IOSR Journal of Computer Engineering (IOSR-JCE)
e-ISSN: 2278-0661, p- ISSN: 2278-8727Volume 16, Issue 2, Ver. III (Mar-Apr. 2014), PP 58-63
www.iosrjournals.org

Role of Adjacency Matrix in Graph Theory

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Abstract: *Graph theory is an applied branch of the mathematics which deals the problems, with the help of graphs. There are many applications of graph theory to a wide variety of subjects which include Operations Research, Physics, Chemistry, Economics, Genetics, Sociology, Computer Science, Engineering, Mechanical Engineering and the other branches of science. A graph can be represented inside a computer by using the adjacency matrix. We define adjacency matrix and observed based on adjacency matrix. We prove theorem of adjacency matrix and give example. Also draw a graph of adjacency matrix. We derive algorithm to found new adjacency matrix after fusion and fusion for connectedness. Also use fusion algorithm to check the connectedness.*

Keywords: *Adjacency matrix, Binary matrix, Graph, Mathematical Induction, Fusion.*

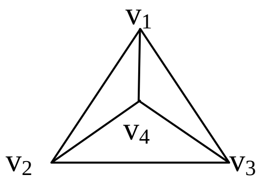
I. Introduction

Adjacency Matrix: Let G a graph with n vertices 1,2,…,n. The adjacency matrix of G with respect to this particular listing of n vertices is the n × n matrix and denoted by X(G) and defined as

$$X(G) = [x_{ij}] = \begin{cases} 1, & \text{if } (v_i , v_j) \text{ is an edge, i.e. } v_i \text{ is adjacent to } v_j \\ 0, & \text{if there is no edge between } v_i \text{ and } v_j \end{cases}$$

Elements of the adjacency matrix are either 0 or 1. Such a matrix sometimes also called as a bit matrix or as Boolean matrix.

If there exists an edge between vertex v_i and v_j where i is a row and j is a column then value of x_{ij}= 1 and if there is no edge between vertex v_i and v_j, then value of x_{ij}= 0.



Adjacency matrix

$$X(G) = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Observations based on adjacency matrix X(G):

(1) The adjacency matrix X(G) of graph G is symmetric matrix that mean $x_{ij} = x_{ji}$ for all value of i and j.

(2) The entries along the principal diagonal of X(G) are all zero if and only if the graph has no self-loops. A self loop at the i^{th} vertex corresponds to $x_{ij} = 1$.

(3) From the adjacency matrix we get no information about the parallel edges or multiple loops, so we ignore the parallel edges in the definition of X(G).

(4) Number of vertices having self-loop equal to number of non zero diagonal entries in matrix X(G).

(5) A graph G is disconnected and is in two components G_1 and G_2 if and only if its adjacency matrix X(G) can be partitioned as

$$X(G) = \left(\begin{array}{c|c} X(G_1) & 0 \\ \hline 0 & X(G_2) \end{array}\right)$$

where $X(G_1)$ is the adjacency matrix of the component G_1 and $X(G_2)$ is that of the component G_2. The partitioning clearly gives that there exists no edge joining any vertex in subgroup G_1 to any vertex in subgroup G_2.

(6) Permutation of rows or columns corresponds to the relabeling of vertices.

(7) If two rows are interchanged the corresponding columns are also interchanged. Hence two graphs G_1 and G_2 with no parallel edges are isomorphic if and only if their adjacency matrices $X(G_1)$ and $X(G_2)$ are related.

$$X(G_2) = R^{-1}. X(G_2) . R$$

Where R is a permutation matrix.

(8) ij^{th} entry of $[X(G)]^m$ gives number of walks length m from i^{th} vertex to the j^{th} vertex in the graph G and hence the smallest index m such that ij^{th} entry is non zero gives the distance from i^{th} vertex to the j^{th} vertex.

(9) Number of 1's in a row (or in a column) gives the degree of the vertex corresponds to the row(or column), counting diagonal elements twice. In general deg(v) = number of 1's in a off diagonal row + 2 × diagonal entry. Counting Path Between vertices: The number of path between two vertices in a graph can be determined using its adjacency matrix.

II. Theorem and Examples

Theorem: Let G be a graph with adjacency matrix X(G) with respect to the ordering $v_1, v_2, \ldots, v_n$(with directed or undirected edges, with multiple edges and loops allowed). The number of different path of length r from v_i to v_j, where r is a positive integer, equal the $(i , j)^{th}$ entry of X^r.

Proof: This theorem will be proved using mathematical induction. Let G be a graph with adjacency matrix X(G) (assuming an ordering, $v_1, v_2, \ldots, v_n$ of the vertex of G). The number of paths from v_i to v_j of length r from v_i to v_j, where r is a positive integer, equal the $(i , j)^{th}$ entry of X(G), since this entry is the number of edges from v_i to v_j.

Assume that the $(i , j)^{th}$ entry of X^r is the number of different paths of length r from v_i to v_j. This is the inductions hypothesis. Since $X(G)^{r+1} = X(G)^r. X(G)$ the $(i , j)^{th}$ entry of $X(G)^{r+1}$ equals

$$b_{i1} a_{i2} + b_{i2} a_{2j} + \ldots + b_{in} a_{nj}$$

where b_{ik} is the $(i , j)^{th}$ entry of $X^r(G)$. By the induction hypothesis, b_{ik} is the number of paths of length r from v_i to v_k.

A path of length r+1 from v_i to v_j is made up of a path of length r from v_i to some intermediate vertex v_k and an edge from v_k to v_j. By the product rule for counting, the number of such paths is the product of the number of paths of length r from v_i to v_k, namely, b_{ik} and the number of edges from v_k to v_j, namely a_{kj}. When these products are added for the possible intermediate vertices v_k, the desired results follows by the sum rule for counting.

Example1: A graph has the following adjacency matrix. Check whether it is connected.

$$X(G) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution: Since here n = 5, we look at $B = X + X^2 + X^3 + X^4$.

We find, X^2 , X^3 and X^4 by the matrix multiplication.

$$X^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}, \quad X^3 = \begin{pmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 3 & 3 & 0 \end{pmatrix}, \quad X^4 = \begin{pmatrix} 2 & 1 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 5 & 4 & 0 \\ 0 & 0 & 4 & 5 & 0 \\ 3 & 3 & 0 & 0 & 6 \end{pmatrix}$$

$$B=\begin{pmatrix}0&0&1&0&1\\0&0&0&1&0\\0&0&0&0&1\\0&1&0&0&1\\0&0&1&1&0\end{pmatrix}+\begin{pmatrix}1&0&0&0&1\\0&1&0&0&1\\0&0&2&1&0\\0&0&1&2&1\\1&1&0&0&2\end{pmatrix}+\begin{pmatrix}0&0&2&1&0\\0&0&1&2&0\\2&1&0&0&3\\1&2&0&0&3\\0&0&3&3&0\end{pmatrix}+\begin{pmatrix}2&1&0&0&3\\1&2&0&0&3\\0&0&5&4&0\\0&0&4&5&0\\3&3&0&0&6\end{pmatrix}$$

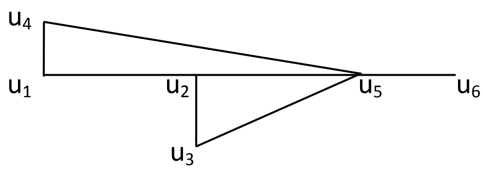
$$=\begin{pmatrix}3&1&3&1&4\\1&3&1&3&4\\3&1&7&5&4\\1&3&5&7&4\\4&4&4&4&8\end{pmatrix}$$

Since B has no non zero entry off the main diagonal. Hence the graph is connected.

Example2: Draw a graph whose adjacency matrix is

$$X(G)=\begin{array}{c}\\ v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6\end{array}\begin{array}{c}\begin{array}{cccccc}v_1&v_2&v_3&v_4&v_5&v_6\end{array}\\ \begin{pmatrix}0&1&0&1&0&0\\1&0&1&0&1&0\\0&1&0&0&1&0\\1&0&0&0&1&0\\0&1&1&1&0&1\\0&0&0&0&1&1\end{pmatrix}\end{array}$$

Solution: The graph G is



Algorithm: To find new adjacency matrix after fusion:

Step 1. Change u's row to the sum of u's row with v's row and (symmetrically) change u's column to the sum of u's column with v's column.

Step 2. Delete the row and column corresponding to v. The resulting matrix is the adjacency matrix of the new graph G.

Algorithm: Fusion Algorithm for Connectedness:

Step1. Replace G by its underlying simple graph. To get adjacency matrix of new graph just replace all non-zero entries off the diagonal by 1 and make all entries on the diagonal 0. Denote the underlying simple graph also as G.

Step2. Fuse vertex v_1 to the first of the vertices v_2 …. v_n with which it is adjacent to give a new graph, also denoted by G, in which the new vertex is also denoted by v_i.

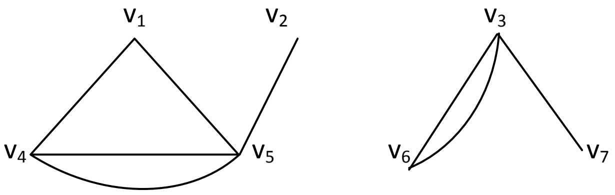
Step3. The above two step process gives the adjacency matrix X(G).

Step4. Repeat steps 1 and 2 with v_1 will v_2 is not adjacent to any of the other vertices.

Step5. Repeat step 2 and 4 on the vertex v_2 of the last graph and then on all remaining vertices of the resulting graphs.

The final graph is empty and the number of its (isolated) vertices is the number of connected components of the initial graph G.

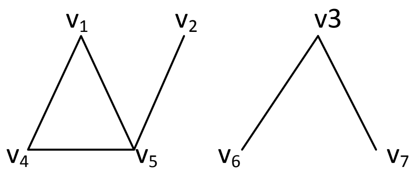
Example3: Given below is the adjacency matrix of graph G with seven vertices listed as v_1, v_2, v_3, v_4, v_5, v_6, v_7. Use fusion algorithm to check the connectedness.



Solution: Adjacency matrix of graph G

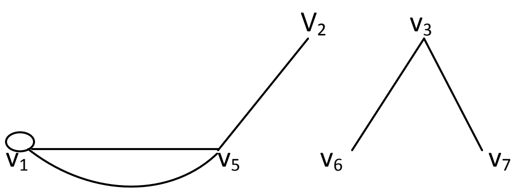
$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Step1: Replacing G by its underlying simple graph



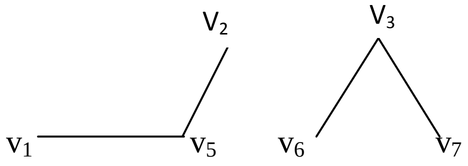
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Fusing V_1 with V_4



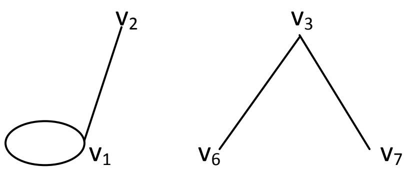
$$\begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Repeat step1 on graph obtained in Step2.



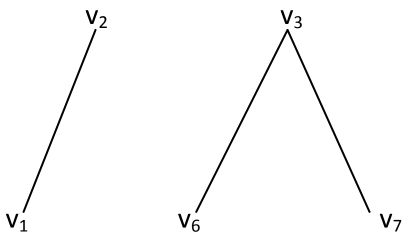
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Fusing v_1 to v_5



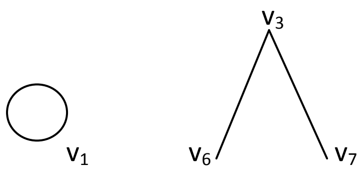
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Repeating Step 1



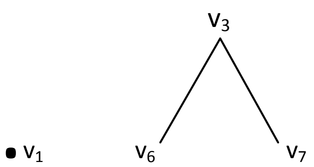
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Fusing v_1 with v_2



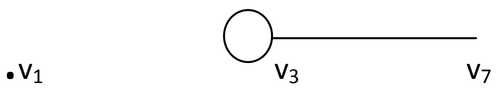
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Repeating Step 1

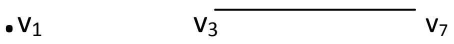


$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Fusing v_3 with v_6



$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Fusing v_3 with v_7

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Repeating Step 1

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since the final adjacency matrix is 2×2 null matrix. We conclude that original graph G has 2 connected components.

IV. Concluding Remarks

Thus we proved the number of different path of length r from v_i to v_j of a graph with adjacency matrix. We also checked the graph is connected to given adjacency matrix. Through adjacency matrix, we have drawn a graph G which is u_1, u_2, u_3, u_4, u_5, u_6. Using fusion algorithm to checked connectedness and found adjacency matrix is a 2×2 null matrix. So graph has 2 connected components. Thus the graph represented by adjacency matrix**.**

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