

# Robustly Strategic Consumption-Portfolio Rules with Informational Frictions\*

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May 12, 2015

## Abstract

This paper provides a tractable continuous-time constant-absolute-risk averse (CARA)-Gaussian framework to explore how the interactions of fundamental uncertainty, model uncertainty due to a preference for robustness (RB), and state uncertainty due to information-processing constraints (rational inattention or RI) affect strategic consumption-portfolio rules and precautionary savings in the presence of uninsurable labor income. Specifically, after solving the model explicitly, I compute and compare the elasticities of strategic asset allocation and precautionary savings to risk aversion, robustness, and inattention. Furthermore, for plausibly estimated and calibrated model parameters, I quantitatively analyze how the interactions of model uncertainty and state uncertainty affect the optimal share invested in the risky asset, and show that they can provide a potential explanation for the observed stockholding behavior of households with different education and income levels.

*JEL Classification Numbers: C61, D81, E21.*

**Keywords:** Robustness, Model Uncertainty, Rational Inattention, Uninsurable Labor Income, Strategic Asset Allocation, Precautionary Savings.

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\*I am grateful for Ricardo Lagos (editor), an associate editor, and two anonymous referees for many constructive suggestions and comments. I also thank Evan Anderson, Rhys Bidder, Jaroslav Borovička, Zhiwu Chen, Max Croce, Richard Dennis, Martin Ellison, Ken Kasa, Tom Sargent, Chris Sims, Neng Wang, Shenghao Zhu, and seminar and conference participants at Seoul National University, National University of Singapore, the 19th International Conference of Computing in Economics and Finance, and the Asian Meeting of the Econometric Society for helpful comments and discussions related to this paper. The financial support from the General Research Fund (GRF, No. HKU791913) in Hong Kong is acknowledged. The usual disclaimer applies.

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*“To form a long-term portfolio, investors must first think systematically about their preferences and about the constraints they face. . . . One of the most interesting challenges of the 21st century will be to develop systems, combining the scientific knowledge of financial economists with information technology and the human wisdom of financial planners, to help investors carry out the task of strategic asset allocation.” – John Y. Campbell (Invited Address to AEA and AFA, January 4, 2002)*

## 1. Introduction

Intertemporal consumption-saving and portfolio choice is a fundamental topic in modern economics. In the real world, ordinary investors face pervasive uncertainty, and have to make consumption-saving-investment decisions in environments in which they are not only uncertain about both the present or future states of the world (e.g., equity returns and uninsurable labor income), but are also concerned about the structure of the model economy. It is therefore critical for us to understand how rational investors make optimal financial decisions when facing various types of risks and uncertainty. This paper provides a tractable continuous-time constant absolute risk aversion (CARA)-Gaussian framework to explore how investors make strategic consumption-saving-asset allocation decisions when they face both *fundamental uncertainty* (e.g., labor income uncertainty or uncertainty about the equity return) and *induced uncertainty*. In this paper, we define induced uncertainty as the interaction of model uncertainty due to a preference for robustness and state uncertainty due to limited information-processing capacity.<sup>1</sup>

For most individual investors, human wealth, the expected present value of their current and future labor income, constitutes a major fraction of their total wealth. However, moral hazard and adverse selection problems prevent the emergence of markets that can insure investors against their idiosyncratic labor income. Such market incompleteness has stimulated substantial research interest in the behavior of precautionary saving. Both theoretical and empirical studies support that we need to take the precautionary saving motive into account when modeling consumption and saving behavior.<sup>2</sup> The recent empirical evidence on household portfolios in the U.S. and major European countries has also stimulated research in generalizing the single asset precautionary saving model to allow for portfolio choice between risky and risk-free financial assets.<sup>3</sup> For example, Heaton and Lucas (2000) studied how the presence of background risks influences portfolio allocations. They found that labor income is the most important source of wealth and labor income risk is weakly positively correlated with equity returns. Viceira (2001) examined the effects of labor income risk on optimal consumption and portfolio choice for both employed and retired investors. Campbell (2006) outlined the field of household finance and argued that some house-

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<sup>1</sup>Here we label model uncertainty or state uncertainty “induced uncertainty” because it is induced by the interactions of the preference for robustness or information-processing constraints with fundamental uncertainty.

<sup>2</sup>See, for example, Caballero (1990), Carroll and Samwick (1998), and Kennickell and Lusardi (2005).

<sup>3</sup>The empirical research on household portfolios documents that the stock market participation rate was increasing in the U.S. and Europe and the importance of the precautionary saving motive for portfolio choice. See Guiso, Jappelli, and Terlizzese (1996) and Luigi, Haliassos, and Jappelli (2002).

holds make serious investment mistakes, which lead to nonparticipation in risky asset markets and underdiversification of risky portfolios. Wang (2009) examined optimal consumption-saving and asset allocation when consumers cannot observe their income growth.<sup>4</sup> These studies mainly consider two key aspects of labor income risk: the variance and persistence of labor income and the correlation between labor income and the equity return. In the presence of labor income, there is an income-hedging demand when the equity return is correlated with labor income.<sup>5</sup> In addition, non-tradable labor income leads to precautionary savings by interacting with risk aversion when it is not perfectly correlated with the equity return.

In this paper, the household portfolio choice problem is revisited by assuming that individual investors not only face fundamental uncertainty but also face induced uncertainty (model uncertainty and state uncertainty). Model uncertainty and state uncertainty arise from two major types of incomplete information: one is incomplete information about the distribution of the state transition equation, and the other is incomplete information about the true level of the state.<sup>6</sup> Hansen and Sargent (1995) first introduced the preference for robustness (RB, a concern for model misspecification) into linear-quadratic-Gaussian (LQG) economic models.<sup>7</sup> In robust control problems, agents are concerned about the possibility that their true model is misspecified in a manner that is difficult to detect statistically; consequently, they choose their decisions as if the subjective distribution over shocks was chosen by an evil agent to minimize their utility. As discussed in Hansen, Sargent, and Tallarini (HST, 1999) and Luo and Young (2010), RB models can produce precautionary savings even within the class of LQG models, which leads to analytical simplicity.<sup>8</sup> Sims (2003) first introduced information-processing constraint (rational inattention or RI) into economics and argued that it is a plausible method for introducing sluggishness, randomness, and delay into economic models. In his formulation agents have finite Shannon channel capacity, limiting their ability to process signals about the true state. As a result, a shock to the state induces only gradual responses by individuals. Another important implication of rational inattention is that attention is a scarce resource that is important for productivity. Specifically, people may be less productive if they are worrying about problems at home or distracted by other events that reduce productivity. In other words, people would be more productive at work if they have higher income and can own more distraction-saving goods and services at home (e.g., a good baby sitter).<sup>9</sup> Because RI intro-

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<sup>4</sup>See Campbell and Viceira (2002) for a recent survey on this topic.

<sup>5</sup>For example, if the labor income risk is positively correlated with the shock to the equity return, the equity is less desirable because it offers a bad hedge against negative labor income shocks.

<sup>6</sup>In Section 5, I also examine the implications of another well-adopted type of imperfect state observation (incomplete information about income) for optimal consumption-portfolio rule.

<sup>7</sup>See Hansen and Sargent (2007) for a textbook treatment on robustness.

<sup>8</sup>Luo, Nie, and Young (2012) briefly discussed the differences between CARA and RB within the discrete-time setting. Although both RB and CARA preferences (i.e., Caballero 1990 and Wang 2004, 2009) increase the constant precautionary savings demand, they have distinct implications for the marginal propensity to consume out of permanent income (MPC).

<sup>9</sup>See Banerjee and Mullainathan (2008) for a discussion on the relationship among limited attention, productivity, and income distribution.

duces additional uncertainty, the endogenous noise due to finite capacity, into economic models, RI by itself creates an additional demand for robustness. The distinction between these two types of informational frictions can be seen from the following continuous-time transition equation of the true state ( $s_t$ ):

$$ds_t = (As_t + Bc_t) dt + \sigma dB_t,$$

where  $s_t$  and  $c_t$  are state and control variables, respectively;  $A$ ,  $B$ , and  $\sigma$  are constant coefficients; and  $B_t$  is a standard Brownian motion. Under RB, agents do not know the true data generating process driven by the random innovation ( $B_t$ ), whereas agents under RI cannot observe the true state ( $s_t$ ) perfectly.

As the first contribution of this paper, a continuous-time theoretical framework is constructive in which there are (i) two fundamental risks: uninsurable labor income and the equity return, (ii) two types of induced uncertainty: model uncertainty (MU) due to the preference for robustness and state uncertainty (SU) due to rational inattention, and (iii) CARA utility. The main reason that I adopt the CARA utility specification is for technical convenience.<sup>10</sup> It is shown that the models with these features can be solved explicitly. In particular, when introducing state uncertainty due to RI, I derive the continuous-time version of the information-processing constraint (IPC) proposed in Sims (2003), and find the explicit expressions for the stochastic properties of the RI-induced noise and the Kalman filtering equation. This paper is therefore closely related to the literature on imperfect information, learning, asset allocation and asset pricing (see Genotte 1986, Lundtofte 2008, and Wang 2009).<sup>11</sup>

Second, after solving the models explicitly, we can exactly inspect the mechanism through which these two types of induced uncertainty interact and affect different types of demand for the risky asset (i.e., the standard speculation demand and the income-hedging demand), the precautionary saving demand, and consumption dynamics.<sup>12</sup> In particular, I find that optimal allocation in the risky asset and the precautionary saving demand are more sensitive to CARA than RB, and are more sensitive to RB than RI when investors are not highly information-constrained.

Third, after calibrating the RB parameter using the detection error probabilities (DEP, or  $p$ ), I

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<sup>10</sup>A few papers find closed-form solutions in CARA models with uninsurable labor income. For example, Svensson and Werner (1993) study an infinite horizon consumption-investment model with normally distributed income. It is well known that there is no closed-form solution if we move away from the CARA specification (e.g., if we adopt the constant relative risk aversion or CRRA utility) and explicitly model uninsurable labor income in the infinite-horizon consumption-portfolio choice model in the vein of Merton (1969, 1971). It is also worth noting that Bliss and Panigirtzoglou (2004) found evidence from option prices that the CARA specification provides a better representation of preferences than the CRRA specification.

<sup>11</sup>It is also shown that the RI model is equivalent to the traditional signal extraction (SE) model with exogenously specified noises in the sense that they lead to the same model dynamics when the signal-to-noise ratio (SNR) and finite capacity satisfy some restriction. In other words, we can provide a microfoundation (limited information-processing constraint) for the exogenously specified SNR in the traditional SE models.

<sup>12</sup>Uppal and Wang (2002), Maenhout (2004, 2006), Cao, Wang, and Zhang (2005), Liu, Pan, and Wang (2005), and Liu (2010) examined how model uncertainty affects portfolio choices and/or asset prices.

explore how RI affects the calibrated parameter value of RB for given values of  $p$ . Specifically, I find that in the presence of model uncertainty, the correlation between the equity return and undiversified labor income not only affects the hedging demand for the risky asset, but also affects its standard speculation demand. The key reason is that given the same value of  $p$ , the correlation between labor income and the equity return increases the calibrated value of RB and thus reduces the optimal share invested in the risky asset.

Finally, the model presented in this paper has some testable implications. As discussed in Haliassos and Bertaut (1995), Haliassos and Michaelides (2000), and Campbell (2006), the empirical evidence on the correlation between labor income and equity returns for different population groups is difficult to reconcile with the observed stockholding behavior. Davis and Willen (1999) estimated that the correlation is between 0.1 and 0.3 for college-educated males and is about  $-0.25$  for male high school dropouts. Heaton and Lucas (1999) found that the correlation between the entrepreneurial risk and the equity return was about 0.2. Since negative correlation between earnings and equity returns implies increased willingness to invest in the risky asset, less-educated investors should be more heavily invested in the stock market while college graduates and entrepreneurs should put less wealth in the stock market. In contrast, the empirical evidence on stock market participation shows a significantly positive correlation between education level and stockholding.<sup>13</sup> I show that incorporating induced uncertainty due to the interaction of RB and RI can have the potential to help reconcile the model with the empirical evidence. Specifically, *poorer and less well-educated* investors probably face greater induced uncertainty (state uncertainty and model uncertainty); consequently, they rationally choose to invest less in the stock market even if the correlation between their labor income and equity returns is negative and they have stronger incentive to hedge against their earnings risk.

This paper contributes to the literature on incomplete-markets consumption-saving-portfolio decisions under uncertainty, and is closely related to Maenhout (2004) and Wang (2009). Maenhout (2004) explored how model uncertainty due to a preference for robustness affects optimal portfolio choice, and showed that robustness significantly reduces the demand for the risky asset and increases the equilibrium equity premium. Wang (2009) studied the effects of incomplete information about the income growth rate on the agent's consumption, saving, and portfolio choice in an incomplete-market economy. He found that the estimation risk arising from the agent's learning process leads to additional precautionary savings demand and the agent can partially hedge against both the income risk and estimation risk by investing in the risky asset. Unlike Maenhout (2004), the present paper explores how the interaction of model uncertainty and state uncertainty affects the consumption-saving-portfolio decisions in the presence of uninsurable labor income. The model presented in this paper can therefore be used to study the relationship between the cor-

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<sup>13</sup>Haliassos and Bertaut (1995) found that the share invested in the stock market is substantially larger among those with at least a college degree compared to those with less than high school education at all income levels.

relation between the labor income risk and the equity return risk and the stockholding behavior. Unlike Wang (2009), this paper considers model uncertainty due to robustness. In addition, the state uncertainty considered in this paper is not only from the income process but also from the equity return. Finally, this paper is also related with the work on robust/risk-sensitive/rational inattention permanent income models such as Hansen, Sargent, and Tallarini (1999), Luo (2008), and Luo and Young (2010). The key difference between this paper and the papers mentioned above is that they adopted the linear-quadratic permanent income framework with constant asset returns to study consumption and saving dynamics and did not consider the portfolio choice problem. In addition, they can establish the observational equivalence between the discount factor and the robustness preference. In contrast, in the present paper, this observational equivalence breaks down in the presence of the portfolio choice.

This remainder of the paper is organized as follows. Section 2 presents the setup of a continuous-time consumption and portfolio choice model with uninsurable labor income. Section 3 introduces RB into the benchmark model and examines the theoretical and empirical implications of RB on consumption-portfolio rules and precautionary savings using calibrated RB parameters. Section 4 examines how the interactions of RB and RI due to limited information-processing constraint affect robustly strategic consumption-portfolio rules. Section 5 discusses another type of informational frictions, incomplete information about individual income components, and compares it with the RI hypothesis. Section 6 concludes.

## 2. A Continuous-time Consumption-Portfolio Choice Model with Uninsurable Labor Income

In this paper, we follow Wang (2009) and consider a continuous-time version of the Caballero-type model (1990) with portfolio choice. The typical consumer facing uninsurable labor income in the model economy makes optimal consumption-saving-asset allocation decisions. Specifically, we assume that the consumer can access: one risk-free asset and one risky asset, and also receive uninsurable labor income. Labor income ( $y_t$ ) is assumed to follow a continuous-time AR(1) (Ornstein-Uhlenbeck) process:

$$dy_t = \rho \left( \frac{\mu}{\rho} - y_t \right) dt + \sigma_y dB_{y,t}, \quad (1)$$

where the unconditional mean and variance of income are  $\bar{y} = \mu / \rho$  and  $\sigma_y^2 / (2\rho)$ , respectively; the persistence coefficient  $\rho$  governs the speed of convergence or divergence from the steady state;<sup>14</sup>  $B_{y,t}$  is a standard Brownian motion on the real line  $\mathcal{R}$ ; and  $\sigma_y$  is the unconditional volatility of the

<sup>14</sup>If  $\rho > 0$ , the income process is stationary and deviations of income from the steady state are temporary; if  $\rho \leq 0$ , income is non-stationary. The  $\rho = 0$  case corresponds to a simple Brownian motion without drift. The larger  $\rho$  is, the less  $y$  tends to drift away from  $\bar{y}$ . As  $\rho$  goes to  $\infty$ , the variance of  $y$  goes to 0, which means that  $y$  can never deviate from  $\bar{y}$ .

income change over an incremental unit of time.

The agent can invest in both a risk-free asset with a constant interest rate  $r$  and a risky asset (i.e., the market portfolio) with a risky return  $r_t^e$ . The instantaneous return  $dr_t^e$  of the risky market portfolio over  $dt$  is given by

$$dr_t^e = (r + \pi) dt + \sigma_e dB_{e,t}, \quad (2)$$

where  $\pi$  is the market risk premium;  $B_{e,t}$  is a standard Brownian motion; and  $\sigma_e$  is the standard deviation of the market return. Let  $\rho_{ye}$  be the contemporaneous correlation between the labor income process and the return of the risky asset. When  $\rho_{ye} = 0$ , the labor income risk is idiosyncratic and is uncorrelated with the risky market return; when  $\rho_{ye} = 1$ , the labor income risk is perfectly correlated with the risky market return. The agent's financial wealth evolution is then given by

$$dw_t = (rw_t + y_t - c_t) dt + \alpha_t (\pi dt + \sigma_e dB_{e,t}), \quad (3)$$

where  $\alpha_t$  denotes the amount of wealth that the investor allocates to the market portfolio at time  $t$ .

The typical consumer is assumed to maximize the following expected lifetime utility:

$$E_0 \left[ \int_{t=0}^{\infty} \exp(-\delta t) u(c_t) dt \right],$$

subject to (3). The utility function takes the CARA form:  $u(c_t) = -\exp(-\gamma c_t) / \gamma$ , where  $\gamma > 0$  is the coefficient of absolute risk aversion.<sup>15</sup> To simplify the model, we define a new state variable,  $s_t$ :

$$s_t \equiv w_t + h_t,$$

where  $h_t$  is human wealth at time  $t$  and is defined as the expected present value of current and future labor income discounted at the risk-free interest rate  $r$ :

$$h_t \equiv E_t \left[ \int_t^{\infty} \exp(-r(s-t)) y_s ds \right].$$

For the given income process, (1),  $h_t = \frac{1}{r+\rho} y_t + \frac{\mu}{r(r+\rho)}$ .<sup>16</sup> Following the state-space-reduction approach proposed in Luo (2008) and using this new state variable, we can rewrite (3) as

$$ds_t = (rs_t - c_t + \pi \alpha_t) dt + \sigma dB_t, \quad (4)$$

<sup>15</sup>It is well-known that the CARA utility specification is tractable for deriving the consumption function or optimal consumption-portfolio rules in different settings. See Merton (1969), Caballero (1990), Svensson and Werner (1993), and Wang (2004, 2009).

<sup>16</sup>Here we need to impose the restriction that  $r > -\rho$  to guarantee below the finiteness of human wealth.

where  $\sigma dB_t = \sigma_e \alpha_t dB_{e,t} + \sigma_s dB_{y,t}$ ,  $\sigma_s = \sigma_y / (r + \rho)$ , and

$$\sigma = \sqrt{\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye} \sigma_s \sigma_e \alpha_t} \quad (5)$$

is the unconditional variance of the innovation to  $s_t$ .<sup>17</sup>

In this benchmark full-information rational expectations (FI-RE) model, we assume that the consumer trusts the model and observes the state perfectly, i.e., there is no model uncertainty and no state uncertainty. The value function is denoted by  $J(s_t)$ . The Hamilton-Jacobi-Bellman (HJB) equation for this optimizing problem can be written as:

$$0 = \sup_{c_t, \alpha_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) \right],$$

where

$$\mathcal{D}J(s_t) = J_s(rs_t - c_t + \pi \alpha_t) + \frac{1}{2} J_{ss} (\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye} \sigma_s \sigma_e \alpha_t). \quad (6)$$

Finally, the transversality condition (TVC),  $\lim_{t \rightarrow \infty} E |\exp(-\delta t) J_t| = 0$ , holds (see Appendix 7.1 for the proof). Solving the above HJB subject to (4) leads to the following optimal portfolio-consumption rules:<sup>18</sup>

$$\alpha = \frac{\pi}{r\gamma\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \quad (7)$$

and

$$c_t = rs_t + \frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\gamma\sigma_e^2} - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} - \Gamma, \quad (8)$$

where

$$\Gamma \equiv \frac{1}{2} r\gamma \left(1 - \rho_{ye}^2\right) \sigma_s^2, \quad (9)$$

is the investor's precautionary saving demand. The first term in (7) is the standard speculation demand for the risky asset, which is positively correlated with the risk premium of the risky asset over the risk-free asset and is negatively correlated with the degree of risk aversion and the variance of the return to the risky asset. The second term in (7) is the labor income hedging demand of the risky asset. When  $\rho_{ye} \neq 0$ , i.e., the income shock is not purely idiosyncratic, the desirability of the risky asset depends not only on its expected excess return relative to its variance, but also on its ability to hedge consumption against bad realizations of labor income. Following the literature of precautionary savings, we measure the demand for precautionary saving as the amount of saving

<sup>17</sup>The main advantage of this state-space-reduction approach is to allow us to solve the model with both model uncertainty and state uncertainty explicitly and help better inspect the mechanism by which the informational frictions interact and affect optimal consumption-portfolio rules. It is worth noting that if we only consider model uncertainty, the reduced univariate model and the original multivariate model are equivalent in the sense that they lead to the same consumption-portfolio rules. The detailed proof is available from the online appendix.

<sup>18</sup>This solution is similar to that obtained in Model I of Wang (2009).



due to the interaction of the degree of risk aversion and non-diversifiable labor income risk.<sup>19</sup> If labor income is perfectly correlated with the return to the risky asset (i.e.,  $\rho_{ye} = \pm 1$ ), the market is complete and the consumer can fully hedge his or her labor income risk; consequently, his or her demand for precautionary saving is 0.

### 3. Incorporating Model Uncertainty due to Robustness

#### 3.1. Modeling Robustness

As argued in Hansen and Sargent (2007), the simplest version of robustness considers the question of how to make optimal decisions when the decision-maker does not know the true probability model that generates the data. The main goal of introducing robustness is to design optimal policies that not only work well when the reference (or approximating) model governing the evolution of the state variables is the true model, but also perform reasonably well when there is some type of model misspecification. To introduce robustness into our model proposed above, we follow the continuous-time methodology proposed by Anderson, Hansen, and Sargent (2003) (henceforth, AHS) and adopted in Maenhout (2004) to assume that consumers are concerned about the model misspecifications and take Equation (4) as the approximating model.<sup>20</sup> The corresponding distorting model can thus be obtained by adding an endogenous distortion  $v(s_t)$  to (4):

$$ds_t = (rs_t - c_t + \pi\alpha_t) dt + \sigma(\sigma v(s_t) dt + dB_t). \quad (10)$$

As shown in AHS (2003), the objective  $\mathcal{D}J$  defined in (6) plays a crucial role in introducing robustness.  $\mathcal{D}J$  can be thought of as  $E[dJ]/dt$  and is easily obtained using Itô's lemma. A key insight of AHS (2003) is that this differential expectations operator reflects a particular underlying model for the state variable. The consumer accepts the approximating model, (4), as the best approximating model, but is still concerned that it is misspecified. He or she therefore wants to consider a range of models (i.e., the distorted model, (10)) surrounding the approximating model when computing the continuation payoff. A preference for robustness is then achieved by having the agent guard against the distorting model that is reasonably close to the approximating model. The drift adjustment  $v(s_t)$  is chosen to minimize the sum of (i) the expected continuation payoff adjusted to reflect the additional drift component in (10) and (ii) an entropy penalty:

$$\inf_v \left[ \mathcal{D}J + v(s_t) \sigma^2 J_s + \frac{1}{\vartheta_t} \mathcal{H} \right], \quad (11)$$

<sup>19</sup>Note that hedging with the risky asset ( $\rho_{ye} \neq 0$ ) reduces the consumer's precautionary saving demand.

<sup>20</sup>As argued in Hansen and Sargent (2007), the agent's commitment technology is irrelevant under RB if the evolution of the state is backward-looking. We therefore do not specify the commitment technology of the consumer in the RB models of this paper.

where the first two terms are the expected continuation payoff when the state variable follows (10), i.e., the alternative model based on drift distortion  $v(s_t)$ ,  $\mathcal{H} = (v(s_t)\sigma_s)^2/2$  is the relative entropy or the expected log likelihood ratio between the distorted model and the approximating model and measures the distance between the two models,<sup>21</sup> and  $1/\vartheta_t$  is the weight on the entropy penalty term.<sup>22</sup>  $\vartheta_t$  is fixed and state-independent in AHS (2003); whereas it is state-dependent in Maenhout (2004). The key reason for using a state-dependent counterpart  $\vartheta_t$  in Maenhout (2004) is to assure the homotheticity or scale invariance of the decision problem with the CRRA utility function.<sup>23</sup> In this paper, we also specify that  $\vartheta_t$  is state-dependent ( $\vartheta(s_t)$ ) in the CARA-Gaussian setting. The main reason for this specification is to guarantee homotheticity, which makes robustness not diminish as the value of the total wealth increases.<sup>24</sup> Note that the evil agent's minimization problem, (11), becomes invariant to the scale of total resource,  $s_t$  when using the state-dependent specification of  $\vartheta_t$ .

Applying these results in the above model yields the following HJB equation under robustness:

$$\sup_{c_t, \alpha_t} \inf_{v_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) + v(s_t) \sigma^2 J_s + \frac{1}{2\vartheta(s_t)} v^2(s_t) \sigma^2 \right]. \quad (12)$$

Finally, the transversality condition (TVC),  $\lim_{t \rightarrow \infty} E |\exp(-\delta t) J_t| = 0$ , holds at optimum. Solving first for the infimization part of (12) yields:

$$v^*(s_t) = -\vartheta(s_t) J_s,$$

where  $\vartheta(s_t) = -\vartheta/J(s_t) > 0$  and  $\vartheta$  is a constant (see Appendix 7.1 for the derivation). Following Uppal and Wang (2003) and Liu, Pan, and Wang (2005), here we can also define "1/J(s\_t)" in the  $\vartheta(s_t)$  specification as a *normalization* factor that is introduced to convert the relative entropy (i.e., the distance between the approximating model and the distorted model) to units of utility so that it is consistent with the units of the expected future value function evaluated with the distorted model. It is worth noting that adopting a slightly more general specification,  $\vartheta(s_t) = -\varphi\vartheta/J(s_t)$  where  $\varphi$  is a constant, does not affect the main results of the paper. The reason is as follows. We can simply define a new constant,  $\tilde{\vartheta} = \varphi\vartheta$ , and  $\tilde{\vartheta}$ , rather than  $\vartheta$ , will enter the decision rules. Using

<sup>21</sup>Define  $q_t \equiv -\int_0^t v(s_t)\sigma_s dB_s - \frac{1}{2}\int_0^t (v(s_t)\sigma_s)^2 ds$ , we have the following Radon-Nikodym derivative:

$$\frac{dQ}{dP} \Big|_{\mathcal{F}_t=q_t}$$

for each time  $t$ , where  $Q$  and  $P$  are the distributions of the distorted model and the approximating model, respectively.

<sup>22</sup>The last term in (11) is due to the consumer's preference for robustness. Note that the  $\vartheta_t = 0$  case corresponds to the standard expected utility case. This robustness specification is called the *multiplier (or penalty) robust control* problem. We will discuss another closely related robustness specification, the *constraint robust control* problem, in the next subsection. See AHS (2003) and Hansen, Sargent, Turmuhambetova, and Williams (2006) (henceforth, HSTW) for detailed discussions on these two robustness specifications.

<sup>23</sup>See Maenhout (2004) for detailed discussions on the appealing features of "homothetic robustness".

<sup>24</sup>Note that the impact of robustness wears off if we assume that  $\vartheta_t$  is constant. It is clear from the procedure of solving the robust HJB proposed. (See Appendix 7.1 for the details.)

a given detection error probability, we can easily calibrate the corresponding value of  $\tilde{\vartheta}$  that affects the optimal consumption-portfolio rules.<sup>25</sup>

Because  $\vartheta(s_t) > 0$ , the perturbation adds a negative drift term to the state transition equation because  $J_s > 0$ . Substituting for  $v^*$  in (12) gives:

$$\sup_{c_t, \alpha_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + (rs_t - c_t + \pi \alpha_t) J_s + \frac{1}{2} \sigma^2 J_{ss} - \frac{1}{2} \vartheta(s_t) \sigma^2 J_s^2 \right]. \quad (13)$$

### 3.2. Theoretical Implications

Following the standard procedure, we can then solve (13) and obtain the optimal consumption-portfolio rules under robustness. The following proposition summarizes the solution:

**Proposition 1.** *Under robustness, the optimal consumption and portfolio rules are*

$$c_t^* = rs_t + \frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\tilde{\gamma}\sigma_e^2} - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} - \Gamma, \quad (14)$$

and

$$\alpha^* = \frac{\pi}{r\tilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \quad (15)$$

respectively, where the effective coefficient of absolute risk aversion  $\tilde{\gamma}$  is defined as:  $\tilde{\gamma} \equiv (1 + \vartheta)\gamma$ ,  $\Psi = (\delta - r) / (r\gamma)$  captures the dissavings effect of relative impatience, and the precautionary savings demand,  $\Gamma$ , is:

$$\Gamma = \frac{1}{2} r \tilde{\gamma} (1 - \rho_{ye}^2) \sigma_s^2. \quad (16)$$

Finally, the worst possible distortion can be written as:

$$v^* = -r\gamma\vartheta. \quad (17)$$

*Proof.* See Appendix 7.1. ■

From (14), it is clear that robustness does not change the marginal propensity to consume out of permanent income (MPC), but affects the amount of precautionary savings ( $\Gamma$ ). In other words, in the continuous-time setting, consumption is not sensitive to unanticipated income shocks. This conclusion is different from that obtained in the discrete-time robust-LQG permanent income model, in which the MPC is increased via the interaction between RB and income uncertainty and consumption is more sensitive to unanticipated shocks.<sup>26</sup>

<sup>25</sup>See Section 7.2 for the detailed procedure to calibrate the value of  $\vartheta$  using the detection error probabilities.

<sup>26</sup>See HST (1999) and Luo and Young (2010) for detailed discussions on how RB affects consumption and precautionary savings in the discrete-time robust-LQG models.

Expression (15) shows that RB reduces the optimal speculation demand by a factor,  $1 + \vartheta$ , but does not affect the hedging demand of the risky asset. In other words, RB increases the relative importance of the income hedging demand to the speculation demand by increasing the effective coefficient of absolute risk aversion ( $\tilde{\gamma}$ ). Expression (16) shows that the precautionary savings premium increases with the degree of robustness ( $\vartheta$ ) by increasing the value of  $\tilde{\gamma}$  and interacting with two types of fundamental uncertainty: labor income uncertainty ( $\sigma_s^2$ ) and the correlation between labor income and the equity return ( $\rho_{ye}$ ). An interesting question here is the relative importance of RB ( $\vartheta$ ) and CARA ( $\gamma$ ) in determining the precautionary savings premium, holding other parameters constant. We can use the elasticities of precautionary saving as a measure of their importance. Specifically, using (16), we have the following proposition:

**Proposition 2.** *The relative sensitivity of precautionary saving to robustness (RB,  $\vartheta$ ) and CARA ( $\gamma$ ) can be measured by:*

$$\mu_{\gamma\vartheta} \equiv \frac{e_\gamma}{e_\vartheta} = \frac{1 + \vartheta}{\vartheta} > 1, \quad (18)$$

where  $e_\vartheta \equiv \frac{\partial \Gamma/\Gamma}{\partial \vartheta/\vartheta} =$  and  $e_\gamma \equiv \frac{\partial \Gamma/\Gamma}{\partial \gamma/\gamma}$  are the elasticities of the precautionary saving demand to RB and CARA, respectively. (18) means that the precautionary savings demand is more sensitive to absolute risk aversion measured by  $\gamma$  than RB measured by  $\vartheta$ . Note that Expression (18) can also measure the relative sensitivity of portfolio choice to RB and CARA.

*Proof.* Using (15) and (16), the proof is straightforward. ■

Using (18), it is simple to show that  $\partial \mu_{\gamma\vartheta} / \partial \vartheta > 0$ , which means that  $\mu_{\gamma\vartheta}$  is increasing with the degree of RB,  $\vartheta$ . To fully explore the quantitative effects of robustness on portfolio choice and precautionary saving, we need to calibrate  $\vartheta$  using the detection error probability approach (DEP) proposed in Hansen, Sargent, and Wang (henceforth, HSW, 2002), AHS (2003), and Hansen and Sargent (Chapter 9, 2007). In the next subsection, we will examine the relative importance of RB to CARA quantitatively after calibrating  $\vartheta$  using the U.S. data.

**Proposition 3.** *The observational equivalence between the discount factor and robustness established in the discrete-time Hansen-Sargent-Tallarini (1999) does not hold in our continuous-time CARA-Gaussian model.*

Hansen, Sargent, and Tallarini (1999) (henceforth, HST) show that the discount factor and the concern about robustness are observationally equivalent in the sense that they lead to the same consumption and investment decisions in a discrete-time LQG permanent income model. The reason is that introducing a concern about robustness increases savings in the same way as increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RS or

RB on consumption and investment.<sup>27</sup> In contrast, in the continuous-time CARA-Gaussian model with portfolio choice discussed in this section, the observational equivalence between the discount rate and the RB parameter no longer holds. This result can be readily obtained by inspecting the explicit expressions of consumption, precautionary savings, and portfolio choice, (14)-(16). The main reason for this result is that the preference for robustness governed by  $\vartheta$  affects the portfolio rule, while the discount rate ( $\delta$ ) has no impact on the portfolio rule in this Merton-type solution. It is well known that this type of intertemporal consumption-portfolio choice models leads to myopic portfolio rules, and the discount rate does not play a role in affecting asset allocation. It is straightforward to show that once we rule out the risky asset from our model, we can re-establish the observational equivalence between robustness and patience. Specifically, for given  $r$  and  $\gamma$ , when  $\delta = r + 0.5 (r\gamma)^2 \vartheta \sigma_s$ , this RB model is observationally equivalent to the FI-RE model with a lower discount rate ( $\delta = r$ ) in the sense that they lead to the same consumption-saving decisions.<sup>28</sup> In summary, although both the discount rate ( $\delta$ ) and  $\vartheta$  affect the constant term in the consumption function, and their observational equivalence can be established in the sense that they generate the same value of the constant term, they imply different portfolio choices and thus break down the observational equivalence between  $\delta$  and  $\vartheta$ .<sup>29</sup>

Following HSTW (2006) and Hansen and Sargent (2007), we could use the following *constraint* formulation of the above RB problem:

$$\sup_{c_t, \alpha_t} \inf_{v_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) + v(s_t) \sigma^2 J_s \right], \quad (19)$$

subject to

$$\frac{1}{2} (v(s_t) \sigma)^2 \leq \eta, \quad (20)$$

where  $\eta > 0$  measures the consumer's tolerance for model misspecification. It is clear from the above constraint that the worst-case distortion is:

$$v^*(s_t) = -\frac{\sqrt{2\eta}}{\sigma} < 0.$$

Substituting this expression into (19), we can reduce the robust HJB equation to the following HJB equation:

$$\sup_{c_t, \alpha_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) - \sqrt{2\eta} \sigma J_s \right]. \quad (21)$$

<sup>27</sup>As shown in HST (1999), the two models have different implications for asset prices because continuation valuations would change as one alters the values of the discount factor and the robustness parameter within the observational equivalence set.

<sup>28</sup>Note that in the case without the risky asset, the consumption function and the precautionary saving demand become  $c_t^* = r s_t + \Psi - \Gamma$  and  $\Gamma = 0.5 r \gamma \sigma_s^2$ , respectively.

<sup>29</sup>Note that the precautionary saving demand is included in the constant term.

However, given that  $\sigma = \sqrt{\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye} \sigma_s \sigma_e \alpha_t}$ , since both  $\sigma$  and  $\sigma^2$  appear in (21), there is no explicit solution for (21). As argued in Lei (2001) and HSTW (2006), the risk is second-order in the multiplier formulation because the robustness parameter interacts with the risk-aversion parameter, and they are multiplied by the variance  $\sigma^2$ . In other words, the second-order risk aversion is now enhanced by the presence of robustness measured by  $\vartheta$ . In contrast, in the constraint formulation, the robustness term,  $\sqrt{2\eta}\sigma$ , is proportional to the standard deviation  $\sigma$ . In other words,  $\eta$  measures the amount of the first-order risk aversion. In the online appendix, I show that the multiplier and constraint formulations are observationally equivalent in the sense that they lead to the same portfolio rule and the same level of the worst-case distortion when  $\vartheta$  and  $\eta$  satisfy some restriction in special cases in which  $\rho_{ye} = \pm 1$  or  $\sigma_s = 0$ . To keep the model tractable, following the literature I focus on the multiplier formulation of RB in the subsequent analysis.

### 3.3. Quantitative Implications

To fully explore how RB affects the joint behavior of portfolio choice, consumption, and labor income, we adopt the calibration procedure outlined in HSW (2002) and AHS (2003) to calibrate the value of the RB parameter ( $\vartheta$ ) that governs the degree of robustness. Specifically, we calibrate  $\vartheta$  by using the method of detection error probabilities (DEP) that is based on a statistical theory of model selection. We can then infer what values of  $\vartheta$  imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by  $p$  is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard many models, implying that the cloud of models surrounding the approximating model is large. In this case, it is easier for the consumer to distinguish the two models (see Appendix 7.2 for the detailed calibration procedure using the value of  $p$ ).

Using the data set documented in Campbell (2003), we set the parameter values for the processes of returns, volatility, and consumption as follows:  $\mu = 0.08$ ,  $r = 0.02$ ,  $\delta = 0.02$ , and  $\sigma_e = 0.156$ . For the labor income process, we follow Wang (2009) and set that  $\sigma_y = 0.1$ . When  $\rho = 0$ , i.e., when labor income follows a Brownian motion, we can compute that  $\sigma_s = 5$ . Figure 1 illustrates how DEP ( $p$ ) varies with the value of  $\vartheta$  for different values of  $\gamma$ . We can see from the figure that the stronger the preference for robustness (higher  $\vartheta$ ), the less the  $p$  is. For example, when  $\gamma = 2$ ,  $p = 10\%$  when  $\vartheta = 1.4$ , while  $p = 16\%$  when  $\vartheta = 1$ .<sup>30</sup> Both values of  $p$  are reasonable as argued in AHS (2002), HSW (2002), Maenhout (2004), and Hansen and Sargent (Chapter 9, 2007).

Figures 2 and 3 illustrate how  $p$  varies with the value of  $\vartheta$  for different values of  $\sigma_s$  and  $\rho_{ye}$ , respectively.<sup>31</sup> These figures also show that the higher the value of  $\vartheta$ , the less the  $p$  is. In addition,

<sup>30</sup>Caballero (1990) and Wang (2004) also set  $\gamma = 2$ .

<sup>31</sup>Note that since  $\sigma_s = \sigma_y / (r + \rho)$ , the value of  $\sigma_s$  can measure the persistence ( $\rho$ ) and volatility ( $\sigma_y$ ) of the labor income process.

to calibrate the same value of  $p$ , less values of  $\sigma_s$  (i.e., more volatile or higher persistent labor income processes) or higher values of  $\rho_{ye}$  lead to higher values of  $\vartheta$ . The intuition behind this result is that  $\sigma_s$  and  $\vartheta$  have opposite effects on  $\bar{v}$  (this is clear from (64) in Appendix 7.2). To keep the same value of  $p$ , if one parameter increases, the other one must reduce to offset its effect on  $\bar{v}$ . As emphasized in Hansen and Sargent (2007), in the robustness model,  $p$  is the deep model parameter governing the preference for RB, and  $\vartheta$  reflects the effect of RB on the model's behavior. Combining these facts with the expression for robust portfolio rule, (15), we can see that an increase in  $\rho_{ye}$  not only reduces the hedging demand directly, but also reduces the standard speculation demand of the risky asset by affecting the calibrated values of  $\vartheta$  using the same values of  $p$ . In contrast, an increase in  $\sigma_s$  reduces the hedging demand, but increases the speculation demand. It is worth noting that in the RB model  $p$  can be used to measure the amount of model uncertainty, whereas  $\vartheta$  is used to measure the degree of the agent's preference for RB. If we keep  $p$  constant when recalibrating  $\vartheta$  for different values of other parameters, the amount of model uncertainty is held constant, i.e., the set of distorted models with which we surround the approximating model does not change. In contrast, if we keep  $\vartheta$  constant,  $p$  will change accordingly when the values of other parameters change. In other words, the amount of model uncertainty is "elastic" and will change accordingly when the fundamental factors change.

From the expression for robust portfolio rule, (15), we can see that plausible values of RB can significantly affect the share invested in the risky asset. Figure 4 shows how the robust portfolio rule varies with the degree of RB for different values of  $\rho_{ye}$ . It clearly shows that  $\alpha^*$  decreases with the value of  $\vartheta$  for different values of  $\rho_{ye}$ . In addition, it is also clear from the same figure that  $\alpha^*$  decreases with  $\rho_{ye}$  for a given value of  $\vartheta$ . The intuition behind this result is the same as that in the FI-RE case: When the labor income risk becomes more positively correlated with the shock to the equity return, the equity is less desirable and the agent thus invests less in it.

Figure 4 also illustrates how the precautionary saving demand ( $\Gamma$ ) varies with the degree of RB for different values of  $\rho_{ye}$ . It clearly shows that  $\Gamma$  increases with the value of  $\vartheta$  for different values of  $\rho_{ye}$  and RB has a very significant impact on precautionary savings. For example, when  $\rho_{ye} = 0.35$ ,  $\Gamma = 0.63$  when  $\vartheta = 1$ , while  $\Gamma = 0.96$  when  $\vartheta = 1.4$ . Furthermore, we can see that  $\Gamma$  decreases with  $\rho_{ye}$  for a given value of  $\vartheta$ . The intuition for this result is that the higher the value of  $\rho_{ye}$ , the more important the hedging demand for the equity, and thus the less demand for precautionary savings.

## 4. Incorporating State Uncertainty

### 4.1. Information-Processing Constraint

So far we have considered the case in which the consumer can observe the state perfectly. In this section, we consider a situation in which the typical consumer with the preference for robust-

ness cannot observe the state ( $s$ ) perfectly due to finite information-processing capacity (rational inattention, or RI). In other words, the typical consumer can neither observe  $s_t$  nor can he or she observe the source of innovation  $dB_t$ , included in the state transition equation, (4):

$$ds_t = (rs_t - c_t + \pi\alpha_t) dt + \sigma dB_t. \quad (22)$$

Following Kasa (2006) and Reis (2011), we assume that the consumer observes only a noisy signal containing imperfect information about  $s_t$ :

$$ds_t^* = s_t dt + d\zeta_t, \quad (23)$$

where  $\zeta_t$  is the noise shock, and is a Brownian motion with mean 0 and variance  $\Lambda$  (in the RI setting, the variance,  $\Lambda$ , is a choice variable for the agent). Note that here we assume that the consumer receives signals on  $s_t dt$  rather than on  $ds_t$ . As emphasized in Sims (1998) and discussed in Kasa (2006) and Reis (2011), the latter specification is not suitable to model state uncertainty due to finite capacity because this specification means that in any finite interval, arbitrarily large amounts of information can be passed through the consumer's channel. In addition, following the RI literature, we assume that  $\zeta_t$  is independent of the Brownian motion governing the fundamental shock,  $B_t$ .<sup>32</sup>

To model RI due to finite capacity, we follow Sims (2003) and impose the following constraint on the consumer's information-processing ability:

$$\mathcal{H}(s_{t+\Delta t} | \mathcal{I}_t) - \mathcal{H}(s_{t+\Delta t} | \mathcal{I}_{t+\Delta t}) \leq \kappa \Delta t, \quad (24)$$

where  $\kappa$  is the consumer's information channel capacity;  $\mathcal{H}(s_{t+\Delta t} | \mathcal{I}_t)$  denotes the entropy of the state prior to observing the new signal at  $t + \Delta t$ ; and  $\mathcal{H}(s_{t+\Delta t} | \mathcal{I}_{t+\Delta t})$  is the entropy after observing the new signal.  $\kappa$  imposes an upper bound on the amount of information – that is, the change in the entropy – that can be transmitted in any given period. Formally, entropy is defined as the expectation of the negative of the (natural) log of the density function of a random variable  $s$ ,  $-E[\ln(f(s))]$ . For example, the entropy of a discrete distribution with equal weight on two points is simply  $E[\ln_2(f(X))] = -0.5 \ln(0.5) - 0.5 \ln(0.5) = 0.69$ , and the unit of information contained in this distribution is 0.69 “nats” or 1 bit.<sup>33</sup> In this case, an agent can remove all uncertainty about  $s$  if the capacity devoted to monitoring  $s$  is  $\kappa = 1$  bit. Since imperfect observations on the state lead to welfare losses, the information-processing constraint must be binding. In other words, rational investors use all of their channel capacity,  $\kappa$ , to reduce the uncertainty upon new observations. To

<sup>32</sup>In the traditional signal extraction literature, sometimes it is assumed that the fundamental shock and the noise shock (or measure errors) are correlated. In real systems, we do observe correlated shocks and noises. See Stengel (Chapter 4, 1994) for a discussion on correlated process and measurement noise.

<sup>33</sup>For alternative bases for the logarithm, the unit of information differs; with log base 2 the unit of information is the “bit” and with base 10 it is a “dit” or a “hartley.”



apply this information constraint to the state transition equation, we first rewrite (22) in the time interval of  $[t, t + \Delta t]$ :<sup>34</sup>

$$s_{t+\Delta t} = \rho_{0,t} + \rho_1 s_t + \rho_2 \sqrt{\Delta t} \varepsilon_{t+\Delta t}, \quad (25)$$

where  $\rho_{0,t} = (-c_t + \pi \alpha_t) (1 - \exp(r\Delta t)) / (-r\Delta t)$ ,  $\rho_1 = \exp(r\Delta t)$ ,  $\rho_2 = \sigma \sqrt{(1 - \exp(2r\Delta t)) / (-2r\Delta t)}$ , and  $\varepsilon_{t+\Delta t}$  is the time- $(t + \Delta t)$  standard normal distributed innovation to permanent income. Taking conditional variances on both sides of (25) and substituting it into (24), we have

$$\ln(\rho_1^2 \Sigma_t + \rho_2^2) - \ln(\Sigma_{t+\Delta t}) = 2\kappa \Delta t, \quad (26)$$

which reduces to

$$\dot{\Sigma}_t = 2(r - \kappa) \Sigma_t + \sigma^2,$$

as  $\Delta t \rightarrow 0$ , where  $\Sigma_t = E_t [(s_t - \hat{s}_t)^2]$  the conditional variance at  $t$  (see Appendix 7.3 for a proof).

In the steady state in which  $\dot{\Sigma}_t = 0$ , the steady state conditional variance can be written as:<sup>35</sup>

$$\Sigma = \frac{\sigma^2}{2(\kappa - r)}. \quad (27)$$

To make optimal decisions, the consumer is required to filter in the optimal way the value of  $s_t$  using the observed  $s_t^*$ . Although the setting of our CARA-Gaussian model is not a typical tracking problem, the filtering problem in this model could be similar to the tracking problem proposed in Sims (2003, 2010). Specifically, we may think that the model with imperfect state observations can be decomposed into a two-stage optimization procedure:<sup>36</sup>

1. The optimal filtering problem determines the optimal evolution of the perceived (estimated) state;
2. The optimal control problem in which the decision makers treat the perceived state as the underlying state when making optimal decisions.

Here we assume ex post Gaussian distributions and Gaussian noise but adopt exponential or CARA preferences. See Peng (2004), Van Nieuwerburgh and Veldkamp (2009, 2010), and Mondria (2010) for this specification. Because both the optimality of ex post Gaussianity and the standard Kalman filter are based on the linear-quadratic-Gaussian (LQG) specification, the applications of these results in the RI models with CARA preferences are only approximately valid.

<sup>34</sup>Note that here we use the fact that  $\Delta B_t = \varepsilon_t \sqrt{\Delta t}$ , where  $\Delta B_t$  represents the increment of a Wiener process.

<sup>35</sup>Note that here we need to impose the restriction  $\kappa - r > 0$ . If this condition fails, the state is not stabilizable and the conditional variance diverges.

<sup>36</sup>See Liptser and Shirayev (2001) for a textbook treatment on this topic and an application in a precautionary saving model in Wang (2004).

In stage 1, consumers need to estimate the unobserved state ( $s_t$ ) using its prior distribution and all processed and available information (i.e., their noisy observations,  $\mathcal{F}_t = \left\{ s_j^* \right\}_{j=0}^t$ ). Specifically, consumers rationally compute the conditional distribution of the unobserved state and represent the original optimization problem as a Markovian one. Given the Gaussian prior  $s_0 \sim N(\hat{s}_0, \Sigma_0)$ , finding the posterior distribution of  $s_t$  becomes a standard filtering problem that can be solved using the Kalman-Bucy filtering method. Specifically, the optimal estimate for  $s_t$  given  $\mathcal{F}_t = \left\{ s_j^* \right\}_{j=0}^t$  in the mean square sense coincides with the conditional expectation:  $\hat{s}_t = E_t[s_t]$ , where  $E_t[\cdot]$  is based on  $\mathcal{F}_t$ . Applying Theorem 12.1 in Liptser and Shiryaev (2001), we can obtain the filtering differential equations for  $\hat{s}_t$  and  $\Sigma_t$  as follows:

$$d\hat{s}_t = (r\hat{s}_t - c_t + \pi\alpha_t) dt + K_t d\eta_t, \quad (28)$$

$$\dot{\Sigma}_t = -\Lambda K_t^2 + 2r\Sigma_t + \sigma^2, \quad (29)$$

given  $s_0 \sim N(\hat{s}_0, \Sigma_0)$ , where

$$K_t = \frac{\Sigma_t}{\Lambda} \quad (30)$$

is the Kalman gain and

$$d\eta_t = \sqrt{\Lambda} dB_t^*, \quad (31)$$

with mean  $E[d\eta_t] = 0$  and  $\text{var}(d\eta_t) = \Lambda dt$ , where  $B_t^*$  is a standard Brownian motion and  $\Lambda$  is to be determined. Note that  $\eta_t$  is a Brownian motion with mean 0. Although the Brownian variable,  $\zeta_t$ , is not observable, the innovation process,  $\eta_t$ , is observable because it is derived from observable processes (i.e.,  $ds_t^*$  and  $(r\hat{s}_t - c_t + \pi\alpha_t) dt$ ). In this case, the path of the conditional expectation,  $\hat{s}_t$ , is generated by the path of the innovation process,  $\eta_t$ . In the steady state, we have the following proposition:

**Proposition 4.** *Given finite capacity  $\kappa$ , in the steady state, the evolution of the perceived state can be written as:*

$$d\hat{s}_t = (r\hat{s}_t - c_t + \pi\alpha_t) dt + \hat{\sigma} dB_t^*, \quad (32)$$

where

$$\hat{\sigma} \equiv \Sigma / \sqrt{\Lambda} = f(\kappa) \sigma, \quad (33)$$

$f(\kappa) = \sqrt{\frac{\kappa}{\kappa-r}} > 1$  (i.e., the standard deviation of the estimated state is greater than that of the true state),

$$\Lambda = \frac{\sigma^2}{4\kappa(\kappa-r)} \quad (34)$$

is the steady state conditional variance, and

$$K = 2\kappa \quad (35)$$

is the corresponding Kalman gain.

*Proof.* In the steady state in which  $\dot{\Sigma}_t = 0$ , substituting the definition of the Kalman gain, (30), into  $-\Lambda K_t^2 + 2r\Sigma_t + \sigma^2 = 0$  and using  $\Sigma = \frac{\sigma^2}{2(\kappa-r)}$ , we can easily obtain that:

$$\Lambda = \frac{\sigma^2}{4\kappa(\kappa-r)} \text{ and } K = 2\kappa.$$

■

It is worth noting that the above RI case can be observationally equivalent to the traditional signal extraction (SE) model with exogenously specified noises (i.e., the steady state variance of the noise  $\Lambda$  or the signal-to-noise ratio (SNR)  $\sigma^2/\Lambda$  are specified exogenously) in the sense that they lead to the same model dynamics when the signal-to-noise ratio and finite capacity satisfy some restriction.<sup>37</sup> In other words, RI can provide a microfoundation for the exogenously specified SNR in the traditional SE models.

In the RI literature, to explain the observed aggregate fluctuations and the effects of monetary policy on the macroeconomy, the calibrated values of  $K$  are well below 1 (the FI-RE case). For example, Adam (2007) found  $K = 0.4$  bits based on the response of aggregate output to monetary policy shocks. Luo (2008) found that if  $K = 0.5$  bits, the otherwise standard permanent income model generates realistic relative volatility of consumption to income.

#### 4.2. Interaction between Model Uncertainty and State Uncertainty

In this section, we assume that the typical consumer not only cannot observe the state perfectly, but also has concerns about the innovation to perceived permanent income. In the model with both state uncertainty and model uncertainty, the prior variance of  $s$ ,  $\sigma^2$ , is affected by the optimal portfolio choice,  $\alpha^*$ , which is to be determined after solving the whole model with both model uncertainty ( $\vartheta$ ) and state uncertainty ( $\kappa$  or SNR). Given the value of  $\kappa$ , the value of the variance of the noise ( $\Lambda$ ) should also be endogenously determined by  $\alpha^*$ . The following is the two-stage procedure to solve the optimization problem of the consumer under both model uncertainty (MU) and state uncertainty (SU):

1. First, given finite SNR, we guess that the optimal portfolio choice under MU and SU is time-invariant, i.e.,  $\alpha_t = \alpha$ . Consequently,  $\sigma = \sqrt{\sigma_e^2 \alpha^2 + \sigma_s^2 + 2\rho_{ye} \sigma_s \sigma_e \alpha}$  is also time-invariant. The consumer with imperfect information about the state (SNR  $> 0$ ) understands that he or she cannot observe  $s_t$  perfectly and needs to use the Kalman filter, (28), to update the perceived state when making decisions. In other words, (28) is regarded as the approximating model

<sup>37</sup>See Appendix 7.5 for the detailed discussion.

in this MU-SU model. The consumer solves the following HJB:

$$\sup_{c_t, \alpha} \inf_{v_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(\hat{s}_t) + \mathcal{D}J(\hat{s}_t) + v(\hat{s}_t) \sigma^2 J_s + \frac{1}{2\vartheta} v(\hat{s}_t)^2 \sigma^2 \right], \quad (36)$$

subject to the distorted model:

$$d\hat{s}_t = (r\hat{s}_t - c_t + \pi\alpha) dt + \hat{\sigma} \left( \hat{\sigma} v(\hat{s}_t) dt + d\tilde{B}_t \right), \quad (37)$$

where  $\hat{\sigma} \equiv f(\kappa) \sigma$  and  $f(\kappa) = \sqrt{\frac{\kappa}{\kappa-r}} > 1$ , and the transversality condition (TVC),  $\lim_{t \rightarrow \infty} E \exp(-\delta t) J(\hat{s}_t) = 0$ , holds.

2. Second, after solving for optimal consumption and portfolio rules under RB and RI, we can verify whether or not the resulting portfolio rule is time-variant. If yes, our guess in the first step is correct and can thus rationalize the above procedure that we used to derive the stochastic property of the endogenous noise,  $\Lambda$ . The key reason is that time-invariant  $\alpha$  yields time-invariant variance of the fundamental shock ( $\sigma$ ).

The following proposition summarizes the solution to (36)-(37):

**Proposition 5.** *Given  $\vartheta$  and  $K$ , the optimal consumption and portfolio rules under robustness are*

$$c_t^* = r\hat{s}_t + \frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\gamma\sigma_e^2} - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} - \Gamma \quad (38)$$

and

$$\alpha^* = \frac{\pi}{r\tilde{\gamma}f(\kappa)\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \quad (39)$$

respectively, where  $\hat{s}_t$  is governed by (37), we use the fact that  $\hat{\sigma}^2 = f(\kappa)^2 \sigma^2$ , and  $\tilde{\gamma} \equiv (1 + \vartheta) \gamma$ , and the precautionary savings premium,  $\Gamma$ , is

$$\Gamma = \frac{1}{2} r \tilde{\gamma} f(\kappa)^2 (1 - \rho_{ye}^2) \sigma_s^2. \quad (40)$$

Finally, the worst possible distortion can be written as  $v^* = -r\gamma\vartheta$ .

*Proof.* See Appendix 7.4. ■

From (38), it is clear that robustness does not change the marginal propensity to consume (MPC) out of perceived permanent income ( $\hat{s}_t$ ), but affects the amount of precautionary savings. In other words, in the continuous-time setting, consumption is not sensitive to unanticipated income shocks. This conclusion is different from that obtained in the discrete-time robust-LQG permanent income model in which the MPC is increased via the interaction between RB and income

uncertainty, and consumption is more sensitive to unanticipated shocks.<sup>38</sup>

Given (38)-(40), it is straightforward to show that the observational equivalence between the discount rate and the RB parameter no longer holds in the MU-SU model. The reason is the same as that in the model without SU:  $\vartheta$  affects optimal portfolio choice via increasing  $\tilde{\gamma}$ , whereas  $\delta$  does not enter the portfolio rule. In other words, although both the discount factor ( $\exp(-\delta)$ ) and  $\gamma$  increase the precautionary savings premium and their observational equivalence can be established in the sense that they generate the same value of  $\Gamma$ , they imply different portfolio choices. Comparing (14) with (38), it is clear that the certainty equivalence principle holds in this model, i.e., the consumption function under SU can be easily obtained by replacing the true state with the perceived state.

Expression (39) shows that finite capacity ( $\kappa$ ) affects the speculation demand invested in the risky asset (the first term in (39)). Given that  $f(\kappa) = \sqrt{\kappa/(\kappa - r)} > 1$ , we can see that SU reduces the share invested in the risky asset. The intuition behind this result is that consumption reacts to the income and asset return shocks gradually and with delay due to extracting useful information about the true state from noisy observations. In other words, SU and MU affect the optimal portfolio choice in the same direction. Figure 5 illustrates how strategic portfolio rule ( $\alpha^*$ ) varies with the degree of SU ( $\kappa$ ) for different plausible values of  $\gamma$ . It clearly shows that  $\alpha^*$  decreases with the value of  $\kappa$  for any given value of  $\gamma$ . In addition, it is also clear from the same figure that  $\alpha^*$  decreases with  $\gamma$  for a given value of  $\kappa$ , which is consistent with the result obtained in the model without SU.

Expression (40) shows that the precautionary savings demand increases with the degree of SU governed by  $f(\kappa)$ . Figure 5 also illustrates how the precautionary saving demand ( $\Gamma$ ) varies with the degree of SU for different plausible values of  $\gamma$ . It clearly shows that  $\Gamma$  increases with the value of  $\kappa$  for different values of  $\gamma$ , and SU has a significant impact on precautionary savings.

As in the previous section, we use the elasticities of precautionary saving to changes in the degrees of MU and SU as a measure of their importance. Specifically, using (40), we have the following proposition:

**Proposition 6.** *The relative sensitivity of precautionary saving to model uncertainty (MU,  $\vartheta$ ) and state uncertainty (SU,  $\kappa$ ) can be measured by:*

$$\mu_{\vartheta\kappa} \equiv -\frac{e_{\vartheta}}{e_{\kappa}} = \frac{\vartheta}{1 + \vartheta} \frac{\kappa - r}{r}, \quad (41)$$

where  $e_{\vartheta} \equiv \frac{\partial \Gamma / \Gamma}{\partial \vartheta / \vartheta}$  and  $e_{\kappa} \equiv \frac{\partial \Gamma / \Gamma}{\partial \kappa / \kappa}$  are the elasticities of precautionary saving to model uncertainty and state

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<sup>38</sup>See HST (1999) and Luo and Young (2010) for detailed discussions on how RB affects consumption and precautionary savings in the discrete-time LQG permanent income models.

uncertainty, respectively. Furthermore, when

$$\kappa < (\geq) r \left( 2 + \frac{1}{\vartheta} \right),$$

the precautionary saving demand is more sensitive to state uncertainty than model uncertainty. In other words, when finite capacity is sufficiently low, the precautionary saving demand becomes more sensitive to state uncertainty.

*Proof.* The proof is straightforward using (41) and the facts that  $\mu_{\vartheta\kappa} \equiv \left( \frac{\partial \Gamma/\Gamma}{\partial \vartheta/\vartheta} \frac{\partial \tilde{\vartheta}/\tilde{\vartheta}}{\partial \vartheta/\vartheta} \right) / \left( \frac{\partial \Gamma/\Gamma}{\partial f/f} \frac{\partial f/f}{\partial \kappa/\kappa} \right)$ , where  $\tilde{\vartheta} = 1 + \vartheta$ . ■

Using (41), it is straightforward to show that:

$$\frac{\partial \mu_{\vartheta\kappa}}{\partial \vartheta} > 0 \text{ and } \frac{\partial \mu_{\vartheta\kappa}}{\partial \kappa} > 0,$$

which mean that  $\mu_{\vartheta\kappa}$  is increasing with the degree of RB,  $\vartheta$ , while is decreasing with the degree of SU (i.e., less values of  $\kappa$ ).

Using (38) and (39), we can obtain the stochastic properties of the joint dynamics of consumption, labor income, and the equity return. The following proposition summarizes the major results on the effects of RB on the joint behavior of consumption, labor income, and the equity return:

**Proposition 7.** *Given  $\vartheta$  and  $\kappa$ , the expected growth of consumption is:*

$$g_c \equiv \frac{E[dc_t^*]}{dt} = -\frac{\delta - r}{r\gamma} + \frac{1}{2}rf(\kappa)^2 \tilde{\gamma} \left( 1 - \rho_{ye}^2 \right) \sigma_s^2 + \frac{\pi^2}{2rf(\kappa) \tilde{\gamma} \sigma_e^2}, \quad (42)$$

the volatility of consumption growth is  $\text{var}(dc_t^*) = r^2 f(\kappa)^2 \sigma^2$ , where  $\sigma$  is given in (5) and  $\alpha^*$  is given in (39), the relative volatility of consumption growth to income growth is:

$$\mu \equiv \frac{\text{sd}(dc_t^*)}{\text{sd}(dy_t)} = \frac{rf(\kappa)}{r + \rho} \sqrt{1 - \rho_{ye}^2 + \left[ \frac{\pi(r + \rho)}{\tilde{\gamma} \sigma_e \sigma_y} \right]^2}, \quad (43)$$

and the contemporaneous correlation between consumption growth and the equity return is:

$$\rho_{cy} \equiv \text{corr}(dc_t^*, dy_t) = f(\kappa) \frac{\left( 1 - \rho_{ye}^2 \right) \sigma_s + \pi \rho_{ye} / (r \tilde{\gamma} \sigma_e)}{\sqrt{\left( 1 - \rho_{ye}^2 \right) \sigma_s^2 + \pi^2 / (r \tilde{\gamma} \sigma_e)^2}}. \quad (44)$$

*Proof.* See Appendix 7.4. ■

Expression (42) clearly shows that both RB and RI can affect the expected consumption growth by interacting with two sources of fundamental uncertainty: (i) labor income uncertainty ( $\sigma_s^2$ ) and (ii) asset return uncertainty ( $\sigma_e^2$ ). Specifically, we have

$$\frac{\partial g_c}{\partial \vartheta} > 0 \text{ if } \vartheta > \frac{\pi}{r\gamma\sqrt{f(\kappa)^3}\sqrt{1-\rho_{ye}^2\sigma_s\sigma_e}} - 1.$$

Using the same parameter values above, we can compute that RB can increase the expected growth rate if  $\vartheta$  is greater than 0.4 when  $\kappa = 0.1$  (here we set  $\rho_{ye} = 0.18$ ). In contrast, RB can increase the expected growth rate if  $\vartheta$  is greater than 0.76 when  $\kappa = 0.3$ . Furthermore, we have

$$\frac{\partial g_c}{\partial \kappa} < 0 \text{ if } \kappa > \underline{\kappa} \equiv \frac{r}{1 - \left\{ \pi^2 / \left[ 2(r\tilde{\gamma})^2 (1 - \rho_{ye}^2) \sigma_s^2 \sigma_e^2 \right] \right\}^{-2/3}}.$$

Because  $\underline{\kappa}$  is negative for the plausible parameter values, SU (less  $\kappa$ ) can always increase the expected growth rate (see Figure 6).

From Expression (43), we can see that RB reduces the relative volatility of consumption growth to income growth by increasing  $\tilde{\gamma}$  and reducing the optimal share invested in the risky asset. This result is different from that obtained in the permanent income model in which RB increases the relative volatility of consumption growth to income growth by strengthening the consumption sensitivity to income shocks.<sup>39</sup> It is also clear from (43) that SU measured by  $f(\kappa)$  increases the relative volatility. This effect is similar to that obtained in the discrete-time permanent income model (see Luo (2008) for a proof on how SU due to RI increases the relative volatility of consumption growth to income growth at the individual level). Figure 6 illustrates how  $g_c$  varies with the degree of SU ( $\kappa$ ) for different plausible values of  $\gamma$ . It is clear from the figure that the quantitative impact of SU on  $g_c$  is much stronger than that of MU on  $g_c$ .

Since  $|\rho_{ye}| \leq 1$ , we have:

$$\frac{\partial \rho_{cy}}{\partial \vartheta} > 0,$$

which means that RB raises the contemporaneous correlation between consumption growth and income growth. In addition,  $\rho_{cy} = \frac{1}{\sqrt{1+\pi^2/(r\tilde{\gamma}\sigma_s\sigma_e)^2}}$  when labor income is purely idiosyncratic, i.e.,  $\rho_{ye} = 0$ , while  $\rho_{cy} = 1$  when the income risk and the return risk are perfectly correlated. From (44), it is obvious that SU increases  $\rho_{cy}$  (see Figure 6).

<sup>39</sup>See Luo and Young (2010) for a proof that RB worsens the PIH model's prediction on the relative volatility of consumption growth to income growth.

### 4.3. Quantitative Implications

In this section, we adopt the same calibration procedure as in the last section to calibrate the value of  $\vartheta$  for a given DEP,  $p$ , in the MU-SU model.<sup>40</sup> Using the same parameter values as in the last section, Figure 7 illustrates how  $p$  varies with the value of  $\vartheta$  for different values of  $\kappa$ . We can see from the figure that the stronger the preference for robustness (higher  $\vartheta$ ), the less the  $p$  is. Tables 1 and 2 report how different values of  $\kappa$  affect calibrated values of  $\vartheta$ , optimal allocation in the risky asset ( $\alpha^*$ ), the relative importance of the income hedging demand to the speculation demand ( $|\alpha_h^*|/\alpha_s^*$ ), and precautionary saving demand ( $\Gamma$ ) for different values of  $\rho_{ye}$  and  $\sigma_s$ , respectively.<sup>41</sup> Specifically, for given values of  $\sigma_s$  and  $\rho_{ye}$ , when  $\kappa$  decreases (i.e., more information-constrained), the calibrated value of  $\vartheta$  increases; consequently, the optimal share invested in the risky asset decreases and the relative importance of the income hedging demand to the speculation demand increases. In addition, the precautionary saving demand decreases with the value of  $\kappa$ .

From Table 1, we can see that for given values of  $\kappa$ , the precautionary saving demand decreases with the correlation between the equity return and labor income risk ( $\rho_{ye}$ ), holding other factors constant, which is consistent with what we obtained in the MU model. The intuition is that the higher the correlation coefficient, the less demand for the risky asset and thus precautionary saving. In Table 2, we can see that as labor income becomes more volatile, the optimal allocation in the risky asset increases and the precautionary saving demand decreases. The reason is that the higher the value of  $\sigma_s$ , the less the calibrated value of  $\vartheta$ , holding other factors fixed; consequently, the effective coefficient of absolute risk aversion decreases, and thus the optimal share increases and the precautionary saving demand decreases.

### 4.4. Empirical Implications

As discussed in Haliassos and Michaelides (2000) and Campbell (2006), the empirical evidence on the correlation between labor income and equity returns for different population groups is difficult to reconcile with the observed stockholding behavior. Davis and Willen (1999) estimated that the correlation is between 0.1 and 0.3 for college-educated males and is only about  $-0.25$  for male high school dropouts. Heaton and Lucas (1999) found that the correlation between the entrepreneurial risk and the equity return was about 0.2. Since negative correlation between earnings and equity returns implies increased willingness to invest in the risky asset, less educated investors should be more heavily invested in the stock market while college graduates and entrepreneurs should put less wealth in the stock market. In contrast, the empirical evidence on stock market participation shows a significant correlation between the education level and stockholding. For example, Table 3 in Haliassos and Bertaut (1995) showed that the share invested in the stock market is substan-

<sup>40</sup>See Appendix 7.6 for the detailed calibration procedure in this case.

<sup>41</sup>Here the values of  $\rho_{ye}$  are set to be 0, 0.18, and 0.35 according to Campbell and Viceira (Chapter 6, 2002), and the values of  $\sigma_y$  are chosen to be the same as those used in Wang (2009).



tially larger among those with at least a college degree compared to those with less than a high school education *at all income levels*.<sup>42</sup> Furthermore, for any given education group, the share invested in the stock market is increasing with the income percentile. In other words, people with the same educational attainment level and higher income invest more in the stock market. They also mentioned that more educated groups have higher information-processing capacities.<sup>43</sup> This empirical evidence is consistent with our limited capacity theory. People with low income own less distraction-saving goods and services at home (e.g., they cannot afford a good baby sitter); consequently, they invested less in the stock market because they face greater state uncertainty.<sup>44</sup> In addition, people with higher education may have more efficient information-processing ability and thus face lower transition errors, which leads to higher effective channel capacity. Finally, people with higher education probably have more and better knowledge about the model economy, and are thus less concerned about the model specification.<sup>45</sup>

Our model with both RB and RI can have the potential to reconcile the model with the empirical evidence. Specifically, poorer and less well-educated investors probably face greater state uncertainty and model uncertainty, respectively; consequently, they rationally choose to invest less in the stock market even if the correlation between their labor income and equity returns is negative and they have stronger incentive to hedge against their earnings risk.<sup>46</sup> For example, the optimal amount invested in the risky asset ( $\alpha^*$ ) of a typical well-educated investor is 26.78 when  $\vartheta = 1.5$ ,  $\kappa = 1$ , and  $\rho_{ye} = 0.18$ .<sup>47</sup> In contrast, the optimal amount invested in the risky asset ( $\alpha^*$ ) of a typical less well-educated investor is 22.95 when  $\vartheta = 3$ ,  $\kappa = 0.1$ , and  $\rho_{ye} = -0.25$ . In other words, although the negative correlation between earnings and equity returns increases the willingness of low-educated investors to invest in the risky asset, well-educated investors invest more wealth in the risky asset because they have more knowledge about the structure of the model and higher information-processing capacities, and face less model uncertainty and state uncertainty. In summary, the introduction of induced uncertainty can offer a potential explanation for the two seemingly contradictory observations: the correlation between labor income and equity returns and the stockholding behavior of less educated and well-educated investors.

#### 4.5. Policy Implications

In this section, we discuss the effect of changes in the labor income tax rate on investors' precautionary saving and strategic portfolio choice under MU and SU. Elmendorf and Kimball (2000)

<sup>42</sup>Mankiw and Zeldes (1991) obtained the similar result using the PSID data.

<sup>43</sup>See their brief discussion on this in Section II.C.

<sup>44</sup>As argued in Banerjee and Mullainathan (2008), attention is a scarce resource that is important for labor productivity and income distribution.

<sup>45</sup>Although both of the factors, education and income, affect model and state uncertainty facing investors, it seems that education is more important in affecting model uncertainty and income is more important in affecting state uncertainty.

<sup>46</sup>As documented in Campbell (2006), there is some evidence that households understand their own limitations and constraints, and avoid investment opportunities for which they feel unqualified.

<sup>47</sup>In this section, we also set  $\mu = 0.08$ ,  $r = 0.02$ ,  $\sigma_s = 5$ , and  $\sigma_e = 0.156$ .

found that given decreasing absolute prudence (e.g., CRRA utility), even when labor income risk increases overall saving, it tends to lower investment in the risky asset. They also argued that realistic increases in the marginal tax rate on labor income can cause large enough reductions in the after-tax labor income risk, which leads to significant increases in investment in the risky asset. Using the same policy experiment conducted as in their paper, we also consider the situation in which the marginal tax rate on labor income ( $\tau$ ) is increased from 0 to 10%. In this case, the labor income risk measured by  $(1 - \tau) \sigma_y$  is reduced from  $\sigma_y$  to  $0.9\sigma_y$ . Using the expressions for optimal portfolio choice and precautionary saving, (39) and (40), it is clear that the change in the tax rate leads to an increase in the risky investment and a reduction in precautionary savings, holding other parameter values fixed.

Specifically, when the labor income risk is reduced from  $\sigma_y$  to  $0.9\sigma_y$ , the value of human wealth is also reduced from  $\sigma_s$  to  $0.9\sigma_s$ . Consequently,  $\Gamma$  is reduced from  $0.5r\tilde{\gamma}f(\kappa)^2(1 - \rho_{ye}^2)\sigma_s^2$  to about  $0.4r\tilde{\gamma}f(\kappa)^2(1 - \rho_{ye}^2)\sigma_s^2$ . The presence of induced uncertainty measured by  $(1 + \vartheta)f(\kappa)^2 > 1$  can amplify the impact of this taxation policy on the precautionary saving demand. For example, when  $r = 0.02$ ,  $\vartheta = 1.5$  and  $\kappa = 0.2$ ,  $(1 + \vartheta)f(\kappa)^2 = 2.78$ . In other words, the policy impact on precautionary saving is almost tripled under MU and SU.

From (39), it is clear that the labor income risk does not directly interact with induced uncertainty due to the interaction of RB and RI because  $\sigma_s$  does not enter the standard speculation demand function and the induced uncertainty term  $((1 + \vartheta)f(\kappa))$  does not enter the income-hedging demand function. However, the presence of induced uncertainty can offset the impact of the taxation policy on the optimal risky investment when  $\rho_{ye}$  is positive. The reason is that the taxation policy reduces the hedging demand while the presence of induced uncertainty reduces the standard speculation demand. Therefore, the policy impact on optimal portfolio choice can be mitigated under MU and SU.

## 5. Comparison with Incomplete Information about Individual Labor Income

In this section, we consider another widely-adopted type of informational frictions: incomplete information about the income process, and compare its implications for robustly strategic consumption-portfolio rules implications with that of RI we considered in the preceding section.<sup>48</sup> Specifically, following Muth (1960), Quah (1990), Pischke (1995), and Wang (2004), we assume that where there are two individual components in the income process, agents can only observe the total income but have no way to distinguish the two individual components. Mathematically, we assume that labor income ( $y_t$ ) has two distinct components ( $y_{1,t}$  and  $y_{2,t}$ ):

$$y_t = y_{1,t} + y_{2,t},$$

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<sup>48</sup>Muth (1960) first considered this type of incomplete information when exploring the link between rational expectations and the geometrically declining weighted sum of past and current incomes.

where

$$dy_{1,t} = (\mu_1 - \rho_1 y_{1,t}) dt + \sigma_1 dB_{1,t}, \quad (45)$$

$$dy_{2,t} = (\mu_2 - \rho_2 y_{2,t}) dt + \rho_{12} \sigma_2 dB_{1,t} + \sqrt{1 - \rho_{12}^2} \sigma_2 dB_{2,t}, \quad (46)$$

and  $\rho_{12}$  is the instantaneous correlation between the two individual components,  $y_{1,t}$  and  $y_{2,t}$ .<sup>49</sup> All the other notations are similar to what we used in our benchmark model. Without a loss of generality, we assume that  $\rho_1 < \rho_2$  and  $\sigma_1 > \sigma_2$ . In other words, the first income component is more persistent and volatile than the second component.<sup>50</sup> It is straightforward to show that if both components in the income process are observable, this model is essentially the same as our benchmark model with a univariate income process. In this incomplete-information case, we need to use the filtering technique to obtain the best estimates of the unobservable income components first and then solve the optimization problem given the estimated income components. Following the same technique adopted in Wang (2004), in the steady state in which the conditional variance-covariance matrix is constant, we can obtain the following updating equations for the conditional means of  $(y_{1,t}, y_{2,t})$ :

$$d \begin{pmatrix} \hat{y}_{1,t} \\ \hat{y}_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 - \rho_1 \hat{y}_{1,t} \\ \mu_2 - \rho_2 \hat{y}_{2,t} \end{pmatrix} dt + \begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \end{pmatrix} dZ_t, \quad (47)$$

where  $\hat{y}_{i,t} = E_t [y_{i,t}]$  for  $i = 1, 2$ ,  $dZ_t \equiv \{dy_t - [(\mu_1 + \mu_2) + (\rho_2 - \rho_1) \hat{y}_{1,t} - \rho_2 y_{2,t}] dt\} / \sigma$  is a “constructed” innovation process using total income and the perceived individual components;  $Z_t$  is a standard Brownian motion; and  $\sigma = \sqrt{\sigma_1^2 + 2\sigma_{12} + \sigma_2^2}$ .  $\hat{\sigma}_1$  and  $\hat{\sigma}_2$  are the standard deviations of  $d\hat{y}_{1,t}$  and  $d\hat{y}_{2,t}$  respectively:

$$\hat{\sigma}_1 = \frac{1}{\sigma} [(\rho_2 - \rho_1) \Sigma_{11} + \sigma_1^2 + \sigma_{12}] \quad \text{and} \quad \hat{\sigma}_2 = \frac{1}{\sigma} [-(\rho_2 - \rho_1) \Sigma_{11} + \sigma_2^2 + \sigma_{12}],$$

where

$$\Sigma_{11} = \frac{1}{(\rho_2 - \rho_1)^2} \left( \sqrt{\Theta^2 + \sigma_1^2 \sigma_2^2 (\rho_2 - \rho_1)^2 (1 - \rho_{12}^2)} - \Theta \right) \quad (48)$$

is the steady state conditional variance of  $y_{1,t}$ ,  $\Theta = \rho_1 \sigma_2^2 + \rho_2 \sigma_1^2 + (\rho_1 + \rho_2) \sigma_{12}$ , and  $\sigma_{12} = \rho_{12} \sigma_1 \sigma_2$ .<sup>51</sup> It is worth noting that for this bi-variate Gaussian income specification,  $\Sigma_{11}$  can fully characterize

<sup>49</sup>Muth (1960), Quah (1990), and Pischke (1995) considered a more special two-component income specification (one component is iid, and the other component is a random walk) in a discrete-time setting. Specifically, Quah (1990) showed that this two-component income specification provides a potential resolution to the excess smoothness puzzle in the standard permanent income model if the relative importance of transitory to permanent components is large.

<sup>50</sup>We may think of the first and second components as the aggregate (and persistent) and idiosyncratic (and transitory) components, respectively.

<sup>51</sup>The detailed derivation of the expression for  $\Sigma_{11}$  is similar to that in Wang (2004) and is available from the author upon request. It is straightforward to show that when  $|\rho_{12}| = 1$  or  $\rho_1 = \rho_2$ , the values of  $\Sigma_{11}$  are constant and are independent of the persistence and volatility parameters in the income processes.

the estimation risk induced by partially observed income.<sup>52</sup> Figure 8 illustrates how  $\Sigma_{11}$  varies with  $\rho_2$  and  $\sigma_2/\sigma_1$ .<sup>53</sup> It clearly shows that given the persistence and volatility coefficients of  $y_{1,t}$ , the estimation risk increases with the persistence and volatility of  $y_{2,t}$  (i.e., the less  $\rho_2$  and the higher  $\sigma_2/\sigma_1$ ).

Following the same procedure that I used in the preceding sections, we can solve this IC model with RB. The following proposition summarizes the solution to the above problem:

**Proposition 8.** *Given  $\vartheta$ , the optimal consumption-portfolio rules under RB and IC are:*

$$c_t^* = r \left[ w_t + \frac{1}{r + \rho_1} \left( \hat{y}_{1,t} + \frac{\mu_1}{r} \right) + \frac{1}{r + \rho_2} \left( \hat{y}_{2,t} + \frac{\mu_2}{r} \right) \right] \quad (49)$$

$$+ \left[ 1 - \frac{1}{2} \left( \frac{1 + \vartheta / (r\gamma)}{1 + \vartheta} \right) \right] \frac{\pi^2}{r(1 + \vartheta) \gamma \sigma_e^2} - \frac{\pi \rho_{ey}}{\sigma_e} \left( \frac{\hat{\sigma}_1}{r + \rho_1} + \frac{\hat{\sigma}_2}{r + \rho_2} \right) + \Psi - \Gamma,$$

$$\alpha^* = \frac{\pi}{r\gamma(1 + \vartheta) \sigma_e^2} - \frac{\rho_{ey} \sigma_e}{\sigma_e^2} \left( \frac{\hat{\sigma}_1}{r + \rho_1} + \frac{\hat{\sigma}_2}{r + \rho_2} \right) \quad (50)$$

respectively, where  $\Psi = (\delta - r) / (r\gamma)$  captures the dissavings effect of relative impatience, and

$$\Gamma = \frac{1}{2} r \tilde{\gamma} (1 - \rho_{ey}^2) \left( \frac{\hat{\sigma}_1}{r + \rho_1} + \frac{\hat{\sigma}_2}{r + \rho_2} \right)^2 \quad (51)$$

is the precautionary savings demand, where  $\tilde{\gamma} \equiv \gamma \left( 1 + \frac{\vartheta}{r\gamma} \right)$ .

*Proof.* See Online Appendix. ■

Comparing (39) with (50), it is clear that RI and IC have distinct implications on robustly strategic asset allocation. Specifically, RI affects the speculation demand invested in the risky asset via  $f(\kappa) = \sqrt{\kappa / (\kappa - r)} > 1$ , whereas IC has no impact on the speculation demand (the first term in (50)). Furthermore, RI has no impact on the intertemporal hedging demand, whereas IC affects this demand via changing the volatility of perceived permanent income  $\left( \frac{\hat{\sigma}_1}{r + \rho_1} + \frac{\hat{\sigma}_2}{r + \rho_2} \right)$ . The main reason behind these results is that RI is applied to the state variable  $s$  that summarizes all of the relevant information in the state vector  $(w, y)$ . In contrast, IC is only applied to the labor income process, and consumers under IC can observe financial wealth perfectly.

<sup>52</sup>Note that the conditional variance-covariance matrix of  $(y_{1,t} \ y_{2,t})$  can be written as:

$$\Sigma \equiv \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = E \left[ \begin{pmatrix} y_{1,t} - \hat{y}_{1,t} \\ y_{2,t} - \hat{y}_{2,t} \end{pmatrix} \begin{pmatrix} y_{1,t} - \hat{y}_{1,t} & y_{2,t} - \hat{y}_{2,t} \end{pmatrix} \right] = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \Sigma_{11}.$$

<sup>53</sup>Here we set  $\rho_1 = 0$ ,  $\sigma_1 = 0.05$ , and  $\rho_{12} = 0$ . In other words, the first income component is a unit root and the two components are independent. The pattern of the figure does not change if these parameters change. The only exception is the  $\rho_{12} = \pm 1$  case. In this specification,  $\Sigma_{11} = 0$  because the two components are perfectly correlated and the bivariate income specification is essentially the same as the univariate income specification.

Furthermore, comparing (40) with (51), we can see that RI and IC affect the precautionary saving demand via distinct channels. Specifically, RI increases precautionary savings by introducing the  $f(\kappa)$  factor, whereas IC increases precautionary savings by increasing the variance of perceived permanent income from  $\left(\frac{\sigma_1}{r+\rho_1}\right)^2 + 2\frac{\rho_{12}\sigma_1\sigma_2}{(r+\rho_1)(r+\rho_2)} + \left(\frac{\sigma_2}{r+\rho_2}\right)^2$  to  $\left(\frac{\hat{\sigma}_1}{r+\rho_1} + \frac{\hat{\sigma}_2}{r+\rho_2}\right)^2$ .<sup>54</sup>

## 6. Conclusion

This paper has developed a tractable continuous-time CARA-Gaussian framework to explore how induced uncertainty due to the interaction of RB and RI affects strategic consumption-portfolio rules, precautionary savings, and consumption dynamics in the presence of uninsurable labor income. Specifically, I explored the relative sensitivity of strategic consumption-portfolio rules and precautionary savings with respect to the two types of induced uncertainty: (i) model uncertainty due to robustness and (ii) state uncertainty due to limited information-processing capacity, as well as risk aversion. In addition, I argued that both model uncertainty and state uncertainty are important for us to understand and design optimal household portfolios. In particular, I found that these two types of induced uncertainty reduce the optimal share invested in the risky asset, and thus can offer a potential explanation for two seemingly contradictory observations in the data: the negative correlation between labor income and equity returns and the low stock market participation rate of the less educated and lower income households.

## 7. Appendix

### 7.1. Solving the RB Model

The Bellman equation associated with the optimization problem is

$$J(s_t) = \sup_{c_t, \alpha_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) + \exp(-\delta dt) J(s_{t+dt}) \right],$$

subject to (10), where  $J(s_t)$  is the value function. The HJB equation for this problem is then

$$0 = \sup_{c_t, \alpha_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) \right],$$

---

<sup>54</sup>Note that

$$\begin{aligned} \Delta &\equiv \left( \frac{\hat{\sigma}_1}{r+\rho_1} + \frac{\hat{\sigma}_2}{r+\rho_2} \right)^2 - \left[ \left( \frac{\sigma_1}{r+\rho_1} \right)^2 + 2\frac{\rho_{12}\sigma_1\sigma_2}{(r+\rho_1)(r+\rho_2)} + \left( \frac{\sigma_2}{r+\rho_2} \right)^2 \right] \\ &= 2r\Sigma_{11} \left[ \frac{(\rho_2 - \rho_1)}{(r+\rho_1)(r+\rho_2)} \right]^2 > 0. \end{aligned}$$

where  $\mathcal{D}J(s_t) = J_s(rs_t - c_t + \pi\alpha_t) + \frac{1}{2}J_{ss}(\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t)$ . Under RB, the HJB can be written as

$$\sup_{c_t, \alpha_t} \inf_{v_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) + v(s_t) \sigma^2 J_s + \frac{1}{2\vartheta(s_t)} v(s_t)^2 \sigma^2 \right]$$

subject to the distorting equation, (10). Solving first for the infimization part of the problem yields

$$v^*(s_t) = -\vartheta(s_t) J_s.$$

Given that  $\vartheta(s_t) > 0$ , the perturbation adds a negative drift term to the state transition equation because  $J_s > 0$ . Substituting for  $v^*$  in the robust HJB equation gives:

$$\sup_{c_t, \alpha_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + (rs_t - c_t + \pi\alpha_t) J_s + \frac{1}{2} \sigma^2 J_{ss} - \frac{1}{2} \vartheta(s_t) \sigma^2 J_s^2 \right]. \quad (52)$$

Performing the indicated optimization yields the first-order conditions for  $c_t$  and  $\alpha_t$ :

$$c_t = -\frac{1}{\gamma} \ln(J_s), \quad (53)$$

$$\alpha_t = \frac{\pi J_s + \rho_{ye} \sigma_s \sigma_e (J_{ss} - \vartheta J_s^2)}{(\vartheta J_s^2 - J_{ss}) \sigma_e^2}. \quad (54)$$

Substitute (53) and (54) back into (52) to arrive at the partial differential equation

$$0 = -\frac{J_s}{\gamma} - \delta J + \left( rs_t + \frac{1}{\gamma} \ln(J_s) + \pi\alpha_t \right) J_s + \frac{1}{2} (J_{ss} - \vartheta J_s^2) \sigma^2, \quad (55)$$

where  $\sigma^2 = \sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye} \sigma_s \sigma_e \alpha_t$ . Conjecture that the value function is of the form

$$J(s_t) = -\frac{1}{\alpha_1} \exp(-\alpha_0 - \alpha_1 s_t),$$

where  $\alpha_0$  and  $\alpha_1$  are constants to be determined. Using this conjecture, we obtain that  $J_s = \exp(-\alpha_0 - \alpha_1 s_t) > 0$  and  $J_{ss} = -\alpha_1 \exp(-\alpha_0 - \alpha_1 s_t) < 0$ . Further more, we guess that  $\vartheta(s_t) = -\frac{\vartheta}{J(s_t)} = \frac{\alpha_1 \vartheta}{\exp(-\alpha_0 - \alpha_1 s_t)} > 0$ . (55) can thus be reduced to

$$-\delta \frac{1}{\alpha_1} = -\frac{1}{\gamma} + \left[ rs_t - \left( \frac{\alpha_0}{\gamma} + \frac{\alpha_1}{\gamma} s_t \right) + \frac{\pi (\pi - \rho_{ye} \sigma_s \sigma_e \alpha_1 (1 + \vartheta))}{(1 + \vartheta) \alpha_1 \sigma_e^2} \right] - \frac{1}{2} \alpha_1 (1 + \vartheta) (\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye} \sigma_s \sigma_e \alpha_t)$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$\alpha_1 = r\gamma, \quad (56)$$

$$\alpha_0 = \frac{\delta}{r} - 1 + \frac{\pi [\pi - \rho_{ye} \sigma_s \sigma_e r\gamma (1 + \vartheta)]}{(1 + \vartheta) r \sigma_e^2} - \frac{1}{2} (1 + \vartheta) r\gamma^2 (\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye} \sigma_s \sigma_e \alpha_t), \quad (57)$$

where  $\alpha^* = \frac{\pi}{(1+\vartheta)\gamma r\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}$ . Substituting these back into the first-order condition (53) yields the consumption function, (14), in the main text. Using (57) and  $\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t = \left(1 - \rho_{ye}^2\right)\sigma_s^2 + \frac{\pi^2}{(r\tilde{\gamma})^2\sigma_e^2}$ , we can obtain the expression for the precautionary savings premium, (16), in the main text.

When  $\rho_{ye} = 1$ , we have

$$\begin{aligned}\frac{\alpha_0}{\gamma} &= \frac{\delta}{r} - 1 + \frac{\pi \left[ \pi - \rho_{ye}\sigma_s\sigma_e r\gamma (1 + \vartheta) \right]}{(1 + \vartheta) r\sigma_e^2} - \frac{1}{2} (1 + \vartheta) r\gamma^2 (\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t) \\ &= \frac{\delta - r}{r\gamma} + \frac{\pi^2}{(1 + \vartheta)\gamma r\sigma_e^2} - \frac{\pi\rho_{ye}\sigma_s\sigma_e r\gamma (1 + \vartheta)}{(1 + \vartheta)r\gamma\sigma_e^2} - \frac{1}{2} (1 + \vartheta) r\gamma \left( \frac{\pi}{(1 + \vartheta)\gamma r\sigma_e} \right)^2 \\ &= \frac{\delta - r}{r\gamma} + \frac{\pi^2}{2(1 + \vartheta)\gamma r\sigma_e^2} - \frac{\pi\sigma_s}{\sigma_e}\end{aligned}$$

Finally, we check if the investor's transversality condition (TVC),  $\lim_{t \rightarrow \infty} E[\exp(-\delta t) |J(s_t)|] = 0$ , is satisfied. Substituting the consumption-portfolio rules,  $c_t^*$  and  $\alpha^*$ , into the state transition equation for  $s_t$  yields:

$$ds_t = A dt + \sigma dB_t,$$

where  $A = -\frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\gamma\sigma_e^2} + \frac{1}{2}r\tilde{\gamma} \left(1 - \rho_{ye}^2\right)\sigma_s^2$  under the approximating model. This Brownian motion with drift can be rewritten as:

$$s_t = s_0 + At + \sigma(B_t - B_0), \quad (58)$$

where  $B_t - B_0 \sim N(0, t)$ . Substituting (58) into  $E[\exp(-\delta t) |J(s_t)|]$  yields:

$$\begin{aligned}E[\exp(-\delta t) |J(s_t)|] &= \frac{1}{\alpha_1} E[\exp(-\delta t - \alpha_0 - \alpha_1 s_t)] \\ &= \frac{1}{\alpha_1} \exp\left(E[-\delta - \alpha_0 - \alpha_1 s_t] + \frac{1}{2} \text{var}(\alpha_1 s_t)\right) \\ &= \frac{1}{\alpha_1} \exp\left(-\delta t - \alpha_0 - \alpha_1(s_0 + At) + \frac{1}{2}\alpha_1^2\sigma^2 t\right) \\ &= |J(s_0)| \exp\left(-\left(\delta + \alpha_1 A - \frac{1}{2}\alpha_1^2\sigma^2\right) t\right)\end{aligned}$$

where  $|J(s_0)| = \frac{1}{\alpha_1} \exp(-\alpha_0 - \alpha_1 s_0)$  is a positive constant and we use the facts that  $s_t - s_0 \sim N(At, \sigma^2 t)$ . Therefore, the TVC is satisfied if and only if the following condition holds:

$$\delta + \alpha_1 A - \frac{1}{2}\alpha_1^2\sigma^2 = r + \frac{\pi^2}{2\sigma_e^2} + \frac{1}{2}(r\gamma)^2 \left[ (1 + \vartheta) \left(1 - \rho_{ye}^2\right)\sigma_s^2 - (\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t) \right] > 0. \quad (59)$$

In the FI-RE case in which  $\vartheta = 0$ , this condition reduces to:  $r + \pi^2 / (2\sigma_e^2) - (r\gamma)^2 (\rho_{ye}\sigma_s + \sigma_e\alpha^*)^2 / 2 > 0$ . Using the parameter values we consider in the text, it is straightforward to show that the TVC is always satisfied in both the FI-RE and RB models. It is straightforward to show that the TVC still

holds under the distorted model in which  $A = -\frac{\delta-r}{r\gamma} + \frac{\pi^2}{2r\gamma\sigma_\varepsilon^2} + \frac{1}{2}r\tilde{\gamma} \left(1 - \rho_{ye}^2\right) \sigma_s^2 - r\gamma\vartheta\sigma^2$ .

## 7.2. Calibrating the Robustness Parameter

The value of  $p$  is determined by the following procedure. Let model  $P$  denote the approximating model, (4):

$$ds_t = (rs_t - c_t + \pi\alpha_t) dt + \sigma dB_t,$$

and model  $Q$  be the distorted model, (10):

$$ds_t = (rs_t - c_t + \pi\alpha_t) dt + \sigma(\sigma v(s_t) dt + dB_t).$$

Define  $p_P$  as

$$p_P = \text{Prob} \left( \ln \left( \frac{L_Q}{L_P} \right) > 0 \middle| P \right), \quad (60)$$

where  $\ln \left( \frac{L_Q}{L_P} \right)$  is the log-likelihood ratio. When model  $P$  generates the data,  $p_P$  measures the probability that a likelihood ratio test selects model  $Q$ . In this case, we call  $p_P$  the probability of the model detection error. Similarly, when model  $Q$  generates the data, we can define  $p_Q$  as

$$p_Q = \text{Prob} \left( \ln \left( \frac{L_P}{L_Q} \right) > 0 \middle| Q \right). \quad (61)$$

Given initial priors of 0.5 on each model and that the length of the sample is  $N$ , the detection error probability,  $p$ , can be written as:

$$p(\vartheta; N) = \frac{1}{2} (p_P + p_Q), \quad (62)$$

where  $\vartheta$  is the robustness parameter used to generate model  $Q$ . Given this definition, we can see that  $1 - p$  measures the probability that econometricians can distinguish the approximating model from the distorted model.

The general idea of the calibration procedure is to find a value of  $\vartheta$  such that  $p(\vartheta; N)$  equals a given value (for example, 10%) after simulating model  $P$ , (4), and model  $Q$ , (10).<sup>55</sup> In the continuous-time model with the iid Gaussian specification,  $p(\vartheta; N)$  can be easily computed. Because both models  $P$  and  $Q$  are arithmetic Brownian motions with constant drift and diffusion coefficients, the log-likelihood ratios are Brownian motions and are normally distributed random variables. Specifically, the logarithm of the Radon-Nikodym derivative of the distorted model ( $Q$ ) with respect to the approximating model ( $P$ ) can be written as

$$\ln \left( \frac{L_Q}{L_P} \right) = - \int_0^N \bar{v} dB_s - \frac{1}{2} \int_0^N \bar{v}^2 ds, \quad (63)$$

<sup>55</sup>The number of periods used in the calculation,  $N$ , is set to be the actual length of the data we study. For example, if we consider the post-war U.S. annual time series data provided by Robert Shiller from 1946 – 2010,  $T = 65$ .



where

$$\bar{v} \equiv v^* \sigma = -r\gamma\vartheta \sqrt{\sigma_e^2 \alpha^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha}. \quad (64)$$

Similarly, the logarithm of the Radon-Nikodym derivative of the approximating model ( $P$ ) with respect to the distorted model ( $Q$ ) is

$$\ln \left( \frac{L_P}{L_Q} \right) = \int_0^N \bar{v} dB_s + \frac{1}{2} \int_0^N \bar{v}^2 ds. \quad (65)$$

Using (60)-(65), it is straightforward to derive  $p(\vartheta; N)$ :

$$p(\vartheta; N) = \Pr \left( x < \frac{\bar{v}}{2} \sqrt{N} \right), \quad (66)$$

where  $x$  follows a standard normal distribution. From the expressions of  $\bar{v}$ , (64), and  $p(\vartheta; N)$ , (66), we can show that the value of  $p$  is decreasing with the value of  $\vartheta$  because  $\partial \alpha^* / \partial \vartheta < 0$ .

### 7.3. Deriving Continuous-time IPC

The IPC,

$$\ln(\rho_1^2 \Sigma_t + \rho_2^2) - \ln \Sigma_{t+\Delta t} = 2\kappa \Delta t,$$

can be rewritten as

$$\ln \left( \exp(2r\Delta t) \Sigma_t + \frac{1 - \exp(2r\Delta t)}{-2r\Delta t} \Delta t \sigma^2 \right) - \ln \Sigma_{t+\Delta t} = 2\kappa \Delta t,$$

which can be reduced to

$$\Sigma_{t+\Delta t} - \Sigma_t = [\exp(2(r - \kappa)\Delta t) - 1] \Sigma_t + \frac{\exp(2(r - \kappa)\Delta t) - \exp(-2\kappa\Delta t)}{2r} \sigma^2.$$

Dividing  $\Delta t$  on both sides of this equation and letting  $\Delta t \rightarrow 0$ , we have the following updating equation for  $\Sigma_t$ :

$$\dot{\Sigma}_t = \lim_{\Delta t \rightarrow 0} \frac{\Sigma_{t+\Delta t} - \Sigma_t}{\Delta t} = 2(r - \kappa) \Sigma_t + \sigma^2.$$

### 7.4. Solving the RB-RI Model

The Bellman equation associated with the optimization problem under SU is

$$J(\hat{s}_t) = \sup_{c_t, \alpha_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) + \exp(-\delta dt) J(\hat{s}_{t+dt}) \right],$$

subject to (37), where  $J(\hat{s}_t)$  is the value function. The HJB equation for this problem can thus be written as

$$0 = \sup_{c_t, \alpha_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(\hat{s}_t) + \mathcal{D}J(s_t) \right],$$

where

$$\mathcal{D}J(\hat{s}_t) = J_{\hat{s}}(r\hat{s}_t - c_t + \pi\alpha_t) + \frac{1}{2}J_{\hat{s}\hat{s}}f(\kappa)^2(\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t).$$

Here we use the facts that  $\hat{\vartheta} \equiv f(\kappa)\sigma$  and  $\sigma = \sqrt{\sigma_e^2\alpha^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha}$ . Under MU and SU, the HJB can be written as

$$\sup_{c_t, \alpha_t} \inf_{v_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(\hat{s}_t) + \mathcal{D}J(\hat{s}_t) + v(\hat{s}_t)\hat{\vartheta}^2 J_{\hat{s}} + \frac{1}{2\vartheta(s_t)}v(\hat{s}_t)^2\hat{\vartheta}^2 \right]$$

subject to the distorting equation, (37). Solving first for the infimization part of the problem yields:  $v^*(\hat{s}_t) = -\vartheta(\hat{s}_t)J_{\hat{s}}$ . Given that  $\vartheta(\hat{s}_t) > 0$ , the perturbation adds a negative drift term to the state transition equation because  $J_{\hat{s}} > 0$ . Substituting for  $v^*$  in the robust HJB equation gives:

$$\sup_{c_t, \alpha_t} \left[ -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(\hat{s}_t) + (r\hat{s}_t - c_t + \pi\alpha_t)J_{\hat{s}} + \frac{1}{2}\hat{\vartheta}^2 J_{\hat{s}\hat{s}} - \frac{1}{2}\vartheta(\hat{s}_t)\hat{\vartheta}^2 J_{\hat{s}}^2 \right]. \quad (67)$$

Performing the indicated optimization yields the first-order conditions for  $c_t$  and  $\alpha_t$ :

$$c_t = -\frac{1}{\gamma} \ln(J_{\hat{s}}), \quad (68)$$

$$\alpha_t = \frac{\pi J_{\hat{s}}/f(\kappa) + \rho_{ye}\sigma_s\sigma_e(J_{\hat{s}\hat{s}} - \vartheta J_{\hat{s}}^2)}{(\vartheta J_{\hat{s}}^2 - J_{\hat{s}\hat{s}})\sigma_e^2}. \quad (69)$$

Substitute (68) and (69) back into (67) to arrive at the partial differential equation

$$0 = -\frac{J_{\hat{s}}}{\gamma} - \delta J + \left( r\hat{s}_t + \frac{1}{\gamma} \ln(J_{\hat{s}}) + \pi\alpha_t \right) J_{\hat{s}} + \frac{1}{2}f(\kappa)^2(J_{\hat{s}\hat{s}} - \vartheta J_{\hat{s}}^2)\sigma^2.$$

Conjecture that the value function is of the form

$$J(\hat{s}_t) = -\frac{1}{\alpha_1} \exp(-\alpha_0 - \alpha_1\hat{s}_t),$$

where  $\alpha_0$  and  $\alpha_1$  are constants to be determined. Using this conjecture, we obtain  $J_{\hat{s}} = \exp(-\alpha_0 - \alpha_1\hat{s}_t) > 0$  and  $J_{\hat{s}\hat{s}} = -\alpha_1 \exp(-\alpha_0 - \alpha_1\hat{s}_t) < 0$ . Furthermore, we guess that  $\vartheta(\hat{s}_t) = -\frac{\hat{\vartheta}}{J(\hat{s}_t)} = \frac{\alpha_1\hat{\vartheta}}{\exp(-\alpha_0 - \alpha_1\hat{s}_t)} > 0$ . Substituting these expressions into (55) yields:

$$\begin{aligned} -\delta \frac{1}{\alpha_1} &= -\frac{1}{\gamma} + \left\{ r\hat{s}_t - \left( \frac{\alpha_0}{\gamma} + \frac{\alpha_1\hat{s}_t}{\gamma} \right) + \frac{\pi [\pi/f(\kappa) - \rho_{ye}\sigma_s\sigma_e\alpha_1(1+\vartheta)]}{(1+\vartheta)\alpha_1\sigma_e^2} \right\} \\ &\quad - \frac{1}{2}f(\kappa)^2\alpha_1(1+\vartheta)(\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t) \end{aligned}$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$\alpha_1 = r\gamma, \quad (70)$$

$$\alpha_0 = \frac{\delta - r}{r} + \frac{\pi [\pi / f(\kappa) - \rho_{ye} \sigma_s \sigma_e r \alpha (1 + \vartheta)]}{(1 + \vartheta) r \sigma_e^2} - \frac{1}{2} r f(\kappa)^2 (1 + \vartheta) \gamma^2 (\sigma_e^2 \alpha_t + \sigma_s^2 + 2\rho_{ye} \sigma_s \sigma_e \alpha_t). \quad (71)$$

Substituting (70) and (71) into (68) and (69) yields the optimal portfolio and consumption rule, (39) and (38), respectively, in the main text. Using (71) and that fact that  $\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye} \sigma_s \sigma_e \alpha_t = (1 - \rho_{ye}^2) \sigma_s^2 + \frac{\pi^2}{r^2 \tilde{\gamma}^2 f(\kappa)^2 \sigma_e^2}$ , we can obtain Expression (40) in the main text.

Using (38) and (39), we have  $dc_t^* = r d\hat{s}_t$  and  $\text{var}(dc_t^*) = r^2 f(\kappa)^2 \sigma^2$ . The relative volatility of consumption growth to income growth can thus be written as

$$\mu \equiv \frac{\text{sd}(dc_t^*)}{\text{sd}(dy_t)} = r f(\kappa) \sqrt{\frac{1 - \rho_{ye}^2}{(r + \rho)^2} + \frac{\pi^2}{\tilde{\gamma}^2 \sigma_e^2 \sigma_y^2}}.$$

The contemporaneous covariance between consumption growth and income growth is

$$\text{cov}(dc_t^*, dy_t) = r f(\kappa) \text{cov}\left(\alpha^* \sigma_e dB_{e,t} + \frac{1}{r + \rho} \sigma_y dB_{y,t}, \sigma_y dB_{y,t}\right) = r f(\kappa) \left(\frac{1}{r + \rho} \sigma_y^2 + \alpha^* \rho_{ye} \sigma_e \sigma_y\right),$$

which implies that

$$\rho_{cy} \equiv \text{corr}(dc_t^*, dy_t) = f(\kappa) \frac{\sigma_s + \alpha^* \rho_{ye} \sigma_e}{\sigma}.$$

Substituting  $\sigma = \sqrt{(1 - \rho_{ye}^2) \sigma_s^2 + \frac{\pi^2}{(r\tilde{\gamma})^2 f(\kappa)^2 \sigma_e^2}}$  and  $\alpha^* = \frac{\pi}{r\tilde{\gamma} f(\kappa)^2 \sigma_e^2} - \frac{\rho_{ye} \sigma_s \sigma_e}{\sigma_e^2}$  into this expression leads to (44) in the main text.

Finally, using the derived consumption-portfolio rules and the value function, we can use the same procedure as that in Appendix 7.1 to verify that the transversality condition (TVC),  $\lim_{t \rightarrow \infty} E |\exp(-\delta t) J(\hat{s}_t)| = 0$ , holds under RB and RI.

## 7.5. The Equivalence between Rational Inattention and Signal Extraction with Exogenous Noises

Dividing  $\Lambda$  on both sides of (29), we obtain the following differential Riccati equation governing the evolution of  $K_t$ :

$$\dot{K}_t = -K_t^2 + 2rK_t + \frac{\sigma^2}{\Lambda}, \quad (72)$$

where  $\sigma^2/\Lambda$  is the signal-to-noise ratio (SNR) in this problem. In the steady state, we have the following proposition for this signal extraction case with exogenous noises:

**Proposition 9.** *Given SNR ( $\sigma^2/\Lambda$ ), in the steady state, the evolution of the perceived state can be written*

as

$$d\hat{s}_t = (r\hat{s}_t - c_t + \pi\alpha_t) dt + \hat{\sigma}dB_t^*,$$

where

$$\hat{\sigma} \equiv K\sqrt{\Lambda} = g(\tau)\sigma, \quad (73)$$

$$K = r + \sqrt{r^2 + \frac{\sigma^2}{\Lambda}}, \quad (74)$$

$g(\tau) \equiv r\sqrt{\tau} + \sqrt{1 + r^2\tau} > 1$ , and  $\tau \equiv 1/\text{SNR} = \Lambda/\sigma^2$ . Furthermore, if SNR and  $\kappa$  satisfy the following equality:

$$\text{SNR} = 4\kappa(\kappa - r),$$

then the RI and SE cases are observationally equivalent in the sense that they lead to the same model dynamics.

*Proof.* In the steady state in which  $\dot{K}_t = 0$ , solving the following algebraic Riccati equation,

$$-K_t^2 + 2rK_t + \frac{\sigma^2}{\Lambda} = 0,$$

yields the steady state Kalman gain:

$$K = r + \sqrt{r^2 + \frac{\sigma^2}{\Lambda}}. \quad (75)$$

and steady state conditional variance:  $\Sigma = K\Lambda$ . ■

## 7.6. Calibrating the Robustness Parameter in the RB-RI Model

In this MU-SU model, let model  $P$  denote the approximating model, (32) and model  $Q$  be the distorted model, (37). Because both models  $P$  and  $Q$  are arithmetic Brownian motions with constant drift and diffusion coefficients under MU-SU, the log-likelihood ratios are normally distributed random variables. Consequently, the logarithm of the Radon-Nikodym derivative of the distorted model ( $Q$ ) with respect to the approximating model ( $P$ ) can be written as

$$\ln\left(\frac{L_Q}{L_P}\right) = -\int_0^N \bar{v}dB_s - \frac{1}{2}\int_0^N \bar{v}^2ds, \quad (76)$$

where

$$\bar{v} \equiv v^*\hat{\sigma} = -r\gamma\vartheta\sqrt{\sigma_e^2\alpha^{*2} + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha^*}. \quad (77)$$

Similarly, the logarithm of the Radon-Nikodym derivative of the approximating model ( $P$ ) with respect to the distorted model ( $Q$ ) is

$$\ln \left( \frac{L_P}{L_Q} \right) = \int_0^N \bar{v} dB_s + \frac{1}{2} \int_0^N \bar{v}^2 ds. \quad (78)$$

Given (76) and (78), it is straightforward to derive  $p(\vartheta; N)$ :

$$p(\vartheta; N) = \Pr \left( x < \frac{\bar{v}}{2} \sqrt{N} \right), \quad (79)$$

where  $x$  follows a standard normal distribution.

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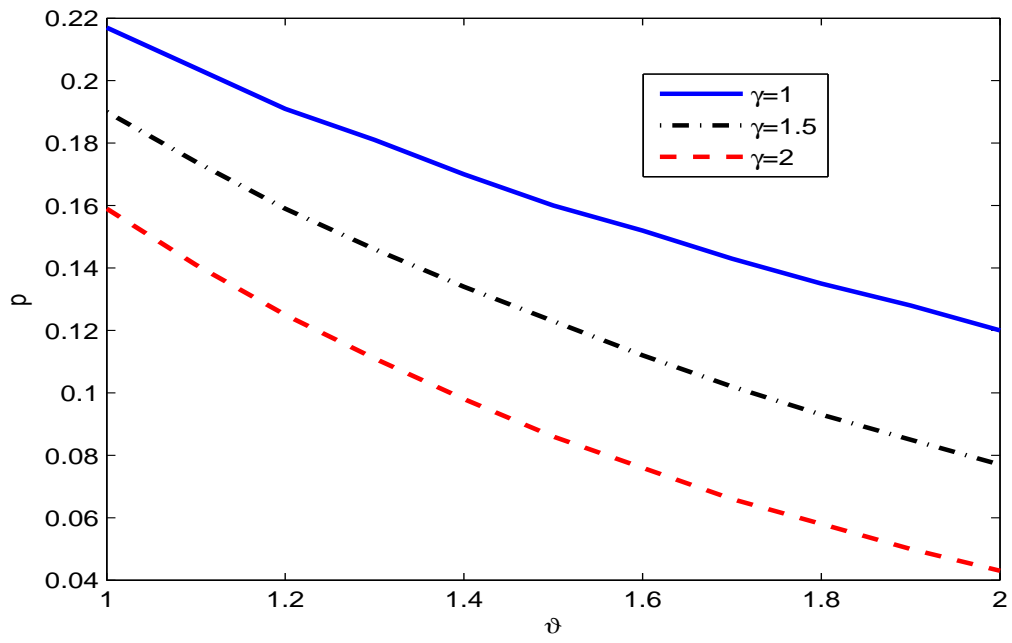


Figure 1. Relationship between  $\vartheta$  and  $p$

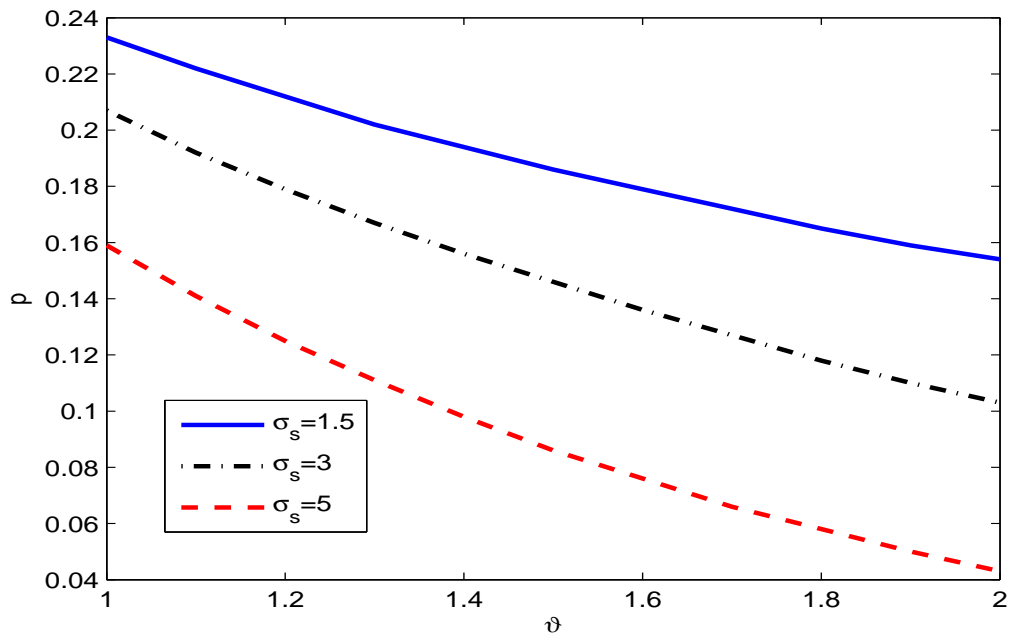


Figure 2. Relationship between  $\vartheta$  and  $p$

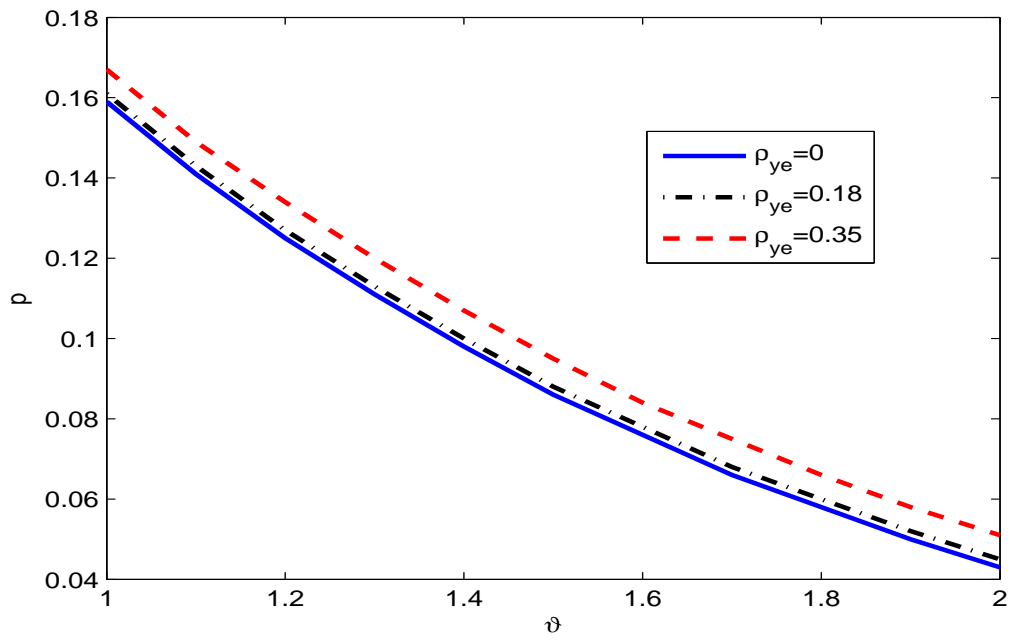


Figure 3. Relationship between  $\vartheta$  and  $p$

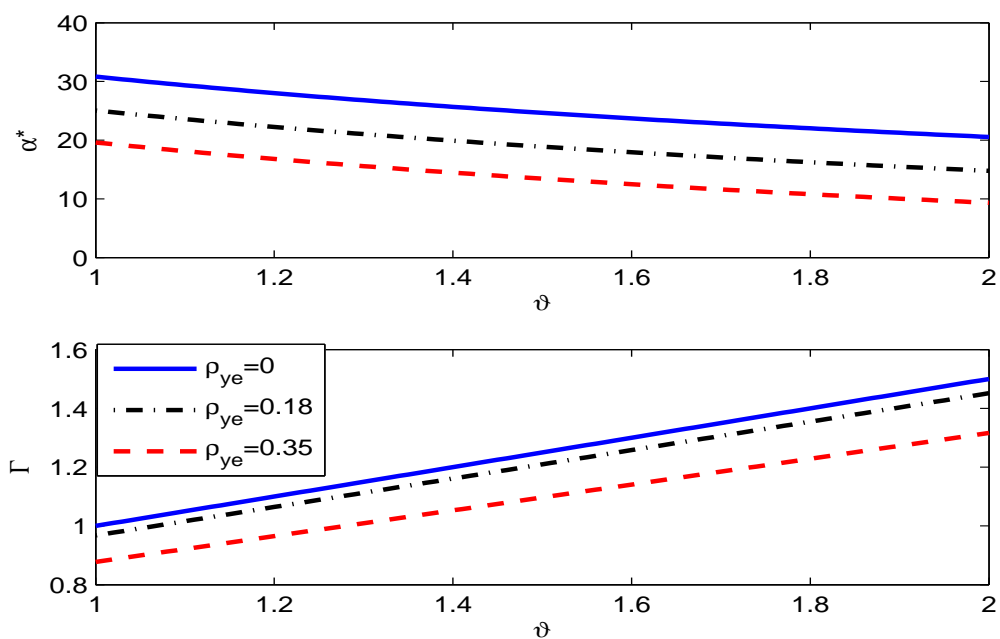


Figure 4. Robust Portfolio Rule and Precautionary Savings

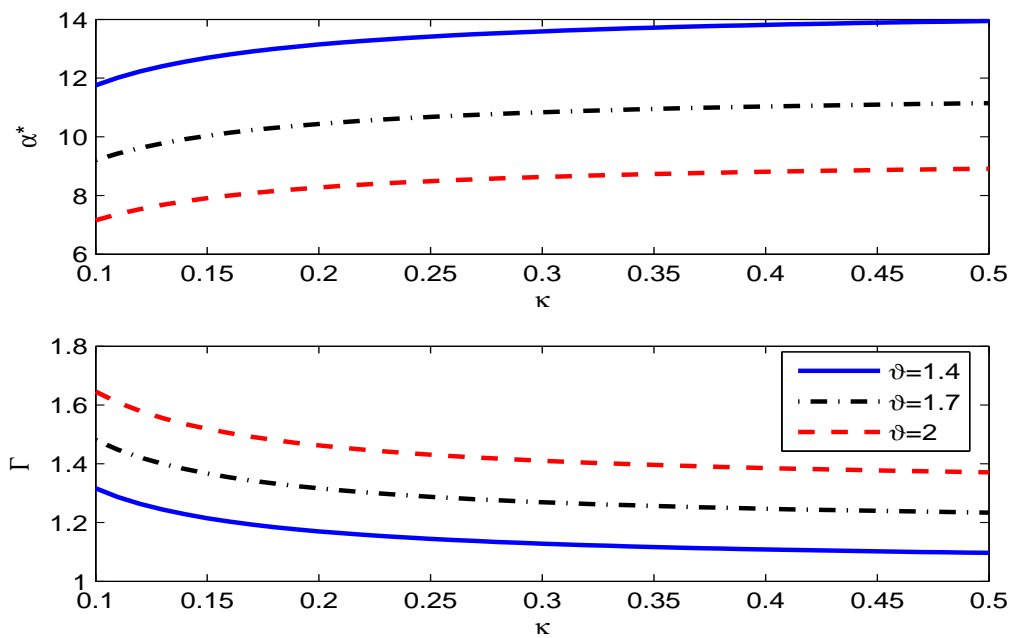


Figure 5. Robust Portfolio Rule and Precautionary Savings under SU

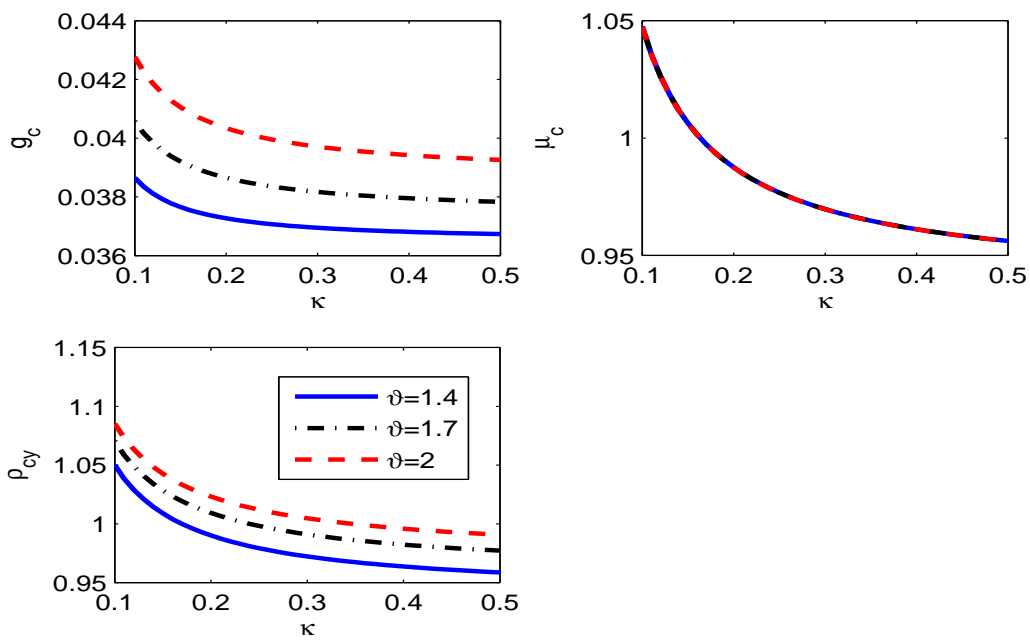


Figure 6. Effects of MU and SU on Consumption Dynamics

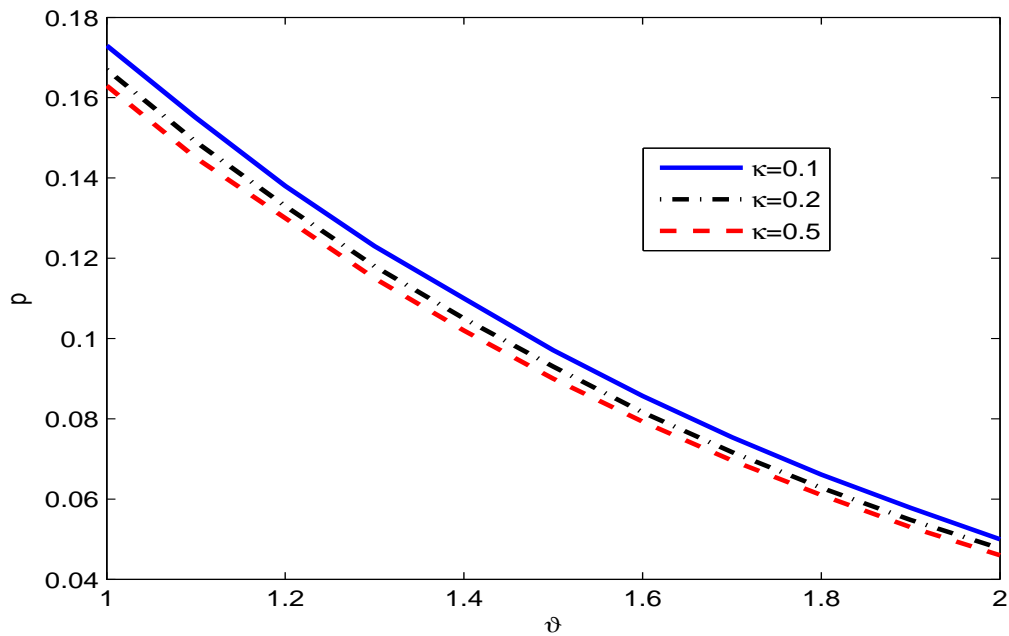


Figure 7. Relationship between  $\vartheta$  and  $p$  under SU

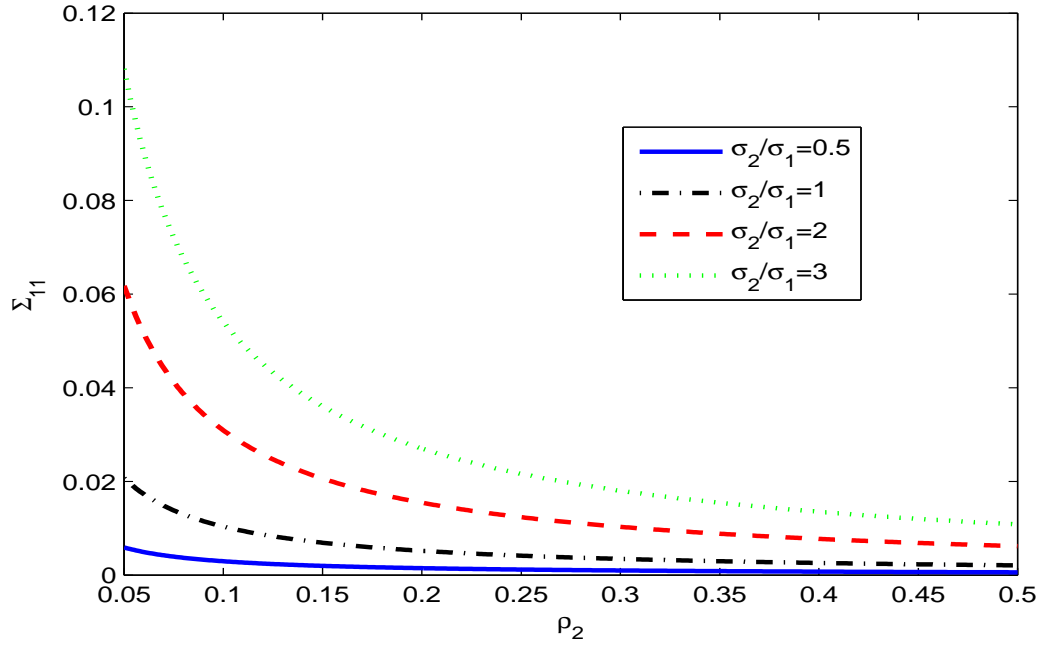


Figure 8. Effects of Incomplete Information about Income on Estimation Risk

Table 1. Implications of the correlation on  $\alpha^*$  and  $\Gamma$  under MU and SU ( $p = 10\%$ ,  $\sigma_s = 5$ )

		$\rho_{ye} = 0$	$\rho_{ye} = 0.18$	$\rho_{ye} = 0.35$
$\kappa = 0.1$	$\vartheta$	1.455	1.475	1.536
	$\alpha^*$	22.46	16.51	10.52
	$\alpha_s^*$	22.46	22.28	21.74
	$\alpha_h^*$	0	-5.77	-11.22
	$ \alpha_h^*  / \alpha_s^*$	0	0.259	0.516
	$\Gamma$	1.534	1.497	1.391
$\kappa = 0.2$	$\vartheta$	1.418	1.437	1.495
	$\alpha^*$	24.18	18.23	12.22
	$\alpha_s^*$	24.18	23.99	23.44
	$\alpha_h^*$	0	-5.77	-11.22
	$ \alpha_h^*  / \alpha_s^*$	0	0.240	0.479
	$\Gamma$	1.364	1.31	1.216
$\kappa = 0.5$	$\vartheta$	1.396	1.415	1.47
	$\alpha^*$	25.21	19.24	13.23
	$\alpha_s^*$	25.21	25.01	24.45
	$\alpha_h^*$	0	-5.77	-11.22
	$ \alpha_h^*  / \alpha_s^*$	0	0.231	0.459
	$\Gamma$	1.279	1.217	1.129
$\kappa = 1$	$\vartheta$	1.389	1.407	1.462
	$\alpha^*$	25.54	19.58	13.57
	$\alpha_s^*$	25.54	25.35	24.78
	$\alpha_h^*$	0	-5.77	-11.22
	$ \alpha_h^*  / \alpha_s^*$	0	0.228	0.453
	$\Gamma$	1.253	1.188	1.102

**Table 2.** Implications of income uncertainty on  $\alpha^*$  and  $\Gamma$  under MU and SU ( $p = 10\%$ ,  $\rho_{ye} = 0.18$ )

		$\sigma_s = 4$	$\sigma_s = 5$	$\sigma_s = 6$
$\kappa = 0.1$	$\vartheta$	1.770	1.475	1.265
	$\alpha^*$	15.29	16.51	17.42
	$\alpha_s^*$	19.90	22.28	24.34
	$\alpha_h^*$	-4.62	-5.77	-6.92
	$ \alpha_h^*  / \alpha_s^*$	0.232	0.259	0.284
	$\Gamma$	1.675	1.497	1.370
$\kappa = 0.2$	$\vartheta$	1.715	1.437	1.238
	$\alpha^*$	16.92	18.23	19.21
	$\alpha_s^*$	21.54	23.99	26.13
	$\alpha_h^*$	-4.62	-5.77	-6.92
	$ \alpha_h^*  / \alpha_s^*$	0.214	0.240	0.265
	$\Gamma$	1.460	1.310	1.203
$\kappa = 0.5$	$\vartheta$	1.682	1.415	1.222
	$\alpha^*$	17.90	19.24	20.26
	$\alpha_s^*$	22.52	25.01	27.18
	$\alpha_h^*$	-4.62	-5.77	-6.92
	$ \alpha_h^*  / \alpha_s^*$	0.205	0.231	0.255
	$\Gamma$	1.352	1.217	1.120
$\kappa = 1$	$\vartheta$	1.671	1.407	1.216
	$\alpha^*$	18.23	19.58	20.61
	$\alpha_s^*$	22.84	25.35	27.54
	$\alpha_h^*$	-4.62	-5.77	-6.92
	$ \alpha_h^*  / \alpha_s^*$	0.202	0.228	0.251
	$\Gamma$	1.319	1.188	1.094