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ROBUSTNESS OF THE BLACK AND SCHOLES FORMULA

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Consider an option on a stock whose volatility is unknown and stochastic. An agent assumes this volatility to be a specific function of time and the stock price, knowing that this assumption may result in a misspecification of the volatility. However, if the misspecified volatility dominates the true volatility, then the misspecified price of the option dominates its true price. Moreover, the option hedging strategy computed under the assumption of the misspecified volatility provides an almost sure one-sided hedge for the option under the true volatility. Analogous results hold if the true volatility dominates the misspecified volatility. These comparisons can fail, however, if the misspecified volatility is not assumed to be a function of time and the stock price. The positive results, which apply to both European and American options, are used to obtain a bound and hedge for Asian options.

KEY WORDS: option pricing, hedging strategies, stochastic volatility

1. INTRODUCTION

Since the development of the Black–Scholes option pricing formula (Black and Scholes 1973), practitioners have used it extensively, even to evaluate options whose underlying asset (hereafter called the "stock") is known to not satisfy the Black–Scholes hypothesis of a deterministic volatility. In this paper, we provide conditions under which the Black–Scholes formula is robust with respect to a misspecification of volatility. We extend the well-known property of the option price being an increasing function of a deterministic volatility to a comparison of the price of contingent claims associated with two different stochastic volatilities. Our principal assumptions are that the contingent claims have convex payoffs and the only source of randomness in the misspecified volatility is a dependence on the current price of the stock. Under these assumptions, if the misspecified volatility dominates (respectively, is dominated by) the true volatility, then the contingent claim price corresponding to the misspecified volatility dominates (respectively, is dominated by) the true contingent claim price. A counterexample, based on ideas by M. Yor and reported in

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Section 4, shows that in the absence of our assumptions, option prices may fail to compare in any reasonable way.

The key to the results reported here is that for both European and American contingent claims with convex payoffs, if the interest rate process is deterministic and the stock volatility depends only on time and the current price of the stock, then the price of the contingent claim is a convex function of the price of the stock. This result was previously obtained by Bergman, Grundy, and Wiener (1996), who reported the European version of it in a detailed study of the properties of European contingent claims. The proof in Bergman et al. proceeds through an analysis of the parabolic partial differential equation satisfied by the contingent claim price. We obtain the result using the theory of stochastic flows and the Girsanov Theorem. Hobson (1996) has subsequently simplified these arguments, using stochastic coupling.

Avellaneda, Levy, and Paras (1995); Avellaneda and Paras (1996); and Avellaneda and Lewicki (1996) obtained pricing and hedging bounds in markets with bounds on uncertain volatility. Carr (1993) derived formulas for "the Greeks" associated with an option in a constant-coefficient market by differentiating the Black–Scholes equation. Martini (1995) has applied semigroup methods to the problem of misspecified volatility.

The present paper examines the performance of a hedging portfolio derived from misspecified volatility, and in this respect it is an extension of El Karoui and Jeanblanc-Picqué (1990). We find that, under our assumptions, if the misspecified volatility dominates (respectively, is dominated by) the true volatility, then the self-financing value of the misspecified hedging portfolio exceeds (respectively, is exceeded by) the payoff of the contingent claim at expiration. We obtain this result also for American contingent claims.

When the volatility of the underlying stock is allowed to be random in a path-dependent way, the price of a European call can fail to be convex in the stock price. We provide an example of this in which the stock price is continuous and driven by a single Brownian motion, and the volatility depends on the initial stock price and on the driving Brownian motion. Moreover, the volatility increases with increasing initial stock price. Nonetheless, the price of the call is neither increasing nor convex in the initial stock price.

Bergman et al. (1996) provide a similar example, but with volatility decreasing with increasing initial stock price. They also show that dependence of the volatility on a second Brownian motion or jumps in the stock price can lead to nonincreasing, nonconvex European call prices.

In a discrete-time, finite-state model, Levy and Levy (1988) established option pricing and hedging bounds similar to ours. In their paper, the robustness is with respect to the range of values attained by the stock. Lyons (1995) has considered nonconvex options in a model with uncertain volatility. Lo (1987) has provided an upper bound for the option prices over all possible distributions of the terminal stock price with fixed mean and variance. Hull and White (1987) obtained an explicit formula for the price of an option on a stock whose volatility is stochastic and independent of the Brownian motion driving the stock price. Neither Lo nor Hull and White consider hedging portfolios.

This paper is organized as follows. Section 2 provides the basic model and definitions. In order to introduce some notation, Section 3 reviews the constant-coefficient Black and Scholes result. Section 4 shows that option prices can behave badly when the stock has stochastic, path-dependent volatility. Under the assumption that the interest rate is deterministic and the misspecified stock price process is Markov, the key to the positive results is the convexity of the contingent claim price as a function of the misspecified stock price. This result is proved in Section 5 using the theory of stochastic flows. Section 6 then proves a main result, the comparison of European contingent claim prices and performance of hedging portfolios under a misspecification of the stochastic volatility of the stock. In Section 7 we use a change of numéraire to extend these results to cover the case of random interest rate. We take up the study of American contingent claims in Section 8, first providing conditions under which these reduce to European contingent claims. Section 9 establishes the convexity of the price of American contingent claims, and Section 10 presents the comparison of American contingent claims prices and performance of hedging portfolios. Section 11 concludes with two examples in which our bounds provide useful information. In one case, we provide upper and lower bounds on a European call on the arithmetic average of two independent stock price processes. In the other case, we obtain an upper bound on the price of an Asian option. In both cases, the hedging portfolios associated with the bounds are shown to perform as expected.

2. BASIC ASSUMPTIONS AND DEFINITIONS

We consider a continuous-time economy with a positive finite horizon T. Two assets are traded continuously in a frictionless market. We denote by $M(t)$ the price at time t of the money market, and by $S(t)$ the price at time t of the stock. We assume there do not exist arbitrage opportunities, and we adopt the strong version of this assumption, namely the existence of a martingale ("risk-neutral") probability measure. More precisely, we make the following assumption.

HYPOTHESIS 2.1. *The money market price M and the stock price S are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to a filtration $\{\mathcal{F}(t); 0 \le t \le T\}$. Furthermore,*

$$M(t) = e^{\int_0^t r(u)\,du}, \tag{2.1}$$

$$dS(t) = S(t)[r(t)\,dt + \sigma(t)\,dW(t)], \tag{2.2}$$

where $\{W(t); 0 \le t \le T\}$ is a one-dimensional Brownian motion adapted to $\{\mathcal{F}(t); 0 \le t \le T\}$, where the interest rate process *r is deterministic and satisfies $\int_0^T |r(t)|\,dt < \infty$, and where the* volatility process *σ is nonnegative, adapted to $\{\mathcal{F}(t); 0 \le t \le T\}$, and satisfies $\int_0^T \sigma^2(t)\,dt < \infty$ almost surely. Finally, we assume that the local martingale*

$$\frac{S(t)}{M(t)} = S(0)\exp\left\{\int_0^t \sigma(u)\,dW(u) - \frac{1}{2}\int_0^t \sigma^2(u)\,du\right\}, \qquad 0 \le t \le T, \tag{2.3}$$

is in fact a square-integrable martingale, i.e., S/M is a martingale and

$$\mathbb{E}\frac{S^2(t)}{M^2(t)} < \infty, \qquad 0 \le t \le T.$$

We do not assume that the market is complete. In particular, the filtration $\{\mathcal{F}(t); 0 \le t \le T\}$ may be strictly larger than the filtration generated by the Brownian motion $\{W(t); 0 \le t \le T\}$. In Section 7, we relax Hypothesis 2.1 by allowing the interest rate to be stochastic.

REMARK 2.2. It is often the case that $W(t) = W_0(t) + \int_0^t \lambda(u)\,du$, where λ is the *market price of risk* for the stock price process and W_0 is a one-dimensional Brownian motion on $(\Omega, \mathcal{F})$ under the probability measure $\mathbb{P}_0$ related to $\mathbb{P}$ by the formula

$$\frac{d\mathbb{P}}{d\mathbb{P}_0} = \exp\left\{-\int_0^T \lambda(t)\,dW_0(t) - \frac{1}{2}\int_0^T \lambda^2(t)\,dt\right\}. \tag{2.4}$$

In this setup, equation (2.2) can be rewritten as

$$dS(t) = S(t)[(r(t) + \sigma(t)\lambda(t))\,dt + \sigma(t)\,dW_0(t)],$$

so that under $\mathbb{P}_0$ the mean rate of return on the stock is not necessarily equal to the interest rate.

DEFINITION 2.3. A *portfolio process* $\{\Delta(t), 0 \le t \le T\}$ is a bounded adapted process. Given a (nonrandom) initial portfolio value $\Pi_\Delta(0)$, the *self-financing value* of a portfolio process Δ is the solution of the linear stochastic differential equation

$$d\Pi_\Delta(t) = r(t)\,[\Pi_\Delta(t) - \Delta(t)S(t)]\,dt + \Delta(t)\,dS(t), \tag{2.5}$$

which is

$$\Pi_\Delta(t) = M(t)\left[\Pi_\Delta(0) + \int_0^t \Delta(u)\,d\left(\frac{S(u)}{M(u)}\right)\right]. \tag{2.6}$$

REMARK 2.4. Because $S(t)/M(t)$ is assumed to be a square-integrable martingale and Δ is bounded, the discounted portfolio value $\Pi_\Delta(t)/M(t)$ is also a square-integrable martingale.

DEFINITION 2.5. A *payoff function* is a convex function h, defined on $(0,\infty)$ and having bounded one-sided derivatives, that is,

$$|h'(x\pm)| \le C \quad \forall x > 0 \tag{2.7}$$

for a positive constant C. A (non–path-dependent) *European contingent claim* is a contract that pays $h(S(T))$ at time T. (For example, a European call is characterized by $h(x) = (x-K)^+$ and a European put by $h(x) = (K-x)^+$, where $K > 0$ is the exercise price.) A *price process* for a European contingent claim is any adapted process $\{P(t); 0 \le t \le T\}$ satisfying

$$P(T) = h(S(T)) \quad \text{a.s.} \tag{2.8}$$

Because the market under consideration is not necessarily complete, the arbitrage price of a European contingent claim is not necessarily defined. Furthermore, we shall often be working with a misspecified stock volatility, in which case the arbitrage price, even if it is defined, is likewise misspecified. For these reasons, we have adopted the very weak definition of a price process in Definition 2.5. Of course, if the market is complete, the *arbitrage price* of the European contingent claim is given by

$$P_E(t) \triangleq M(t)\mathbb{E}\left[\frac{h(S(T))}{M(T)}\middle|\mathcal{F}(t)\right], \qquad 0 \le t \le T. \tag{2.9}$$

DEFINITION 2.6. Let h be a payoff function for a European contingent claim. Let P be a price process for this contingent claim and let Δ be a portfolio process. The *tracking error*[1] associated with (P, Δ) is defined to be the process $e(t) \triangleq \Pi_\Delta(t) - P(t)$, where Π_Δ and P are related by the initial condition $\Pi_\Delta(0) = P(0)$. If the *discounted tracking error* $e(t)/M(t)$ is:

(i) identically equal to zero, then (P, Δ) is said to be a *replicating strategy*;
(ii) nondecreasing, then (P, Δ) is said to be a *superstrategy*;
(iii) nonincreasing, then (P, Δ) is said to be a *substrategy*.

A hedger who incorrectly estimates the volatility of the stock underlying a European contingent claim will incorrectly compute the contingent claim price and hedging portfolio. Let (P, Δ) be the result of such a computation. Suppose the hedger begins with a portfolio whose initial value is $\Pi_\Delta(0) = P(0)$ and uses the portfolio process Δ. At expiration, the hedger will have a portfolio valued at $\Pi_\Delta(T)$. If (P, Δ) is a superstrategy, then the discounted tracking error e/M is nondecreasing, and because $e(0) = 0$, we have

$$\Pi_\Delta(T) = P(T) + e(T) \ge h(S(T)). \tag{2.10}$$

In other words, the hedger has successfully hedged a short position in the contingent claim. Moreover, because $\Pi_\Delta(t)/M(t)$ is a martingale, $P(t)/M(t)$ is a supermartingale and consequently satisfies

$$\frac{P(t)}{M(t)} \ge \mathbb{E}\left[\frac{h(S(T))}{M(T)}\middle|\mathcal{F}(t)\right], \qquad 0 \le t \le T. \tag{2.11}$$

In particular,

$$P(0) \ge \mathbb{E}[h(S(T))/M(T)]. \tag{2.12}$$

A substrategy (actually, the negative of the portfolio process of a substrategy) hedges a long position, and inequalities (2.11) and (2.12) are reversed. A replicating strategy hedges a short position, its negative hedges a long position, and inequalities (2.11) and (2.12) become equalities. If the market is complete, then there exists a unique replicating strategy for the European contingent claim; the price process for this strategy is given by (2.9).

[1] The cost process introduced by Föllmer and Schweizer (1991) is the opposite of the tracking error.

3. CLASSICAL BLACK–SCHOLES

In this section, we set the stage by briefly reviewing properties of the classical Black–Scholes model. Let h be a payoff function, and assume that $r \in \mathbb{R}$ and $\sigma > 0$ are constant. For $t \in [0, T]$, we have

$$S(T) = S(t) \exp\left[\sigma(W(T) - W(t)) + \left(r - \frac{1}{2}\sigma^2\right)(T - t)\right].$$

Because $(W(T) - W(t))/\sqrt{T-t}$ is a standard normal random variable independent of $\mathcal{F}(t)$, the arbitrage price process (2.9) for the option with payoff function h is

$$P_E(t) = \mathcal{BS}(T - t, S(t); r, \sigma), \tag{3.1}$$

where

$$\begin{aligned} \mathcal{BS}(\tau, x; r, \sigma) &= e^{-r\tau}\,\mathbb{E}\left[h\left(x \exp\left(\sigma\sqrt{\tau}X + \left(r - \frac{1}{2}\sigma^2\right)\tau\right)\right)\right] \\ &= e^{-r\tau}\int_{-\infty}^{\infty} h(xe^y)\Phi'\left(y; \left(r - \frac{1}{2}\sigma^2\right)\tau, \sigma\sqrt{\tau}\right)dy \\ &= e^{-r\tau}\int_0^{\infty} h(z)\Phi'\left(\log z; \log x + \left(r - \frac{1}{2}\sigma^2\right)\tau, \sigma\sqrt{\tau}\right)\frac{dz}{z}, \end{aligned} \tag{3.2}$$

X is a standard normal random variable, and $\Phi'(y; \mu, \rho)$ is the normal density with mean μ and standard deviation ρ, the derivative with respect to y of the cumulative normal distribution $\Phi(y; \mu, \rho)$ with the same mean and standard deviation.

By differentiation of the second improper integral in (3.2), one can verify that $\mathcal{BS}(\cdot, \cdot; r, \sigma)$ is in $C^{1,2}((0, \infty)^2)$ and for $\tau > 0$, $x > 0$, we have

$$\begin{aligned} \frac{\partial}{\partial\tau}\mathcal{BS}(\tau, x; r, \sigma) &= \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\mathcal{BS}(\tau, x; r, \sigma) + rx\frac{\partial}{\partial x}\mathcal{BS}(\tau, x; r, \sigma) \\ &\quad - r\mathcal{BS}(\tau, x; r, \sigma), \end{aligned} \tag{3.3}$$

$$x^2\frac{\partial^2}{\partial x^2}\mathcal{BS}(\tau, x; r, \sigma) = \frac{1}{\sigma\tau}\frac{\partial}{\partial\sigma}\mathcal{BS}(\tau, x; r, \sigma). \tag{3.4}$$

It is clear that $\mathcal{BS}(0, x; r, \sigma) = h(x)$ for all $x > 0$. Differentiation of the first improper integral in (3.2) shows that

$$\frac{\partial}{\partial x}\mathcal{BS}(\tau, x; r, \sigma) = e^{-r\tau}\int_{-\infty}^{\infty} h'(xe^y)\, e^y\, \Phi'\left(y; \left(r - \frac{1}{2}\sigma^2\right)\tau, \sigma\sqrt{\tau}\right)dy,$$

where h' is defined almost everywhere and is bounded and nondecreasing because h is convex and satisfies (2.7). Therefore, $(\partial/\partial x)\mathcal{BS}(\tau, x; r, \sigma)$ is bounded and nondecreasing in x, and, in particular,

$$\frac{\partial^2}{\partial x^2}\mathcal{BS}(\tau, x; r, \sigma) \geq 0, \quad \tau > 0,\ x > 0.$$

From (3.4), we then have

$$\frac{\partial}{\partial \sigma}\mathcal{BS}(\tau, x;\ r, \sigma) \geq 0. \tag{3.5}$$

Applying Itô's formula to (3.1) and using (3.3), it is easy to verify that with

$$\Delta(t) = \frac{\partial}{\partial x}\mathcal{BS}(T - t, S(t);\ r, \sigma),$$

the pair (P_E, Δ) is a replicating strategy. The process Δ is bounded.

REMARK 3.1. In the special case $h(x) = (x - K)^+$ of the European call, (3.2) becomes the well-known Black–Scholes formula,

$$\mathcal{BS}(\tau, x;\ r, \sigma) = x\Phi(d^+(\tau, x; r, \sigma); 0, \sigma\sqrt{\tau}) - Ke^{-r\tau}\Phi(d^-(\tau, x; r, \sigma); 0, \sigma\sqrt{\tau}),$$

where $d^{\pm}(\tau, x; r, \sigma) = \log(x/K) + r\tau \pm \frac{1}{2}\sigma^2\tau$. Furthermore, $\frac{\partial}{\partial x}\mathcal{BS}(\tau, x;\ r, \sigma) = \Phi(d^+(\tau, x;\ r, \sigma); 0, \sigma\sqrt{\tau})$ and the mapping $\sigma \mapsto (\partial/\partial x)\mathcal{BS}(\tau, x;\ r, \sigma)$ is strictly increasing on $[0, \sigma_0]$ and strictly decreasing on $[\sigma_0, \infty)$, where $\sigma_0 \triangleq \sqrt{\frac{2}{\tau}(\log(x/K) + r\tau)}$.

4. A STOCHASTIC VOLATILITY COUNTEREXAMPLE

The remainder of this paper concerns models with stochastic volatility. We obtain positive results when the misspecified volatility is random only through dependence on the current stock price. In this section, we consider more general stochastic volatility. We assume that the market is complete, so that the arbitrage price is defined by (2.9), and we show that when volatility is stochastic in a path-dependent manner, the value of a European call can decrease with increasing volatility, and can even decrease with increasing stock price. The example provided below, which exhibits these counterintuitive behaviors, is derived from ideas of M. Yor.

We set $r = 0$. Let $a > 0$ be fixed, define $T_a = \inf\{t \geq 0; W(t) = a\}$, and set

$$\sigma(t) = \mathbb{1}_{\{W(t) < S(0)\}}\mathbb{1}_{\{t \leq T_a\}}.$$

Note that σ is a nondecreasing function of the initial stock price $S(0)$. Set $K = ae^a$. The value at time zero of a European call with strike price K and expiration time 1 is

$$v(x) = \mathbb{E}\left(S^x(1) - K\right)^+, \tag{4.1}$$

where $S^x(0) = x$ and $dS^x(t) = \sigma(t) S^x(t)\, dW(t)$. Here we use the notation $v(x)$ rather than $P_E(0)$ of (2.9) in order to explicitly indicate the dependence on the initial stock price x.

THEOREM 4.1. *The European call value* v *of (4.1) satisfies* $\lim_{x\downarrow 0} v(x) = 0$, $v(a) = 0$, *and* $v(x) > 0$ *for all* $x \in (0, a)$.

According to Theorem 4.1, if we take an initial stock price $x \in (0, a)$, then not only does $S^a(0)$ strictly exceed $S^x(0)$, but also the volatility associated with the stock price process $S^a(\cdot)$ is at least as great as the volatility associated with the stock price process $S^x(\cdot)$, at all times, almost surely. Nonetheless, $v(x) > v(a)$; that is, the European call on the stock with initial price x is strictly more valuable at the initial time than the European call on the stock with initial price a.

It is important to note that the option under consideration in Theorem 4.1 is a standard European call, not a barrier option. The volatility is zero whenever the driving Brownian motion exceeds the initial stock price or has reached the level a, but the stock price itself can exceed these quantities without the volatility vanishing. Moreover, one could create a similar example in which the volatility never vanishes by using as volatility $1\!\!1_{\{W(t)<S(0)\}} 1\!\!1_{\{t\le T_a\}} + \epsilon$. For sufficiently small positive ϵ, it will still be possible to find $0 < x < a$ such that $v(x) > v(a)$.

Proof of Theorem 4.1. According to Tanaka's formula (Karatzas and Shreve 1991, Prop. 6.8 and Thm. 6.22, Chap. 3), for $x > 0$ we have

$$\begin{aligned} &-(W(1\wedge T_a) - x)^- \\ &\qquad = -x + \int_0^{1\wedge T_a} 1\!\!1_{\{W(u)<x\}}(W(u))\, dW(u) - L\,(1\wedge T_a; x)\,, \end{aligned}$$

where $L(t; x)$ denotes the local time of W at x up to time t. Therefore, the stock price with initial condition x is given by

$$\begin{aligned} S^x(1) &= x\ \exp\left[\int_0^1 \sigma(u)\, dW(u) - \frac{1}{2}\int_0^1 \sigma^2(u)\, du\right] \\ &= x\ \exp\left[L\,(1\wedge T_a; x) + x - (W(1\wedge T_a) - x)^- - \frac{1}{2}\int_0^1 \sigma^2(u)\, du\right] \\ &\le x\ \exp\left[L\,(1\wedge T_a; x) + x\right]. \end{aligned}$$

In particular, $S^a(1) \le ae^a = K$ almost surely, so $v(a) = 0$. Furthermore, $v(x) \le \mathbb{E}S^x(1) = x$, which shows that $\lim_{x\downarrow 0} v(x) = 0$.

Now fix $x \in (0, a)$. On the set $\{T_a \le 1\}$, we have

$$S^x(1) \ge x \exp\left[L(T_a; x) + x - \frac{1}{2}\right].$$

But $L(T_a; x)$ is unbounded above on the set $\{T_a \le 1\}$, because a Brownian motion can spend substantial Lebesgue time in an arbitrarily small band about x and still reach a prior to time 1. (A rigorous proof of the unboundedness of $L(T_a; x)$ on $\{T_a \leq 1\}$ is provided in Appendix A.) We have then $\mathbb{P}(S^x(1) > K) > 0$, which implies $v(x) > 0$. $\square$

We also have the following counterexample, in which two stock prices have the same initial condition and the European call on one of them has a convex value function. Consider, in addition to the stock price S^x above, a stock price $\widetilde{S}^x$, starting at $x \in (0, a)$ and having volatility $\tilde{\sigma}(t) = \mathbb{1}_{\{t \leq T_a\}}$. The value of the European call on this stock is

$$\tilde{v}(x) = E\left[(xe^{W_{1\wedge T_a} - (1/2)(1\wedge T_a)} - K)^+\right],$$

and differentiation under the expectation leads the formula

$$\tilde{v}'(x) = \mathbb{E}\left[e^{W_{1\wedge T_a} - (1/2)(1\wedge T_a)} \mathbb{1}_{\{xe^{W_{1\wedge T_a} - (1/2)(1\wedge T_a)} \geq K\}}\right].$$

Thus $\tilde{v}$ is convex. Nevertheless, despite the fact that $\tilde{\sigma}(t) \geq \sigma(t)$, we have $\tilde{v}(x) = 0 < v(x)$ for all $x \in (0, a)$.

5. CONVEXITY OF EUROPEAN CONTINGENT CLAIM VALUES

For the remainder of this paper, in addition to the true volatility process σ, we shall have a misspecified volatility γ, which we allow to be stochastic only through dependence on the current stock price. This type of dependence prevents the anomalous behavior of the previous section, so that comparisons of σ and γ lead to comparisons of contingent claim prices and performance of hedging portfolios.

HYPOTHESIS 5.1. *Let* $\gamma: [0, T] \times (0, \infty) \mapsto \mathbb{R}$ *be continuous and bounded above. Assume moreover that* $(\partial/\partial s)[s\gamma(t, s)]$ *is continuous in* (t, s) *and Lipschitz continuous and bounded in* $s \in (0, \infty)$, *uniformly in* $t \in [0, T]$.

In place of (2.2), we shall now consider a misspecified stock price process governed by

$$dS_\gamma^x(t) = S_\gamma^x(t)\left[r(t)\,dt + \gamma(t, S_\gamma^x(t))\,dW(t)\right], \tag{5.1}$$

where the subscript γ records that this price process is being generated by the misspecified volatility γ and the superscript x records the initial condition $S_\gamma^x(0) = x$, $x > 0$. The interest rate $r(\cdot)$ is still deterministic. Let a payoff function h be given, and define the (misspecified) value of the contingent claim to be

$$v_\gamma(x) = \frac{1}{M(T)}\mathbb{E}h\left(S_\gamma^x(T)\right), \quad x > 0.$$

If the stock price really were governed by (5.1), the market would be complete and $v_\gamma(x)$ would be the arbitrage price of the contingent claim.

In later sections, we consider the value of the contingent claim at times other than zero; in this section we simplify notation by considering the value at time zero only. The following convexity result holds at all times, even though we state and prove it only at time zero. This result has been previously proved, using different methodology, by Bergman et al. (1996); see Section 1 of this paper for more details.

THEOREM 5.2. *Under Hypothesis 5.1 and the assumption of a deterministic interest rate $r(\cdot)$, the European contingent claim value v_γ is convex.*

Proof. In this proof, we suppress the subscript γ. Using the theory of stochastic flows (see, e.g., Kunita 1990, Thm. 4.7.2; Doss 1977), we may choose versions of $\{S^x(t); 0 \le t \le T\}$ which, for each $t \in [0, T]$ and each $\omega \in \Omega$, are diffeomorphisms in x from $(0, \infty)$ to $(0, \infty)$. The process $D^x(t) = (\partial/\partial x)S^x(t)$ has initial condition $D^x(0) = 1$ and satisfies

$$dD^x(t) = D^x(t)\left[r(t)\,dt + \frac{\partial}{\partial s}\rho\left(t, S^x(t)\right)\,dW(t)\right],$$

where $\rho(t,s) \triangleq s\gamma(t,s)$. Therefore, $D^x(t) = M(t)\zeta^x(t)$, where

$$\zeta^x(t) = \exp\left[\int_0^t \frac{\partial}{\partial s}\rho\left(u, S^x(u)\right)\,dW(u) - \frac{1}{2}\int_0^t \left(\frac{\partial}{\partial s}\rho\left(u, S^x(u)\right)\right)^2 du\right],$$

is a strictly positive martingale. Define a new probability measure $\mathbb{P}^x$ on $(\Omega, \mathcal{F})$ by $d\mathbb{P}^x/d\mathbb{P} \triangleq \zeta^x(T)$. According to Girsanov's Theorem, under $\mathbb{P}^x$ the process

$$W^x(t) = W(t) - \int_0^t \frac{\partial}{\partial s}\rho\left(u, S^x(u)\right)\,du$$

is a Brownian motion.

Let $x > 0$, $y > 0$ be given with $x \ne y$. We have

$$\frac{S^y(t) - S^x(t)}{M(t)} = y - x + \int_0^t \frac{1}{M(u)}\left[\rho\left(u, S^y(u)\right) - \rho\left(u, S^x(u)\right)\right]\,dW(u),$$

and so $\varphi(t) \triangleq \mathbb{E}[(S^y(t) - S^x(t))/M(t)]^2$ satisfies

$$\begin{aligned}\varphi(t) &\le 2(y-x)^2 + 2\mathbb{E}\left[\int_0^t \frac{1}{M(u)}\left[\rho\left(u, S^y(u)\right) - \rho\left(u, S^x(u)\right)\right]\,dW(u)\right]^2 \\ &\le 2(y-x)^2 + 2K^2\int_0^t \varphi(u)\,du,\end{aligned}$$

where K is a bound on $(\partial/\partial s)\rho$. Gronwall's inequality implies

$$\text{(5.2)} \qquad \begin{aligned} \mathbb{E}\left(S^y(t) - S^x(t)\right)^2 &= M^2(t)\varphi(t) \\ &\leq 2(y-x)^2 M^2(t) e^{2K^2 t}, \qquad 0 \leq t \leq T. \end{aligned}$$

We are now prepared to differentiate v. Let $h'(x-)$ and $h'(x+)$ denote the respective left and right derivatives of the convex function h. These are defined, nondecreasing, and left- and right-continuous, respectively. Let $0 < x < y$ be given, and consider the difference quotient

$$\frac{v(y) - v(x)}{y - x} = \frac{1}{M(T)(y-x)} \mathbb{E}\left[h(S^y(T)) - h\left(S^x(T)\right)\right].$$

Because $x < y$, we must have $S^x(T) \leq S^y(T)$ almost surely. Indeed, if $\tau = T \wedge \inf\{t \geq 0;\, S^x(t) = S^y(t)\}$ were strictly smaller than T, then strong uniqueness for (5.1) would imply $S^x(t) = S^y(t)$ for all $t \in [\tau, T]$. Consequently, the above difference quotient is bounded above by $1/M(T)$ times the expectation of $h'\left(S^y(T)+\right)\left[(S^y(T) - S^x(T))/y - x\right]$, and for $y \in (x, \infty)$, (2.7) and (5.2) show that this is a uniformly integrable collection of random variables. It follows that

$$\begin{aligned} \limsup_{y \downarrow x} \frac{v(y) - v(x)}{y - x} &\leq \mathbb{E}\left[h'\left(S^x(T)+\right)\zeta^x(T)\right] \\ &= \mathbb{E}^x\left[h'\left(S^x(T)+\right)\right], \end{aligned}$$

where $\mathbb{E}^x$ is the expectation corresponding to $\mathbb{P}^x$. Similarly, we have

$$\liminf_{y \downarrow x} \frac{v(y) - v(x)}{y - x} \geq \mathbb{E}^x\left[h'(S^x(T)+)\right].$$

We conclude that $v'(x+) = \mathbb{E}^x\left[h'\left(S^x(T)+\right)\right]$. An analogous argument shows that $v'(x-) = \mathbb{E}^x\left[h'\left(S^x(T)-\right)\right]$.

To remove the dependence on x of the expectation operators in the formulas for $v'(x\pm)$, we rewrite (5.1) as

$$dS^x(t) = S^x(t) r(t)\, dt + \rho\left(t, S^x(t)\right) \frac{\partial}{\partial s}\rho\left(t, S^x(t)\right) dt + \rho\left(t, S^x(t)\right) dW^x(t),$$

a stochastic differential equation for which uniqueness in law holds. Consider $\widetilde{S}^x$ given by

$$d\widetilde{S}^x(t) = \widetilde{S}^x(t) r(t)\, dt + \rho(t, \widetilde{S}^x(t)) \frac{\partial}{\partial s}\rho(t, \widetilde{S}^x(t))\, dt + \rho(t, \widetilde{S}^x(t))\, dW(t),$$

with initial condition $\widetilde{S}^x(0) = x$. The process $\widetilde{S}^x$ has the same distribution under $\mathbb{P}$ as the process S^x under $\mathbb{P}^x$. Thus we may rewrite the formulas for $v'(x\pm)$ as

$$v'(x\pm) = \mathbb{E}\left[h'\left(\widetilde{S}^x(T)\pm\right)\right], \quad x > 0. \tag{5.3}$$

If $0 < x < y$, then by the uniqueness argument used earlier to show that $S^x(T) \le S^y(T)$, we conclude that $\widetilde{S}^x(T) \le \widetilde{S}^y(T)$ almost surely. Because $h'(x\pm)$ is nondecreasing, $v'(x\pm)$ must be as well. □

From (5.3) and (2.7), we have immediately the following corollary.

COROLLARY 5.3. *Under Hypothesis 5.1 and the assumption of a deterministic interest rate, the convex European contingent claim value function v_γ satisfies*

$$|v'_\gamma(x\pm)| \le C \quad \forall x > 0.$$

6. EUROPEAN CONTINGENT CLAIM BOUNDS AND HEDGES

In addition to Hypotheses 2.1 and 5.1, in this section we shall impose the following condition.

HYPOTHESIS 6.1. *The functions $r\colon [0,T] \mapsto \mathbb{R}$ and $\gamma\colon [0,T]\times(0,\infty) \mapsto [0,\infty)$ are Hölder continuous.*

In place of (5.1), we use the notation $S_\gamma^{t,x}$ to denote the misspecified price process when the initial condition is $S_\gamma^{t,x}(t) = x$; that is,

$$dS_\gamma^{t,x}(u) = S_\gamma^{t,x}(u)\left[r(u)\,du + \gamma(u, S_\gamma^{t,x}(u))\,dW(u)\right], \qquad t \le u \le T. \tag{6.1}$$

The misspecified value at time t of the contingent claim with payoff function h is

$$v_\gamma(t,x) = \mathbb{E}e^{-\int_t^T r(u)\,du} h(S_\gamma^{t,x}(T)), \qquad 0 \le t \le T, x > 0. \tag{6.2}$$

Our hypotheses guarantee that v_γ is in $C([0,T]\times(0,\infty)) \cap C^{1,2}([0,T)\times(0,\infty))$ and

$$\mathcal{L}_\gamma v_\gamma(t,x) = 0, \qquad 0 \le t < T, x > 0, \tag{6.3}$$

where

$$\mathcal{L}_\gamma f(t,x) \triangleq r(t)f(t,x) - \frac{\partial}{\partial t}f(t,x) - \frac{1}{2}\gamma^2(t,x)x^2\frac{\partial^2}{\partial x^2}f(t,x) \\ -r(t)x\frac{\partial}{\partial x}f(t,x). \tag{6.4}$$

Furthermore, $(\partial/\partial x)v_\gamma$ is bounded (Corollary 5.3).

We now consider a hedger who believes the stock price dynamics are given by (6.1), when in fact the true stock price dynamics are given by (2.2) with the true volatility $\sigma(\cdot)$ satisfying the conditions of Hypothesis 2.1. Observing the true stock price $S(t)$, the hedger (incorrectly) computes the contingent claim price to be $P_\gamma(t) \stackrel{\Delta}{=} v_\gamma(t, S(t))$ and uses the hedging portfolio $\Delta_\gamma(t) \stackrel{\Delta}{=} (\partial/\partial x)v_\gamma(t, S(t))$. Beginning with the initial value $v_\gamma(0, S(0))$, this hedger's self-financing portfolio value will evolve according to the formula (cf. (2.5)):

$$d\Pi_{\Delta_\gamma}(t) = r(t)\Pi_{\Delta_\gamma}(t)\,dt + \Delta_\gamma(t)[dS(t) - r(t)S(t)\,dt], \tag{6.5}$$

whereas Itô's rule and (6.4) show that the (incorrectly) computed value of the contingent claim is governed by

$$\begin{aligned} dP_\gamma(t) &= r(t)P_\gamma(t)\,dt + \Delta_\gamma(t)[dS(t) - r(t)S(t)\,dt] \\ &\quad + \frac{1}{2}\left[\sigma^2(t) - \gamma^2(t, S(t))\right] S^2(t)\frac{\partial^2}{\partial x^2}v_\gamma(t, S(t))\,dt. \end{aligned} \tag{6.6}$$

The tracking error $e_\gamma(t) \stackrel{\Delta}{=} \Pi_{\Delta_\gamma}(t) - P_\gamma(t)$ is thus given by

$$e_\gamma(t) = \frac{1}{2}M(t)\int_0^t \frac{1}{M(u)}\left[\gamma^2(u, S(u)) - \sigma^2(u)\right] S^2(u)\frac{\partial^2}{\partial x^2}v_\gamma(u, S(u))\,du. \tag{6.7}$$

THEOREM 6.2. *Assume Hypotheses 2.1, 5.1, and 6.1. If*

$$\sigma(t) \le \gamma(t, S(t)) \tag{6.8}$$

for Lebesgue–almost all $t \in [0, T]$, almost surely, then $(P_\gamma, \Delta_\gamma)$ is a superstrategy, and

$$\Pi_{\Delta_\gamma}(T) \ge h(S(T)), \tag{6.9}$$

$$v_\gamma(0, S(0)) \ge \mathbb{E}[h(S(T))/M(T)]. \tag{6.10}$$

If

$$\sigma(t) \ge \gamma(t, S(t)) \tag{6.11}$$

for Lebesgue–almost all $t \in [0, T]$, almost surely, then $(P_\gamma, \Delta_\gamma)$ is a substrategy and inequalities (6.9) and (6.10) are reversed.

Proof. It is clear from the definitions that $P_\gamma(T) = v_\gamma(T, S(T)) = h(S(T))$. The theorem follows immediately from (6.7) and the convexity of $v_\gamma(u, \cdot)$ (Theorem 5.2). Inequalities (6.9) and (6.10) are restatements of (2.10) and (2.12). □

REMARK 6.3. If v is not convex or if neither (6.8) nor (6.11) is almost everywhere satisfied, (6.7) is still valid and the tracking error is a finite-variation process. If both $\sigma(t)$ and $\gamma(t)$ are deterministic and we replace (6.8) by the weaker integrated volatility condition

$$\int_0^T \sigma^2(t)\,dt \le \int_0^T \gamma^2(t)\,dt,$$

then (6.10) holds but (6.9) can fail. This is discussed more fully in Appendix B.

7. STOCHASTIC INTEREST RATE AND CHANGE OF NUMÉRAIRE

In this section only, we relax Hypothesis 2.1 by assuming that the interest rate r is a stochastic process. More precisely, in place of Hypothesis 2.1, in this section we shall assume the following.

HYPOTHESIS 7.1. *The money market price M and the stock price S are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to a filtration $\{\mathcal{F}(t); 0 \le t \le T\}$. Furthermoree,*

$$M(t) = e^{\int_0^t r(u)\,du}, \tag{7.1}$$

$$dS(t) = S(t)[r(t)\,dt + \sigma(t)\,dW(t)], \tag{7.2}$$

where $\{W(t); 0 \le t \le T\}$ is a one-dimensional Brownian motion adapted to $\{\mathcal{F}(t); 0 \le t \le T\}$, where the interest rate process r *is adapted to $\{\mathcal{F}(t); 0 \le t \le T\}$ and satisfies $\int_0^T |r(t)|\,dt < \infty$ almost surely, and where the* volatility process σ *is nonnegative, adapted to $\{\mathcal{F}(t); 0 \le t \le T\}$, and satisfies $\int_0^T \sigma^2(t)\,dt < \infty$ almost surely.*

We define the positive martingale

$$Z(t) \triangleq \frac{\mathbb{E}\left[e^{-\int_0^T r(u)\,du} \,\middle|\, \mathcal{F}(t)\right]}{\mathbb{E}e^{-\int_0^T r(u)\,du}}, \qquad 0 \le t \le T.$$

We can use Z to change from $\mathbb{P}$ to an equivalent probability measure $\widehat{\mathbb{P}}$ according to the formula

$$\widehat{\mathbb{P}}(A) \triangleq \int_A Z(t)\,d\mathbb{P} \qquad \forall A \in \mathcal{F}(t), \quad 0 \le t \le T.$$

We call $\widehat{P}$ the *forward measure*. We also define the *zero-coupon bond price process*

$$B(t) \triangleq \mathbb{E}\left[e^{-\int_t^T r(u)\,du} \,|\mathcal{F}(t)\right], \qquad 0 \le t \le T,$$

which is related to Z by the formula

$$Z(t) = \frac{B(t)}{B(0)M(t)}, \qquad 0 \le t \le T.$$

Because $S(t)/M(t)$ is a martingale, we have immediately the following result.

LEMMA 7.2. *Under the forward measure $\widehat{P}$, the forward stock price process*

$$\widehat{S}(t) \triangleq \frac{S(t)}{B(t)}, \qquad 0 \le t \le T,$$

is a martingale relative to the filtration $\{\mathcal{F}(t); 0 \le t \le T\}$.

HYPOTHESIS 7.3. *The bond price process B (and hence the forward stock price process $\widehat{S}$) is continuous. Furthermore, the forward stock price process $\widehat{S}$ is square integrable, and its quadratic variation $\langle \widehat{S} \rangle$ is absolutely continuous with respect to Lebesgue measure; that is,*

$$\langle \widehat{S} \rangle(t) = \int_0^t \frac{\partial}{\partial u} \langle \widehat{S} \rangle(u)\, du, \qquad 0 \le t \le T. \tag{7.3}$$

REMARK 7.4. If the filtration $\{\mathcal{F}(t); 0 \le t \le T\}$ is the augmentation by $\mathbb{P}$-null sets of the filtration generated by a one- or multidimensional Brownian motion, then $\widehat{S}$ is continuous and equation (7.3) holds. This is because S/M and Z are local martingales under $\mathbb{P}$, and any local martingale relative to the filtration generated by a Brownian motion has a stochastic integral representation with respect to that Brownian motion. From this, a representation of $\widehat{S}(t) = S(t)/(B(0)Z(t)M(t))$ can be obtained as the sum of a Lebesgue integral and an Itô integral whose integrator is the Brownian motion generating the filtration. The continuity of $\widehat{S}$ follows, as does the absolute continuity of the quadratic variation of the local $\mathbb{P}$-martingale part of $\widehat{S}$. The change to $\widehat{\mathbb{P}}$ does not affect this quadratic variation.

In addition to regarding $\widehat{S}(t)$ as the forward price of the stock, we can interpret it as the price of the stock denominated in units of the zero-coupon bond. Making this change of numéraire, we show below that we can reduce the present situation to the case of zero interest rate. For a more detailed exposition of change of numéraire, introduced in a particular case by Merton (1973), one can consult Jamshidian (1989) and El Karoui, Geman, and Rochet (1995).

We set

$$\widehat{\sigma}(t) \triangleq \frac{1}{\widehat{S}(t)} \sqrt{\frac{d}{dt} \langle \widehat{S} \rangle(t)}.$$

From Hypothesis 7.3, on an enlarged probability space there is a Brownian motion $\widehat{W}$ such that

$$d\widehat{S}(t) = \widehat{S}(t)\widehat{\sigma}(t)\, d\widehat{W}(t) \tag{7.4}$$

(see Ikdea and Watanabe 1981, Chap. II, Thm. 7.1′, or Karatzas and Shreve 1991, Chap. 3, Thm. 4.2).

As the risk-free investment instrument, we take the zero-coupon bond denominated in units of the zero-coupon bond. The price of this instrument is always 1, which corresponds to an identically zero interest rate. Because this instrument is available, the probability measure $\widehat{\mathbb{P}}$ which renders $\widehat{W}$ in (7.4) into a Brownian motion is a "risk-neutral" probability measure under the new numéraire.

Consider now a hedger who invests in the stock and the zero-coupon bond. Suppose he (incorrectly) believes that when the stock is denominated in units of the zero-coupon bond, its volatility at time t is $\gamma(t, \widehat{S}(t))$, where $\gamma\colon [0,T]\times(0,\infty)\mapsto[0,\infty)$ satisfies Hypotheses 5.1 and 6.1. In other words, he believes the stock price process, denominated in units of the zero-coupon bond, is given by

$$d\widehat{S}_{\gamma}^{t,x}(u) = \widehat{S}_{\gamma}^{t,x}(u)\gamma(u, \widehat{S}_{\gamma}^{t,x}(u))\,d\widehat{W}(u), \qquad t \le u \le T, \tag{7.5}$$

with $\widehat{S}_{\gamma}^{t,x}(t) = x$. Observing $\widehat{S}(t)$ at time t, this hedger would compute the value, denominated in terms of the zero-coupon bond, of the European contingent claim to be $\widehat{P}_{\gamma}(t) \triangleq \widehat{v}_{\gamma}(t, \widehat{S}(t))$, where

$$\widehat{v}_{\gamma}(t, x) \triangleq \widehat{\mathbb{E}}h(T, \widehat{S}_{\gamma}^{t,x}(T)). \tag{7.6}$$

Here we are using the fact that $B(T) = 1$, and so $\widehat{S}(T) = S(T)$, the actual stock price at time T. Believing that $\widehat{S}(T) = \widehat{S}_{\gamma}^{t,x}(T)$, the hedger uses the latter random variable on the right-hand side of (7.6). The hedger holds $\widehat{\Delta}_{\gamma}(t) \triangleq (\partial/\partial x)\widehat{v}(t, \widehat{S}(t))$ units of stock, where each unit has value $B(t)$, and starting from $\widehat{\Pi}_{\Delta_{\gamma}}(0) = \widehat{v}_{\gamma}(0, \widehat{S}(0))$ units, this generates the self-financing portfolio value (denominated in units of zero-coupon bond) of (cf. (2.6))

$$\widehat{\Pi}_{\widehat{\Delta}_{\gamma}}(t) = \widehat{v}_{\gamma}(0, \widehat{S}(0)) + \int_0^t \widehat{\Delta}_{\gamma}(u)\,d\widehat{S}(u).$$

The tracking error is $\widehat{e}_{\gamma}(t) = \widehat{\Pi}_{\Delta_{\gamma}}(t) - \widehat{P}_{\gamma}(t)$. We have the following corollary to Theorem 6.2.

THEOREM 7.5. *Assume Hypotheses 5.1, 7.1, and 7.3 and assume that γ is Hölder continuous. We have $\widehat{P}_{\gamma}(T) = h(S(T))$ almost surely. If $\widehat{\sigma}(t) \le \gamma(t, \widehat{S}(t))$ for Lebesgue–almost all $t \in [0, T]$, almost surely, then the tracking error is nondecreasing. If $\widehat{\sigma}(t) \ge \gamma(t, \widehat{S}(t))$ for Lebesgue–almost all $t \in [0, T]$, almost surely, then the tracking error is nonincreasing.*

REMARK 7.6. The function $\widehat{v}_{\gamma}$ of (7.6) solves the heat equation

$$-\frac{\partial}{\partial t}\widehat{v}_{\gamma}(t, x) = \frac{1}{2}\gamma^2(t, x)x^2\frac{\partial^2}{\partial x^2}\widehat{v}_{\gamma}(t, x), \qquad 0 \le t \le T,\ x > 0,$$

obtained by setting $r \equiv 0$ in (6.3). If γ is a function of t alone, then

$$\widehat{v}_\gamma(t,x) = \mathcal{BS}(T-t, x; 0, \Gamma(t)/\sqrt{T-t}),$$

where $\Gamma(t) \triangleq (\int_t^T \gamma^2(u)\,du)^{1/2}$.

8. SUFFICIENT CONDITIONS FOR REDUCTION OF AMERICAN TO EUROPEAN CONTINGENT CLAIMS

We next turn our attention to American contingent claims. In this section, we note that for a large class of American contingent claims, the American pricing and hedging problem reduces to the European pricing and hedging problem. In Sections 9 and 10 we take up the task of obtaining pricing and hedging bounds on American contingent claims that do not reduce to European ones.

Still working under Hypothesis 2.1, we recall that the *Snell envelope* of $\{h(S(t))/M(t);\ 0 \le t \le T\}$ is the smallest supermartingale that dominates this process, and is given by

$$\operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\frac{h(S(\tau))}{M(\tau)} \,\middle|\, \mathcal{F}(t) \right], \qquad 0 \le t \le T,$$

where $\mathcal{T}_t$ denotes the set of stopping times τ satisfying $t \le \tau \le T$ almost surely. In a complete market, the arbitrage price process P_A for the American contingent claim with payoff function h is (from Bensoussan 1984; Karatzas 1988; Myneni 1992; Kramkov and Vishnyakov 1994):

$$P_A(t) \triangleq M(t) \cdot \operatorname{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\frac{h(S(\tau))}{M(\tau)} \,\middle|\, \mathcal{F}(t) \right], \qquad 0 \le t \le T. \tag{8.1}$$

An optimal exercise time is

$$D \triangleq \inf\{t \in [0,T];\ P_A(t) = h(S(t))\}, \tag{8.2}$$

and the process $\{P_A(t \wedge D)/M(t \wedge D), 0 \le t \le T\}$ is a martingale.

Under certain conditions,

$$P_A(t) = M(t)\,\mathbb{E}\left[\frac{h(S(T))}{M(T)} \,\middle|\, \mathcal{F}(t) \right], \qquad 0 \le t \le T,$$

and the price processes for the European and American contingent claims agree. We give a sufficient condition for this, which includes, of course, the case $h(x) = (x-K)^+$ of the call option.

THEOREM 8.1. *Assume Hypothesis 2.1. In a complete market, if $r(t) \ge 0$ for all $t \in [0,T]$ and $h(0) = 0$, then the European and American contingent claim price processes coincide.*

Proof. We show that $h(S)/M$ is a submartingale, and the Optional Sampling Theorem applied to (8.1) yields the result. For any $\alpha \geq 1,\ x > 0$, the convexity of h implies

$$h(x) \leq \frac{1}{\alpha} h(\alpha x) + \frac{\alpha - 1}{\alpha} h(0) = \frac{1}{\alpha} h(\alpha x).$$

Therefore, for $0 \leq u < t \leq T$,

$$\begin{aligned} h(S(u)) &\leq e^{-\int_u^t r(y)\,dy} h\left(e^{\int_u^t r(y)\,dy} S(u)\right) \\ &= \frac{M(u)}{M(t)} h\left(M(t)\mathbb{E}\left[\frac{S(t)}{M(t)}\,\middle|\,\mathcal{F}(u)\right]\right) \\ &= \frac{M(u)}{M(t)} h(\mathbb{E}[S(t)|\mathcal{F}(u)]) \\ &\leq \frac{M(u)}{M(t)} \mathbb{E}[h(S(t))|\mathcal{F}(u)], \end{aligned} \tag{8.3}$$

where the last step is Jensen's inequality. □

REMARK 8.2. If h is not linear (e.g., $h(x) = (x - K)^+$) and the stock price has support on all of $(0, \infty)$ (e.g., geometric Brownian motion), then the Jensen inequality in (8.3) is strict. This implies that for all $u \in [0, T)$,

$$P_A(u) = M(u)\,\mathbb{E}\left[\frac{h(S(T))}{M(T)}\,\middle|\,\mathcal{F}(u)\right] > h(S(u)),$$

so that $D = T$ almost surely.

9. CONVEXITY OF AMERICAN CONTINGENT CLAIM VALUES

In addition to Hypothesis 2.1, in this section we assume Hypothesis 5.1 and the following.

HYPOTHESIS 9.1. *The interest rate r is nonnegative and the payoff function h is bounded from below.*

Because of Hypothesis 5.1, the misspecified stock price given by (5.1) is Markov, a fact we exploit below. Because the misspecified volatility is a function only of time and the misspecified stock price process, the market in the misspecified stock price process is complete. Given a time $t \in [0, T]$ and a stock price $x > 0$, the (misspecified) value of the American contingent claim with payoff h is

$$v_\gamma(t, x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-\int_t^\tau r(y)\,dy} h(S_\gamma^{t,x}(\tau))\right], \qquad 0 \leq t \leq T,\ x > 0, \tag{9.1}$$

where we have used the notation of (6.1). For fixed (t, x), the process

$$(9.2) \qquad e^{-\int_t^u r(y)\,dy}\, v_\gamma(u, S_\gamma^{t,x}(u)), \qquad t \le u \le T,$$

is the smallest supermartingale that dominates

$$e^{-\int_t^u r(y)\,dy}\, h(S_\gamma^{t,x}(u)), \qquad t \le u \le T.$$

This assertion is just the claim made in the second paragraph of Section 8, except there the initial time was zero and here it is t.

We construct v_γ by an iterative process found in El Karoui (1981) and adapted to the situation at hand. Let $g\colon [0, T] \times (0, \infty) \mapsto \mathbb{R}$ be any function satisfying the following four conditions.

Conditions 9.2.

(a) g is jointly Borel measurable;
(b) there exist positive constants C_1 and C_2 such that

$$-C_1 \le g(t, x) \le C_1 + C_2 x, \quad \forall x > 0;$$

(c) g is *lower semicontinuous from the right*; that is, for every $(t, x) \in [0, T] \times (0, \infty)$ and every sequence $\{(t_n, x_n)\}_{n=1}^{\infty} \subset [0, T] \times (0, \infty)$ with $t_n \downarrow t$ and $x_n \to x$, we have

$$g(t, x) \le \liminf_{n\to\infty} g(t_n, x_n)\,;$$

(d) for every $t \in [0, T]$, the function $g(t, x)$ is convex in x.

For g satisfying Conditions 9.2 and for $u \in [0, T]$, we define

$$(L_u g)(t, x) = e^{-\int_t^u r(y)\,dy} \mathbb{E} g(u, S_\gamma^{t,x}(u)), \qquad 0 \le t \le u, x > 0.$$

We also define the operator K by

$$(9.3) \qquad (Kg)(t, x) = \sup_{u\in[t,T]} (L_u g)(t, x), \qquad 0 \le t \le T, x > 0.$$

LEMMA 9.3. *If g satisfies Conditions 9.2, then Kg does also.*

Proof. According to the theory of stochastic flows (e.g., Doss 1977; Kunita 1990), we may choose versions of the processes $(S_\gamma^{t,x}(u)\,;\ t \le u \le T)$ so that for each u and $\omega \in \Omega$, the mapping $(t, x) \mapsto S_\gamma^{t,x}(u)$ is continuous. This, combined with Fatou's Lemma and properties (b) and (c) for g, shows that the mapping $u \mapsto (L_u g)(t,x)$ is lower semicontinuous from the right. Therefore, without loss of generality we may restrict the supremum in (9.3) to rational $u \in [0,T]$. This shows that Kg is Borel measurable.

From (b) and the supermartingale property for $(e^{-\int_t^u r(y)\,dy} S_\gamma^{t,x}(u)\,;\ t \le u \le T)$, we see that $L_u g$ satisfies (b), and hence Kg does as well. The convexity of $(L_u g)(t,\cdot)$ follows from Theorem 5.2. The supremum of convex functions is convex, which gives us the convexity of Kg.

It remains to show that Kg is lower semicontinuous from the right. Let us fix $(t,x) \in [0,T) \times (0,\infty)$ and let $t_n \downarrow t$, $x_n \to x$. If $(Kg)(t,x) = g(t,x)$, then

$$(Kg)(t,x) = g(t,x) \le \liminf_{n\to\infty} g(t_n,x_n) \le \liminf_{n\to\infty} (Kg)(t_n,x_n).$$

On the other hand, if $Kg(t,x) > g(t,x)$, then for each $\epsilon > 0$, there exists $u \in (t,T]$ such that $(Kg)(t,x) - \epsilon \le (L_u g)(t,x)$. For large enough n, we have $u \in [t_n,T]$, and so

$$(L_u g)(t,x) \ \le\ \liminf_{n\to\infty} (L_u g)(t_n,x_n) \le \liminf_{n\to\infty} (Kg)(t_n,x_n). \qquad \square$$

We now take $g(t,x) = h(x)$ for all $t \in [0,T]$, $x \in [0,\infty)$. Because $Kg \ge g$ for any g, we have $K^{n+1}h \ge K^n h$, where K^n denotes the n-fold iterate of K. We can thus define

$$w \triangleq \lim_{n\to\infty} K^n h = \sup_n K^n h.$$

It is easily verified that w satisfies Conditions 9.2.

THEOREM 9.4. *The function w is the smallest fixed point of K dominating h. Moreover, w is the function v_γ defined by (9.1).*

Proof. We have $w \ge K^{n+1}(w) = K(K^n w)$. Letting $n \to \infty$, we obtain $w \ge Kw$. The reverse inequality is trivial.

If u is a fixed point of K dominating h, then $u = K^n u \ge K^n h$. Letting $n \to \infty$, we obtain $u \ge w$.

Fix (t,x) and consider $X(u) = e^{-\int_t^u r(y)\,dy}\, w(u, S_\gamma^{t,x}(u))$. For $t \le u_1 \le u_2 \le T$, we have

$$\begin{aligned}
\mathbb{E}[X(u_2)|\mathcal{F}(u_1)] &= e^{-\int_t^{u_1} r(y)\,dy}\mathbb{E}\left[e^{-\int_{u_1}^{u_2} r(y)\,dy}\, w(u_2, S_\gamma^{t,x}(u_2)) \,\middle|\, \mathcal{F}(u_1)\right] \\
&= e^{-\int_t^{u_1} r(y)\,dy} (L_{u_2} w)(u_1, S_\gamma^{t,x}(u_1)) \\
&\le e^{-\int_t^{u_1} r(y)\,dy} (Kw)(u_1, S_\gamma^{t,x}(u_1)) \\
&= X(u_1).
\end{aligned}$$

Thus, X is a supermartingale dominating $e^{-\int_t^u r(y)\,dy}\, h(S_\gamma^{t,x}(u))$, and so must dominate $e^{-\int_t^u r(y)\,dy}\, v_\gamma(u, S_\gamma^{t,x}(u))$ as well. In particular, $w(t,x) = X(0) \ge v_\gamma(t,x)$.

For the reverse inequality, we observe from the supermartingale property for

$$e^{-\int_t^u r(y)\,dy} v_\gamma(u, S^{t,x}(u))$$

that $(L_u v_\gamma)(t,x) \le v_\gamma(t,x)$, and hence $K v_\gamma \le v_\gamma$. Therefore, v_γ is a fixed point of K, and being a fixed point of K, v_γ must dominate w. □

COROLLARY 9.5. *Under Hypotheses 2.1, 5.1, and 9.1, the American contingent claim value v_γ of (9.1) is convex.*

10. AMERICAN CONTINGENT CLAIM BOUNDS AND HEDGES

Now let $\gamma: [0,T] \times (0,\infty) \mapsto [0,\infty)$ satisfy Hypothesis 5.1. A hedger who believes the stock price volatility is given by γ will price and hedge the American contingent claim as described in Section 6, but now with

$$v_\gamma(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-\int_t^\tau r(y)\,dy} h(S_\gamma^{t,x}(\tau)) \right]. \tag{10.1}$$

If the hedger is short the contingent claim, then he does not know when it will be exercised and must be prepared to hedge all the way to time T. If the hedger is long the contingent claim, he will exercise it at time

$$D_\gamma = \inf\{t \in [0,T];\ v_\gamma(t, S(t)) = h(S(t))\}, \tag{10.2}$$

and so only needs to hedge until this time. These observations motivate the following definition.

DEFINITION 10.1. Let h be a payoff function. A *price process* for the American contingent claim with payoff function h is any adapted process $\{P(t); 0 \le t \le T\}$ satisfying $P(t) \ge h(S(t))$, $0 \le t \le T$, and $P(T) = h(S(T))$, almost surely. Let P be a price process and Δ a portfolio process. The *tracking error* associated with (P, Δ) is defined to be the process $e(t) \triangleq \Pi_\Delta(t) - P(t)$, where Π_Δ and P are related by the initial condition $\Pi_\Delta(0) = P(0)$. We say that (P, Δ) is a *superstrategy* for the American option if the discounted tracking error $e(t)/M(t)$ is nondecreasing for $0 \le t \le T$. We say that (P, Δ) is a *substrategy* for the American option if the discounted tracking error is nonincreasing for $0 \le t \le D$, where

$$D \triangleq T \wedge \inf\{t \in [0,T]; P(t) = h(S(t))\}.$$

Under suitable regularity conditions, the function v_γ of (10.1) is characterized by the variational inequality

$$\min\{\mathcal{L}_\gamma v_\gamma(t,x),\ v_\gamma(t,x) - h(x)\} = 0, \qquad 0 \le t \le T, x > 0, \tag{10.3}$$

$$v_\gamma(T,x) = h(x), \quad x > 0,$$

where $\mathcal{L}_\gamma f$ is defined by (6.4). Moreover, $[0,T]\times(0,\infty)$ separates into two regions:

$$\mathcal{C}_\gamma = \{(t,x)\,;\, v_\gamma(t,x) > h(x)\}, \qquad \mathcal{E}_\gamma = \{(t,x)\,;\, v_\gamma(t,x) = h(x)\}.$$

Under Hypothesis 5.1 on γ, the strict ellipticity of $\mathcal{L}_\gamma$ implies that v_γ is $C^{1,2}$ in $\mathcal{C}_\gamma$. There can be points a of discontinuity of h', but in such a case, the segment $[0,T]\times\{a\}$ must lie in $\mathcal{C}_\gamma$; if it did not, Meyer's (1976) convex function extension of Itô's formula applied to the supermartingale $e^{-\int_0^t r(u)du} v_\gamma(t, S_\gamma(t))$ would produce a singularly continuous, strictly increasing local time term, which would violate the supermartingale property for the process (9.2). Therefore, v_γ is also C^1 in $\mathcal{E}_\gamma$. Finally, it is generally the case that v_γ is C^1 across the boundary between $\mathcal{C}_\gamma$ and $\mathcal{E}_\gamma$, a property known as the "principle of smooth fit." Rather than undertake a more technical discussion of the regularity of v_γ, we shall simply assume what we need.

HYPOTHESIS 10.2. *The function v_γ is C^1 on $[0,T]\times(0,\infty)$, $(\partial^2/\partial x^2)v_\gamma(t,x)$ is piecewise continuous in x for each $t \in [0,T]$, and is bounded uniformly in $(t,x) \in [0,T]\times(0,\infty)$, and v_γ satisfies (10.3) everywhere $\mathcal{L}_\gamma v_\gamma$ is defined.*

Hypothesis 10.2 permits the application of Itô's rule to v_γ. One can show this by modifying v_γ to obtain a smooth function, applying Itô's rule to this smooth function, and then passing to the limit.

THEOREM 10.3. *Assume Hypotheses 2.1, 5.1, 9.1, and 10.2. Let $P_\gamma(t) = v_\gamma(t, S(t))$ and $\Delta_\gamma(t) = (\partial/\partial x)v_\gamma(t, S(t))$ be as in Section 6, but with v_γ now defined by (10.1). If*

$$\sigma(t) \le \gamma(t, S(t)), \qquad 0 \le t \le T, \text{ a.s.,}$$

then $(P_\gamma, \Delta_\gamma)$ is a superstrategy for the American option. If

$$\sigma(t) \ge \gamma(t, S(t)), \qquad 0 \le t \le D, \text{ a.s.,}$$

then $(P_\gamma, \Delta_\gamma)$ is a substrategy for the American option.

Proof. From Itô's rule and (6.5), we have

$$\begin{aligned} de_\gamma(t) \;=\; & \left[r(t)e_\gamma(t) + \mathcal{L}_\gamma v_\gamma(t, S(t))\right] dt \\ & + \frac{1}{2}\left[\gamma^2(t,S(t)) - \sigma^2(t)\right] S^2(t)\frac{\partial^2}{\partial x^2} v_\gamma(t, S(t))\, dt. \end{aligned}$$

If $\gamma \ge \sigma$, then Corollary 9.5 and the inequality $\mathcal{L}_\gamma v \ge 0$ from (10.3) imply that the discounted tracking error

$$\frac{e(t)}{M(t)} = \int_0^t \frac{1}{M(u)} \mathcal{L}_\gamma v_\gamma(u, S(u))\,du + \frac{1}{2}\int_0^t \frac{1}{M(u)} \left[\gamma^2(u, S(u)) - \sigma^2(u)\right] S^2(u) \frac{\partial^2}{\partial x^2} v_\gamma(u, S(u))\,du$$

is nondecreasing. If $\gamma \leq \sigma$, we use the fact that $\mathcal{L}_\gamma v_\gamma(u, S(u)) = 0$ for $0 \leq u \leq D$ to show that $e(t)/M(t)$ is nonincreasing for $0 \leq t \leq D$. □

11. EXAMPLES

11.1. Arithmetic Mean

Suppose we have a complete market with two risky assets, whose dynamics are

$$dS_i(t) = S_i(t)[r\,dt + \sigma_i\,dW_i(t)], \qquad i = 1, 2,$$

where W_1 and W_2 are independent Brownian motions, and where the interest rate r and the volatilities $\sigma_1 > 0$ and $\sigma_2 > 0$ are constant. The arithmetic mean $S_3(t) \triangleq (S_1(t) + S_2(t))/2$ satisfies

$$dS_3(t) = rS_3(t)\,dt + \frac{1}{2}[\sigma_1 S_1(t)\,dW_1(t) + \sigma_2 S_2(t)\,dW_2(t)].$$

Defining a new Brownian motion

$$W_3(t) = \int_0^t \frac{\sigma_1 S_1(u)\,dW_1(u) + \sigma_2 S_2(u)\,dW_2(u)}{\sqrt{\sigma_1^2 S_1^2(u) + \sigma_2^2 S_2^2(u)}},$$

and setting

$$\sigma(t) = \frac{\sqrt{\sigma_1^2 S_1^2(t) + \sigma_2^2 S_2^2(t)}}{S_1(t) + S_2(t)},$$

we may rewrite this in the usual form

$$dS_3(t) = S_3(t)\,[r\,dt + \sigma(t)\,dW_3(t)],$$

but with stochastic volatility. Indeed, $\sigma(t)$ is not even a function of $S_3(t)$.

Consider a European call on S_3 with expiration time T and strike price K. The arbitrage price of this call at time zero is $P_E(0) = \mathbb{E}[e^{-rT}(S_3(T) - K)^+]$, a quantity that is difficult to compute. However, with $\alpha = \sigma_1\sigma_2/\sqrt{\sigma_1^2 + \sigma_2^2}$, and $\beta = \sigma_1 \vee \sigma_2$, we have $\alpha \leq \sigma(t) \leq \beta$ for all $t \in [0, T]$, almost surely. Theorem 6.2 implies

$$\mathcal{BS}(T, S_3(0); r, \alpha) \leq P(0) \leq \mathcal{BS}(T, S_3(0); r, \beta),$$

where explicit formulas for the bounds are provided in Remark 3.1. Moreover, a hedger who sells the option for $\mathcal{BS}(T, S_3(0); r, \beta)$ and uses the hedging portfolio

$$\Delta_\beta(t) = (\partial/\partial x)\mathcal{BS}(T - t, S_3(t); r, \beta)$$

is guaranteed to have at least $(S_3(T) - K)^+$ at time T and to be overhedged at each time $t \in [0, T)$. A hedger who borrows $\mathcal{BS}(T, S_3(0); r, \alpha)$ to buy the option and uses the hedging portfolio $\Delta_\alpha(t) = -(\partial/\partial x)\mathcal{BS}(T - t, S_3(t); r, \alpha)$ is guaranteed to have accumulated no more than $(S_3(T) - K)^+$ in debt at time T.

One could also produce a lower bound on the option by noting that the geometric mean $S_4(t) = \sqrt{S_1(t)S_2(t)}$ always lies below S_3, and so

$$\mathbb{E}\left[e^{-rT}(S_4(T) - K)^+\right] \le P_E(0).$$

Furthermore, S_4 is a geometric Brownian motion, satisfying

$$dS_4(t) = S_4(t)\left[\left(r - \frac{\sigma_1^2 + \sigma_2^2}{8}\right) dt + \frac{1}{2}\sqrt{\sigma_1^2 + \sigma_2^2}\, dW_4(t)\right],$$

where W_4 is the Brownian motion

$$W_4(t) = \int_0^t \frac{\sigma_1\, dW_1(t) + \sigma_2\, dW_2(t)}{\sqrt{\sigma_1^2 + \sigma_2^2}}.$$

Now $\mathbb{E}[e^{-rT}(S_4(T) - K)^+]$ is not a Black–Scholes price because S_4 does not have mean rate of return r under $\mathbb{P}$. However, setting $\mu = r - (\sigma_1^2 + \sigma_2^2)/8$, we may use the Black–Scholes formula to compute

$$\begin{aligned}\mathbb{E}\left[e^{-rT}(S_4(T) - K)^+\right] &= e^{(\mu - r)T}\mathbb{E}\left[e^{-\mu T}(S_4(T) - K)^+\right]\\ &= e^{(\mu - r)T}\mathcal{BS}(T, S_4(0); \mu, \alpha).\end{aligned}$$

Because the volatilities of S_3 and S_4 are noncomparable, we can make no claim concerning investment in S_3 according to the hedging strategy

$$\Delta(t) = e^{(\mu - r)t}\frac{\partial}{\partial x}\mathcal{BS}(T - t, S(t); \mu, \alpha)$$

derived from S_4.

11.2. Asian Options

An Asian option pays off a time-average of the stock price. We consider Asian options which, at time T, pay

$$\left(\frac{1}{\theta}\int_{T-\theta}^{T} S(u)\,du - K\right)^{+},$$

where $K > 0$ and $\theta \in (0, T]$. We take S to be a geometric Brownian motion; that is,

$$dS(t) = S(t)\,[r\,dt + \sigma\,dW(t)],$$

where the interest rate r and the volatility $\sigma > 0$ are constant. The arbitrage price process for an Asian option is

$$P_{As}(t) = \mathbb{E}\left[e^{-r(T-t)} \left(\frac{1}{\theta}\int_{T-\theta}^{T} S(u)\,du - K\right)^{+} \middle| \mathcal{F}(t)\right], \qquad 0 \le t \le T.$$

This quantity can be explicitly computed (Yor 1992), but is quite complicated. Geman and Yor (1992, 1993) compute the moments of all orders of $(1/\theta)\int_{T-\theta}^{T} S(u)\,du$ and obtain the Laplace transform of the Asian option price process. Kemna and Vorst (1992) in a discrete-time case and Bouaziz, Bryis, and Crouhy (1994) in a continuous-time case provide bounds for the price of an Asian option, but do not consider a hedging strategy.

We apply the methodology of this paper to give a price bound and properties of the associated hedging strategy. The key idea is to find another underlying asset whose price at time T is $(1/\theta)\int_{T-\theta}^{T} S(u)\,du$.

LEMMA 11.1. *The process* $X(t) = e^{-r(T-t)}\mathbb{E}[(1/\theta)\int_{T-\theta}^{T} S(u)\,du \mid \mathcal{F}(t)]$ *is given by*

$$X(t) = \rho(t)S(t) + \frac{e^{-r(T-t)}}{\theta}\int_{T-\theta}^{t\vee(T-\theta)} S(u)\,du, \qquad 0 \le t \le T\,, \tag{11.1}$$

where

$$\rho(t) = \frac{1 - e^{-rT + r(t\vee(T-\theta))}}{r\theta}.$$

This process satisfies the stochastic differential equation

$$dX(t) = rX(t)\,dt + \sigma\rho(t)S(t)\,dW(t). \tag{11.2}$$

Proof. Fubini's Theorem implies $X(t) = (1/\theta)\int_{T-\theta}^{T}\mathbb{E}[S(u)|\mathcal{F}(t)]\,du$. For $t \le u$, $\mathbb{E}[S(u)|\mathcal{F}(t)] = e^{r(u-t)}S(t)$, whereas for $t \ge u$, $\mathbb{E}[S(u)|\mathcal{F}(t)] = S(u)$. The lemma follows by straightforward computation. □

Rewriting (11.2) as

$$dX(t) = X(t)\left[r\,dt + \frac{\sigma S(t)\rho(t)}{X(t)}\,dW(t)\right],$$

we identify the volatility of X as $[\sigma S(t)\rho(t)]/X(t)$, a positive quantity that is dominated by σ because of (11.1). Theorem 6.2 (with $\sigma(t) = \sigma S(t)\rho(t)/X(t)$ and $\gamma(t,x) \equiv \sigma$) implies

$$P_{As}(0) = \mathbb{E}[e^{-r(T-t)}(X(T) - K)^+] \le \mathcal{BS}(T, X(0); r, \sigma).$$

We thus obtain the continuous-time analogue of the discrete-time result of Kemna and Vorst (1992) that the price of an Asian option is dominated by a Black–Scholes price. The continuous-time result was also obtained by Geman and Yor (1993), who established the necessity of an assumption on the risk-neutral drift for the property to be true. Note, however, that $X(0) = \rho(0)S(0) = (1/r\theta)(1 - e^{-r\theta})S(0)$.

Theorem 6.2 also leads to a hedging strategy, as we now explain.

THEOREM 11.2. *For the Asian option, the pair of processes*

$$\begin{aligned} P^*(t) &= \mathcal{BS}(T-t, X(t);\ r, \sigma), \\ \Delta^*(t) &= \rho(t)\frac{\partial}{\partial x}\mathcal{BS}(T-t, X(t);\ r, \sigma) \end{aligned}$$

is a superstrategy. (Here $\mathcal{BS}(\cdot,\cdot;\cdot,\cdot)$ is as in Remark 3.1.)

Proof. We observe first that

$$P^*(T) = (X(T) - K)^+ = \left(\frac{1}{\theta}\int_{T-\theta}^{T} S(u)\,du - K\right)^+,$$

which is the modification of (2.8) appropriate for an Asian option. We next appeal to Theorem 6.2, applied to the "stock price" X, whose volatility is bounded by σ. This corollary shows that if we hold

$$\Delta_\sigma(t) = \frac{\partial}{\partial x}\mathcal{BS}(T-t, X(t);\ r, \sigma)$$

"shares" of X at each time t, then we have nondecreasing discounted tracking error $e^{-rt}e^*(t) = e^{-rt}[\Pi^*(t) - P^*(t)]$ associated with the self-financing portfolio value given by $\Pi^*(0) = P^*(0)$ and

(11.3) $$d\Pi^*(t) = [\Pi^*(t) - \Delta_\sigma(t)X(t)]r(t)\,dt + \Delta_\sigma(t)\,dX(t).$$

In integral form, (11.3) is

$$\begin{aligned}\Pi^*(t) &= M(t)\left[P^*(0) + \int_0^t \frac{1}{M(u)}\Delta_\sigma(u)\rho(u)\sigma S(u)\,dW(u)\right]\\ &= M(t)\left[P^*(0) + \int_0^t \frac{1}{M(u)}\Delta^*(u)\sigma S(u)\,dW(u)\right],\end{aligned}$$

which is the self-financing value of the portfolio that holds $\Delta^*(u)$ shares of the real stock S at each time u. □

APPENDIX A: UNBOUNDEDNESS OF LOCAL TIME

The purpose of this appendix is to show that for $0 < x < a$, the local time $L(T_a; x)$ is unbounded on the set $\{T_a \le 1\}$, a fact used in the proof of Theorem 4.1.

PROPOSITION A.1. *Let W be a Brownian motion with $W(0) = 0$, let $b > 0$ be given, and set $T_b = \inf\{t \ge 0;\ W(t) = b\}$. Let $L(t)$ denote the local time of W at 0. Then, for each $r > 0$ and $\lambda > 0$, we have*

A.1
$$\mathbb{P}(T_b \le \tau, L(T_b) \ge \lambda) > 0.$$

Proof. The Laplace transform of the distribution of $(T_b, L(T_b))$ is (see Karatzas and Shreve 1991, Chap. 6, Prob. 4.4):

$$\mathbb{E}\exp(-\alpha T_b - \gamma L(T_b)) = \frac{\sqrt{2\alpha}}{(\gamma + \sqrt{2\alpha})\sinh(b\sqrt{2\alpha}) + \sqrt{2\alpha}\cosh(b\sqrt{2\alpha})}$$

for $\alpha > 0$, $\gamma > 0$, which implies that

$$\mathbb{E}\exp(-\gamma L(T_b)) = \frac{1}{1 + b\gamma}, \quad \gamma > 0,$$

so that the density of $L(T_b)$ is

$$\mathbb{P}(L(T_b) \in d\ell) = \frac{1}{b}e^{-\ell/b}, \quad \ell > 0.$$

Let $\mathbb{P}(T_b \in dt | L(T_b) = \ell)$ be a regular conditional distribution, and define

$$g(\alpha, \ell) = \int_0^\infty e^{-\alpha t}\mathbb{P}(T_b \in dt | L(T_b) = \ell)\,dt, \quad \alpha > 0.$$

Let $\rho > 1/b$ be given, and set $\gamma = \rho - 1/b$. Then

$$\begin{aligned}\int_0^\infty & e^{-\rho\ell} g(\alpha,\ell)d\ell \\ &= b\int_0^\infty\int_0^\infty e^{-\alpha t-\gamma\ell}\mathbb{P}(T_b\in dt|L(T_b)=\ell)\mathbb{P}(L(T_b)\in d\ell) \\ &= b\,\mathbb{E}\exp(-\alpha T_b-\gamma L(T_b)) \\ &= \frac{b\sqrt{2\alpha}}{(\rho-\frac{1}{b}+\sqrt{2\alpha})\sinh(b\sqrt{2\alpha})+\sqrt{2\alpha}\cosh(b\sqrt{2\alpha})}.\end{aligned}$$

This Laplace transform formula implies

$$g(\alpha,\ell) = \frac{b\sqrt{2\alpha}}{\sinh(b\sqrt{2\alpha})}\exp\left(-\left[\sqrt{2\alpha}-\frac{1}{b}+\sqrt{2\alpha}\coth(b\sqrt{2\alpha})\right]\ell\right).$$

Let $\tau > 0$ and $\lambda > 0$ be given. To prove (A.1), it suffices to show that $\mathbb{P}(T_b \le \tau | L(T_b) = \ell) > 0$ for all $\ell \in [\lambda, \lambda+1]$. But, for $\alpha > 0$,

$$\begin{aligned}\mathbb{P}(T_b\le\tau|L(T_b)=\ell) &= \int_0^\tau \mathbb{P}(T_b\in dt|L(T_b)=\ell) \\ &\ge g(\alpha,\ell)-\int_\tau^\infty e^{-\alpha t}\mathbb{P}(T_b\in dt|L(T_b)=\ell) \\ &\ge g(\alpha,\ell)-e^{-\alpha\tau}\mathbb{P}(T_b\ge\tau|L(T_b)=\ell),\end{aligned}$$

which implies

$$(1-e^{-\alpha\tau})\mathbb{P}(T_b\le\tau\,|\,L(T_b)=\ell)\ge g(\alpha,\ell)-e^{-\alpha\tau}.$$

For α sufficiently large, the right-hand side is positive for all $\ell \in [\lambda, \lambda+1]$. □

COROLLARY A.2. *Let W be a Brownian motion with $W(0) = 0$, let $a > 0$ be given, and set $T_a = \inf\{t \ge 0; W(t) = a\}$. Fix $x \in (0, a)$ and let $L(t; x)$ denote the local time of W at x. Then, for each $\lambda > 0$, we have*

$$P(T_a \le 1, L(T_a; x) \ge \lambda) > 0.$$

Proof. According to the strong Markov property, the process

$$W^x(t) = W(t+T_x) - x, \quad t \ge 0,$$

is a Brownian motion independent of $T_x = \inf\{t \ge 0; W(t) = x\}$. Set $b = a - x$ and $T_b^x = \inf\{t \ge 0,\ W^x(t) = b\}$, so that $T_a = T_x + T_b^x$. The local time $L^x(t)$ of W^x at zero up to time t is the local time $L(T_x + t; x)$ of W at x up to time $T_x + t$. For each $\lambda > 0$, we have

$$\begin{aligned} P(T_a \le 1, L(T_a; x) \ge \lambda) &\ge P\left(T_x \le \frac{1}{2}, T_b^x \le \frac{1}{2}, L^x(T_b^x) \ge \lambda\right) \\ &= P\left(T_x \le \frac{1}{2}\right) P\left(T_b^x \le \frac{1}{2}, L^x(T_b^x) \ge \lambda\right) \\ &> 0. \end{aligned}$$ □

APPENDIX B: DOMINATION OF THE INTEGRAL OF VOLATILITY MAY NOT PERMIT HEDGING

We assume in this appendix that the volatility process $\sigma(t)$ is a nonnegative, square-integrable, deterministic function of t, and we define

$$\Sigma(t) \triangleq \left(\int_t^T \sigma^2(u)\, du\right)^{1/2}, \qquad 0 \le t \le T.$$

To avoid trivialities, we assume that $\Sigma(t)$ is strictly positive for all $t \in [0, T]$. We let $\gamma(t)$ be another nonnegative, square-integrable, deterministic function of t, and define

$$\Gamma(t) \triangleq \left(\int_t^T \gamma^2(u)\, du\right)^{1/2}, \qquad 0 \le t \le T.$$

In this appendix, we study the misspecified price and hedging strategy for a European contingent claim under the assumption

$$\text{(B.1)} \qquad \Sigma(t) \le \Gamma(t), \qquad 0 \le t \le T.$$

(For example, we might have $\sigma(t) = \sqrt{T-t}$ and $\gamma(t) = \sqrt{t}$.) As we shall see, this assumption ensures that $v(t, x) \le v_\gamma(t, x)$ for all $0 \le t \le T$, $x > 0$ (see below or (6.2) for definitions), but it does not guarantee that the hedging portfolio associated with v_γ protects a short position in the claim. It particular, the expected tracking error under the risk-neutral probability measure is nonnegative (see (B.6)), but the expected tracking error under the market probability measure can be negative (see Remark B.4). This latter fact shows that the actual tracking error can be negative with positive probability, and hence the short position is not hedged.

In order to simplify notation, we assume throughout that $r = 0$. The case of $r(t)$ being a deterministic function follows easily. We adopt the notation

$$\text{(B.2)} \qquad L_\gamma(t, T) \triangleq \exp\left[\int_t^T \gamma(u)\, dW(u) - \frac{1}{2}\Gamma^2(t)\right],$$

so that (see (6.1), (6.2), (3.2))

$$
\begin{aligned}
S_\gamma^{t,x}(T) &= xL_\gamma(t,T), \\
v_\gamma(t,x) &\triangleq \mathbb{E}h(xL_\gamma(t,T)) \qquad \text{(B.3)}\\
&= \mathcal{BS}(T-t,x;0,\Gamma(t)/\sqrt{T-t}), \\
\frac{\partial}{\partial x}v_\gamma(t,x) &= \mathbb{E}[L_\gamma(t,T)h'(xL_\gamma(t,T))]. \qquad \text{(B.4)}
\end{aligned}
$$

When γ in (B.2)–(B.4) is replaced by σ, we drop the subscript, writing simply $L(t,T)$ and $v(t,x)$ rather than $L_\sigma(t,T)$ and $v_\sigma(t,x)$. Under condition (B.1), we have immediately from (3.5) that

$$
v(t,S(t)) \le v_\gamma(t,S(t)), \qquad 0 \le t \le T, \text{ a.s.}
$$

The tracking error associated with the portfolio process $\Delta_\gamma(t) \triangleq (\partial/\partial x)v_\gamma(t,S(t))$ and the price process $P_\gamma(t) \triangleq v_\gamma(t,S(t))$ is

$$
e_\gamma(t) \triangleq \Pi_{\Delta_\gamma}(t) - P_\gamma(t), \qquad 0 \le t \le T, \tag{B.5}
$$

and because $\Pi_{\Delta_\gamma}(t)$ is a martingale, we have

$$
\begin{aligned}
\mathbb{E}e_\gamma(T) &= \Pi_{\Delta_\gamma}(0) - \mathbb{E}h(S(T)) \qquad \text{(B.6)}\\
&= v_\gamma(0,S(0)) - v(0,S(0)) \\
&\ge 0.
\end{aligned}
$$

The result (B.6) is unsatisfying because the expection is computed under the risk-neutral probability measure rather than the market probability measure. To study the expectation under the market measure, we assume (see Remark 2.2) that $W(t) = W_0(t) + \int_0^t \lambda(u)\,du$, where the market price of risk λ is a square-integrable deterministic function and W_0 is a Brownian motion under the market probability measure $\mathbb{P}_0$ related to $\mathbb{P}$ by (2.4). We may write

$$
dS(t) = S(t)[\sigma(t)\lambda(t)\,dt + \sigma(t)\,dW_0(t)].
$$

We define

$$
\begin{aligned}
A(t) &\triangleq \exp\left(\int_0^t \sigma(u)\lambda(u)\,du\right), \\
L_0(t,T) &\triangleq \exp\left[\int_t^T \sigma(u)\,dW_0(u) - \frac{1}{2}\Sigma(t,T)\right], \\
L_{0,\gamma}(t,T) &\triangleq \exp\left[\int_t^T \gamma(u)\,dW_0(u) - \frac{1}{2}\Gamma(t,T)\right],
\end{aligned}
$$

and for $\tau \in [0, T]$, we set

$$\beta(t;\tau) \triangleq \begin{cases} \sigma(t), & 0 \le t \le \tau, \\ \gamma(t), & \tau < t \le T, \end{cases}$$
$$B(t;\tau) \triangleq \left(\int_t^T \beta^2(u;\tau)\, du \right)^{1/2}, \qquad 0 \le t \le T.$$

Under $\mathbb{P}_0$, the processes $L_0(t, T)$ and $L_{0,\gamma}(t, T)$ have the same distributions as the respective processes $L(t, T)$ and $L_\gamma(t, T)$ under $\mathbb{P}$. Also,

$$S(t) = S(0)A(t)L_0(0, t), \tag{B.7}$$
$$dS(t) = S(0)A(t)\, dL_0(0, t) + S(0)L_0(0, t)\, dA(t). \tag{B.8}$$

Let $\mathbb{E}_0$ denote the expectation under $\mathbb{P}_0$.

PROPOSITION B.1. *Irrespective of whether the condition (B.1) holds, the expectation of the tracking error under the market measure is given by*

$$\begin{aligned} \mathbb{E}_0 e_\gamma(T) &= v_\gamma(0, S(0)) - v(0, S(0)) \\ &\quad + S(0) \int_0^T \left[\frac{\partial}{\partial x} \mathcal{BS}(T, S(0)A(\tau); 0, B(0;\tau)/\sqrt{T}) \right. \\ &\qquad \left. - \frac{\partial}{\partial x} \mathcal{BS}(T, S(0)A(\tau); 0, \Sigma(0)/\sqrt{T}) \right] dA(\tau). \end{aligned} \tag{B.9}$$

Proof. From (2.6), (B.7), (B.8), the $\mathbb{P}_0$-martingale property for $L_0(0, t)$, and (B.4), we have

$$\begin{aligned} &\mathbb{E}_0 \Pi_{\Delta_\gamma}(T) \\ &= \Pi_{\Delta_\gamma}(0) + \mathbb{E}_0 \int_0^T \Delta_\gamma(t)\, dS(t) \\ &= v_\gamma(0, S(0)) + \mathbb{E}_0 \int_0^T S(0)L_0(0,\tau) \frac{\partial}{\partial x} v_\gamma(\tau, S(0)A(\tau)L_0(0,\tau))\, dA(\tau) \\ &= v_\gamma(0, S(0)) + \mathbb{E} \int_0^T S(0)L(0,\tau) \frac{\partial}{\partial x} v_\gamma(\tau, S(0)A(\tau)L(0,\tau))\, dA(\tau) \\ &= v_\gamma(0, S(0)) \\ &\quad + \mathbb{E} \int_0^T S(0)L(0,\tau) \mathbb{E}\left[L_\gamma(\tau, T) h'(S(0)A(\tau)L(0,\tau)L_\gamma(\tau, T)) \middle| \mathcal{F}(\tau) \right] dA(\tau) \\ &= v_\gamma(0, S(0)) + \mathbb{E} \int_0^T S(0) L_{\beta(\cdot;\tau)}(0, T) h'(S(0)A(\tau)L_{\beta(\cdot;\tau)}(0, T))\, dA(\tau) \end{aligned}$$

$$
\begin{aligned}
&= v_\gamma(0, S(0)) + \int_0^T S(0)\frac{\partial}{\partial x} v_{\beta(\cdot;\tau)}(0, S(0)A(\tau))\, dA(\tau) \\
&= v_\gamma(0, S(0) + S(0)\int_0^T \frac{\partial}{\partial x}\mathcal{BS}(T, S(0)A(\tau); 0, B(0;\tau)/\sqrt{T})\, dA(\tau).
\end{aligned}
$$

On the other hand, using the integration formula

$$
h(yA(T)) = h(y) + \int_0^T yh'(yA(t))\, dA(t),
$$

we have

$$
\begin{aligned}
\mathbb{E}_0 P_\gamma(T) &= \mathbb{E}_0 h(S(0)A(T)L_0(0,T)) \\
&= \mathbb{E}_0 h(S(0)L_0(0,T)) + \mathbb{E}_0 \int_0^T S(0)L_0(0,T)h'(S(0)A(t)L_0(0,T))\, dA(t) \\
&= \mathbb{E}h(S(0)L(0,T)) + \mathbb{E}\int_0^T S(0)L(0,T)h'(S(0)A(t)L(0,T))\, dA(t) \\
&= v(0, S(0)) + S(0)\int_0^T \frac{\partial}{\partial x} v(0, S(0)A(t))\, dA(t) \\
&= v(0, S(0)) + S(0)\int_0^T \frac{\partial}{\partial x}\mathcal{BS}(T, S(0)A(t); 0, \Sigma(0)/\sqrt{T})\, dA(t).
\end{aligned}
$$

The proposition now follows from (B.5). □

REMARK B.2. The risk premium λ influences the expected tracking error $E_0 e_\gamma(T)$ in two ways. First, the risk premium is implicit in $S(0)$, the market price of the stock, although this dependence is not modeled here. Secondly, the risk premium appears explicitly in the integrator $A(\cdot)$ of (B.9). Note that the integrand in (B.9) involves $A(\cdot)$ as well as the true and misspecified volatilities, with a switch from one to the other.

COROLLARY 13.3. *Consider the case of a European call; that is,* $h(x) = (x - K)^+$, *for some* $K > 0$. *Assume that (B.1) holds,* $\Sigma(0) = \Gamma(0)$, *and the market price of risk* λ *is nonnegative. If*

$$
B^2(0;\tau) \le 2\log\frac{S(0)}{K} + 2\int_0^T \sigma(u)\lambda(u)\, du, \qquad 0 \le \tau \le T, \tag{B.10}
$$

then $\mathbb{E}_0 e_\gamma(T) \ge 0$. *However, if*

$$
\Sigma^2(0) \ge 2\log\frac{S(0)}{K} + 2\int_0^T \sigma(u)\lambda(u)\, du, \tag{B.11}
$$

then $\mathbb{E}_0 e_\gamma(T) \le 0$.

Proof. Inequality (B.1) implies

$$\Sigma(0)/\sqrt{T} \leq B(0;\tau)/\sqrt{T}, \qquad 0 \leq \tau \leq T.$$

Under (B.10), we have in addition that

$$B(0;\tau)/\sqrt{T} \leq \sqrt{\frac{2}{T}\log\frac{S(0)A(\tau)}{K}}, \qquad 0 \leq \tau \leq T,$$

and Remark 3.1 shows that

$$\text{(B.12)} \quad \frac{\partial}{\partial x}\mathcal{BS}(T, S(0)A(\tau);\ 0, B(0;\tau)/\sqrt{T}) \geq \frac{\partial}{\partial x}\mathcal{BS}(T, S(0)A(\tau); 0, \Sigma(0)/\sqrt{T}).$$

Because $A(\cdot)$ is nondecreasing, we conclude from Proposition B.1 that

$$\mathbb{E}_0 e_\gamma(T) \geq v_\gamma(0, S(0)) - v(0, S(0)) = 0.$$

Under (B.11), inequality (B.12) is reversed, and we obtain

$$\mathbb{E}_0 e_\gamma(T) \leq v_\gamma(0, S(0)) - v(0, S(0)) \ = \ 0. \qquad \square$$

REMARK B.4. Condition (B.10) holds if the call is initially deep in the money, whereas (B.11) holds if the call is deep out of the money. In the latter case, the strict inequality $E_0 e_\gamma(T) < 0$ can easily hold, despite (B.1).

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