ROBUSTNESS STUDIES IN ADAPTIVE CONTROL*

by

James Krause, Michael Athans, Shankar S. Sastry and Lena Valavani
Laboratory for Information and Decision Systems
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

ABSTRACT

This paper examines robustness issues in Model Reference Adaptive Control systems in the presence of unmodeled dynamics and output disturbances. We present an approximate technique, trend analysis, by which we can study the evolution of the parameter error trajectory under periodic excitation. This analysis provides new insights upon the size and spectral content of the excitation sufficient to guarantee local stability.

I. INTRODUCTION

Several similar Model Reference Adaptive Controllers (MRAC's) have been shown to be globally stable under certain restrictive assumptions, including the assumption that the order and relative degree of the plant are exactly known, and that no disturbances are present ([1]-[3]). Under certain "sufficiently rich" excitation conditions, the origin is globally asymptotically stable for the adaptive controller parameter error ([5],[6]).

When the restrictive assumptions are violated, as they always are in practice, no proof of stability exists [4],[9],[10]. Furthermore, instability can occur under excitations which are "sufficiently rich" in the sense mentioned above. A stronger definition of sufficient richness is required to guarantee stability of the adaptive controller in the presence of unmodeled dynamics and disturbances.

The key factor to the stability of the adaptive controller is the time-evolution of the parameter error vector. This time-evolution is described by a complicated set of time-varying nonlinear differential equations, putting a closed-form analytic solution of reach.

This paper presents an approximate analysis technique for studying the long term trends of the parameter error vector trajectory for an adaptive controller under periodic excitation. Certain measures of the error of this technique have been proven to approach zero as the adaptive gain (which controls the rate of adaptation) approaches zero. The relevant theorems are stated here and proven in [8]. Thus the analysis technique can be termed a trend analysis for slowly adapting controllers.

The trend analysis provides a vector field, defined as a function of the command input, which approximates (in a long term sense) the time-derivative of the parameter error vector. From this vector field one can determine potential limit sets of the parameter error vector, and associated regions of attraction and approximate rates of convergence. New excitation conditions are defined which are sufficient for stability of the adaptive controller in the presence of unmodeled dynamics. Direction of future research are indicated which are required to enable analytic evaluation of some of the sufficient condition for a controller with a given adaptive gain.

At the present state of our studies, the new richness conditions provide substantial insight into the type and size of excitation required for stability. With reasonable designer-known information, one can determine a desirable frequency range for the command input energy, and the most desirable subset of this range. One can also estimate the amplitude of command input required to prevent a dangerous slow drift of the parameter error due to an output disturbance of any frequency.

We present the problem and establish notation in Section II. The approximate analysis technique is presented in Section III, and theorems which justify its use are stated. Sufficient excitation conditions for stability are given and discussed. The primary insights and results derived from the analysis are briefly stated and discussed.

II. THE ADAPTIVE CONTROL SYSTEM

The designer assumes that the low frequency portion of the single-input-single-output plant can be described by

$$y(s) = \hat{G}_p(s)u(s) \tag{1}$$

where $\hat{G}_p(s)$ is of order n. The poles and zeroes of $\hat{G}_p(s)$ are unknown but are assumed to lie within some known bounds.

The actual plant input/output relationship is

$$y(s) = G_p(s)u(s) + d_o(s) \tag{2}$$

where

$$G_p(s) = \hat{G}_p(s)\ (1+E_p(s)) \tag{3}$$

The quantity $d_o(s)$ is an output disturbance, and $E_p(s)$ is a modeling error due to neglected high-frequency plant dynamics. An upper bound $\bar{E}_p(\omega)$ on the magnitude of $E_p(j\omega)$ is generally known in practice [7].

A model output is constructed:

$$y_M(s) = G_M(s)r(s) \tag{4}$$

where r(s) is the command input.

The plant control input for adaptive control is

$$u(t) = \underline{W}^T(t)\underline{k}(t) \tag{5}$$

where $\underline{k}(t)$ is a 2n-vector of adjustable parameters, and

$$W(t) = H(t) * \begin{bmatrix} r(t) \\ u(t) \\ y(t) \end{bmatrix}, \tag{6}$$

* Research supported by ONR/N00014-82-K-0582 (NR 606-003), NSF/ECS-8210960 and NASA/NGL-22-009-124.

To appear, Proc. 22nd IEEE Conf. on Decision and Control, San Antonio, TX, Dec. 1983.

where $*$ denotes convolution and $H(t)$ is the impulse response of a known causal linear time-invariant system. The vector $\underline{W}(t)$ is constructed by the adaptive controller.

If $\underline{k}(t)$ is assigned a constant value $\underline{k}_o$, a time-invariant closed-loop plant results.

Classical robustness techniques can be used to determine which values of $\underline{k}_o$ result in guaranteed stability of the closed-loop plant. Define $\underline{K}_s$ to be the set of all such stabilizing $\underline{k}_o$.

The adjustable parameter vector $\underline{k}(t)$ can be considered a sum of a constant desired value and an error:

$$\underline{k}(t) = \underline{k}^* + \tilde{\underline{k}}(t) \tag{7}$$

$\underline{k}^*$ is such that $\tilde{\underline{k}}(t)=0$ and $E_p(s)=0$ results in $y(s)/r(s) = y_M(s)/r(s)$. With an appropriately chosen model, such model matching can result in a desirable performance improvement over a nonadaptive alternative design [8].

An error signal is defined:

$$\begin{aligned} e(t) &= y(t) - y_M(t) \\ &= T_1(t)*(\underline{w}^T(t)\tilde{\underline{k}}(t))+T_2(t)*r(t)+T_3(t)*d_0(t) \end{aligned} \tag{8}$$

where $T_1(t)$, $T_2(t)$, and $T_3(t)$ are impulse responses of causal linear systems. The laplace transform $T_1(s)$ of $T_1(t)$ is of the form

$$T_1(s) = T_o(s)(1+E_M(s)) \tag{9}$$

where $T_o(s)$ is a strictly positive real known linear system. $E_M(s)$ is unknown but shares the characteristic shape of $E_p(s)$. By choosing a grid on the space of possible plant parameters, one can numerically map the known upper bound on $|E_p(j\omega)|$ to an upper bound on $|E_M(j\omega)|$, $|T_2(j\omega)|$, and $|T_3(j\omega)|$. When $E_p(\omega)=0$, $E_M(j\omega) = T_2(j\omega)=0$. These upper bounds are important in the application of the trend analysis.

The parameter update law is

$$\dot{\underline{k}}(t) = \dot{\tilde{\underline{k}}}(t) = -\gamma\underline{w}(t)e(t) \tag{10}$$

where the adaptive gain γ is any strictly positive scalar constant.

Equations (8) and (10) characterize the parameter error behavior as a function of $\underline{w}(t)$. When $E_p(s)=0$ and $d_o(s)=0$, the adaptive control system is stable in the sense that $e(t)\in L_2$ and all signals are uniformly bounded. With nonzero unmodeled dynamics, no stability proof has been given, and potential instability has been demonstrated [4],[9],[10].

III. PARAMETER ERROR TRENDS WITH PERIODIC EXCITATION

A. Introduction

It is shown in this section that instability is a potential but not necessary consequence of unmodeled high frequency dynamics. Whether or not instability occurs is largely a function of the command input. Low frequency components of the command aid in correct parameter adjustment, while high frequency components contribute to misadjustment. An output disturbance at any frequency contributes to a hazardous "drift" of the parameter error. An excitation condition is given which is both necessary and sufficient for local (with respect to the parameter error) stability of the adaptive controller. Rate of adaptation is also a factor; an excitation condition which is sufficient for the stability of a slowly adapting system may not be sufficient for the stability of a rapidly adapting system.

B. Presentation and Validation of the Trend Analysis

The trajectory of the adjustable parameter vector $\underline{k}(t)$ is critical to the stability of the system. If $\tilde{\underline{k}}(t)$ approaches a limit inside $\underline{K}_s$, the system is stable. If $\underline{k}(t)$ strays outside of $\underline{K}_s$, a rapid onset of instability may occur and has been observed.

The differential equations describing the time evolution of $\underline{k}(t)$ are highly nonlinear and time-varying, and cannot be solved analytically, hence we employ approximate analysis techniques. A vector $\underline{k}'(t)$ is defined which has a much simpler trajectory and which represents the long term trends of the vector $\underline{k}(t)$. The accurracy of the trend analysis is shown to improve as the rate of adaptation is reduced.

The trend vector trajectory is a function of the system excitation. The command input studied is periodic and of the form

$$r(t) = r_o + \sum_{i=1}^{\ell} r_i \sin(\omega_i t+\eta_i); \ell<\infty \tag{11}$$

To show the relationship between the trajectory of the trend vector $\underline{k}'(t)$ and the actual trajectory of $\underline{k}(t)$, and to aid the definition of the trend, we first present the limiting case in which the rate of adaptation is zero:

$$\underline{k}(t) = \underline{k}_o \in \underline{K}_s \quad \forall t \tag{12}$$

The usual parameter update law (10) may be modified to function as an observer:

$$\dot{\hat{\underline{k}}}(t) = -\gamma\underline{w}^T(t)e(t) \tag{13}$$

With $r(t)$ as in (11), equations (8), (13) and the identity

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \tag{14}$$

yield

$$\dot{\hat{\underline{k}}} = \gamma(\underline{A}_{dc}+\underline{A}_{ac}(t))\tilde{\underline{k}}_o + \gamma\underline{b}_{dc} + \gamma\underline{b}_{ac}(t) \tag{15}$$

where $\underline{A}_{dc}$, $\underline{A}_{ac}(t)$ are 2nx2n and $\underline{b}_{dc}$, $\underline{b}_{ac}(t)$ are 2nx1. The elements of $\underline{A}_{dc}$ and $\underline{b}_{dc}$ are constants, and the elements of $\underline{A}_{ac}(t)$ and $\underline{b}_{ac}(t)$ are sums of sinusoids of nonzero frequency. $\underline{A}_{dc}$ and $\underline{b}_{dc}$ are readily-computed functions of accessible signals ($\underline{w}(t)$ which is a function of $r(t)$ and $y(t)$) and as such can be considered user-known. Aiding computation and analysis is the property that each of the frequency components present in $r(t)$ contributes independently to $\underline{A}_{dc}$ and $\underline{b}_{dc}$.

In the absence of an output measurement, $\underline{A}_{dc}$ and $\underline{b}_{dc}$ are a function of $r(t)$ and $\tilde{\underline{k}}_o$, and are defined for all $r(t)$ of the form of equation (11) and all $\tilde{\underline{k}}_o \in K_s$. Exact computation of $\underline{A}_{dc}$ and $\underline{b}_{dc}$ is no longer so easy. Nominal values of $\underline{A}_{dc}$ and $\underline{b}_{dc}$ can be computed *a priori* for each possible $\tilde{\underline{k}}_o \in K_s$. The actual values of $\underline{A}_{dc}$ $\underline{b}_{dc}$ will be a perturbed version of these nominal values. A bound on the perturbation can be calculated. The sign of the real part of the eigenvalues of the computed $\underline{A}_{dc}$ can never be changed by inclusion of any stable unmodeled dynamics, but the presence of unmodeled dynamics in $T_1(s)$ can cause such a change.

To avoid confusion between the various sources of error, we will subsequently ignore the issue of *a priori* calculation of $\underline{A}_{dc}$ and $\underline{b}_{dc}$ without an output measurement, and shall concentrate only on the errors relevant to the stability of the system. For notational simplicity, the dependence of $\underline{A}_{dc}$ and $\underline{b}_{dc}$, and later Δ and Δ', on either $r(t)$, $y(t)$ or $r(t)$, $\tilde{\underline{k}}_o(t)$, shall usually not be shown explicitly.

Define the trend

$$\dot{\hat{\underline{k}}}'(t) = \gamma \underline{A}_{dc}\,\tilde{\underline{k}}_o + \gamma \underline{b}_{dc}$$
$$\hat{\underline{k}}'(t_o) = \hat{\underline{k}}(t_o) \qquad (16)$$

All theorems to follow are proven in [8].

Theorem 1: There exists a $T>0$ such that $\hat{\underline{k}}'(t_o+mT)=\hat{\underline{k}}(t_o+mT)$ for all positive integers m.

Theorem 2: If $\underline{A}_{dc}\tilde{\underline{k}}_o + \underline{b}_{dc} \neq 0$, then

$$\lim_{t_1\to\infty} \frac{||\hat{\underline{k}}(t_o+t_1)-\hat{\underline{k}}'(t_o+t_1)||_1}{||\hat{\underline{k}}'(t_o+t_1)||_1} = 0$$

and $||\hat{\underline{k}}(t_o+t_1)-\hat{\underline{k}}(t_o+t_1)||_1$ is uniformly bounded.

The adaptive control problem is fundamentally different than the parameter observer problem above. Thus while Theorems 1 and 2 and eqn. (15) aid in the understanding of the trend analysis approach, they do not provide justification for use of the trend analysis in the control problem. We now provide such justification.

Let the parameter update law be given by eqn. (10). Let $\underline{A}_{dc}$ and $\underline{b}_{dc}$ be computed as before, assuming constant parameters. Define the trend vector

$$\dot{\tilde{\underline{k}}}' = \gamma \underline{A}_{dc}\tilde{\underline{k}}' + \gamma \underline{b}_{dc}$$
$$\tilde{\underline{k}}'(t_o) = \tilde{\underline{k}}(t_o) \in K_s \qquad (17)$$

Consider any T satisfying theorem 1.

Define
$$\underline{\Delta} = \tilde{\underline{k}}(t_o+T) - \tilde{\underline{k}}(t_o)$$
$$\underline{\Delta}' = \tilde{\underline{k}}'(t_o+T) - \tilde{\underline{k}}(t_o) \qquad (18)$$

Theorem 3: Given any $\delta>0$ and choice of a ρ-norm there exists a $\gamma_o>0$ such that for all $\gamma<\gamma_o$

$$\frac{||\underline{\Delta}'-\underline{\Delta}||_p}{||\underline{\Delta}'||_p} < \delta$$

Corollary: Given any δ there exists a γ_1 such that for all $\gamma<\gamma_1$

$$1-\frac{|\langle\underline{\Delta}',\underline{\Delta}\rangle|}{||\underline{\Delta}'||_2||\underline{\Delta}||_2} < \delta$$

where $\langle\cdot,\cdot\rangle$ denotes the dot product.

Theorem 3 is somewhat analogous to Theorem 1. Rather than exact matching of the trend and the actual trajectory at a selected point in time, we now have near matching with a fractional error which can be made arbitrarily small. The corollary states that the angle between the vector directions Δ and Δ' can be made arbitrarily small by sufficiently slow adaptation.

Theorem 3 treats the error at a specific point in time. For the intermediate points, we have the following:

Theorem 4: Given a choice of a ρ-norm, there exists a constant ε such that

$$||\tilde{\underline{k}}'(t)-\tilde{\underline{k}}(t)||_p < \gamma\varepsilon; \quad t\in[t_o,t_o+T] .$$

Corollary: If the sequence $\{\tilde{\underline{k}}(t_o+mT)\}, m=\{1,2,3...\}$ converges to a limit $\underline{k}_L$, then $\tilde{\underline{k}}(t)$ converges to a ball defined by

$$B = \{\tilde{\underline{k}}(t) \,\big|\, ||\tilde{\underline{k}}(t)-\underline{k}_L||_p < \gamma\varepsilon\}$$

for some constant ε.

We have thus provided analytic justification for the use of a trend analysis in the study of an adaptive controller. The trend analysis can be made arbitrarily accurrate by choice of a suitably small adaptive gain γ, which is the designer's prerogative.

C. Sufficient Conditions for Stability

We now proceed to discuss the qualities of the trend itself and to present sufficient conditions for stability of the adaptive system.

Given $r(t)$ only, $\underline{\Delta}'$ is a continuous function of $\tilde{\underline{k}}(t_o)$. Consider the trend vector field $F(\tilde{\underline{k}}_o,r,\gamma)$ defined on $K_s \in R^{2n}$ by associating the corresponding $\underline{\Delta}'(\tilde{\underline{k}}_o,r,\gamma)$ with each point $\tilde{\underline{k}}_o$ in K_s.

As γ approaches zero, the direction of $\underline{\Delta}'$ approaches a limit (see equation (17)):

$$\lim_{\gamma\to 0} \frac{\underline{\Delta}'}{||\underline{\Delta}'||_2} = \underline{\Delta}_L' = \frac{\underline{A}_{dc}\tilde{\underline{k}}(t_o)+\underline{b}_{dc}}{||\underline{A}_{dc}\tilde{\underline{k}}(t_o)+\underline{b}_{dc}||_2} \qquad (19)$$

By the corollary to Theorem 3, the direction of $\underline{\Delta}$ approaches this same limit as γ approaches zero. Let $\underline{F}_L(\tilde{\underline{k}}_o,r)$ be defined by associating the corresponding vector $\underline{\Delta}_L'(\tilde{\underline{k}}_o,r)$ with each point $\tilde{\underline{k}}_o$ in K_s.

We now state, without proof, the first sufficient richness condition of this paper:

Condition 1:

$r(t)$ is sufficient for <u>local stability in the limit</u> as γ approaches zero if the system

$$\dot{\underline{x}} = \underline{F}_L(\underline{x},r) \qquad (20)$$

has a locally stable equilibrium point.

For the case of a specific choice of a nonzero γ, we must take into account the error of the approximate trend analysis. Because this can be done a variety of ways, a variety of sufficient conditions can be generated. Two will be presented here.

In the process we will define constant upper bounds on various types of errors. A discussion of the computation of such bounds shall be given later.

Given any scalar constant ρ, and vector constant $\underline{x}$, define

$$\beta_1 = \{\tilde{\underline{k}}: ||\tilde{\underline{k}}-\underline{x}||_2 \leq \rho_1\} \qquad (21)$$

$$z_1 = \max_{\tilde{\underline{k}}_o\in\beta_1} ||\underline{\Delta}'(\tilde{\underline{k}}_o)-\underline{\Delta}(\tilde{\underline{k}}_o)||_2 \qquad (22)$$

$$\beta_2 = \{\tilde{\underline{k}}: ||\tilde{\underline{k}}-\underline{x}||_2 \leq \rho-z_1\} \qquad (23)$$

$$z_2 = \max_{\substack{\tilde{\underline{k}}_o\in\beta_1 \\ t_1\in[t_o,t_o+T]}} ||\tilde{\underline{k}}'(t_1)-\tilde{\underline{k}}(t_1)||_2 \qquad (24)$$

$$V = \{\underline{v}: \underline{v}\in R^{2n};\ ||\underline{v}||_2 \leq 1\} \qquad (25)$$

$$\underline{K}_s' = \{\tilde{\underline{k}}_o: \tilde{\underline{k}}_o+z_2\underline{v}\in K_2,\ \ \underline{v}\in V\} \qquad (26)$$

The bounds z_1 and z_2 are known to exist for sufficiently small γ, and in fact approach zero as γ approaches zero.

Condition 2:

r(t) is sufficient for local stability of the adaptive controller if there exists a $\rho_1>0$ and an $\underline{x}$ such that $\beta_1 \in K'_s$ and

$$(\underline{\tilde{k}}_o + \underline{\Delta}'(\underline{\tilde{k}}_o)) \in \beta_2 \; : \; \underline{\tilde{k}}_o \in \beta_1 \tag{27}$$

Partial proof: The use of K'_s rather than K_s in the condition is a subtlety that shall not be discussed here.

Denote

$$\underline{\tilde{k}}(t_o) = \underline{\tilde{k}}_o \tag{28}$$

$$\underline{\tilde{k}}(t_o+T) = \underline{\tilde{k}}_1 = \underline{\tilde{k}}_o+\Delta \tag{29}$$

$$\underline{\tilde{k}}'_1 = \underline{\tilde{k}}_o + \Delta' \tag{30}$$

If condition 2 is satisfied, then

$$||\underline{\tilde{k}}'_1-\underline{x}||_2 \le \rho_1-z_1 \tag{31}$$

Thus

$$\begin{aligned} ||\underline{\tilde{k}}_1-\underline{x}||_2 &\le ||\underline{\tilde{k}}_1-\underline{\tilde{k}}'_1||_2 + ||\underline{\tilde{k}}'_1-\underline{x}||_2 \\ &< z_1 + \rho_1-z_1 = \rho_1 \end{aligned} \tag{32}$$

Therefore

$$\underline{\tilde{k}}_m \in \beta_m; \quad m=0,1,2,\ldots \tag{33}$$

and β_1 is a locally stable region for $\underline{\tilde{k}}(t)$. Completion of the proof would involve a demonstration that equation (33) and $\beta_1 \in K_s$ implies boundedness of other system signals.

In effect, condition 2 requires that F be oriented inward around the boundary of a region, and that this inward orientation exceed the error bound using a particular standard of measurement. Condition 3 to follow has a similar interpretation but employs a different standard of measurement.

For any positive $\rho_3<\rho_1$, define

$$\beta_3 = \{\underline{\tilde{k}}: ||\underline{\tilde{k}}-\underline{x}||_2 \le \rho_3\} \tag{34}$$

$$\beta_4 = \{\underline{\tilde{k}}: ||\underline{\tilde{k}}-\underline{x}||_2 \le z_2+\rho_3\} \tag{35}$$

$$z_3 = \max_{\underline{\tilde{k}}_o \in \beta_1} \arccos \frac{\langle \underline{\Delta}'(\underline{\tilde{k}}_o,r), \underline{\Delta}(\underline{\tilde{k}}_o,r)\rangle}{||\underline{\Delta}'(\underline{\tilde{k}}_o,r)||_2 \, ||\underline{\Delta}(\underline{\tilde{k}}_o,r)||_2} \tag{36}$$

That is, z_3 is an upper bound on the angle between $\underline{\Delta}'$ and $\underline{\Delta}$ for all $\underline{\tilde{k}}_o$ in β_1. For a sufficiently small γ, z_3 exists; as γ approaches zero, z_3 approaches zero.

Condition 3:

r(t) is sufficient for local stability of the adaptive controller if there exists ρ_1, ρ_3 and $\underline{x}$ such that $0<\rho_3<\rho_1$, $\beta_1 \in K'_s$ and

$$\arccos \frac{\langle(\underline{x}-\underline{\tilde{k}}_o), \underline{\Delta}'(\underline{\tilde{k}}_o,r)\rangle}{||(\underline{x}-\underline{\tilde{k}}_o)||_2 \, ||\underline{\Delta}'||_2} < 90°-z_3$$

for all $\underline{k}_o \in \beta_1 \cap \sim\beta_3$ where $\sim\beta_3$ denotes the complement of β_3.

Moreover, any trajectory of $\underline{\tilde{k}}(t)$ which enters β_1 converges to β_4.

The proof is simple and is left to the reader. A few words must now be said about the practical application of these conditions, lest the reader be mislead as to the present state of our research.

Testing of condition 1 for stability in the limit is currently feasible. Equilibrium points $\underline{\tilde{k}}_E$ are defined by

$$F_L(\underline{\tilde{k}}_E,r) = 0 \tag{37}$$

Since $F_L(\underline{\tilde{k}}_K,r)$ is continuous, one can check for local stability of an equilibrium point through linearization.

Testing of conditions 2 and 3 is not feasible without further research. Computation of the bounds z_1, z_2, and z_3 is not currently possible. The theorems contained herein guarantee that any desired value for any of these bounds can be achieved through the use of some sufficiently small γ, but the value of γ required is not clear due to the nature of the proofs.

The stabilization technique employed in [8] involves selection of a reference input which is sufficient for stability in the limit as γ approaches zero, followed by an arbitrary selection of a "small" value for γ. Simulations indicate that 1) this process is often very overconservative, and 2) the meaning of "small" changes substantially with different plants, models, and excitations. Tight bounds on z_1, z_2, z_3 as a function of these different elements shows promise of allowing the use of much a larger γ (hence faster adaptation) with confidence.

D. Trend Analysis Results

The trend analysis technique constitutes an analytic quantification of the relationship between parameter convergence and the spectral components of the excitation. As such it has provided new information and insights into the stability of model reference adaptive control systems. Some of these results shall be stated here without derivation.

Though the stability problems of adaptive controllers has been well publicized recently, it is worth emphasizing that the trend analysis provides analytic verification of the potential instability of MRAC's. It also provides verification of a potential stability of the controllers and a means of obtaining stability without a modification of the basic algorithm. One can determine a periodic excitation which is capable of stabilizing the system, and add this excitation to the command input.

The excitation should be chosen to provide stability in the limit (condition 1), and allow wide margins for the errors with the unknown bounds z_1, z_2, and z_3. Determination of such an excitation is simplified by an empirically observed property of the matrix $\underline{A}_{dc}(\underline{\tilde{k}},r)$. Given r, $\underline{A}_{dc}$ and $\underline{b}_{dc}$ vary relatively slowly with $\underline{\tilde{k}}$, such that the properties of $F_L(\underline{\tilde{k}},r,\ )$ in some neighborhood of an equilibrium point $\underline{\tilde{k}}_E$ are revealed by considering the F_L resulting from the re-defined trend

$$\begin{aligned} \dot{\underline{\tilde{k}}}' &= \underline{A}_{dc}(\underline{\tilde{k}}_E,r)\underline{\tilde{k}}' + \underline{b}_{dc} \\ \underline{\tilde{k}}'(t_o) &= \underline{\tilde{k}}_o \approx \underline{\tilde{k}}_E \end{aligned} \tag{38}$$

Hence we have, as a minimal condition for asymptotic stability in the limit of an equilibrium point $\underline{\tilde{k}}_E$, the requirement that all the eigenvalues of $\underline{A}_{dc}(\underline{\tilde{k}}_E,r)$ lie in the open left half plane.

We can place further requirements on $\underline{A}_{dc}$ (and hence indirectly on r(t)) to allow a margin for errors of the type appearing in conditions 2 and 3. For example, a large margin for z_3 of equation (36) would be provided by an $\underline{A}_{dc}$ which gives a small value (large in the negative direction) to z_4, where

$$z_4 = \max_{x} \frac{\underline{x}^T \underline{A}_{dc} \underline{x}}{\underline{x}^T \underline{x}} \tag{39}$$

For the case of two adjustable parameters, the smallest values of z_4 are obtained by choice of a command with frequency content concentrated near the model bandwidth. Spectral components of the command above a threshold frequency ω_T serve to increase z_4. Acting alone these high frequency components yield an $\underline{A}_{dc}$ with positive eigenvalues.

The threshold frequency ω_T is the frequency at which $T_1(j\omega)$ has a phase of 90° (see equation (8)). This has an interesting correspondence to the strictly positive real requirement placed on $T_1(s)$ in nominal-system stability proofs [1].

Under the mild assumption that the designer knows 1) an upper bound on the magnitude of the open loop unmodeled dynamics as a function of frequency, and 2) bounds on the low frequency plant parameters (modeled uncertainty as opposed to unmodeled dynamics), the designer can determine a lower bound on ω_T. Thus, the designer can determine a priori a low frequency band in which command energy is guaranteed to aid in parameter identification and stability of the adaptive controller. This is useful in determining the desired frequency content of a dither signal added to the command to maintain stability, and also suggests the desirability of prefiltering the command input to remove high frequency content.

The amplitude of low frequency command required to maintain stability is largely a function of the spectrum of the output disturbance. Through algebraic manipulation, the effect of the output disturbance can be entirely contained in $\underline{b}_{dc}$. Thus an output disturbance shifts the equilibrium point of the parameter error vector without seriously affecting the local stability or instability of the equilibrium point. The danger is that the equilibrium point may be shifted to the edge of K_s and beyond, where the trend is not defined. In this situation we typically see the parameter error vector follow its predicted trend while in K_s, and then rapidly progress toward infinity shortly after leaving K_s.

The frequency content of the disturbance is not a critical issue except for the fact that disturbance rejection varies with frequency. If unopposed by proper command-generated excitation, output disturbance at any frequency contributes to a drift in the parameter error such that the loop gain increases without bound.

For a given output disturbance, a sufficiently large low frequency reference input will give the system described by equation (19) a stable equilibrium point in K_s. That is, local stability in the limit as γ approaches zero will be achieved. Given a sinusoidal reference of a specified low frequency, the designer can conservatively determine an amplitude sufficient to guarantee stability of the system using the following information: 1) bounds on the low frequency plant parameters, 2) the classical (gain margin) robustness constraint, and 3) the model choice.

In summary, one can state that the trend analysis reveals the need for a different perspective on "sufficient richness of excitation" than that present in [5] and [6]. When unmodeled dynamics are considered, the frequency of excitation becomes an issue. The presence of output disturbances at any frequency creates amplitude constraints on the excitation. More than just highlighting these issues, the trend analysis provides a tool for designing a remedy to the stability problem through the filtering of the command input and the addition of a persistent excitation to the command.

IV. CONCLUSIONS

For several promising adaptive algorithms, in the absence of unmodeled dynamics and output disturbances, weak richness conditions on the excitation guarantee convergence of the adaptive controller parameter error to zero. Under more realistic assumptions, convergence to zero parameter error must be given up. To achieve the weaker goal of stability and bounded parameter error without a modification of the algorithm, one must impose stronger conditions on the excitation. One can develop such conditions through the approximate analysis technique sketched in this paper. Valuable insights have resulted as to the type and size of excitation required. Further research is required to enhance the usefulness of the new richness conditions.

REFERENCES

[1] K. Narendra and L. Valavani, "Stable Adaptive Controller Design-Direct Control," IEEE Trans. on Auto. Control, Vol. AC-23, August 1978, pp. 570-583.

[2] K. Narendra, Y. Lin, and L. Valavani, "Stable Adaptive Controller Design, Part II: Proof of Stability," IEEE Trans. on Auto. Control, Vol. AC-25, June 1980, pp. 440-448.

[3] S. Morse, "Global Stability of Parameter-Adaptive Control Systems," IEEE Trans. on Auto. Control, Vol. AC-25, June 1980, pp. 433-439.

[4] C. Rohrs, "Adaptive Control in the Presence of Unmodeled Dynamics," Ph.D. Thesis, M.I.T., September 1982.

[5] A.P. Morgan and K.S. Narendra, "On the Stability of Nonautonomous Differential Equations d/dt(x)=[A+B(t)]x, with Skew Symmetric Matrix B(t)," SIAM Journal of Control and Optimization, Vol. 15, No. 1, January 1977, pp. 163-176.

[6] B.D.O. Anderson, "Exponential Stability of Linear Equations Arising in Adaptive Identification," IEEE Trans. on Auto. Control, Vol. AC-22, Feb. 1977, pp. 83-88.

[7] J. Doyle and G. Stein, "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis", IEEE Trans. on Auto. Control, Vol. AC-26, Feb. 1981, pp. 4-16.

[8] J.M. Krause, "Adaptive Control in the Presence of High Frequency Modelling Errors," S.M. Thesis, MIT, May 1983.

[9] C.E. Rohrs, L. Valavani, M. Athans, and G. Stein, "Robustness of Adaptive Control Algorithms in the Presence of Unmodeled Dynamics," Proc. 21st IEEE Conf. on Decision and Control, Orlando, Fla. Dec. 1982, pp. 3-11.

[10] C.E. Rohrs, L. Valavani, M. Athans and G. Stein, "Robustness of Continuous-Time Adaptive Control Algorithms in the Presence of Unmodeled Dynamic," Paper LIDS-P-1302, MIT, Cambridge, Mass. April 1983 (submitted to IEEE Trans. on Auto. Control).