

V. Spin-Up Maneuvers

The method outlined earlier for the design of minimum-time rest-to-rest maneuvers can be used in the design of a control profile for minimum-time spin-up maneuvers. Unlike the preceding case, we need to cancel only one pole at the origin of the system transfer function. It should be noted that since we have one less constraint as compared to the rest-to-rest case, the sign of the gain

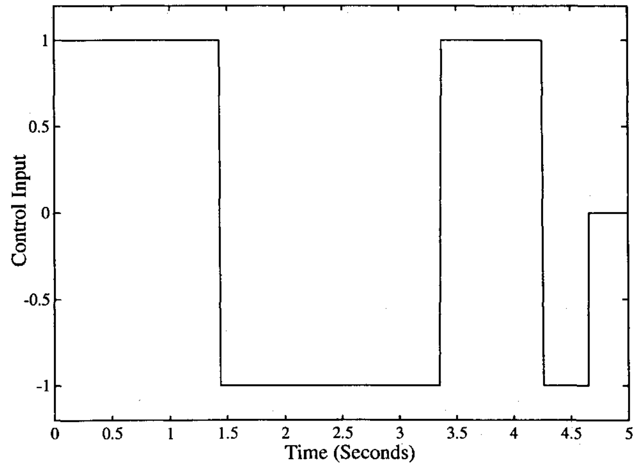

Fig. 11 Time-optimal control profile for a spring-mass-dashpot.

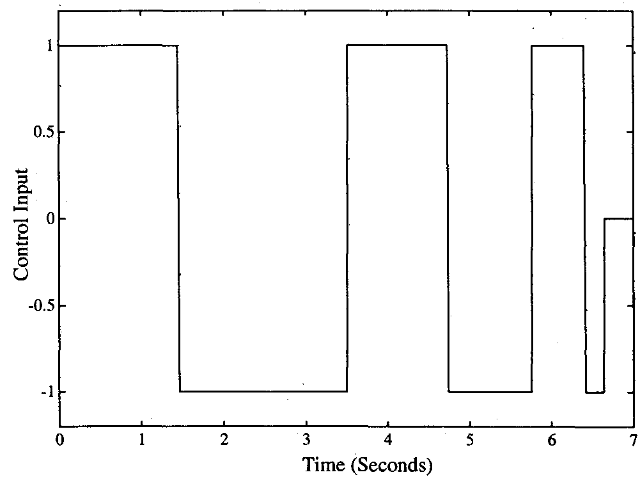

Fig. 12 Robust time-optimal control profile for a spring-mass-dashpot.

= 0 and has the characteristic of being a mirror image about the half maneuver time.[14]

$$1 + 2\sum_{i=1}^{n}(-1)^i e^{-sT_i} - 2\sum_{i=1}^{n}(-1)^i e^{-s(2T_{n+1}-T_i)} - e^{-2sT_{n+1}} \tag{60}$$

To cancel the undamped poles at $s = \pm j\omega_i$, $i = 1$ to n, we require

$$1 + 2\sum_{i=1}^{n}(-1)^i e^{-\sigma T_i}\cos(\omega T_i) - 2\sum_{i=1}^{n}(-1)^i e^{-\sigma(2T_{n+1}-T_i)} \times \cos[\omega(2T_{n+1}-T_i)] - e^{-2\sigma T_{n+1}}\cos(2\omega T_{n+1}) = 0 \tag{61}$$

and

$$2\sum_{i=1}^{n}(-1)^i e^{-\sigma T_i}\sin(\omega T_i) - 2\sum_{i=1}^{n}(-1)^i e^{-\sigma(2T_{n+1}-T_i)} \times \sin[\omega(2T_{n+1}-T_i)] - e^{-2\sigma T_{n+1}}\sin(2\omega T_{n+1}) = 0 \tag{62}$$

which can be simplified, respectively, to

$$2\sin(\omega T_{n+1})\left\{-2\sum_{i=1}^{n}(-1)^i \sin[\omega(T_{n+1}-T_i)] - \sin(\omega T_{n+1})\right\} = 0 \tag{63}$$

and

$$2\cos(\omega T_{n+1})\left\{-2\sum_{i=1}^{n}(-1)^i \sin[\omega(T_{n+1}-T_i)] - \sin(\omega T_{n+1})\right\} = 0 \tag{64}$$

From Eqs. (63) and (64), we have

$$-2\sum_{i=1}^{n}(-1)^i \sin[\omega(T_{n+1}-T_i)] - \sin(\omega T_{n+1}) = 0 \tag{65}$$

Finally, to meet the velocity boundary conditions, we need

$$\dot{\theta}(t_f) = \phi_0\left[2T_{n+1} + 2\sum_{i=1}^{n}(-1)^i(2T_{n+1}-T_i) - 2\sum_{i=1}^{n}(-1)^i T_i\right] \tag{66}$$

We now have $n + 1$ equations in $n + 1$ unknown time delays $T_1, T_2, \dots, T_{n+1}$.

We could address the issue of robustness to errors in estimated frequency by additional time delays that would place multiple zeros at the location of the poles of the system. Time-optimal spin-up maneuver of a spacecraft with damped modes would destroy the symmetry of the control profile about the midmaneuver time.

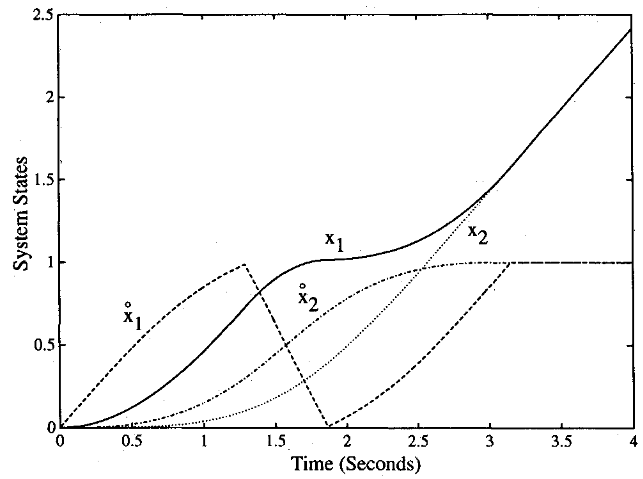

Fig. 13 Response of system to minimum-time spin-up control profile.

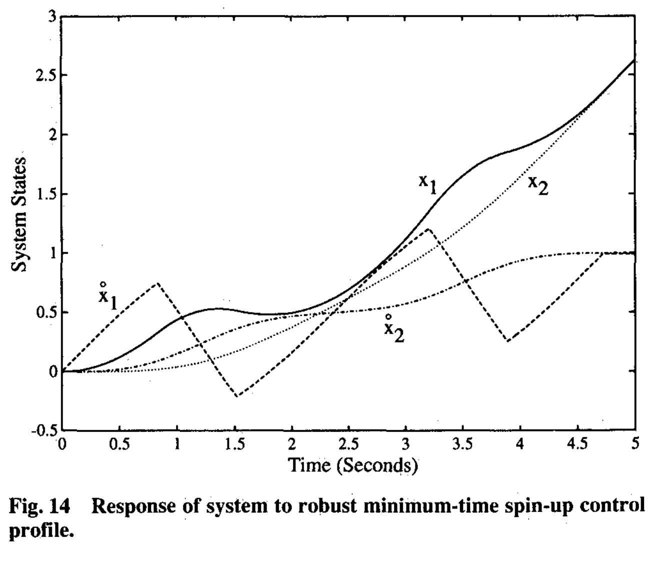

Fig. 14 Response of system to robust minimum-time spin-up control profile.

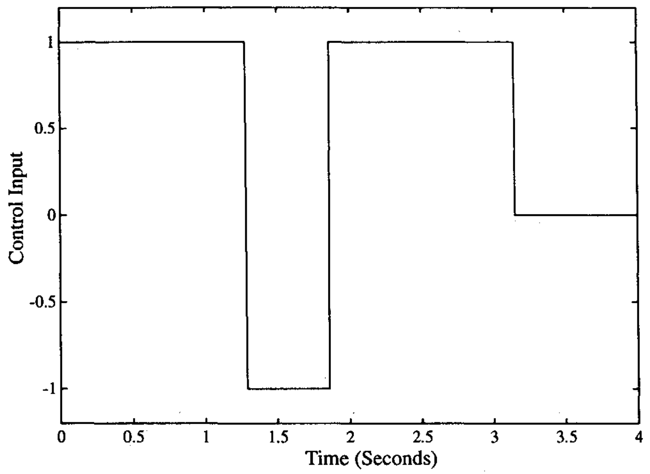

Fig. 15 Time-optimal control profile for a spring-mass system.

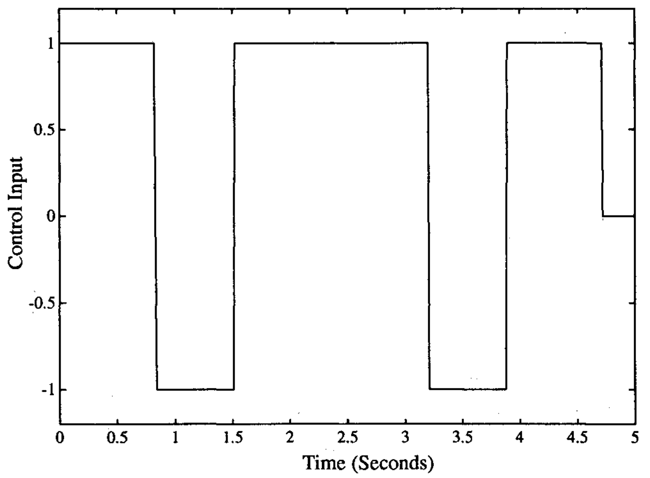

Fig. 16 Robust time-optimal control profile for a spring-mass system.

Thus to design time-optimal control, we would have to formulate a parameter optimization problem with $2n + 1$ time delays, unlike the undamped case, where we were required to determine $n + 1$ time delays. The sufficiency condition can be checked as described in Sec. III. C.

Numerical example 4. We study the two-mass-spring problem (which also represents a flexible spacecraft) described in Sec. III.D. We require the masses to move at a constant velocity of one unit in minimum time. The optimization problem to be solved to arrive at the minimum-time control profile is

$$\min J = (t_f/2)^2 = T_2^2 \tag{67}$$

subject to the constraints

$$-2\sin[\omega_1(T_2 - T_1)] + \sin(\omega_1 T_2) = 0 \tag{68}$$

$$0.7071[2T_2 - 2(2T_2 - T_1) + 2(T_1)] - 1.4142 = 0 \tag{69}$$

$$T_2 > T_1 > 0 \tag{70}$$

The transfer function of the time-delay filter is

$$1 - 2e^{-1.2877s} + 2e^{-[2(1.5755) - 1.2877]s} - e^{-2(1.5755)s} \tag{71}$$

and the robust version for the same is

$$1 - 2e^{-0.8354s} + 2e^{-1.5168s} - 2e^{-[2(2.3627) - 1.5168]s}$$
$$+ 2e^{-[2(2.3627) - 0.8354]s} - e^{-2(2.3627)s} \tag{72}$$

Figures 13 and 14 illustrate the system response to the minimum-time spin-up control profile and the robust control profile, respectively. Figures 15 and 16 are the control profiles for the two cases.

VI. Conclusions

Design of time-optimal control inputs for linear systems has been presented from a frequency domain viewpoint. The design technique involves minimizing the maximum time delay of a time-delay filter subject to the constraint that the output of the time-delay filter saturates the actuator and the zeros of the time-delay filter cancel all of the poles of the system. The generic nature of the design procedure has been illustrated by designing time-optimal control profiles for rest-to-rest and spin-up maneuvers. The issue of robustness to errors in estimated system parameters has been addressed by requiring multiple zeros of the time-delay filter to be located at the estimated locations of the poles of the system. Control profiles for rest-to-rest maneuvers of systems with one and two flexible modes have indicated the requirement of three (four) and five (six) switches (time delays), respectively.

Acknowledgment

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