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Rogue wave-type solutions of the mKdV equation and their relation to known NLSE rogue wave solutions

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Abstract We present the first four exact rational solutions of the set of rational solutions of the modified Korteweg de Vries (mKdV) equation. These solutions can be considered as rogue waves of the corresponding equation. Comparison with rogue wave solutions of the nonlinear Schrödinger equation (NLSE) shows a strong analogy between their characteristics, especially for amplitude-to-background ratio. The new solutions may be useful in the theory of rogue waves in shallow water and for light propagation in cubic nonlinear media involving only a few optical cycles.

Keywords Rogue waves · modified Korteweg de Vries

1 Introduction.

Rogue waves are unexpectedly high amplitude single waves that appear "from nowhere" [1]. Rogue waves are known to appear both in the open ocean and in coastal areas [2]. The physical difference in the two cases is the depth of water. Deep water waves are commonly described by the nonlinear Schrödinger equation (NLSE) [3]. Shallow water waves are described by the Korteweg de Vries (KdV) equation [4, 5]. There are several other model equations for shallow water waves, such as the modified KdV (mKdV) [6] and Camassa-Holm [7] equations. The mKdV equation is also used in the analysis of optical soliton propagation in cubic nonlinear media involving ultrashort pulses where only a few optical cycles are present [8]. Solutions of KdV and mKdV equations are related through the Miura transformation [9].

The first mathematical description of NLSE rogue waves was given in [10]. Namely, it was suggested that rogue waves are the solutions of nonlinear evolution equations that are localized both in space and time. As an example of such treatment, the two lowest order doubly-localized solutions of the nonlinear Schrödinger equation (NLSE) were given in [10]. Indeed, the lowest order solution, namely the 'Peregrine' rational solution has been independently considered as a prototype of an oceanic rogue wave in [11].

There is a set of NLSE solutions with progressively increasing amplitude. There are infinitely many and they can be viewed as higher-order rogue waves [12]. These solutions can be classified according to the number of fundamental rogue waves that they contain [13]. There are also other equations that admit analogous sets of rogue wave solutions. These include the Hirota equation [14,15], Davey-Stewartson equation [16,17], Sasa-Satsuma equation [18,19] and others. These can be extensions of the NLSE [20,21] or belong to other integrable hierarchies.

Standard soliton solutions and multi-soliton solutions of the KdV and mKdV equations are well-known. The subject of rogue waves has only emerged in recent years and it is still not well established. Definitions and mathematical models still need to be clarified. There has been some consideration of 'rogue periodic waves' of the mKdV equation [22]. On the other hand, presently, no (non-periodic) rogue wave solutions are known for KdV and mKdV equations. In this work, we fill this knowledge gap, suggesting that rogue wave solutions do exist, at least in the case of the mKdV equation. Namely, we pro-

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vide such a set of rational solutions to the mKdV equation and compare it to the known rogue wave solutions of the NLSE. There are certain common mathematical features of the solutions that allow us to claim that the set of rational mKdV solutions considered here do represent rogue waves.

Let us start with the case of the NLSE equation [23,24]:

$$i\psi_x + \frac{1}{2}\frac{\partial^2\psi}{\partial t^2} + |\psi|^2\psi = 0, \tag{1}$$

where x is the normalized propagation distance, t is the retarded time in a reference frame moving with the group velocity and the complex function $\psi(x,t)$ is the water wave envelope. This setting of variables is one of the two possible alternatives [1].

The lowest order rogue wave solution of NLSE is known as the 'Peregrine solution':

$$\psi = \left[4\frac{1+2ix}{1+4x^2+4t^2} - 1\right]e^{ix}. \tag{2}$$

It is a rational solution with the lowest order polynomials in the numerator and denominator. Higher order solutions involve polynomials of higher degree. They can be derived using several known techniques. The solution (2) is located on a background $|\psi| = 1$. Relative to this background, it is localized both in time and space. Higher-order solutions have the same property, while the central amplitude increases with the order of the solution [10,12,13].

The mKdV equation has the form:

$$\psi_x + \beta\psi^2\psi_t - \gamma_3\psi_{ttt} = 0, \tag{3}$$

where the variables x and t have the same meaning as in Eq.(1) while $\psi(x,t)$ is a real function describing the wave form directly rather than its envelope. The values β and γ_3 are the nonlinear and dispersion coefficients, respectively. Although they can be eliminated by rescaling time and the amplitude, in the present work, we keep them as independent parameters. The reason is that equation (1) and the complex form of Eq.(3) belong to the same hierarchy of integrable equations [20,21] and the variables in these equations can be related. Below, we present the set of rational solutions of Eq.(3) which take the form of rogue waves with progressively increasing amplitude, and show their connection to the set of rational solutions of the NLSE.

Some rational solutions for the case of mKdV have been given earlier [25–30]. In particular, the first and second order solutions presented here have been given in [30]. Rational mKdV solutions of up to second order, in addition to non-singular complexiton solutions, have also been obtained by Zhaqilao *et al.* in [31]. However, some higher-order solutions presented here, and the finding that they essentially describe rogue waves, are new.

The recent work of Sluniaev and Efim Pelinovsky [32] discloses the role of soliton and breather interactions in the formation of rogue waves within the framework of the integrable modified Korteweg–de Vries (MKdV) equation. The main focus of attention in [32] relates to calculating the amplitudes of the resultant rogue waves. It shows that the focused wave amplitude is exactly the sum of the focusing soliton heights. In other words, it shows that the focusing of solitary waves or/and breathers leads to rogue-wave-type dynamics, representing a new nonlinear mechanism of rogue wave generation. Here, we provide explicit rational solutions of the mKdV equation in order to show that collisions of travelling waves lead to the generation of rogue waves.

The backgrounds of the rogue wave solutions of the NLSE are completely flat, but this is not the case for the rational solutions of the mKdV equation (3). Instead, they have additional "solitons" on a background that extends to infinity in a fixed direction. Nevertheless, for higher order solutions, the central amplitude is higher than both the background and the amplitude of these "solitons". Therefore, the central peak can be viewed as a rogue wave, although it always appears on top of these "solitons", rather than on a flat background. Thus, with these caveats, we can label the new higher-order solutions presented below as "rogue waves". This approach is in agreement with the views expressed in the work of Sluniaev and Efim Pelinovsky [32]. In presenting the new solutions, we limit ourselves to fixed unit backgrounds ($\psi = \pm 1$), since any other background can be obtained by a simple rescaling [24]. The sign of the mKdV solutions can be chosen freely and can be changed to the opposite with no limitation of generality. Presenting rogue wave solutions, we naturally choose the positive sign for the maximum amplitude of the solution. Then the sign of the background may appear positive or negative depending on the order of the solution. This property is similar to the higher-order rogue wave solutions of the NLSE [33].

2 First order rational solutions

The first order rational solution can be written as:

$$\psi_1 = \frac{12\gamma_3}{3\gamma_3 - 2\beta(t-\beta x)^2} - 1, \tag{4}$$

where velocity $v = \beta$. This solution, as well as all others below, can be checked by direct substitution into Eq.(3). The plot of the solution is presented in Fig.1 for a particular choice of the two free parameters. It resembles a soliton with a fixed velocity v. It has a similar 3D profile for any other set of the parameters, except for the case when $v \to 0$. Then the soliton increases its width, being transformed in the actual limit into a new background, $\psi = 3$.

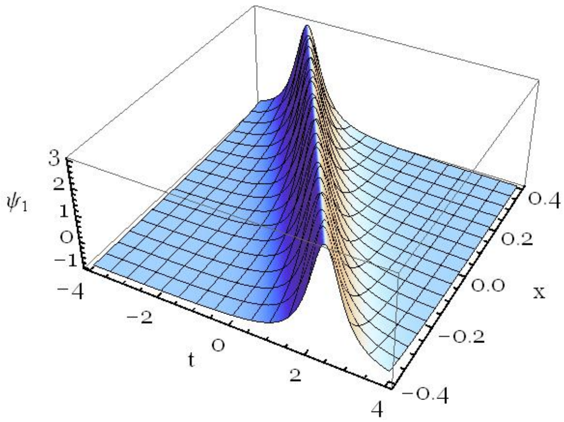

Fig. 1 First order rational solution, Eq.(4), of mKdV. Here $\gamma_3 = 1$ and $\beta = -12$.

The maximal amplitude of the solution is along the line $t = xv$. Remarkably, the maximum $\psi = 3$ is the same as that for the first order NLSE rogue wave (2). However, the background level here is -1. Thus, the maximum is 4 units higher that the background.

3 Second order rational solutions

We omit algebra and present the second order solution in terms of two polynomials, G_2 and D_2:

$$\psi_2 = \frac{12G_2}{D_2} + 1. \tag{5}$$

Here,

$$G_2 = 3 - (6ax + bt)\left[(6ax + bt)^3 + 6(22ax + bt)\right]$$

and

$$\begin{aligned} D_2 = 12ax\big[&243(2a)^4x^4(ax+bt) \\ &+bt\left(3b^4t^4 - 2b^2t^2 + 51\right) \\ &+72a^2btx^2\left(5b^2t^2 - 9\right) \\ &+108a^3x^3\left(15b^2t^2 - 13\right) \\ &+3ax\left(15b^4t^4 - 30b^2t^2 + 139\right)\big] \\ &+b^6t^6 + 3b^4t^4 + 27b^2t^2 + 9. \end{aligned}$$

where the composite coefficients $a = b^3\gamma_3/4$ and

$$b = \sqrt{\frac{-2\beta}{3\gamma_3}}$$

are either both positive or both purely imaginary. In either case, the solution ψ_2 is real, so all real equation parameters are allowed.

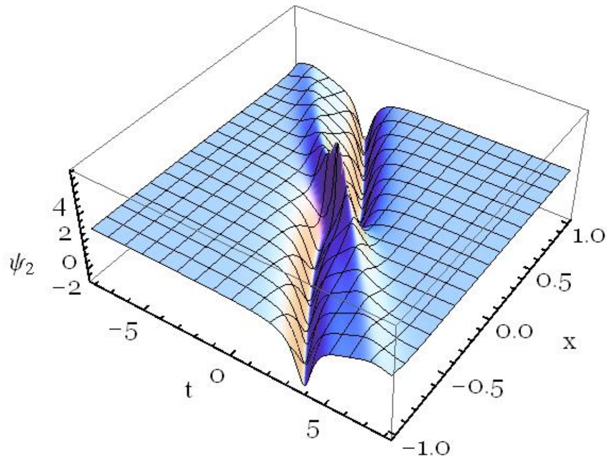

Fig. 2 Second order rational solution, Eq.(5), of mKdV. Here $\gamma_3 = 1$, $\beta = -6$. The minimum is -3 and this occurs near the line $t = -6x$ when $|x|$ is large enough.

The maximum of the solution Eq.(5) at the point $(0, 0)$ is 5. This is the same as the maximum of the second order NLSE rogue wave solution, even though the equation has no NLSE component. The wave profile resembles a peak on top of a moving soliton. As the peak is higher than the soliton, the central bump can be interpreted as a rogue wave.

The profile of the solution Eq.(5) is shown in Fig.2 for $\beta = -6$. The solution looks similar for any other choice of parameters except for the limiting case $\beta \to 0$ when the central bump spreads and approaches a flat background equal to 5. For $\beta = -6$, when x is not very close to zero, there is a 'valley' along the curve which looks roughly like the straight line $t = -6x$. Actually the curve is given by the formula $t = sign(x)|6x|^{1/3} - 6x$.

4 Third order rational solution

The general third order rational solution is:

$$\psi_3 = \frac{24G_3}{D_3} - 1, \tag{6}$$

where

$$\begin{aligned} G_3 = 4\{&800c^3x^3z + 150c^2x^2\left(16z^4 - 8z^2 + 5\right) \\ &+120cxz\left(-16z^4 + 40z^2 + 15\right) \\ &+z^2\big[8z^2(32z^6 + 120z^4 + 420z^2 \\ &-225) - 675\big]\} + 675, \end{aligned}$$

and

$$D_3 = 2025 + 8\{800c^4x^4 - 800c^3x^3z(-3+4z^2) \\ +30c^2x^2(165 - 180z^2 + 240z^4 + 64z^6) \\ +10cxz[-675 + 32z^2(45 - 27z^2 + 8z^6)] \\ +z^2\left[6075 + 2z^2(3375 + 16z^2[585 \\ +135z^2 + 24z^4 + 16z^6])\right]\}.$$

Here, complex coefficients $z = \frac{1}{4}(2bt - cx)$ and

$$b = \sqrt{\frac{-2\beta}{3\gamma_3}},$$

while $c = -3b^3\gamma_3$. Again, a and c are either both positive or both purely imaginary. In either case, the solution ψ_3 is real, so all real equation parameters are allowed.

The third-order solution has higher-order polynomials G_3 and D_3 in comparison to the previous solutions. The plot of this 3-rd order rational solution is shown in Fig.3. The highest point of the plot is at the centre. It is equal to 7. The background is -1, thus, the maximum is 8 units above the background. The peak surely can be considered as a rogue wave. This plot vaguely resembles collision of 2 moving bright solitons while the "solitons" are roughly parallel at infinity.

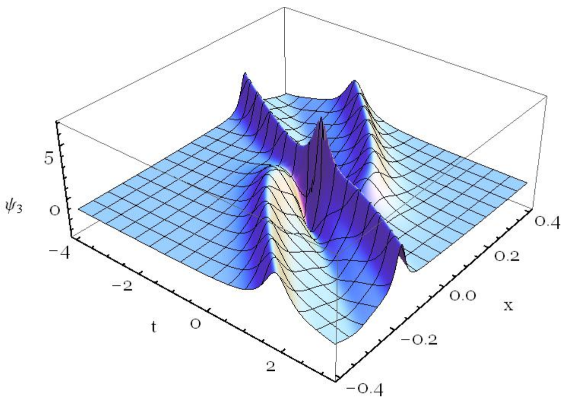

Fig. 3 The plot of the third order rational solution, Eq.(6) when $\gamma_3 = 1$, $\beta = -12$. The composite coefficients are $b = 2\sqrt{2}$ and $c = -48\sqrt{2}$. The minimum is around -3.3 and this occurs at 2 valley points close to the centre. The ridges have heights around 3. The maximum occurs at the origin. It is 7, being the highest point of the solution.

5 Fourth order rational solutions

The exact solutions can be calculated to any order. We restrict ourselves to the fourth-order solution as the expressions become too lengthy for journal presentation. Thus, the general fourth order rational solution is given by

$$\psi_4 = \frac{40G_4}{D_4} + 1, \tag{7}$$

where the bulky polynomials G_4 and D_4 are given in the Appendix. A plot of this solution is presented in Fig.4 for one set of parameters. The maximum of the solution at the origin is 9. It has the same qualitative features for any other set of parameters.

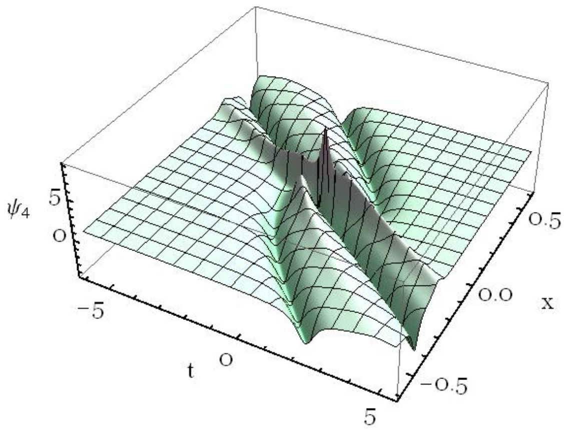

Fig. 4 Fourth order rational solution, Eq.(7), for the set of parameters $\gamma_3 = 1$, $\beta = -12$. The composite coefficient $d = \sqrt{2}$. The maximum is now 9 and the background is $+1$.

6 Common features with the NLSE rogue waves

Figs. 2-4 of the previous Sections demonstrate clearly that the rational solutions have maximum amplitude at the centre. These peaks can be interpreted as the collision point of the localized travelling waves or solitons. Near the maximal wave points, they can also be considered as rogue waves. This interpretation has been suggested in several previous works including [32] for the case of mKdV. A detailed mathematical comparison with the rogue waves of the NLSE shows that this interpretation has deeper roots than it may appear from a first glance. Let us compare the maximal amplitudes relative to the background levels of the solutions. The main parameter of the rogue wave is the wave amplification which is the absolute value of the ratio of the wave maximum to the background level. This ratio for the mKdV turns out to be the same as for the NLSE. A few other mathematical features of the solutions for the two cases also are found to be the same.

In Table 1, we summarize the findings. The first column in the table shows the order of the solution while the second and third columns show the power of the polynomials involved in each solution. The fourth column shows the background level. This is either $+1$ or -1. Finally, the last column provides the maximum value of the solutions.

The most intriguing finding deduced from our calculations is that these rational solutions of 3^{rd} order PDEs, with no NLSE components, still show features of the rogue waves associated with the NLSE. Namely, for the rogue wave of order j, the highest power in the denominator polynomial is $j(j+1)$, the background level is $(-1)^j$ and the central maximum value is $2j+1$. These analytical expressions are shown in the last row of Table 1. Remarkably, they are exactly the same as the values for NLSE rogue waves given earlier [33]. Naturally, we can consider the new solutions being analogs of the NLSE cases. As the equation is real, it directly describes the wave profiles rather than giving NLSE-type envelope functions. We assume that there should be a limiting transition between the solutions of the NLS and the mKdV equations. However, finding this transition is beyond the scope of the present work.

j	$HP(G_j)$	$HP(D_j)$	*background*	*max.*
1	0	2	-1	3
2	4	6	+1	5
3	10	12	-1	7
4	18	20	+1	9
j	$j(j+1)-2$	$j(j+1)$	$(-1)^j$	$2j+1$

Table 1 Main features of rational solutions of order j. Here, $HP(G_j)$ is the highest power occurring in the numerator, while $HP(D_j)$ is the highest power occurring in the denominator.

As we deal with an odd order equation, all the solutions show velocity effects on propagation. The ridges and valleys of the solutions resemble 'solitons' propagating with velocities close to β, and they persist indefinitely. Thus, the rogue waves of these equations have additional tails spreading to infinity. Nevertheless, the central elevation of these solutions, which is higher than the background, may appear at any position along these ridges. This position is normally defined by the initial conditions. Therefore, we can describe the central bumps of these solutions as waves that 'appear from nowhere and disappear without a trace', thus providing an excellent analogy with NLSE rogue waves [10].

7 Conclusions

We have shown that a set of polynomial rational solutions exists for the mKdV as a 3^{rd} order equation. The new solutions possess the main features of rogue waves, just like the rational solutions of the NLSE. For the first order solution, the maximum value occurs along a ridge. For higher-order solutions, their maximum values are higher than the wave amplitudes around them. Each peak is tied to the ridges of the solution, but can be located anywhere along it.

Mathematically, a rogue wave is a solution of an evolution equation which is localized both in time and space and which features a high amplitude peak. The Peregrine solution of the NLSE is one example of a rogue wave that falls into this category. Our second to fourth-order solutions of the mKdV also satisfy this definition. Thus, they can be considered as new examples of rogue waves for this equation. Moreover, as Table 1 shows, some parameters of the rogue wave hierarchies, such as the maximum amplitude relative to the background level, are the same for the two equations. Thus, there are strong reasons to view the family of rational solutions found in our work as rogue waves of the mKdV. The fact, that the solution parameters of these rogue waves, expressed in analytical form, are exactly the same as for the rogue waves of the NLSE may have far-reaching consequences for the relation between the NLSE and the mKdV rogue wave solutions. We leave finding this relation for future analysis.

As the mKdV is integrable, in principle, the whole set of solutions of arbitrary (j^{th}) order could be found in analytical form. As the complexity of the solutions increases dramatically with j, the fifth-order solution is expected to occupy more than a standard journal page. Consequently, we leave presentation of such higher-order solutions for future studies.

One more problem to address is the stability of the solutions. This is a complex problem that can be separated into stability of relative to: (1) perturbations on the solutions, and (2) perturbations on the governing equation itself. The results in each case differ significantly, as the previous studies of the NLSE case reveal [34]. Clearly, this is another complex issue that cannot be fully solved in the frame of a single publication. Rogue waves are limited in both space and time, and so do not remain stable in the sense of a soliton that can persist forever. Thus "robustness" is a better term than "stability" for rogue waves.

The issue of the modulational stability of a plane (background) wave for the mKdV has been studied in [35]. If we convert notation (thus setting the dispersion coefficient, γ_3 to -1), then the results in [35] show that a plane wave is modulationally stable when $\beta < 0$ (i.e. for the 'defocussing' case) and modulationally unstable when $\beta > 0$ (i.e. for the 'focussing' case).

Clearly, interesting topics remain for future work.

Appendix

Here, we provide the polynomials appearing in the fourth-order solution given by Eq.(7). Namely,

$$G_4 = 4465125 - 4\big[12250(16d\,t)^5(dt-6y) \\ -g_1 - g_3 + g_2 + g_0\big],$$

where

$$g_0 = 384d(g_5 - 992250)ty + (g_7 + 355721625)y^2,$$
$$g_1 = 3675(16d\,t)^4\left(32y^6 - 830y^2 - 45\right),$$
$$g_2 = 210(4d\,t)^2\left[8(g_4 + 3019275)y^2 + 14175\right],$$
$$g_3 = 700(8d\,t)^3 y[16\big(112y^6 - 5808y^4 + 882y^2 \\ +50785\big)y^2 + 135135]$$
$$g_4 = 2y^2[16\big(48y^6 + 2456y^4 - 63975y^2 \\ +18585\big)y^2 + 4621925],$$
$$g_5 = y^2\left[8(g_6 - 37241225)y^2 - 158704875\right],$$
$$g_6 = 2y^2[16y^2\big(16y^6 - 840y^4 - 23485y^2 \\ +422275\big) - 2720025],$$
$$g_7 = 16y^2\left[4(g_8 + 335264125)y^2 + 1090648125\right],$$
$$g_8 = 2y^2\left[4(g_9 - 23649465)y^2 + 44942625\right],$$

and

$$g_9 = 8y^2\left[4\left(4y^4 - 69y^2 + 2745\right)y^2 + 209405\right].$$

Now, the polynomial in the denominator is given by

$$D_4 = 8\{700(8d\,t)^3[120d(d_9 + 1575)t \\ +(d_{10} - 656775)y] + d_{11} + d_{12}\} + 22325625,$$

where

$$d_1 = 2y^2[8\left(16y^6 - 920y^4 + 13845y^2 - 15550\right)y^2 \\ +15612625],$$
$$d_2 = y^2\left[8(d_1 - 134765925)y^2 + 4789381625\right],$$
$$d_3 = 2y^2\left[32(d_2 + 4789119125)y^2 + 9531388125\right],$$
$$d_4 = 8y^2\left[8y^2\left(2y^4 - 58y^2 + 175\right) - 115185\right],$$
$$d_5 = y^2\left[2(d_4 + 3754625)y^2 - 54385975\right],$$
$$d_6 = 16y^2\left[16(d_5 - 38463775)y^2 - 73150875\right],$$
$$d_7 = 4y^2[4(768y^8 - 5824y^6 + 501872y^4 \\ -1924860y^2 + 25822265)y^2 + 50654975],$$
$$d_8 = 6(20d\,t)^2\left[4(d_7 + 30131325)y^2 + 1885275\right],$$
$$d_9 = 8y^2\left[16\left(2y^4 - 7y^2 + 700\right)y^2 + 3115\right],$$
$$d_{10} = 4y^2[8y^2\left(32y^6 - 4120y^4 + 15060y^2 - 450835\right) \\ -1269275],$$
$$d_{11} = 120d(d_6 - 82852875)ty + (d_3 + 5529313125)y^2 \\ +d_8,$$

and

$$d_{12} = 61250(16d\,t)^5(dt - 6y)\left(4y^2 + 1\right).$$

Here, $y = d(t + 6\gamma_3 d^2 x)$, with

$$d = \sqrt{\frac{-2\beta}{6\gamma_3}}.$$

As in the previous solutions, d and y are either both positive or both purely imaginary. In either case, the solution ψ_4 is real, so all real equation parameters are allowed.

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