Robustly Stable Multivariate Polynomials

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Dedicated to Ulrich Oberst
on the occasion of his 70th birthday.

***Abstract*— We consider stability and robust stability of polynomials with respect to a given arbitrary disjoint decomposition $\mathbb{C}^n = \Gamma \uplus \Lambda$. The polynomial is called stable if it has no zeros in the region of instability Λ and robustly stable if it is stable and remains so under small variations of its coefficients. Inspired by the article *Robust stability of multivariate polynomials. Part 1: Small coefficient perturbations* by V. L. Kharitonov and J. A. Torres-Muñoz (Multidimens. Systems Signal Process., 10(1):21–32, 1999), we generalise some of their results to arbitrary stability decompositions and develop some fundamental results on robustly stable polynomials. Among them is a characterisation of robust stability in terms of the stability of several other polynomials, which yields a test for robust stability based on stability tests. Finally, we consider the special situation that the region of instability is a Cartesian product and recover some results for the special situations of linear partial differential resp. difference equation with constant coefficients.**

I. Introduction

In this article we investigate robustly stable polynomials in several variables, i.e., a subclass of stable polynomials featuring additional robustness properties.

Stability of a polynomial is always defined with respect to a given disjoint decomposition $\mathbb{C}^n = \Gamma \uplus \Lambda$ of $\mathbb{C}^n$ into two sets Γ and Λ, where n denotes the number of variables. A polynomial $P \in \mathbb{C}[s] = \mathbb{C}[s_1, \ldots, s_n]$ in n variables $s_1, \ldots, s_n$ with complex coeffficients is *stable* with respect to this decomposition $\mathbb{C}^n = \Gamma \uplus \Lambda$ or with respect to Λ if all of its zeros are in Γ, i.e., if

$$\mathrm{V}_{\mathbb{C}^n}(P) \subseteq \Gamma, \text{ or, equivalently, if } \mathrm{V}_{\mathbb{C}^n}(P) \cap \Lambda = \emptyset, \qquad (1)$$

where $\mathrm{V}_{\mathbb{C}^n}(P) = \{z \in \mathbb{C}^n;\ P(z) = 0\}$ denotes the zero set of P. For this reason, the decomposition $\mathbb{C}^n = \Gamma \uplus \Lambda$ is referred to as a *stability decomposition*, the set Γ is called the *region of stability* and Λ the *region of instability* or, in short, just the stable and unstable region. Although the condition $\mathrm{V}_{\mathbb{C}^n}(P) \subseteq \Gamma$ is more common in one-dimensional systems theory, throughout this article we will use the equivalent condition $\mathrm{V}_{\mathbb{C}^n}(P) \cap \Lambda = \emptyset$, because in the important multidimensional examples the unstable region is more easily described than the stable one.

In the univariate case ($n = 1$) the two most prominent examples of stability decompositions are the ones where the region of stability is the open left complex half plane $\{z \in \mathbb{C};\ \Re(z) < 0\}$ resp. the open unit circle $\{z \in \mathbb{C};\ |z| < 1\}$. The first one is used to describe asymptotic stability of solutions of ordinary linear differential equations with constant coefficients algebraically. For positive time, these solutions tend to zero exactly if all the zeros of the characteristic polynomial have real part smaller than zero. The second region of stability serves the same purpose for ordinary difference equations. The polynomials which are stable with respect to these stability decompositions are called Hurwitz resp. Schur polynomials. But the choice of the stability decomposition depends on the application and the properties of the equations one wants to describe. In [18, p. 2] a variation of the stability decomposition for ordinary differntial equations is described, where additional restrictions to the convergence speed of the solutions are incorporated.

For systems of linear partial differntial resp. difference equation mostly the n-fold powers of the respective univariate regions of instability are used. But also here, and even more than in the univariate case, other stability decompositions are of interest too. One reason for this is that in partial differential equations the variables bear different interpretations, e.g., in a wave equation one variable is interpreted as time, the other ones as variables of space and there is no reason that all the variables are to be treated the same when investigating the equation with respect to a certain kind of stability. The wish to treat as many different varieties of stability simultaneously motivates the general definition above.

In applications, the coefficients of the polynomials of interest are often the result of measurements, when analysing a given system, or systems are constructed with a purpose (e.g, stabilisation) according to theoretically computed specifications. Both procedures are bound to be unexact and erroneous, nonetheless, one wishes that a system's properties, e.g. stability, remain the same in spite of those inaccuracies.

To account for this a polynomial is called *robustly stable* with respect to a given stability decomposition if it is stable with respect to the same decomposition and remains so under small variations of its coefficients; for the formal definition see Definition 1.

In the univariate case, a stable polynomial is automatically also robustly stable as long as the region of instability is a closed subset of $\mathbb{C}$, the reason for this being that the zero set of a polynomial in one variable consists of finitely many points and therefore is compact. The same is true, although for a different reason, for the standard stability decompositions for partial difference equations as we will see in Example 21. For partial differential equations, however,

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stability does not imply robust stability. Robust stability of bivariate differential equations involves the *very strict Hurwitz polynomials* which were generalised to an arbitrary number of variables in [11].

The present article is strongly motivated and influenced by the article [11] by V. L. Kharitonov and J. A. Torres-Muñoz. The author encountered this article while working on BIBO stability of partial differential equations [15], [16], where a class of recursively defined polynomials in [11] – the robustly stable polynomials with respect to the standard stability decomposition for partial differential equations – were vital in the proof of a conjecture by E. I. Jury [10]. The main motivation for the research presented here was to interpret the results of [11] geometrically and to find out which of them and how they can be generalised to arbitrary stability decompositions.

There exist many contributions on robust stability with respect to the standard cases for partial differential resp. difference equations – see Examples 20 and 21 for some references –, but, to the knowledge of the author, only in [13] a more general but still restricted class of stability decompositions are considered.

Section II is the core of this article. For general stability decompositions we show some basic properties of robustly stable polynomials, the most important ones summarised in Result 7, where we give three assertions which are equivalent to the robust stability of a polynomial. Section III is of an algorithmic nature: The knowledge gained in Section II allows us to devise a test for robust stability and we will explicate it in detail for the case that the region of instability is a real semi-algebraic set, i.e., can be described by finitely many real polynomial inequalities (identifying $\mathbb{C}^n = \mathbb{R}^{2n}$). Eventually, in Section IV, we will specialise the results of Section II to the case that the region of instability is a Cartesial product, i.e., $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ for $\Lambda_1, \ldots, \Lambda_n \subseteq \mathbb{C}$. We also give a test for robust stability adapted to this particular situation. We conclude the article by focussing on the standard stability decomposition for partial differential resp. difference equations, thus recovering most of the results of [11] and disclosing the connections between the two articles.

This article is an abbreviated version and does not contain any proofs. A complete paper containing all details will be submitted to a journal.

II. Robustly Stable Multivariate Polynomials

In this section we will define robust stability of polynomials and establish some of their properties. In the following we assume a stability decomposition $\mathbb{C}^n = \Gamma \uplus \Lambda$ and its derived notion of stable polynomials.

We need the following notations: For a polynomial $P = \sum_{\mu \in \mathbb{N}^n} p_\mu s^\mu \in \mathbb{C}[s]$ with coefficients $p_\mu \in \mathbb{C}$ – all but finitely many equal to zero – we denote by $\deg(P) \in \mathbb{N}^n$ its *(component-wise) degree*, i.e., the minimal tuple $d \in \mathbb{N}^n$ such that P can be written as $P = \sum_{\mu \leqslant_{\mathrm{cw}} d} p_\mu s^\mu$, where $\mu \leqslant_{\mathrm{cw}} d$ means that for each $i \in \{1, \ldots, n\}$ the relation $\mu_i \leqslant d_i$ is satisfied. We write $\mathrm{c}(P) = (p_\mu)_{\mu \leqslant_{\mathrm{cw}} d} \in \mathbb{C}^d$, where $d := \deg(P)$, for the coefficient vector of P.

Definition 1. Let $P \in \mathbb{C}[s]$ be a polynomial with component-wise degree d. We call the polynomial *robustly stable* with respect to a given stability decomposition if it is stable with respect to the same decomposition and remains so under small variations of its coefficient vector $\mathrm{c}(P) \in \mathbb{C}^d$, or, more precisely, if there exists an open neighbourhood $U \subseteq \mathbb{C}^d$ of $\mathrm{c}(P)$ such that all polynomials with coefficient vector in U are stable.

Thus the robustly stable polynomials are the stable polynomials featuring a *stability radius* greater than zero. The stability radius of a polynomial P is the greatest euclidean ball in $\mathbb{C}^{\deg(P)}$ centred about the polynomial's coefficient vector such that all the polynomials with coefficient vector in the ball are stable [13], [12, Sec. VIII].

Lemma 2. *Let $P \in \mathbb{C}[s]$ be robustly stable with respect to Λ. Then P is also robustly stable with respect to the closure $\mathrm{cl}_{\mathbb{C}^n}(\Lambda)$ of Λ in $\mathbb{C}^n$.*

Our next aim is a stronger version of the lemma above, namely that for robustly stable polynomials P the closures of $\mathrm{V}_{\mathbb{C}^n}(P)$ and Λ are disjoint even if points at infinity are included (Theorem 4).

For this we will use the one-point-compactification $\overline{\mathbb{C}} = \mathbb{C} \uplus \{\infty\}$. This set is a one-dimensional complex manifold which can be identified with the projective line over $\mathbb{C}$ via

$$\begin{array}{ccc} \mathbb{P}_1(\mathbb{C}) & \overset{=}{\longleftrightarrow} & \overline{\mathbb{C}} \\ \mathbb{C}\begin{pmatrix} v \\ w \end{pmatrix} & \longleftrightarrow & \begin{cases} \frac{v}{w}, & \text{if } w \neq 0 \\ \infty, & \text{if } w = 0. \end{cases} \end{array}$$

The atlas composed of the two charts

$$\begin{array}{llll} \varphi_\emptyset = \mathrm{id}\colon & \mathbb{C} & \longrightarrow \mathbb{C}, & x \longmapsto x \quad \text{and} \\ \varphi_{\{1\}} = \mathrm{inv}\colon & \overline{\mathbb{C}} \setminus \{0\} & \longrightarrow \mathbb{C}, & x \longmapsto x^{-1} = \begin{cases} 0 & \text{if } x = \infty \\ \frac{1}{x} & \text{else} \end{cases} \end{array}$$

makes $\overline{\mathbb{C}}$ a holomorphic manifold.

To formulate this for higher dimensions we employ the following notations. Let X be an arbitrary set and $x = (x_1, \ldots, x_n) \in X^n$. Furthermore let $S \subseteq [n] := \{1, \ldots, n\}$ and $S' := [n] \setminus S$ be its complement. We write $x_S := (x_i)_{i \in S}$ and identify $x = (x_S, x_{S'}) \in X^n = X^S \times X^{S'}$.

We consider the n-dimensional manifold $\overline{\mathbb{C}}^n$ equipped with the atlas formed by the charts

$$\begin{array}{rcl} \varphi_S\colon \quad V_S & \longrightarrow & \mathbb{C}^n \\ x = (x_S, x_{S'}) & \longmapsto & (x_S^{-1}, x_{S'}) \end{array}$$

together with their domains of definition $V_S := (\overline{\mathbb{C}} \setminus \{0\})^S \times \mathbb{C}^{S'}$ for every $S \subseteq [n]$. We write for short $x_S^{-1} := (x_i^{-1})_{i \in S}$ component-wise and, as in the one-dimensional case, $\infty^{-1} = 0$. This makes $\overline{\mathbb{C}}^n$ a compact holomorphic manifold. Note that the set $\overline{\mathbb{C}}^n = \left(\overline{\mathbb{C}}\right)^n = \mathbb{P}_1(\mathbb{C})^n$ is different to the complex n-space $\mathbb{P}_n(\mathbb{C}) = \mathbb{P}(\mathbb{C}^{n+1})$ as well as to the one-point-compactification $\mathbb{C}^n \uplus \{\infty\} = \overline{\mathbb{C}^n}$ of $\mathbb{C}^n$.

We denote the closure of a subset M of $\overline{\mathbb{C}}^n$ by $\mathrm{cl}_{\overline{\mathbb{C}}^n}(M)$. Because of the compactness of $\overline{\mathbb{C}}^n$, the set $\mathrm{cl}_{\overline{\mathbb{C}}^n}(M)$ is compact too. Expressing $\mathrm{cl}_{\overline{\mathbb{C}}^n}(M)$ in terms of the charts, we notice first that

$$\mathrm{cl}_{\overline{\mathbb{C}}^n}(M) = \bigcup_{S\subseteq[n]} \mathrm{cl}_{V_S}(M\cap V_S),$$

since the V_i form a covering of $\overline{\mathbb{C}}^n$. Since the charts φ_S are bijective we get

$$\mathrm{cl}_{\overline{\mathbb{C}}^n}(M) = \bigcup_{S\subseteq[n]} \mathrm{cl}_{V_S}(M\cap V_S) = \bigcup_{S\subseteq[n]} \mathrm{cl}_{V_S}(\varphi_S^{-1}(\varphi_S(M\cap V_S))).$$

Using that φ_S^{-1} is a homeomorphism we arrive at

$$\mathrm{cl}_{\overline{\mathbb{C}}^n}(M) = \bigcup_{S\subseteq[n]} \varphi_S^{-1}(\mathrm{cl}_{\mathbb{C}^n}(\varphi_S(M\cap V_S))).$$

For $M\subseteq\mathbb{C}^n$ which is for our purposes the most important case we write

$$\mathrm{cl}_{\overline{\mathbb{C}}^n}(M) = \bigcup_{S\subseteq[n]} \varphi_S^{-1}(\mathrm{cl}_{\mathbb{C}^n}(\varphi_S(M\cap W_S))),$$

where $W_S := V_S\cap\mathbb{C}^n = (\mathbb{C}\setminus\{0\})^S\times\mathbb{C}^{S'}$

The two subsets of $\mathbb{C}^n$ of interest to us are $M=\Lambda$ resp. $M=\mathrm{V}_{\mathbb{C}^n}(P)$. First we focus on the latter. We need a "good description" of the set $\mathrm{cl}_{\mathbb{C}^n}(\varphi_S(\mathrm{V}_{\mathbb{C}^n}(P)\cap W_S))$ which is part of the following lemma.

Definition and Lemma 3. *For a polynomial $P\in\mathbb{C}[s]$ with degree d and a set $S\subseteq[n]$ we define*

$$P_S(s) := s_S^{d_S} P(\varphi_S^{-1}(s)) \in \mathbb{C}[s].$$

The following four assertions hold:

1) $\mathrm{V}_{\mathbb{C}^n}(P_S)\cap W_S = \varphi_S(\mathrm{V}_{\mathbb{C}^n}(P)\cap W_S)$.
2) $\mathrm{cl}_{\mathbb{C}^n}(\mathrm{V}_{\mathbb{C}^n}(P_S)\cap W_S) = \mathrm{V}_{\mathbb{C}^n}(P_S)$.
3) *If P is stable with respect to Λ then P_S is stable with respect to $\varphi_S(\Lambda\cap W_S)$.*
4) *If P is robustly stable with respect to Λ then P_S is robustly stable with respect to $\varphi_S(\Lambda\cap W_S)$.*

Items 1 and 2 of the preceding lemma form the first part of the "good description" of $\mathrm{cl}_{\mathbb{C}^n}(\varphi_S(\mathrm{V}_{\mathbb{C}^n}(P)\cap W_S))$. The inclusion "$\subseteq$" in item 2 is easy to prove whereas the other one is a little bit intricate and some results from algebraic geometry are needed to show it.

Now we can formulate one of the main result of this article.

Theorem 4. *Let $P\in\mathbb{C}[s]$ be robustly stable with respect to the stability decomposition $\mathbb{C}^n=\Gamma\uplus\Lambda$. Then the closures of its vanishing set and of the unstable region in $\overline{\mathbb{C}}^n$ are disjoint, i.e.,*

$$\mathrm{cl}_{\overline{\mathbb{C}}^n}(\mathrm{V}_{\mathbb{C}^n}(P))\cap\mathrm{cl}_{\overline{\mathbb{C}}^n}(\Lambda) = \emptyset. \tag{2}$$

Using the following lemma it is not hard to see that the other implication is also true: If the closures of the vanishing set of a polynomial and of the region of instability are disjoint, then the polynomial is robustly stable.

Lemma 5. *Let M_1 and M_2 be topological spaces and let $N_1\subseteq M_1$ as well as $N_2\subseteq M_2$ be compact subsets. Furthermore, using the product topology on $M_1\times M_2$, let $U\subseteq M_1\times M_2$ be an open set containing $N_1\times N_2$. Then there exist open sets $G\subseteq M_1$ and $H\subseteq M_2$ such that*

$$N_1\times N_2\subseteq G\times H\subseteq U.$$

Theorem 6. *For $d\in\mathbb{N}^n$ let $K\subseteq\mathbb{C}^d$ be compact and $\Lambda\subseteq\mathbb{C}^n$ be a region of instability. If all polynomials P with coefficient vector in K satisfy*

$$\mathrm{cl}_{\overline{\mathbb{C}}^n}(\mathrm{V}_{\mathbb{C}^n}(P))\cap\mathrm{cl}_{\overline{\mathbb{C}}^n}(\Lambda) = \emptyset,$$

then there exist open supersets $U\supseteq K$ and $V\supseteq\Lambda$ such that for all polynomials P with coefficient vector in U the statement

$$\mathrm{cl}_{\overline{\mathbb{C}}^n}(\mathrm{V}_{\mathbb{C}^n}(P))\cap V = \emptyset$$

is true.

In particular, if K is a set containing only one element, this means the following: Let $P\in\mathbb{C}[s]$ be a polynomial such that

$$\mathrm{cl}_{\overline{\mathbb{C}}^n}(\mathrm{V}_{\mathbb{C}^n}(P))\cap\mathrm{cl}_{\overline{\mathbb{C}}^n}(\Lambda) = \emptyset.$$

Then P is robustly stable with respect to Λ. Furthermore, there exists an open superset $V\subseteq\overline{\mathbb{C}}^n$ of $\mathrm{cl}_{\overline{\mathbb{C}}^n}(\Lambda)$, such that P is robustly stable with respect to $V\cap\mathbb{C}^n$, i.e., the region of instability can be slightly enlarged.

From the existence of V follows that every robustly stable polynomial features a stability margin *greater than zero [6], [19], i.e., a minimal distance between the polynomial's zero set and the unstable region.*

We present our results up to now:

Result 7. *Let $\mathbb{C}^n=\Gamma\uplus\Lambda$ be a stability decomposition and $P\in\mathbb{C}[s]$ be a polynomial. Then the following four statements are equivalent:*

1) *P is robustly stable with respect to Λ.*
2) *The closures of $\mathrm{V}_{\mathbb{C}^n}(P)$ and of Λ in $\overline{\mathbb{C}}^n$ are disjoint.*
3) *For all $S\subseteq[n]$ the polynomial P_S is stable with respect to $\mathrm{cl}_{\mathbb{C}^n}(\varphi_S(\Lambda\cap W_S))$.*
4) *There exist open sets $V\subseteq\overline{\mathbb{C}}^n$ containing $\mathrm{cl}_{\overline{\mathbb{C}}^n}(\Lambda)$ such that P is stable with respect to $V\cap\mathbb{C}^n$.*

Item 2 is a geometric characterisation of robust stability, item 3 will be the key for deciding algorithmically whether a polynomial is robustly stable with respect to a given region of instability Λ.

The corollaries below follow easily, one just has to use the appropriate equivalent statement of Result 7.

Corollary 8. *If $\Lambda\subseteq\mathbb{C}^n$ is compact, then $\mathrm{cl}_{\overline{\mathbb{C}}^n}(\Lambda)=\Lambda$. Consequently*

$$\mathrm{cl}_{\overline{\mathbb{C}}^n}(\mathrm{V}_{\mathbb{C}^n}(P))\cap\mathrm{cl}_{\overline{\mathbb{C}}^n}(\Lambda) = \mathrm{V}_{\mathbb{C}^n}(P)\cap\Lambda.$$

This signifies that a polynomial is robustly stable with respect to a compact region of instability if and only if it is stable with respect to the same stability decomposition.

Corollary 9. *Let $P\in\mathbb{C}[s]$, $S\subseteq[n]$ and $a_S\in\mathbb{C}^S$. By $P(a_S,-)\in\mathbb{C}[s_{S'}]$ we denote the polynomial which remains if we fix the variables indexed by S. Furthermore, we write*

$M_{a_S} := \{x_{S'} \in \overline{\mathbb{C}}^{S'}; (a_S, x_{S'}) \in M\}$ *for any set* $M \subseteq \overline{\mathbb{C}}^n$. *Then the following two implications hold:*

1) *If* P *is stable with respect to* Λ, *then* $P(a_S, -)$ *is stable with respect to* Λ_{a_S}.
2) *If* P *is robustly stable with respect to* Λ, *then so is* $P(a_S, -)$ *with respect to* Λ_{a_S}.

Corollary 9 becomes particularly powerful when the unstable region is a Cartesian product or even a power of a one-dimensional complex set as in case of the standard stability decompositions for partial differential resp. difference equations, see Section IV.

Corollary 10. *The set* $\mathscr{R}(\Lambda) \subseteq \mathbb{C}[s]$ *of all robustly stable polynomials with respect to a region of instability* Λ *contains the constant polynomial* 1, *is multiplicatively closed and saturated. "Saturated" signifies that if a product of some polynomials is robustly stable, then the factors have the same property.*

III. An Algorithm for Deciding Robust Stability

In this section we present a method which makes it possible to check robust stability of a polynomial $P \in \mathbb{C}[s]$ for a large class of stability decompositions.

Exploiting the equivalence of items 1 and 3 of Result 7, one can directly formulate the following algorithm, for which no restrictions to the unstable region Λ are required.

Algorithm 11. *Given a region of instability* $\Lambda \subseteq \mathbb{C}^n$, *one can test if a polynomial* $P \in \mathbb{C}[s]$ *is robustly stable with respect to* Λ *by performing the following steps:*

1) *For all* $S \subseteq [n]$ *determine the image* $\varphi_S(\Lambda \cap W_S)$ *under the chart* φ_S *and its closure* $\mathrm{cl}_{\mathbb{C}^n}(\varphi_S(\Lambda \cap W_S))$.
2) *Now – again for all* $S \subseteq [n]$ *– compute the polynomial* P_S *and check if it is stable with respect to* $\mathrm{cl}_{\mathbb{C}^n}(\varphi_S(\Lambda \cap W_S))$.
3) *If all the* P_S *prove stable with respect to their respective stability decomposition, then* P *is stable with respect to* Λ. *Conversely, if one of the* P_S *is not stable, then* P *is not robustly stable.*

When implenting this algorithm, one encounters two problems: First, to be able to handle the unstable region, one has to restrict oneself to a class of sets which contains the images under the φ_S as well as the closures of these images and is still admissible for automated processing, and second, one must be able to perform stability tests with sets of this class.

One possible choice are the real semi-algebraic sets, embedded into $\mathbb{C}^n$. We will describe in detail how the algorithms associated with this class of sets – the theorem of Tarski-Seidenberg and its formulation in the language of logics, namely quantifier elimination – can be employed to handle the problems stated above.

The quantifier elimination methods have been developed starting around 1930; for a synopsis on their history consult [9, p. 165f]. These methods have been in use in systems theory since the mid-1970s, the first time in [1] for output feedback stabilization [1, p. 72ff] and also for stability testing with respect to the closed unit polydisc and the n^{th} power of the open right half-plane [1, p. 76ff]. In [9], quantifier elimination methods were used for testing the stability of ordinary differential and difference equations, it contains also some elaborately explained examples. In 2002 they were employed in [19] for calculations involving the robustness of systems, concretely for the calculations of stability and stabilisability margins with respect to the closed unit polydisc, amongst others.

Our approach differs from the ones in [9] and [19] insofar as we will not assume a special stability decomposition – as long as the unstable region is a semi-algebraic set – and, mainly, that we will use quantifier elimination to plug the computational holes in Algorithm 11, thus reducing robust stability testing to stability testing.

We identify $\mathbb{C} = \mathbb{R}^2$ and assume that the region of instability Λ is a semi-algebraic subset of $\mathbb{R}^{2n}$, i.e., it can be written as the solution set of a finite number of polynomial equations and inequalities over the real numbers. More formally, we assume that there exist finite index sets I and J_i for all $i \in I$ as well as polynomials $Q^{i,j} \in \mathbb{R}[x,y] = \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ such that

$$\Lambda = \bigcup_{i \in I} \bigcap_{j \in J_i} \{(x,y) \in \mathbb{R}^{2n};\ Q^{i,j}(x,y) \quad r^{i,j} \quad 0\} \subseteq \mathbb{R}^{2n} = \mathbb{C}^n, \tag{3}$$

where $r^{i,j} \in \{=, \neq, >, <, \geqslant, \leqslant\}$. The semi-algebraic sets cover all the standard stability decompositions, in particular the standard decompositions for linear partial differential resp. difference equations, see Examples 20 and 21.

Two of the central results of semi-algebraic geometry are the theorem of Tarski-Seidenberg (see, e.g., [3, Thm. 2.3.4 on p. 60]) stating that the image of a semi-algebraic set under a semi-algebraic map – in particular, under a projection – is again semi-algebraic, and [3, Thm. 2.4.4 on p. 69], which consists of an algorithm on how to obtain a description like in equation (3) of the image. With this algorithm, translated into the language of logics, one can find formulas without quantifiers which are equivalent to given first order formulas, with the restriction that all variables have to be real. This motivates the name *quantifier elimination method*. In practice, the implementation is done using *cylindrical algebraic decomposition*. There exist several computer programmes for the purpose of real quantifier elimination, the ones known to the author are QEPCAD [5] and Redlog [7].

For our intended test for robust stability, we need an algorithm to compute a description as in equation (3) for the closure $\mathrm{cl}_{\mathbb{C}^n}(\Lambda) = \mathrm{cl}_{\mathbb{R}^{2n}}(\Lambda)$ and another one for testing whether a polynomial P is stable, i.e., whether $\mathrm{V}_{\mathbb{C}^n}(P) \cap \Lambda$ is empty.

Algorithm 12. *Given a semi-algebraic set* $\Lambda \subseteq \mathbb{R}^{2n}$, *its closure* $\mathrm{cl}_{\mathbb{R}^{2n}}(\Lambda)$ *is semi-algebraic again and a presentation of it like in equation* (3) *can be derived using [3, Thm 2.4.4 on p. 69].*

Algorithm 13. *Let* $P \in \mathbb{C}[s]$ *be a polynomial and* $\Lambda \subseteq \mathbb{C}^n = \mathbb{R}^{2n}$ *be a semi-algebraic set. To test if* $\mathrm{V}_{\mathbb{C}^n}(P) \cap \Lambda$ *is empty proceed as follows:*

Decompose P into its real and imaginary part, i.e., write $P(s) = P^{\mathrm{re}}(x,y) + iP^{\mathrm{im}}(x,y)$, where $s_j = x_j + iy_j$ for all $j \in [n]$ and $P^{\mathrm{re}}, P^{\mathrm{im}} \in \mathbb{R}[x,y]$ are polynomials in $2n$ variables over the real numbers. Using that, one can describe the vanishing set of P as

$$\mathrm{V}_{\mathbb{C}^n}(P) = \{(x,y) \in \mathbb{R}^{2n};\ P^{\mathrm{re}}(x,y) = 0\} \cap \left\{(x,y) \in \mathbb{R}^{2n};\ P^{\mathrm{im}}(x,y) = 0\right\} \subseteq \mathbb{R}^{2n}.$$

Since finite intersections of semi-algebraic sets are again semi-algebraic, $\mathrm{V}_{\mathbb{C}^n}(P)$ and also $\mathrm{V}_{\mathbb{C}^n}(P) \cap \Lambda$ are semi-algebraic sets.

Now consider the projection

$$\mathrm{proj} :\ \mathbb{R}^{2n} \longrightarrow \mathbb{R}^0.$$

onto the set $\mathbb{R}^0$ consisting of only one point. The image of a set A under this map is empty if A is empty and otherwise it is $\mathbb{R}^0$. Compute the image of $\mathrm{V}_{\mathbb{C}^n}(P) \cap \Lambda$ under proj *using [3, Thm 2.4.4 on p. 69]. The polynomial P is stable if the result of this computation is the empty set and unstable otherwise.*

Lemma 14. *Let $\Lambda \subseteq \mathbb{R}^{2n}$ be a semi-algebraic set and $S \subseteq [n]$. With the notations introduced before Definition and Lemma 3, the set $\varphi_S(\Lambda \cap W_S) \subseteq \mathbb{C}^n = \mathbb{R}^{2n}$ is also semi-algebraic and a description for it as in equation* (3) *can be derived.*

Using Algorithms 12 and 13, as well as Lemma 14 it is possible to test a polynomial for robust stability with respect to a semi-algebraic region of instability in a fully automated way. Summarising, we specialise Algorithm 11 to this case.

Algorithm 15. *Let $\Lambda \subseteq \mathbb{C}^n = \mathbb{R}^{2n}$ be a semi-algebraic region of instability. To test if a polynomial $P \in \mathbb{C}[s]$ is robustly stable, proceed as follows:*

1) *For all $S \subseteq [n]$ compute a representation of the form presented in equation* (3) *for $\varphi_S(\Lambda \cap W_S)$ and then for $\mathrm{cl}_{\mathbb{C}^n}(\varphi_S(\Lambda \cap W_S))$ using Lemma 14 and Algorithm 12.*
2) *Now – again for all $S \subseteq [n]$ – divide the polynomial P_S into its real and imaginary part and use Algorithm 13 and check if it is stable with respect to $\mathrm{cl}_{\mathbb{C}^n}(\varphi_S(\Lambda \cap W_S))$.*
3) *If all the P_S prove stable with respect to their respective stability decomposition, then P is stable with respect to Λ. Conversely, if one of the P_S is not stable, then P is not robustly stable.*

Remark 16. The novelty of the algorithm above is that a polynomial is tested for robust stability by testing several polynomials for stability. The main advantage of this algorithm is that it can be applied to a great variety of stability decompositions. It should be noted, however, that the time complexity of stability tests based on the theorem of Tarski-Seidenberg and implemented using cylindrical algebraic decomposition is doubly exponential. This signifies, quoting Anderson et al.: "All this is done with a finite number of rational operations. The finiteness for many problems, however, may be illusory from the practical point of view [...]" [1, p. 69].

To draw a conclusion, if there are other stability tests for special stability decompositions available, e.g., the Routh-Hurwitz criterion or Jury's stability criterion, one should consider employing them too and not blindly use quantifier elimination. However, if more general stability decompositions are involved, quantifier elimination seems to be the method of choice.

Another way to reduce computing time is to make use of structural properties of the region of instability. In the next section, for example, we assume that Λ is a Cartesian product which leads to a simpler criterion for robust stability.

IV. Robust Stability for Products

Here we investigate the special case that the region of instability is a Cartesian product. In Examples 20 and 21 we specialise further to the standard cases for partial differential resp. difference equations. In the first example, we also recover most of the results of [11].

For $i \in [n]$ let $\Lambda_i \subset \mathbb{C}$ be arbitrary non-empty sets and let $\Lambda = \prod_{i=1}^n \Lambda_i = \Lambda_1 \times \cdots \times \Lambda_n$. As in Section II we embed $\mathbb{C}^n$ into $\overline{\mathbb{C}}^n$ and look at the images of $\Lambda \cap W_S$ under the charts φ_S for $S \subseteq [n]$. Since the multidimensional charts φ_S are tensor products of the one-dimensional non-trivial chart inv and the identity map of $\mathbb{P}_1(\mathbb{C})$ from page 2, the images factorise and we get

$$\varphi_S(\Lambda \cap W_S) = \prod_{i\in S} \underbrace{\mathrm{inv}(\Lambda_i \setminus \{0\})}_{=:\widetilde{\Lambda}_i} \times \prod_{i\in S'} \Lambda_i = \widetilde{\Lambda}_S \times \Lambda_{S'}.$$

For a polynomial $P = \sum_{\mu \leqslant_{\mathrm{cw}} d} p_\mu s^\mu \in \mathbb{C}[s]$ with $\deg(P) = d$ and $S \subseteq [n]$ we will use the representation

$$P(s_S, s_{S'}) = \sum_{\mu_S \leqslant_{\mathrm{cw}} d_S} b^S_{\mu_S}(s_{S'}) s_S^{\mu_S} \in \mathbb{C}[s] = (\mathbb{C}[s_{S'}])[s_S],$$

$$\text{where } b^S_{\mu_S}(s_{S'}) = \sum_{\mu_{S'} \leqslant_{\mathrm{cw}} d_{S'}} p_{(\mu_S,\mu_{S'})} s_{S'}^{\mu'_S} \in \mathbb{C}[s_{S'}].$$

The leading terms $b^S_{d_S}(s_{S'})$ will be of special interest to us.

The following theorem gives a further equivalence to robust stability for the special case that Λ is a product.

Theorem 17. *Let $P \in \mathbb{C}[s]$ be a polynomial with $\deg(P) = d$, let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ be a product and denote*

$$T := \{i \in [n];\ \infty \in \mathrm{cl}_{\overline{\mathbb{C}}}(\Lambda_i)\} = \{i \in [n];\ 0 \in \mathrm{cl}_{\mathbb{C}}(\widetilde{\Lambda}_i)\}$$

the set of all indices for which ∞ is contained in the closure of the respective factor of the region of instability. The following two statements are equivalent:

1) *P is robustly stable with respect to Λ.*
2) *For all $S \subseteq T$ the polynomials $b^S_{d_S}$ derived from P are stable with respect to $\mathrm{cl}_{\mathbb{C}^{S'}}(\Lambda_{S'})$.*

Utilising Theorem 17 and Algorithm 15, we furnish a simplified algorithm adapted to the situation of this section.

Algorithm 18. *Let $\Lambda_1, \ldots, \Lambda_n \subseteq \mathbb{C} = \mathbb{R}^2$ be semi-algebraic sets and let $P \in \mathbb{C}[s]$ be a polynomial with degree $\deg(P) = d$. By carrying out the following three steps one can test P for robust stability with respect to $\Lambda = \prod_{j\in[n]} \Lambda_j$.*

1) *For all* $j \in [n]$ *compute semi-algebraic representations of* $\mathrm{cl}_{\mathbb{C}}(\Lambda_i)$ *using Algorithm 12. Determine the set T of all indices j for which* Λ_j *is not bounded.*
2) *For all* $S \subseteq T$ *decompose* $b^S_{d_S}$ *into its real and imaginary part and test it for stability with respect to* $\mathrm{cl}_{\mathbb{C}^{S'}}(\Lambda_{S'})$.
3) *If and only if all of them are stable, the polynomial P is robustly stable with respect to* Λ.

Remark 19. Algorithm 18 does only cover a special case, but there it improves Algorithm 15 in several aspects:

1) The images $\varphi_S(\Lambda_S \cap W_S)$ do not have to be computed any more.
2) Only closures of sets in one complex variable resp. two real variables have to be determined.
3) Instead of testing the 2^n polynomials P_S for stability, one needs to test only the $2^{|T|}$ polynomials $b^S_{d_S}$.

Roughly spoken, one has to work with fewer polynomials in fewer variables. Since the runtime of the algorithms based on the theorem of Tarski-Seidenberg increases disproportionately with the number of variables (see Remark 16), the advantages of this specialised algorithm are not negligible.

In the following we will investigate the standard cases for partial differential resp. difference equations and recover some known results.

Example 20. In the standard situation for partial differential equations the region of instability is the n fold power of the complex closed right half plane, i.e.,

$$\Lambda_1 = \cdots = \Lambda_n = \{z \in \mathbb{C};\ \Re(z) \geqslant 0\} \text{ and } \Lambda = \Lambda_1^n.$$

The polynomials which are stable with respect to this Λ are called *(strict) Hurwitz (stable) polynomials* [8, Def. 3], [4, Def. 6.2] or *strict sense stable polynomials* [11, Def. 3]. Polynomials which are robustly stable with respect to this stability decomposition were investigated by V. L. Kharitonov und J. A. Torres-Muñoz in [11] first. The two authors of [11] use a different terminology, they call the robustly stable polynomials just "stable". Since [11] was the main inspiration for writing the present article, we will now regain the main results of [11] as special cases of our results and thus show exactly where and how the two articles are related.

The Hurwitz stability decomposition has the property that

$$\widetilde{\Lambda}_i = \mathrm{inv}(\Lambda_i \setminus \{0\}) = \Lambda_i \setminus \{0\}$$

and thus $\mathrm{cl}_{\mathbb{C}^n}(\widetilde{\Lambda}_i) = \Lambda_i$ for all $i \in [n]$. Since $0 \in \Lambda_i$ for all $i \in [n]$ the assertion of Theorem 17 in this special case is that a polynomial $P \in \mathbb{C}[s]$ is a robust Hurwitz polynomial in n variables if and only if for all $S \subseteq [n]$ the associated polynomial $b^S_{\deg(P)_S} \in \mathbb{C}[s_{S'}]$ is an Hurwitz polynomial in $n - |S|$ variables. Kharitonov and Torres-Muñoz use the latter statement for defining robustly stable polynomials [11, Def. 4] and thus our definition is one of their main results. In [11, Thm. 19] they show that polynomials P for which all the $b^S_{\deg(P)_S}$ are Hurwitz polynomials are robustly stable. The other direction is proven indirectly for bivariate ($n = 2$) polynomials. This argument holds also for a general number of variables since a robust Hurwitz polynomial in n variables posesses the same property if considered as a polynomial in more than n variables, which is easy to see. The bivariate robustly stable Hurwitz polynomials are found in the literature under the name *very strict Hurwitz polynomials* [11, Rem. 3], [12, Rem. 13].

A weaker form of our result that if a polynomial P is robustly stable with respect to an unstable region Λ then there exists an open superset of the closure of Λ in $\overline{\mathbb{C}}^n$ such that P is stable with respect to that superset (Result 7, implication from item 1 to item 4) can be found in [11, Thm. 23], the full implication is proven in [15, Thm. 9].

Kharitonov and Torres-Muñoz also investigated *wide sense stable* polynomials, i.e., polynomials featuring no zero in the n fold power of the *open* right half plane $\{z \in \mathbb{C};\ \Re(z) > 0\}^n$, see [11, Def. 1]. The authors show in [11, Lem. 6] how to derive from a wide sense stable polynomial which is not strict sense stable by an arbitrarily small variation of its coefficient vector an unstable polynomial. This corresponds to Lemma 2 of this article.

Other results of [11] seem to be specific for the Hurwitz situation, e.g., that the partial derivatives of a robustly stable polynomial are robustly stable again [11, Thm. 20] and that for a robustly stable polynomial $P \in \mathbb{R}[s]$ with *real* coefficient vector all the coefficients have the same sign [11, Thm. 22].

Example 21. The standard stability decomposition for systems of partial difference equations is given by

$$\Lambda_1 = \cdots = \Lambda_n = \{z \in \mathbb{C};\ |z| \geqslant 1\} \text{ and } \Lambda = \Lambda_1^n.$$

The image of these factors under the chart inv are $\widetilde{\Lambda}_i = \{z \in \mathbb{C};\ |z| \leqslant 1,\ z \neq 0\}$ and their closure is

$$\mathrm{cl}_{\mathbb{C}}(\widetilde{\Lambda}_i) = \{z \in \mathbb{C};\ |z| \leqslant 1\},$$

the closed unit disc. Since none of the Λ_i, $i \in [n]$, is bounded, testing a polynomial P for robust stability by Algorithm 18 would consist of checking for all $S \subseteq [n]$ the polynomials $b^S_{\deg(P)_S}$ for stability. However, it suffices to test only one polynomial.

The *component-wise order* of a polynomial $P \neq 0$, denoted $\mathrm{ord}(P)$, is the maximal tuple $r \in \mathbb{N}^n$ such that the polynomial can be written as $P = \sum_{\mu \geqslant_{\mathrm{cw}} r} p_\mu s^\mu$. It gives the maximal power of s which is a factor of P, i.e., $P = s^{\mathrm{ord}(P)} R$ with $R \in \mathbb{C}[s]$ and R relatively prime to all the $s_1, \ldots, s_n$. Consequently, $\mathrm{ord}(R) = 0$.

Let $P \in \mathbb{C}[s]$ be a polynomial and R be such that $P = s^{\mathrm{ord}(P)} R$. It is easily seen that powers of s are robustly stable with respect to Λ. Since the set $\mathscr{R}(\Lambda)$ of all polynomials which are robustly stable with respect to Λ is multiplicatively closed and saturated (Corollary 10), the polynomial P is robustly stable with respect to Λ exactly if R shows the same property.

Now we consider the chart $\varphi_{[n]}$, where all the variables are inverted. Denote

$$\Xi := \mathrm{cl}_{\mathbb{C}^n}(\varphi_{[n]}(\Lambda \cap W_{[n]})) = \{z \in \mathbb{C};\ |z| \leqslant 1\}^n$$

the *closed unit polydisc* and let $Q := R_{[n]}$, i.e., $Q(s) = s^{\deg(R)}R(s^{-1})$. Lemma 3, item 4, states that robust stability of R with respect to Λ implies robust stability of Q with respect to Ξ, and one can show that the converse holds too.

Summing up, P is robustly stable with respect to Λ if and only if $Q \in \mathscr{R}(\Xi)$ and this is the case exactly if Q is stable with respect to Ξ, since the closed unit polydisc is a compact subset of $\mathbb{C}^n$, compare Corollary 8. This signifies that checking P for robust stability with respect to Λ can be done by testing $Q = R_{[n]}$ with $P = s^{\mathrm{ord}(P)}R$ for stability with respect to the closed unit polydisc.

The common formulation of this result is that a polynomial P which is stable with respect to Λ resp. the associated polynomial Q which is stable with respect to the closed unit polydisc have a stability radius or margin which is greater than zero [19, Remark after Def. 1].

In the literature, sometimes the polynomials devoid of zeros in Λ are the stable ones [17, Def. 3], [14, p. 1468], [16, Sec. 5], and sometimes the stable polynomials are the ones with no zeros in the closed unit polydisc Ξ [1], [2, Sec. II], [6], see the introduction of [17] for a discussion and further references. In contrast to the direct approach used in the present article, in [2] as well as in [17] the authors used a Möbius transform to transport their results from the continuous to the discrete case.

V. Conclusion

In this article we have defined robustly stable polynomials with respect to an arbitrary stability decomposition and derived some fundamental results for this class of polynomials. They allowed us to devise a test for robust stability which can be performed in an automated fashion if the region of instability is a semi-algebraic set. Finally, we have investigated the important case that the unstable region is a Cartesian product, in particular the standard stability decompositions for partial differential resp. difference equations.

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