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ROBUSTNESS AND MODELLING ERROR CHARACTERIZATION

by

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ABSTRACT

The results on robustness theory presented here are extensions of those given in [1]. The basic innovation in these new results is that they utilize minimal additional information about the structure of the modelling error as well as its magnitude to assess the robustness of feedback systems for which robustness tests based on the magnitude of modelling error alone are inconclusive.

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I. INTRODUCTION

Determining the robustness[1] of a given feedback control system can be logically divided into two distinct questions: (1) how near instability is the feedback system and (2) given the class of model errors for which the feedback system is stable, does this class include the model errors that can be reasonably expected for this particular system? The first question can be answered exactly by appropriate mathematical analysis once a suitable notion of "nearness to instability" is defined. The second question is, however, a question that requires engineering judgment in the definition of what constitutes a reasonable modelling error. The role of mathematical analysis with respect to question (2) is that of providing a simple characterization of a sufficiently large subclass of modelling errors that do not destabilize the feedback system. Without a simple characterization of this subclass of model errors even the best engineering judgment may not be adequate to answer question (2). Nevertheless, very simple characterizations of model errors that are not destabilizing often lead to results that are not very useful practically because they are too restrictive and the associated subclass of nondestabilizing model errors too small. Therefore, a compromise between the simplicity of the characterization and the extent of the subclass of nondestabilizing model errors that can be considered is necessary.

To this end, this paper explores two ideas. The first is the idea of using various definitions of modelling error to obtain different robustness tests that bound the modelling error magnitude to conclude stability. Obviously some tests will work better than others in particular instances. The various tests are all derived from one fundamental robustness theorem. The second idea is that of improving any of the foregoing tests by using information involving a partial characterization of the structure of the modelling error (in particular, a projection of the error onto a subspace). If the modelling error does not have a particular structure then its magnitude must be somewhat larger in order to destabilize the feedback system than if it did possess such structure.

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1 Robustness as used here refers to robust stability although it can also be used in a broader context to include system performance.

The development of the results on the use of model error structure will proceed first by presenting in Section II a generalized version of a fundamental robustness theorem. Section III gives the basic results from matrix theory that will be used in Section IV. Section IV gives a classification of various robustness tests that have appeared previously in the literature as well as a new one that has not, according to the type of model error they guard against. All these tests are generalized to use model error structure as well as magnitude information via the results of Section III in Section V. Also, an example is given demonstrating the results. All key proofs [2] are given in the Appendix.

II. FUNDAMENTAL CHARACTERIZATION OF ROBUSTNESS

The basic system under consideration is given in Figure 1, where G(s), the loop transfer function matrix, incorporates the open-loop plant dynamics as well as any compensation employed. Due to modelling error or uncertainty the actual loop transfer function matrix is $\tilde{G}(s)$, a perturbed version of G(s). Assume that both G(s) and $\tilde{G}(s)$ have state space representations given respectively by the triples (A,B,C) and $(\tilde{A},\tilde{B},\tilde{C})$ (i.e., $G(s) = C(Is-A)^{-1}B$ and $\tilde{G}(s) = \tilde{C}(Is-\tilde{A})^{-1}\tilde{B}$). Associated with the state space representation of G(s) are the open and closed loop characteristic polynomials, respectively $\phi_{OL}(s)$ and $\phi_{CL}(s)$ defined by

$$\phi_{OL}(s) = \det(sI-A) \tag{2.1}$$

$$\phi_{CL}(s) = \det(sI-A+BC) \tag{2.2}$$

The polynomials $\tilde{\phi}_{OL}(s)$ and $\tilde{\phi}_{CL}(s)$ associated with $(\tilde{A},\tilde{B},\tilde{C})$ are analogously defined.

The following theorem generalizes Theorem 2.2 of [1] and is based on the idea of continuously deforming the multivariable Nyquist diagram for G(s) into the one corresponding to $\tilde{G}(s)$ without passing the locus through the critical point. If this can be done and the number of encirclements of the critical point required for stability by G(s) and $\tilde{G}(s)$ are the same then this perturbation of G(s) will not induce instability. In this theorem we let

D_R denote the Nyquist contour (shown in Figure 2) along which $\det(I+G(s))$ is evaluated and define $G(s,\varepsilon)$ as a matrix of rational transfer functions continuous in s and ε for ε in [0,1] and for all s in D_R that also satisfies the following conditions

$$G(s,0) = G(s) \tag{2.3}$$

and

$$G(s,1) = \tilde{G}(s) \tag{2.4}$$

Theorem 1: The polynomial $\tilde{\phi}_{CL}(s)$ has no CRHP (closed-right-half-plane) zeros if the following conditions hold:

1. (a) $\phi_{OL}(s)$ and $\tilde{\phi}_{OL}(s)$ have the same number of CRHP zeros.

   (b) if $\tilde{\phi}_{OL}(j\omega_o) = 0$ then $\phi_{OL}(j\omega_o) = 0$

   (c) $\phi_{CL}(s)$ has no CRHP zeros

2. $\det[I+G(s,\varepsilon)] \neq 0$ for all ε in [0,1] and for all $s \in D_R$ with R sufficiently large.

Theorem 1 forms the basis for the derivation of all subsequent robustness results. We will subsequently assume that the radius R of the contour D_R is taken sufficiently large so that Theorem 1 may be applied.

Notice, that if $\|G(s,\varepsilon)\|_2 \to 0$ as $|s| \to \infty$ for all ε in [0,1], then condition 2 of Theorem 1 need only be verifed for (s,ε) in $\Omega_R \times [0,1]$ where Ω_R is defined as

$$\Omega_R=\{s \mid s\varepsilon D_R \text{ and } \mathrm{Re}(s)\leq 0\}. \tag{2.5}$$

This will be the case when $G(s,\varepsilon)$ is defined in Section IV because both $\|G(s)\|_2 \to 0$ and $\|\tilde{G}(s)\|_2 \to 0$ as $|s| \to \infty$. The development of robustness tests from Theorem 1 involves the construction of inequalities that can guarantee the nonsingularity of $I+G(s,\varepsilon)$ as in condition 2. Therefore, Section III will develop general matrix theory results that test for singularity of the sum of two matrices.

III. MATRIX THEORY

The purpose of this section is to introduce important tools from matrix theory and present some results that form the backbone of the robustness results of section V. The specific problem considered in this section is the following. Given a nonsingular complex matrix A, find the nearest (in some sense) singular matrix $\tilde{A}$ which belongs to a certain class of matrices. If the error matrix E is defined as $E = \tilde{A} - A$ then the problem may be stated in the following form. Given a nonsingular complex matrix A find the matrix E of minimum norm that makes A + E singular when E is constrained to belong to a certain class of matrices.

A. Singular Valve Decomposition (SVD) [6] and Subspace Projections

Let the complex $n \times m$ matrix A have the SVD

$$A = U\Sigma V^H = \sum_{i=1}^{k} \sigma_i \underline{u}_i \underline{v}_i^H \quad ; \quad k = \min(n,m) \tag{3.1}$$

where V^H denotes the complex conjugate transpose of V, U and V are unitary matrices given by

$$U = [\ \underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n\ ] \tag{3.2}$$

$$V = [\ \underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m\ ] \tag{3.3}$$

and Σ has the singular values of A, denoted σ_i, on its main diagonal (arranged in descending order) as its only nonzero elements. When $n=m$ it can be verified that the n^2 matrices $\underline{u}_i\underline{v}_j^H$, form an orthnormal basis in which we can express an arbitrary $n \times m$ matrix E as

$$E = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \underline{u}_i\underline{v}_j^H, E\rangle\, \underline{u}_i\underline{v}_j^H \tag{3.4}$$

where the innerproduct for matrices is defined by

$$\langle A,B\rangle = \mathrm{tr}(A^H B) \tag{3.5}$$

The matrix $<\underline{u}_i\underline{v}_j^H,E>\underline{u}_i\underline{v}_j^H$ is simply the projection of E onto the subspace spanned by $\underline{u}_i\underline{v}_j^H$ which has magnitude $|<\underline{u}_i\underline{v}_j^H,E>|$.

B. Error Matrix Structure

It is well known that if A + E is singular and $\|E\|_2 = \sigma_{min}(A)$ then $<\underline{u}_n\underline{v}_n^H,E> = -\sigma_n = -\sigma_{min}(A)$. Also if $\|E\|_2 < \sigma_{min}(A)$ then A + E is nonsingular. Now suppose that we construct a constraint set for E so that E cannot have a projection of magnitude σ_n in the most sensitive direction $\underline{u}_n\underline{v}_n^H$. This means that the matrix A + E cannot become singular along the direction $\underline{u}_n\underline{v}_n^H$ and thus $\|E\|_2$ must increase if A+E is to be singular. To find out just how much larger $\|E\|_2$ must become we formulate the constrained optimization problem where we assume A has distinct singular values.

Problem A:

$$\min_{E} \|E\|_2 \qquad \text{s.t. } \det(A+E) = 0 \tag{3.6}$$

$$|<\underline{u}_n\underline{v}_n^H,E>| \leq \phi < \sigma_n$$

Solution to Problem A:

The error matrix E is given by

$$E = U\left[\begin{array}{c|cc} P_s & \multicolumn{2}{c}{0} \\ \hline & \phi & \gamma \\ 0 & \gamma^* & -\phi \end{array}\right]V^H \tag{3.7}$$

where P_s arbitrary but

$$\|P_s\| \leq \sqrt{\sigma_n\sigma_{n-1}+\phi(\sigma_n-\sigma_{n-1})} = \|E\|_2, \tag{3.8}$$

and γ is given by

$$\gamma = \sqrt{(\phi+\sigma_{n-1})(\sigma_n-\phi)}e^{j\theta}, \ \theta \text{ arbitrary} \tag{3.9}$$

and A has the SVD

$$A = U\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \ \sigma_n \end{bmatrix}V^H, \quad \sigma_i > \sigma_{i+1} \tag{3.10}$$

The following theorem follows trivially from the solution of Problem A.

Theorem 2: For square matrices A and E, A+E is nonsingular if

$$\sigma_{max}(E)=\|E\|_2< \sqrt{\sigma_n\sigma_{n-1}+\phi(\sigma_n-\sigma_{n-1})} \tag{3.11}$$

and

$$|<\underline{u}_n\underline{v}_n^H,E>| \leq \phi < \sigma_n \tag{3.12}$$

where $\sigma_{n-1} \geq \sigma_n > 0$ are the two smallest singular values of A and $\underline{u}_n$ and $\underline{v}_n$ are respectively the left and right singular vectors of A corresponding to σ_n. □

Corollary 1: For square matrices A and E, det (A+E) $\neq$ 0

if

$$\sigma_{max}(E)=\|E\|_2< \sqrt{\sigma_n\sigma_{n-1}} \tag{3.13}$$

and

$$<\underline{u}_n\underline{v}_n^H,E> = 0 \quad \square \tag{3.14}$$

Theorem 2 is the key to making use of model error structure in the subsequent robustness tests. Corollary 1 has a very pleasing geometrical interpretation that will be discussed next.

in the 2x2 case and displays the columns of A and $\tilde{A}=A+E$ where $\tilde{A}$ is singular and $\|E\|_2$ a minimum. When the number of orthogonal vectors (i.e. columns of A) is greater than 2, Corollary 1 states that it requires the minimum "effort" to align the two shortest vectors in the set.

D. Examples

To make these results clearer we will illustrate the solutions to the problem of finding the matrices E of minimum spectral norm that make A+E singular under various constraints on the E matrix.

Examples:

Let A be given by

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.18}$$

and consider the various constraints on E.

Unconstrained Case:

$$E = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{E_s} & 0 \\ & & 0 \\ \hline 0 & 0 & -1 \end{array}\right] \tag{3.19}$$

where $\|E_s\|_2 \leq 1$ but otherwise E_s is arbitrary.

$e_{33} = 0$ Case:

$$E = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & 0 & 2e^{j\theta} \\ 0 & 2e^{-j\theta} & 0 \end{bmatrix} \tag{3.20}$$

C. Geometric Interpretation

The nature of the solution to Problem A becomes apparent when the SVD is used to transform the A matrix into a positive definite diagonal matrix. This is accomplished with the following simple lemmma.

Lemma 1: If the SVD of A is given by

$$A = U\Sigma V^H \tag{3.15}$$

with U and V^H unitary and Σ and diagonal then A+E is singular if and only if Σ+P is singular where

$$P = U^H E V \tag{3.16}$$

and furthermore $\|P\|_2 = \|E\|_2$. □

Thus, one may work with Σ and P rather than A and E. Therefore, in the subsequent discussion we will make the assumption that the matrix A is now diagonal and positive definite.

The matrix A is now given by

$$A = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ & & \ddots \\ 0 & & \sigma_n \end{bmatrix} \tag{3.17}$$

where $\sigma_i > \sigma_{i+1}$. If the columns of the matrix A are thought of as a set of n orthogonal vectors of lengths σ_i, then Corollary 1 can be interpreted geometrically in the 2x2 case as the problem of aligning two orthogonal vectors with minimum "effort" without decreasing the length of the shortest vector. Here the "effort" required to align the two vectors is equal to $\|E\|_2 = \sigma_{max}(E)$ where E makes A+E singular. Corollary 1 states that the minimum "effort" required to align the two vectors is equal to the geometric mean of their lengths. Figure 3 graphically illustrates Corollary 1 where

$$|e_{11}| < \|E\|_2 = \frac{\sqrt{10}}{2} \cong 1.58 \qquad (3.25)$$

and e_{11} and θ are otherwise arbitrary.

It is important to point out that we have limited ourselves to constraints on E of a very special form and in general arbitrary constraints on the form of E lead to a mathematical nonlinear programming problem that does not in general have a closed form solution. However, these special form of constraints on E will be useful in obtaining robustness results of Section V.

IV. ROBUSTNESS TESTS AND UNSTRUCTURED MODEL ERROR

In this section, we present theorems that guarantee the stability of the perturbed closed-loop system for different characterizations of model uncertainty (i.e., different types of model error). This is done via Theorem 1 by using a specific error criterion to construct a transfer matrix $G(s,\varepsilon)$ continuous in s and ε on $D_R \times [0,1]$ that satisfies (2.3) and (2.4). Then a simple test bounding the magnitude of the error is devised which guarantees that condition 2 of Theorem 1 is satisfied. This procedure is carried out for four different types of errors. These tests use the magnitude of the modelling error and hence are based on the unstructured part of the model error. These different types of model errors will emphasize different aspects of the difference between the nominal $G(s)$ and $\tilde{G}(s)$ and thus under certain circumstances will give essentially different assessments of the robustness or margin of stability of the feedback control system.

A. Robustness Error Criteria

Four basic input/output types of modelling error can be defined by considering absolute and relative errors between the nominal and perturbed systems and the analogous errors for the inverse nominal and inverse perturbed systems. Let $P(s)$ and $\tilde{P}(s)$ denote the nominal and perturbed open loop plants and $K(s)$ the compensation employed; then $G(s) = P(s)K(s)$ or $K(s)P(s)$ depending on where

where $|e_{11}| \leq 2$ and otherwise e_{11} and θ are arbitrary.

$e_{23} = e_{33} = 0$ Case:

$$E = \begin{bmatrix} 0 & 0 & 3e^{j\theta} \\ 0 & e_{22} & 0 \\ 3e^{-j\theta} & 0 & 0 \end{bmatrix} \tag{3.21}$$

where $|e_{22}| \leq 3$ and otherwise e_{22} and θ are arbitrary.

$e_{13} = e_{23} = e_{33} = 0$ Case:

$$E = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & -4 & 0 \\ e_{31} & 0 & 0 \end{bmatrix} \tag{3.22}$$

where

$$\sqrt{|e_{11}|^2 + |e_{31}|^2} \leq 4 = \|E\|_2 \tag{3.23}$$

but otherwise e_{11} and e_{31} are arbitrary.

$|e_{33}| \leq 1/2$ Case:

$$E = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & 1/2 & 3/2\ e^{j\theta} \\ 0 & 3/2\ e^{-j\theta} & -1/2 \end{bmatrix} \tag{3.24}$$

the loop is broken (arbitrarily for convenience we take G(s) as P(s)K(s) although K(s)P(s) would serve just as well).

Define the errors

$$E_A(s) \triangleq \tilde{P}(s) - P(s) \tag{4.1}$$

$$E_M(s) = [\tilde{P}(s)-P(s)]\, P^{-1}(s) = [\tilde{G}(s)-G(s)]G^{-1}(s) \tag{4.2}$$

$$E_S(s) = \tilde{P}^{-1}(s)-P^{-1}(s) \tag{4.3}$$

$$E_D(s) = P(s)[\tilde{P}^{-1}(s)-P^{-1}(s)] = G(s)[\tilde{G}^{-1}(s)-G^{-1}(s)] \tag{4.4}$$

where the subscripts A, M, S and D refer to addition, multiplication, subtraction and division, respectively, and signify model error type. Solving for the perturbed system $\tilde{P}(s)$ in terms of the nominal P(s) and the error we obtain

$$\tilde{P}(s) = P(s)+E_A(s) \tag{4.5}$$

$$\tilde{P}(s) = [I+E_M(s)]P(s) \tag{4.6}$$

$$\tilde{P}(s) = P(s)[I+E_S(s)P(s)]^{-1} = [I+P(s)E_S(s)]^{-1}P(s) \tag{4.7}$$

$$\tilde{P}(s) = [I+E_D(s)]^{-1}\, P(s) \tag{4.8}$$

Notice that E_S is a feedback type error whereas E_A is a feedforward type error. It is also useful to introduce the matrix L(s) defined by the relationship

$$\tilde{P}(s) = L(s)P(s). \tag{4.9}$$

To utilize Theorem 1 to prove stability we will need to construct two homotopys $G(s,\varepsilon)$ which are given by

$$G(s,\varepsilon) = (1-\varepsilon)G(s) + \varepsilon\tilde{G}(s) \tag{4.10}$$

and

$$G(s,\varepsilon) = [(1-\varepsilon)G^{-1}(s) + \varepsilon\tilde{G}^{-1}(s)]^{-1} \tag{4.11}$$

The first $G(s,\varepsilon)$ is used in the proof of Theorem 3 and the second $G(s,\varepsilon)$ in the proof of Theorem 4 via Theorem 1.

Theorem 3 [3,4,5]: The polynomial $\tilde{\phi}_{CL}(s)$ has no CRHP zeros if the following conditions hold:

1. condition 1 of Theorem 1 holds
2. either (a) or (b) holds for all $s\varepsilon\Omega_R$

   (a) $\sigma_{min}\,[K^{-1}(s)+P(s)] > \sigma_{max}[E_A(s)]$

   (b) $\sigma_{min}\,[I+G^{-1}(s)] > \sigma_{max}[E_M(s)]$ □

Theorem 4: The polynomial $\tilde{\phi}_{CL}(s)$ has no CRHP zeros if the following conditions hold:

1. condition 1 of Theorem 1 holds
2. $L(s)$ has no zero or strictly negative real eigenvalues for any $s\varepsilon\Omega_R$
3. either (a) or (b) holds for all $s\varepsilon\Omega_R$

   (a) $\sigma_{min}[K(s)+P^{-1}(s)] > \sigma_{max}[E_S(s)]$

   (b) $\sigma_{min}[I+G(s)] > \sigma_{max}[E_D(s)]$ □

Remark: If $\sigma_{max}[E_D(s)] < 1$ then condition 2 of Theorem 4 is automatically satisfied.

Observation: The condition that $L(s)$ have no strictly real and negative eigenvalues or be singular can be interpreted in terms of a phase reversal of certain signals between the nominal and perturbed systems or as the introduction of transmission zeros by the modelling error. To make this precise, suppose that for some ω_0 that $L(j\omega_0)\underline{x} = \lambda\underline{x}$ for some complex nonzero vector $\underline{x}$ and some real $\lambda < 0$. Then there exists a vector

$\underline{u}(t)$ of input sinusoids of various phasing and at frequency ω_0 which when applied to the nominal system produces an output $\underline{y}(t)$ and produces an output $\lambda\underline{y}(t)$ when applied to the perturbed system. Thus when λ is negative the phase difference between the sinusoidal outputs of the nominal and perturbed systems is 180^o. If $\lambda=0$ then the perturbed system has transmission zeros at $\pm j\omega_0$. This fact is significant since Theorem 4 can never guarantee stability with respect to model uncertainty when the phase of the system outputs is completely uncertain above some frequency or with respect to sensor or actuator failures in the feedback channels.

B. Interpretation of Robustness Error Criteria

The following interpretation about the usefullness of the four robustness tests of Theorems 3 and 4 can be obtained by examining when their singular value tests are likely to be satisfied. First consider Theorem 3 condition 2(a). When $\sigma_{min}(K^{-1}) >> \sigma_{max}(P)$ so that $\sigma_{max}(G) << 1$ then 2(a) becomes

$$\sigma_{max}(K)\,\sigma_{max}(E_A) < 1 \tag{4.12}$$

which means that after crossover the compensator gain must be sufficiently small. When $\sigma_{min}(P) >> \sigma_{max}(K^{-1})$ so that $\sigma_{min}(G) >> 1$ then 2(a) becomes

$$\sigma_{min}(P) > \sigma_{max}(E_A) \tag{4.13}$$

and hence below crossover the gain of the open loop plant must be sufficiently large. As one can see this one has no control over this last inequality. Thus 2(a) is most useful in the region after crossover where parasitic or neglected dynamics contribute to model error (e.g., higher order structural modes) where we can use K to roll off the loop quickly enough to tolerate these errors.

Next consider 2(b) when $\sigma_{min}(G) >> 1$ then

$$\sigma_{max}(E_M) < 1 \tag{4.14}$$

so that one cannot tolerate as much error before crossover as after when $\sigma_{max}(G) << 1$ and 2(a) becomes

$$\sigma_{max}(G)\,\sigma_{max}(E_M) < 1 \tag{4.15}$$

As in the previous case high frequency unmodelled dynamics will be tolerated as long as the loop is rolled off sufficiently in the vicinity where the unmodelled dynamics have large magnitudes.

In Theorem 4 condition 3(a) becomes

$$\sigma_{min}(K) > \sigma_{max}(E_s) \tag{4.16}$$

when $\sigma_{min}(K) >> \sigma_{max}(P^{-1})$ making $\sigma_{min}(G) >> 1$. Thus in the region before crossover (providing E_s does not induce phase reversals) sufficiently large compensator gains are required to stabilize the system for E_s variations. This type of error is usually associated with parameter changes in system time (4.16) constants and is a mathematical statement of well known fact that high gain reduces the sensitivity of the system to open-loop plant variations. When $\sigma_{min}(P^{-1}) >> \sigma_{max}(K)$ then $\sigma_{max}(G) << 1$ and 3(a) becomes

$$\sigma_{max}(P)\sigma_{max}(E_s) < 1 \tag{4.17}$$

which requires that after crossover the open loop plant must have sufficiently small gain to tolerate this type of error

Finally consider 3(b) when $\sigma_{min}(G) >> 1$ then we obtain

$$\sigma_{min}(G) >> \sigma_{max}(E_D) \tag{4.18}$$

and again high loop gain is an effective means to tolerate large low frequency (nonphase reversing) model errors. After crossover when $\sigma_{max}(G) << 1$ we have no control of the tolerance to this type of model error and 3(b) becomes

$$\sigma_{max}(E_D) < 1. \tag{4.19}$$

In summary, Theorem 3 is most useful for model errors occurring predominantly at high frequency and conversely Theorem 4 is most useful for modelling errors that occur predominantly in the low frequency range and do not produce phase reversals.

All the preceding robustness tests guarantee that stability is preserved by ensuring that the magnitude of the model error (according to some particular error criteria) is sufficiently small. In these tests the model error is unconstrained in its structure, and therefore, these tests guard against any type of model error structure. If all types of model error structure are not possible then these robustness tests may be conservative and methods such as those developed in the next section must be employed to take advantage of some particular aspect of the structure of the model error.

V. ROBUSTNESS ANALYSIS FOR LINEAR SYSTEMS WITH STRUCTURED MODEL ERROR

In this section, the robustness tests of Section IV are refined to distinguish between those model errors which do not destabilize the feedback system and those that do, but both of which have magnitudes larger than the MIMO generalization of the "distance to the critical (-1,0) point". To do this it is necessary to be able to distinguish between model errors that increase the margin of stability for the feedback system and those that decrease it. This cannot be done on the basis of the magnitude of the model error. Therefore, it must be done on the basis of the <u>structure</u> of the model error. However, only a <u>partial characterization</u> of the modelling error is necessary and its structure is constructively produced by the method of analysis used in Section III.

In order to make a practical use of these results that utilize the structure of the model error, it is necessary to determine if the model error of minimum magnitude that will destabilize the feedback system can be guaranteed not to occur. This assessment must be made on the basis of engineering judgment about the type of model uncertainties that are reasonable for the nominal design model representing the physical system. For discussions on how to practically determine what constitutes a reasonable modelling error, the reader is referred to [7] for a discussion of model errors in an automative engine control system and [8] for a similar discussion with regard to power system models.

A. Robustness Test Utilizing Model Error Structure

The robustness test of Theorems 3 and 4 may be improved by excluding modelling errors with significant projections onto subspace spanned by $\underline{u}_n\underline{v}_n^H$, where again $\underline{u}_n\underline{v}_n^H$ are the left and right singular vectors of the appropriate transfer function associated with the particular error type. Dropping the explicit s dependence for notational convenience this may be formalized in the following theorem.

Theorem 5: The polymonial $\tilde{\phi}_{CL}$ has no CRHP zeros if the following conditions hold:

1. condition 1 of Theorem 1 holds

2. $(A,E) = (P^{-1}+K,\ E_A)$ or $(I+G^{-1}, E_M)$ or $(K^{-1}+P, E_S)$ or $(I+G, E_D)$

3. $\lambda(L) \notin (-\infty,0]$ for all $s \varepsilon \Omega_R$ if $A = K^{-1}+P$ or $I+G$

4. For all $s \varepsilon \Omega_R$

   (a) A has the SVD $A = U\Sigma V^H = \sum_i \sigma_i \underline{u}_i \underline{v}_i^H$ where $\sigma_{i+1} > \sigma_i$ for all

   (b) $|\langle \underline{u}_n\underline{v}_n^H, E\rangle| \leq c < \sigma_n$ whenever $\sigma_{max}(E) \geq \sigma_n$

   (c) $\sigma_{max}(E) < [\sigma_n\sigma_{n-1} + c(\sigma_n - \sigma_{n-1})]^{1/2}$

This can be interpreted as a change in the multivariable Nyquist locus. If the magnitude of the error is sufficiently small (condition 4(c)) then $\langle \underline{u}_n\underline{v}_n^H, E\rangle$ adequately predicts in which the direction in which the Nyquist locus moves with respect to the critical point. Thus by restricting the class of modelling errors only slightly the feedback system may tolerate much larger errors if $\sigma_{n-1} >> \sigma_n$. For a more detailed discussion see [9].

Remark: Notice that in this theorem the vectors $\underline{u}_n$ and $\underline{v}_n$ are not specified apriori (i.e., they are not defined until the compensator has been designed) and thus we are not able to directly utilize apriori structural information about E. We can only try to determine from this and other information if $\langle \underline{u}_n \underline{v}_n^H, E \rangle$ represents a reasonable modelling error. Recent results in [10, 11] give a very useful partial mathematical solution to an optimization problem like that of Problem A where the structure of E is prespecified. However, as previously mentioned, a closed form solution does not exist in general for this latter problem.

B. Illustrative Example

Suppose that we wish to determine stability robustness of a 2x2 control system which actually has a loop transfer function matrix $\tilde{G}(s)$ but is represented by the nominal diagonal loop transfer matrix G(s) given by

$$G(s) = \begin{bmatrix} g_{11}(s) & 0 \\ 0 & g_{22}(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+7.5} & 0 \\ 0 & \frac{1}{s+0.5} \end{bmatrix} \tag{5.1}$$

so that the nominal closed-loop system has poles at -8.5 and -1.5. If we use the relative error criterion (5.2) (this is a slightly different definition of error than the one in (4.2) where the error is specified at the output; here it is specified at the input),

$$E(s)=G^{-1}(s)[\tilde{G}(s)-G(s)] = \begin{bmatrix} \frac{\tilde{g}_{11}(s)-g_{11}(s)}{g_{11}(s)} & \frac{\tilde{g}_{12}(s)}{g_{11}(s)} \\ \frac{\tilde{g}_{21}(s)}{g_{22}(s)} & \frac{\tilde{g}_{22}(s)-g_{22}(s)}{g_{22}(s)} \end{bmatrix} \tag{5.2}$$

then the multiplicative uncertainty factor matrix L(s) (perturbing the inputs) is given by

$$L(s)=I+E(s) = \begin{bmatrix} \frac{\tilde{g}_{11}(s)}{g_{11}(s)} & \frac{\tilde{g}_{12}(s)}{g_{11}(s)} \\ \frac{\tilde{g}_{21}(s)}{g_{22}(s)} & \frac{\tilde{g}_{22}(s)}{g_{22}(s)} \end{bmatrix} \tag{5.3}$$

First, we compute $\sigma_{min}(I+G^{-1}(j\omega))$ to determine the magnitude of the smallest destabilizing model error E(s). This is simply given by

$$\sigma_{min}(I+G^{-1}(j\omega)) = \left|1.5+j\omega\right| = \sqrt{(1.5)^2+\omega^2} \geq 1.5 \tag{5.4}$$

because

$$I+G^{-1}(s) = \begin{bmatrix} s+8.5 & 0 \\ 0 & s+1.5 \end{bmatrix}. \tag{5.5}$$

Now suppose that the error in the loop gain of each loop of the feedback system is known within $\pm 50\%$ of the nominal loop gain, that is

$$|e_{11}(j\omega)| \leq 0.5 \tag{5.6}$$

and

$$|e_{22}(j\omega)| \leq 0.5 \tag{5.7}$$

Next, suppose that we are more uncertain about the channel crossfeeds in the sense that we can only assert that

$$|e_{12}(j\omega)| = |\ell_{12}(j\omega)| = \left|\frac{\tilde{g}_{12}(j\omega)}{g_{11}(j\omega)}\right| \leq 2 \tag{5.8}$$

and that

$$|e_{21}(j\omega)| = |\ell_{21}(j\omega)| = \left|\frac{\tilde{g}_{21}(j\omega)}{g_{11}(j\omega)}\right| \leq 2\ . \tag{5.9}$$

It follows from (5.6) and (5.7) that we can bound $|e_{11}(j\omega)|$ and $|e_{22}(j\omega)|$ by 1/2 and thus, by (5.8) and (5.9), we can only conclude that

$$\|E(j\omega)\|_2 = \sigma_{max}[E(j\omega)] < 2.5 \tag{5.10}$$

From (5.10) and (5.4) it is clearly possible to have

$$\sigma_{max}[E(j\omega)] > \sigma_{min}[I+G^{-1}(j\omega)]. \tag{5.11}$$

Therefore, Theorem 3 does not apply. However, we can use Theorem 5 to ensure the stability of the perturbed feedback system. To see this, note that the SVD of $I+G^{-1}(j\omega)$ is given by

$$I+G^{-1}(j\omega) = \begin{bmatrix} e^{j\theta_1(\omega)} & 0 \\ 0 & e^{j\theta_2(\omega)} \end{bmatrix} \begin{bmatrix} |j\omega+8.5| & 0 \\ 0 & |j\omega+1.5| \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.12}$$

$$= U(j\omega)\Sigma(j\omega)V^H(j\omega)$$

where

$$\theta_1(\omega) = \arg[j\omega+8.5] \tag{5.13}$$

and

$$\theta_2(\omega) = \arg[j\omega+1.5] \tag{5.14}$$

Note that condition 4(b) of Theorem 5 can be satisfied with $c(j\omega)=1/2$ since from (5.12) defining $\underline{u}_2(j\omega)$ and $\underline{v}_2(j\omega)$ and from (5.7) bounding $e_{22}(j\omega)$ we have that for all ω

$$|\langle \underline{u}_2(j\omega)\underline{v}_2^H(j\omega), E(j\omega)\rangle| = |\underline{u}_2^H(j\omega)E(j\omega)\underline{v}_2(j\omega)| \tag{5.15}$$

$$= |e_{22}(j\omega)| \leq 1/2$$

Thus, by (5.15) and (5.4) we have

$$\sigma_2(j\omega) \geq 1.5 > 1/2 \geq |\langle \underline{u}_2(j\omega)\underline{v}_2^H(j\omega), E(j\omega)\rangle| \tag{5.16}$$

Next, we calculate the right-hand-side of condition 4(c) of Theorem 5 and a lower bound as follows

$$\Big[\sigma_1(j\omega)\sigma_2(j\omega)+c(j\omega)\,[\sigma_2(j\omega)-\sigma_1(j\omega)]\Big]^{1/2} = \Big[|j\omega+8.5|\,|j\omega+1.5|$$
$$+ 1/2[|j\omega+1.5|-|j\omega+8.5|]\Big]^{1/2} \geq (8.5)(1.5)+\left(\frac{-7}{2}\right) \geq 3. \tag{5.17}$$

Therefore, using (5.10) we have that

$$\sigma_{max}[E(j\omega)] \leq 2.5 < 3 \leq \left[\sigma_1(j\omega)\sigma_2(j\omega)+c(j\omega)[\sigma_2(j\omega)-\sigma_1(j\omega)]\right]^{1/2} \tag{5.18}$$

and so condition 4(c) of Theorem 5 holds. Assuming condition 1 of Theorem 9 holds we have shown that the perturbed feedback system is stable. The next smallest destabilizing error can be calculated using Problem A with $\phi(j\omega)=0$ and $\omega=0$ since $\sigma_{min}(I+G^{-1}(j\omega)) \geq \sigma_{min}(I+G^{-1}(0))=1.5$ and is given by

$$E(0) = \begin{bmatrix} 1/2 & 3 \\ 3 & -1/2 \end{bmatrix} \tag{5.19}$$

which means that L(s) may be taken as the constant matrix L given by

$$L = \begin{bmatrix} 3/2 & 3 \\ 3 & 1/2 \end{bmatrix} \tag{5.20}$$

Thus, we see that (refer to Figs. 4 and 5 crossfeed gain errors of magnitude 3 and loop gain changes of $\pm 50\%$ are required to destabilize the feedback system if we insist that (5.6) and (5.7) must hold.

VI SUMMARY AND CONCLUSION

This paper has discussed two ideas for the robustness analysis of linear-time-invariant systems. The first is that of exploring different types of modelling error and in what cases these might be useful. The second idea focuses on improving any given particular robustness test by placing a weak restriction on the structure of model errors considered. If all reasonable model errors are contained in the slightly smaller set, the size of the tolerable model errors may become much larger than otherwise.

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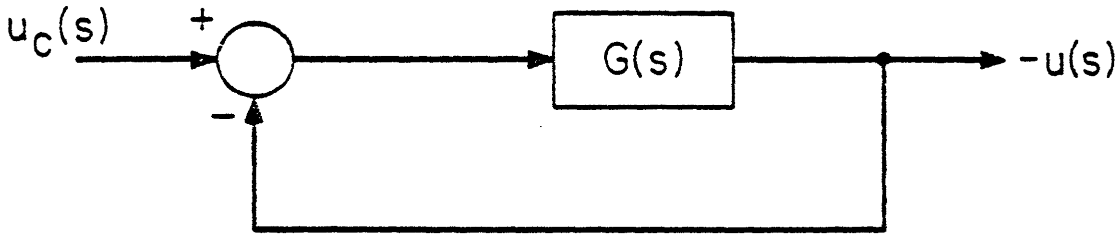

Figure 1: Control system under consideration

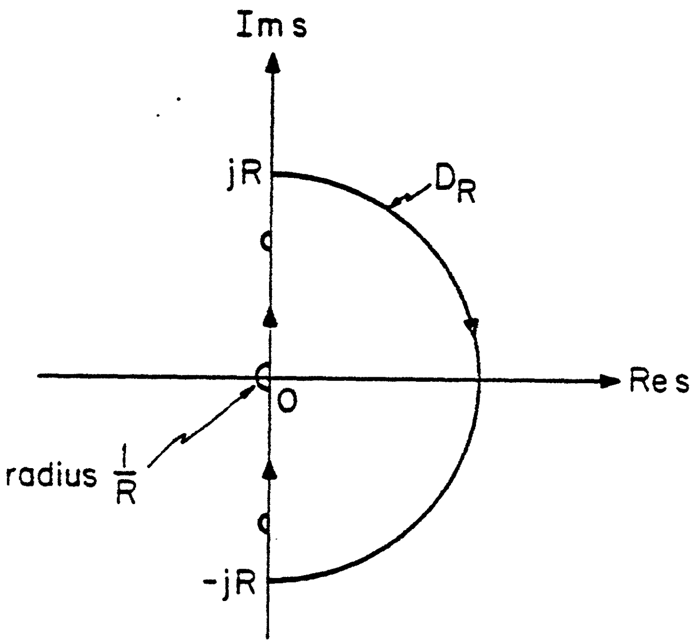

Figure 2: Nyquist contour D_R which avoids $j\omega$-axis zeros of $\phi_{OL}(s)$ by $1/R$ radius indentations.

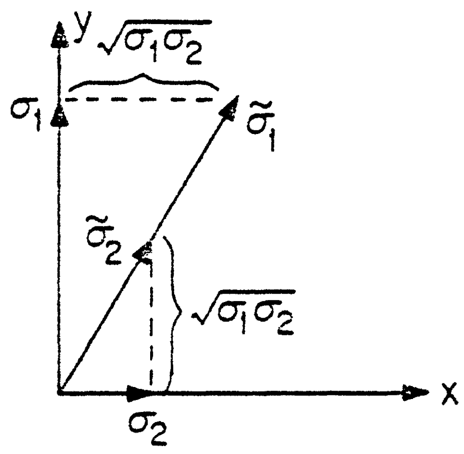

Figure 3: Columns of A and $\tilde{A}$=A+E depicted as vectors aligned with minimum effort.

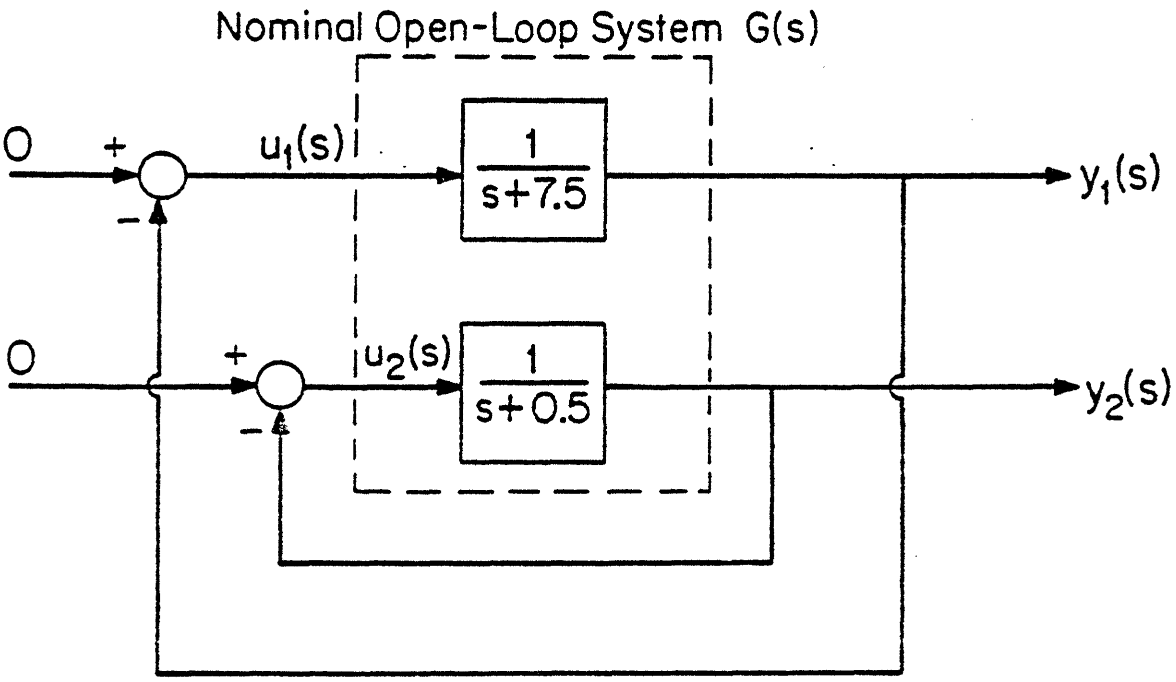

Figure 4: Nominal feedback system (stable).

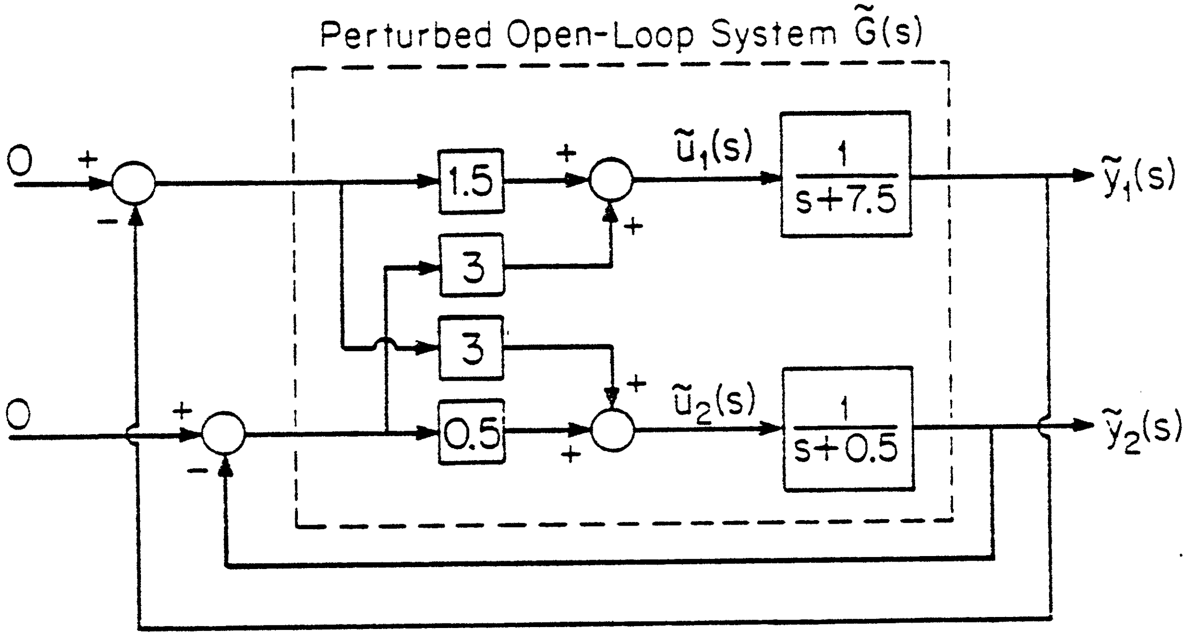

Figure 5: Perturbed feedback system (unstable).

APPENDIX

This appendix gives the necessary proofs involved with the solution to Problem A. We proceed with some definitions followed by six lemmas necessary to prove the solution of Problem A.

<u>Definition</u>: A matrix X belonging to a set S is said to be <u>minimal</u> in S if $\|X\|_2 \leq \|Y\|_2$ for all Y belonging to S.

<u>Sets</u>: In the space of complex matrices define the sets:

$$D_A \triangleq \{E \mid A+E \text{ is rank deficient}\} \tag{A.1}$$

$$P_A \triangleq \{E \mid E \varepsilon D_A \text{ and } |\langle u_n v_n^H, E\rangle| \leq \phi < \sigma_n\} \tag{A.2}$$

$$P_A^H \triangleq \{E \mid E=E^H \text{ and } E \varepsilon P_A\} \tag{A.3}$$

where the matrix A has the SVD given by

$$A = U\Sigma V^H = \sum_{i=1}^{n} \sigma_i \underline{u}_i \underline{v}_i^H, \quad \sigma_i > \sigma_{i+1} > 0 \tag{A.4}$$

where V^H denotes the complex conjugate transpose of V and the vectors $\underline{u}_i$ and $\underline{v}_i$ compose the columns of the unitary matrices U and V respectively. Note that A need not be square.

Lemma A1 gives the form of the unique minimal E in P_A^H when A>0 is diagonal and 2x2. Lemma A2 shows this E is minimal in P_A. Lemma A3 shows this E is the unique minimal element of P_A. Thus Lemmas A1 to A3 give the complete solution to Problem A via Lemma 1 when A is 2x2. Lemmas A4 and A5 extend the solution to the case where A is nx2. Lemma A6 extends the solution to the case where A>0 is nxn and diagonal. Applying Lemma 1 the complete solution to Problem A is obtained.

Lemma A1: Let $A = \mathrm{diag}[\sigma_1, \sigma_2] > 0$ then $E_0 \varepsilon P_A^H$ given by

$$E_0 = \begin{bmatrix} \phi & \gamma \\ \gamma^* & -\phi \end{bmatrix} \tag{A.5}$$

$$\gamma = \sqrt{(\sigma_1+\phi)(\sigma_2-\phi)}\; e^{j\theta}, \ \theta \text{ arbitrary} \tag{A.6}$$

is uniquely minimal in P_A^H.

Proof: Clearly $E_0 \varepsilon P_A^H$ to show that $\| E_0 \|_2 < \| E \|_2$ for all other $E \varepsilon P_A^H$, let E be given by

$$E = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix} \tag{A.7}$$

then $\| E \|_2$ is given by

$$\| E \|_2 = \left| \frac{a+d}{2} \right| + \sqrt{\left(\frac{a+d}{2}\right)^2 + |b|^2} \tag{A.8}$$

and because A+E is rank deficient

$$\| E \|_2 = \left| \frac{a+d}{2} \right| + \sqrt{\left(\frac{a+d}{2}\right)^2 + (\sigma_1+a)(\sigma_2+d)} \tag{A.9}$$

Taking partials of $\| E \|_2$ with respect to a and d we have

$$\frac{\partial \| E \|_2}{\partial a} = 1/2 \ [\mathrm{sgn}(a+d) + z_1] \tag{A.10}$$

$$\frac{\partial \| E \|_2}{\partial d} = 1/2 \ [\mathrm{sgn}(a+d) + z_2] \tag{A.11}$$

where sgn(•) is the usual sign function and

$$z_2 = \frac{\left(\frac{a+d}{2}\right) + \sigma_2}{\sqrt{\left(\frac{a-d}{2}\right)^2 + (\sigma_1+a)(\sigma_2+d)}} \tag{A.12}$$

$$z_2 = \frac{\left(\frac{a + d}{2}\right) + \sigma_1}{\sqrt{\left(\frac{a - d}{2}\right)^2 + (\sigma_1 + a)(\sigma_2 + d)}} \tag{A.13}$$

and $|z_1|<1$ and $|z_2| > 1$ for $|d| < \sigma_2$ and $|a| < \sigma_1$. Thus $\partial\|E\|_2/\partial a$ has the same sign as $a+d$ indicating a global minimum at $a = -d$. Since $|z_2|>1$, $\partial\|E\|_2/\partial d$ is always positive indicating the extremal value $d = -\phi$ minimizes $\|E\|_2$. Therefore the unique minimizing E in P_A^H is E_o.

Lemma A2: Let $A = \text{diag}\,[\sigma_1,\sigma_2,\ldots,\sigma_n]>0$ and let E be minimal in P_A then $E_H = \frac{1}{2}(E+E^H)$ is minimal in P_A.

Proof: Since $E \varepsilon P_A$, $x^H(A + E)x = x^H(A+E^H)x = 0$. This implies $\lambda_{min}(A+E_H) \leq 0$. However if $\lambda_{min}(A+E_H)< 0$ then there exists an $\alpha \,\varepsilon\, (0,1)$ such that $\alpha E_H \varepsilon P_A$ and $\|\alpha E_H\|_2 < \|E\|_2$ contradicting the minimality of E. Thus $\lambda_{min}(A+E_H) = 0$ and E_H is minimal in P_A.

Lemma A3: Let $A = \text{diag}[\sigma_1,\sigma_2]>0$ and let E be minimal in P_A then $E = E^H$.

Proof: For 2x2 matrices E, some simple algebra shows that

$$\|E\|_2^2 = \frac{1}{2}\|E\|_E^2 + \sqrt{\left(\frac{1}{2}\|E\|_E^2\right)^2 - |\det E|^2} \geq \frac{1}{2}\|E\|_E^2 \tag{A.14}$$

where $\|E\|_E^2 = \langle E,E \rangle$.

If E is decomposed as $E = E_H + E_{SH}$ where E_H is hermitian and E_{SH} is skew hermitian then

$$\|E\|_E^2 = \|E_H\|_E^2 + \|E_{SH}\|_E^2$$

and hence

$$\|E\|_2^2 \geq \frac{1}{2}\|E_H\|_E^2 + \frac{1}{2}\|E_{SH}\|_E^2 \tag{A.15}$$

The matrix E_H is uniquely minimal in P_A^H by Lemmas A1 and A2. From the form of E_H required by Lemma A1 we have

$$\|E_H\|_2^2 = \frac{1}{2}\|E_H\|_E^2 \tag{A.16}$$

and hence $\|E_H\|_2^2 < \|E\|_2^2$ if $E_{SH} \neq 0$.

This contradicts the minimality of E and hence $E = E^H = \frac{1}{2}(E+E^H) = E_H$.

Lemma A4: Let $A_1 = \mathrm{diag}[\sigma_1, \sigma_2] > 0$, let

$$A = \left[\frac{A_1}{0}\right] \tag{A.17}$$

and conformably partition E as

$$E = \left[\frac{E_1}{E_2}\right] \tag{A.18}$$

and let E be minimal in P_A then $E_2 = 0$ and E_1 is minimal in P_A .

Proof: Suppose E_1 is not minimal in P_{A_1} then $\|E\|_2 \geq \|E_1\|_2 > \|E_0\|_2$ where E_0 is uniquely minimal in P_{A_1}.

Define E_* as

$$E_* = \left[\frac{E_0}{0}\right] \tag{A.19}$$

then $\|E_*\|_2 = \|E_0\|_2 < \|E\|_2$ and $E_* \varepsilon P_A$ contradicting the minimality of E in P_A. Hence E_1 is minimal in P_{A_1}.

Since $\|E\|_2^2 = \lambda_{max}(E_1^H E_1 + E_2^H E_2)$ and from Lemma A1 $E_1^H E_1 = (\phi^2 + |\gamma|^2) I$ we obtain

$$\|E\|_2^2 = (\phi^2 + |\gamma|^2) + \|E_2\|_2^2 \tag{A.20}$$

Hence $\|E\|_2$ is minimized only when $E_2 = 0$.

<u>Lemma A5</u>: Let A be nx2 with singular values $\sigma_1 > \sigma_2$ then E is uniquely minimal in P_A and given by

$$E = U \begin{bmatrix} \phi & \gamma \\ \gamma^* & -\phi \\ \hline \multicolumn{2}{c}{0} \end{bmatrix} V^H = U E_1 V^H \tag{A.21}$$

where

$$\gamma = \{(\sigma_1 + \phi)(\sigma_2 - \phi)\}^{1/2} e^{j\theta}, \quad \theta \text{ arbitrary} \tag{A.22}$$

and

$$\|E\|_2^2 = \phi^2 + |\gamma|^2 = \sigma_1 \sigma_2 - \phi(\sigma_1 - \sigma_2) \tag{A.23}$$

<u>Proof</u>: A has SVD

$$A = U \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ \hline \multicolumn{2}{c}{0} \end{bmatrix} V^H = U A_1 V^H \tag{A.24}$$

and it is clear (by Lemma 1) allowing unitary transformations) that E_1 is minimal in P_{A_1}, is unique and given by Lemma A4.

Lemma A6: Let $A = \text{diag}[\sigma_1, \sigma_2, \ldots, \sigma_n] > 0$ then the unique minimal E in P_A is given by

$$E = \left[\begin{array}{c|cc} P_s & & 0 \\ \hline 0 & \phi & \gamma \\ & \gamma^* & -\phi \end{array}\right] \tag{A.25}$$

where

$$\gamma = e^{j\theta}\{(\sigma_{n-1}+\phi)(\sigma_n-\phi)\}^{1/2}; \quad \theta \text{ arbitrary} \tag{A.26}$$

with

$$\|P_s\|_2 \leq \|E\|_2 = [\phi^2 + (\sigma_{n-1}+\phi)(\sigma_n-\phi)]^{1/2} \tag{A.27}$$

but P_s is otherwise arbitrary.

Proof: $E \varepsilon P_A$ implies that there exists an $\underline{x} \neq 0$ such that $(A+E)\underline{x} = 0$ where

$$\underline{x}^T = [\underline{x}_1^T, x_n] \tag{A.28}$$

$$\underline{x}_1^T = [x_1, x_2, \ldots, x_{n-1}], \; \|\underline{x}_1\|_2 = 1 \tag{A.29}$$

and where $x_{n-1} \varepsilon [0,1]$. Note $\underline{x}_1 \neq 0$ else $E \not\varepsilon P_A$. Define Z

$$Z = \left[\begin{array}{c|c} \underline{x}_1 & 0 \\ \hline 0 & 1 \end{array}\right] \tag{A.30}$$

then

$$(A+E)\underline{x} = (AZ+EZ)\underline{y} = 0 \tag{A.31}$$

where $\underline{y}^T = [1, x_n]$. Clearly $EZ \varepsilon P_{AZ}$ otherwise $E \not\varepsilon P_A$.

Also from (A.30) we obtain

$$\|EZ\|_2 \leq \|E\|_2 \|Z\|_2 = \|E\|_2. \tag{A.32}$$

Now AZ has the form

$$AZ = \left[\begin{array}{c|c} \underline{w} & 0 \\ \hline 0 & \sigma_n \end{array}\right] \tag{A.33}$$

where

$$\underline{w}^T = [\sigma_1 x_1, \sigma_2 x_2, \ldots, \sigma_{n-1} x_{n-1}] \tag{A.34}$$

and since $\|\underline{x}_1\|_2 = 1$

$$\|\underline{w}\|_2 \geq \sigma_{n-1} \tag{A.35}$$

with equality only when $\underline{x}_1^T = [0,0,\ldots,0,1]$ (recall $x_{n-1} \epsilon [0,1]$).
From Lemma A5,

$$\|EZ\|_2^2 \geq \phi^2 + (\sigma_{n-1} + \phi)(\sigma_n - \phi) \tag{A.36}$$

with equality only when Z is given by

$$Z = \left[\begin{array}{cc} & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \tag{A.37}$$

For this selection of Z in (A.37) E is given by

$$E = [E_1 \,\vdots\, EZ]. \tag{A.38}$$

and thus if $E_1 = 0$ and EZ is minimal in P_{AZ} we then have $\|E\|_2 = \|EZ\|_2 = \{\phi^2 + (\sigma_{n-1} + \phi)(\sigma_n - \phi)\}^{1/2}$ and this choice of E is minimal in P_A. Therefore, Z must be given as in (A.37). Since EZ must be minimal in P_{AZ} making

$$\|E\|_2^2 = \lambda_{max}(E_1{}^H E_1 + (EZ)^H EZ)$$

$$= \left\| E_1{}^H E_1 + \begin{bmatrix} 0 & 0 \\ 0 & \alpha^2 I \end{bmatrix} \right\|_2^2 \qquad \text{(A.39)}$$

for some real $\alpha \neq 0$, E_1 must be the form

$$E_1 = \begin{bmatrix} E_2 \\ 0 \end{bmatrix}, \quad \|E_2\|_2 \leq \|EZ\|_2 \qquad \text{(A.40)}$$

otherwise $\|E\|_2 > \|EZ\|_2$ making E nonminimal. This gives the desired conclusion.

The solution to Problem A follows directly from Lemma A6 and Lemma 1.