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The Role of Semiclassical Description

in the Quantum-like Theory of Light Rays

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Abstract

A procedure alternative to the one *ala Gloge and Marcuse* to perform the *transition* from geometrical optics to the wave optics in paraxial approximation is presented. This is done by employing a recent *deformation* method used to give a *quantum-like* phase-space description of charged-particle beam transport in semiclassical approximation. By taking into account the uncertainty relation (diffraction limit) that holds between the transverse beam spot size and the r.m.s. of the light-ray slopes, the classical phase-space equation for light rays is *deformed* into a von Neumann-like equation which governs the phase-space description of the beam transport in semiclassical approximation. Here, $\hbar$ and the time are replaced by the inverse of the wavenumber, $\lambdabar$, and the propagation coordinate, respectively. In this framework, the corresponding Wigner-like picture is given and the quantum-like corrections for an arbitrary refractive index are considered. In particular, it is shown that the paraxial radiation beam transport can be also described in terms of a fluid motion equation where the pressure term is replaced by a *quantum-like potential* in semiclassical approximation which accounts for the diffraction of the beam. Finally, a comparison of this fluid model with the Madelug's fluid model is performed.

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1 The Bridge between Optics and Mechanics

Since its early formulation, the analogy between Optics and Mechanics showed to be very fruitful to produce very important physical insights. For example, it is well known that this analogy was very important to arrive, passing through the construction of wave mechanics, at the present formulation of Quantum Mechanics that has been recognized a very fundamental theory of the nature. It is worth to mention some very important steps of the development of this analogy.

The first analogy put *geometrical optics* in correspondence with *classical mechanics*, on the basis of the fully similar formulation of Fermat principle and Hamilton principle.

The above analogy has been soon recognized very important in the study of the motion of charged particle in the presence of electromagnetic fields. The natural development of this branch was the formulation of the *electron optics* which has been applied for several very important scientific and technological applications, such as electron microscopy and particle accelerators [1]-[3]. For many years electron optics remained formulated at the level of geometrical optics. Within this framework, the formulation of electron optics can be given in a way fully similar to electromagnetic geometrical optics provided to replace the notion of *light rays* and *refractive index* with *electron rays* and *potential*, respectively.

The analogy has been extended to the wave level, going from Optics to Mechanics by de Broglie [4] and Schrödinger [5], obtaining the wave mechanics and subsequently the Quantum Mechanics. The *transition* from Classical to Wave Mechanics has been induced just considering the relationship between Geometrical and Wave Optics.

The same kind of transition has been performed by Bohr [6] with a formal procedure called *quantization* based mainly on a set of formal prescriptions called *quantization rules* where Planck's constant plays a role. These rules allow to go from the classical formulation of mechanics into another in terms of operators in such a way to obtain an evolution equation for the physical system under consideration.

Within the framework of the above procedures to transit from Classical to Quantum Mechanics, Schrödinger equation has been recognized as the nonrelativistic limit of a more general wave mechanical formulation induced by the correspondence with wave optics [7]: in fact, the nonrelativistic limit of Klein-Gordon equation, which can be put in correspondence with D'Alambert equation, is just the Schrödinger equation.

Going back from Quantum Mechanics to Wave Optics, the above nonrelativistic limit has been considered also for D'Alambert equation, i.e. for Helmohltz equation, by Fock and Leontovich [8] considering the problem of radiation beam propagation through an arbitrary medium. They showed that the equation which governs this propagation is a sort of Schrödinger equation where $\hbar$ and the time are replaced by the inverse of the wave number and the propagation coordinate, respectively. The Schrödinger-like equation of Fock and Leontovich was actually obtained by the electromagnetic wave equation in paraxial approximation which considers the slopes of the light rays, with respect to the propagation coordinate, very small. It is possible to see that this approximation is equivalent to the so-called slowly-varying amplitude approximation, widely used in nonlinear optics [9]-[11] and in plasma physics [12] as well as to the non relativistic limit of the electromagnetic wave equation.

The above correspondence, going back from Quantum Mechanics to Wave Optics, has been extended more recently by Gloge and Marcuse [13] by performing the transition

from Geometrical Optics to Wave Optics in a way fully similar to the one *ala Bohr*. In the formal quantization of Gloge and Marcuse, a set of quantization rules (in which $\hbar$ and the time are replaced by the inverse of the wave number and the propagation coordinate, respectively) are introduced in the Hamiltonian for the electromagnetic rays. The results is the electromagnetic wave equation whose limit, in paraxial approximation, gives the Fock and Leontovich equation.

The procedure of Gloge and Marcuse revealed to be very fruitful because it provided for transferring algorithms and many solutions of quantum mechanics to radiation beam physics, especially for optical fibers [14, 15], coherent and squeezed states theories [16]-[21], Schrödinger cat states [22, 23], and phase-space investigations within a Wigner-like picture [24] in which a quasi-classical distribution, fully similar to quantum Wigner transform [25] governs the paraxial e.m. ray evolution.

In the recent years, the importance to describe, in an unified way, optics of light and optics of electronic rays has been recognized by some author [26] and the importance to transit from *Geometrical Electron Optics* to *Wave Electron Optics*, has been pointed out [27] as a possible development of Electron Optics.

In the recent years as well, a procedure *ala Gloge and Marcuse* has been introduced in Electron Optics to describe the collective behaviour of charged-particle beam transport [28],[29]-[34]. By using some correspondence rules, called *thermal quantization rules*, in which $\hbar$ and the time are replaced by the *beam emittance* [36] and the propagation coordinate, respectively, a quantum-like description of the electronic rays, called Thermal Wave Model (TWM) can be constructed. This procedure, applied in paraxial approximation, allows to get a Schrödinger-like equation for a complex function, the so-called beam wave function (BWF) whose squared modulus is proportional to the beam density.

A novel approach which consists in a *deformation* of the phase-space equation for electronic rays, has allowed to recover TWM, in a way alternative to the one *ala Gloge and Marcuse*, but only in semiclassical approximation [37]. The *transition* from the classical description to the quantum-like one allows for obtaining a von Neumann-like equation which, in turns, provide for a Wigner-like description of charged-particle beam transport in semiclassical approximation.

In this paper, we propose a method, alternative to the one *ala Gloge and Marcuse*, to transit from Geometrical Optics to Wave Optics namely from the classical-like description to the quantum-like description of Light-ray Optics, by using the above deformation procedure employed in Electron Optics. This allows us to get an effective description of Light-ray Optics which shows the role played by the semiclassical approximation in the quantum-like theory of light rays. In the next Section, we briefly review the quantum-like theory of Gloge and Marcuse and in Section 3 we present the classical-like phase-space equation for light rays for an arbitrary refractive index. In Section 4, the deformation procedure is used to transit from the above classical-like phase-space equation to an effective quantum-like equation in semiclassical approximation which formally coincides with von Neumann equation. This *deformed* phase-space description allows us to recover the Wigner-like picture, widely used to describe the electromagnetic beam transport in phase-space [24]. The quantum-like picture of Gloge and Marcuse as well as the Fock-Leontovich Schrödinger-like equation are then recovered in semiclassical approximation. In Section 5 the hierarchy of the moment equations, associated with the von Neumann-like equation, is obtained, and a fluid model, associated with the beam transport in real space, is obtained by truncation of the above hierarchy. In particular, the case of both

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the classical and the semiclassical fluids are considered. In Section 6 this fluid description is compared with the Madelung fluid model, and finally in Section 7 conclusions and remarks are presented.

2 Brief review of the Gloge and Marcuse Quantum-like Theory

Fermat's principle can be formulated in terms of the following least-action principle [13]

$$\delta \int L(x, y, x', y'; z) \, dz = 0 \quad , \tag{1}$$

where $L(x, y, x', y'; z) = n(x, y, z)(1 + x'^2 + y'^2)^{1/2}$ represents the Lagrangian associated with the propagation of light rays through a medium with an arbitrary refractive index $n(x, y, z,)$, where z is the propagation coordinate, x and y denote the transverse (with respect to z) space coordinates and primes denote differentiation with respect to z. According to the hamiltonian terminology, the following generalized momentum p of components

$$p_x = \frac{\partial L}{\partial x'} = n \frac{x'}{(1 + x'^2 + y'^2)^{\frac{1}{2}}} \quad , \tag{2}$$

$$p_y = \frac{\partial L}{\partial y'} = n \frac{y'}{(1 + x'^2 + y'^2)^{\frac{1}{2}}} \quad , \tag{3}$$

and the hamiltonian

$$H = p_x x' + p_y y' - L = \left(n^2 - p_x^2 - p_y^2\right)^{\frac{1}{2}} \quad , \tag{4}$$

can be introduced. In the case of paraxial approximation, for which rays have a direction close to the propagation direction, say z-axis, (2), (3), and (4) become, in normalized form

$$\mathcal{P}_x = \frac{p_x}{n_0} \approx \frac{n}{n_0} x' \quad , \tag{5}$$

$$\mathcal{P}_y = \frac{p_y}{n_0} \approx \frac{n}{n_0} y' \quad , \tag{6}$$

and

$$\mathcal{H} = \frac{H}{n_0} \approx \frac{1}{2} \left(\mathcal{P}_x^2 + \mathcal{P}_x^2\right) + U \quad , \tag{7}$$

where $U \equiv -n/n_0$ (n_0 being the constant average of n close to the z-axis). Consequently, by introducing in (7) the following formal quantization rules

$$\mathcal{P}_x \to -i \, \chi \, \frac{\partial}{\partial x} \quad , \quad \mathcal{P}_y \to -i \, \chi \, \frac{\partial}{\partial y} \quad , \tag{8}$$

and

$$\mathcal{H} \to i \, \chi \, \frac{\partial}{\partial z} \quad , \tag{9}$$

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where $\lambdabar \equiv \lambda/2\pi$ is the inverse of the wavenumber (λ being the wavelength) associated with the e.m. wave, a Schrödinger-like equation for a complex e.m. wave amplitude Φ can be easily obtained, namely [8]

$$i\ \lambdabar\ \frac{\partial}{\partial z}\Phi = -\ \frac{\lambdabar^2}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi + U\Phi \quad. \tag{10}$$

Eq.(10) is referred as to Fock-Leontovich equation [9]. Provided that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Phi(x,y,z)|^2\ dx\ dy\ =\ 1 \quad, \tag{11}$$

$|\Phi(x,y,z)|^2$ gives the normalized e.m. power density as well as the probability density of finding an e.m. ray at (x,y).

Remarkably, the above paraxial approximation is appropriate for describing an e.m. beam travelling along z-axis. Furthermore, the effective potential U (refractive index) can in general depend on $|\Phi|^2$, which in this case accounts for an e.m. beam propagation through a nonlinear medium. Since the evolution of an e.m. beam is governed by a Schrödinger-like equation for the complex e.m. field amplitude Φ, it is easy to prove the following well-known uncertainty relation:

$$W\ P_W \geq \lambdabar \quad, \tag{12}$$

where

$$W \equiv \left[\frac{\int_{-\infty}^{\infty}r^2|\Phi|^2\ d^2r}{\int_{-\infty}^{\infty}|\Phi|^2\ d^2r}\right]^{1/2} \equiv\ <r^2> \tag{13}$$

is the effective beam radius ($r \equiv \sqrt{x^2 + y^2}$) and

$$P_W \equiv \left[\lambdabar^2\frac{\int_{-\infty}^{\infty}|\nabla_{\perp}\Phi|^2\ d^2r}{\int_{-\infty}^{\infty}|\Phi|^2\ d^2r}\right]^{1/2} \quad, \tag{14}$$

is the total e.m. averaged transverse momentum associated with the e.m. beam .

By following [9] and assuming a the refractive index of the form $U(r,z) = \frac{1}{2}K(z)r^2$ (linear lens), one easily obtains

$$\frac{d^2W}{dz^2} + K(z)W - \frac{\lambdabar^2}{W^3} = 0 \quad. \tag{15}$$

In particular for the propagation in vacuo ($K = 0$) :

$$W(z)\ =\ W_0\ \left(1 + \frac{z^2}{z_R^2}\right) \quad, \tag{16}$$

where W_0 is the minimum spot size (waist) and z_R is the so-called *Rayleigh length* (Note that: $W_0^2 = \lambdabar\ z_R$).

It is worth to note that the limits $\lambdabar \to 0$ and $\hbar \to 0$ recover geometrical optics (light-rays equation) and classical mechanics (classical motion equation), respectively. In fact, the physical meaning of $\lambdabar$ is given in terms of diffraction parameter. The condition $\lambdabar \neq 0$ in paraxial approximation is connected to a weak displacement of light rays from the beam propagation direction in such a way to produce a mixing between them. In the exact geometrical optics limit ($\lambdabar = 0$) the ray would be straight lines parallel to the propagation direction when the beam is travelling in vacuo, whilst for $\lambdabar$ finite the rays mixing (diffraction effect) produces a hyperbolic hyperboloid around the z-axis which corresponds to a typical caustic shape described by (16) [38].

3 Classical-like phase-space equation for light rays

In this section we develop the classical-like description of geometric optics in the phase space in terms of a classical phase-space distribution of the light rays in the case of an arbitrary refractive index. We confine our attention on the case of paraxial approximation.

Taking into account this approximation, we describe, according to Eq.s (5) and (6), the geometrical optics of a light beam which propagates along z-axis. We observe that, Hamilton's equation for (7) give (single-light-ray equations):

$$\frac{d\vec{r}_\perp}{dz} = \vec{\mathcal{P}}_\perp \quad , \tag{17}$$

$$\frac{d\vec{\mathcal{P}}_\perp}{dz} = -\vec{\nabla}_\perp U \quad , \tag{18}$$

where $\vec{r}_\perp \equiv x\hat{x} + y\hat{y}$ and $\vec{\mathcal{P}}_\perp \equiv \mathcal{P}_x\hat{x} + \mathcal{P}_y\hat{y}$. Thus, we can associate with a single ray a classical-like particle trajectory. Consequently, Eq.n (18) shows that the refractive index provides to give an effective force on each single light-ray. In addition, it is clear from (5) and (6) that $\mathcal{P}_x$ and $\mathcal{P}_y$ represent a measure of the light-ray slopes with respect to the propagation direction. These slopes are very small in paraxial approximation, in fact

$$\frac{dx}{dz} \equiv x' \ll 1 \ , and \ \frac{dy}{dz} \equiv y' \ll 1 \quad , \tag{19}$$

i.e. $\mathcal{P}_x$ and $\mathcal{P}_y$ account for the small deviation of the rays with respect to the propagation direction. Consequently, as for the particle systems, we may introduce a distribution in the phase space $\rho(x, y, \mathcal{P}_x, \mathcal{P}_y, z)$ which is constant along the characteristics, i.e.

$$\frac{\partial \rho}{\partial z} + \{\rho, H\} = 0 \quad , \tag{20}$$

where $\{..., ...\}$ denotes the classical Poisson's brackets. Eq.n (20) can be explicitly written as:

$$\frac{\partial \rho}{\partial z} + \left(\vec{\mathcal{P}}_\perp \cdot \vec{\nabla}_\perp\right)\rho - \left(\vec{\nabla}_\perp U\right) \cdot \frac{\partial \rho}{\partial \vec{\mathcal{P}}_\perp} = 0 \quad . \tag{21}$$

Let us consider, around the point $(\vec{r}_{\perp 0}, \vec{\mathcal{P}}_{\perp 0})$, the phase-space volume element $d^2 r_\perp d^2 \mathcal{P}_\perp = dx\,dy\,d\mathcal{P}_x d\mathcal{P}_y$. Thus the quantity $\rho\left(\vec{r}_{\perp 0}, \vec{\mathcal{P}}_{\perp 0}, z\right) d^2 r_\perp d^2 \mathcal{P}_\perp$ is the probability to find a light ray at transverse location $\vec{r}_{\perp 0}$ with slope $\vec{\mathcal{P}}_{\perp 0}$, provided that the following normalization condition holds

$$\int \rho\left(\vec{r}_\perp, \vec{\mathcal{P}}_\perp, z\right) d^2 r_\perp d^2 \mathcal{P}_\perp = 1 \quad . \tag{22}$$

Eq.n (21) describes the evolution of the light rays in paraxial approximation and in the geometrical optics context. However, we point out that (21) is still suitable to describe the beam optics beyond the exact geometrical optics limit. In fact, in the case of vacuum, $U = 0$, and in the case of linear focusing (defocusing) devices, $U = k_1 x^2/2 + k_2 y^2/2$, Gaussian beams, whose propagation is affected by the diffraction, can be also described by (21). To give the reader an idea, let us consider a simple 2-D (y-transverse component is neglected for simplicity) focusing/defocusing, infinitely thick (in the both x and z

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directions) device with refractive index of the form $U = k(z)x^2/2$, where $k(z)$ being the strength of the device. Thus, in this case Eq.n (21) becomes:

$$\frac{\partial \rho}{\partial z} + p\frac{\partial \rho}{\partial x} - k(z)x\frac{\partial \rho}{\partial p} = 0 \quad, \tag{23}$$

where, for simplicity, we have put $\mathcal{P}_x \equiv p$.

We look for a solution of (23) of the form

$$\rho(x,p,z) = A\exp\left\{-\frac{1}{B}\left[c(z)x^2 + 2a(z)xp + b(z)p^2\right]\right\} \quad, \tag{24}$$

where A and B are positive constants, and $a(z)$, $b(z)$, $c(z)$ are functions to be determined. By substituting (24) in (23), we obtain the following conditions:

$$c'(z) = 2k(z)a(z) \quad, \tag{25}$$

$$a'(z) = -c(z) + 2k(z)b(z) \quad, \tag{26}$$

and

$$b'(z) = -2a(z) \quad, \tag{27}$$

where primes again stand for derivative with respect to z. The following results thus hold.

(i). Using (25)-(27) it is easy to see that:

$$b(z)c(z) - a(z)^2 = constant \quad. \tag{28}$$

Without loss of generality, we can assume that the constant is $1/4$, i.e.

$$bc - a^2 = \frac{1}{4} \quad. \tag{29}$$

(ii) the normalization condition, applied to (24), gives

$$A = \frac{\sqrt{cb - a^2}}{\pi B} \quad, \tag{30}$$

which, by virtue of assumption (29), becomes

$$A = \frac{1}{2\pi B} \quad. \tag{31}$$

(iii). Defining the second-order moments, $\sigma_x(z)$, $\sigma_p(z)$, and $\sigma_{xp}(z)$, of ρ as

$$\sigma_x^2(z) = \int \rho(x,p,z)x^2 \, dx \, dp \equiv \langle p^2 \rangle \quad, \tag{32}$$

(beam spot-size)

$$\sigma_p^2(z) = \int \rho(x,p,z)p^2 \, dx \, dp \equiv \langle p^2 \rangle \quad, \tag{33}$$

(momentum spread or r.m.s. of ray slopes)

$$\sigma_{xp}(z) = \int \rho(x,p,z)xp \, dx \, dp \equiv \langle xp \rangle \quad, \tag{34}$$

the use of (24), (29), and (31) allow us to obtain

$$b(z) = \frac{\sigma_x^2(z)}{2B} \quad, \quad c(z) = \frac{\sigma_p^2(z)}{2B} \quad, and \quad c(z) = -\frac{\sigma_{xp}(z)}{2B} \quad . \tag{35}$$

Thus, we easily write (29) in the following way:

$$\sigma_x^2(z) \, \sigma_p^2(z) - \sigma_{xp}^2(z) = B^2 = constant \quad . \tag{36}$$

Consequently, the following inequality can be derived:

$$\sigma_x^2(z) \, \sigma_p^2(z) \geq B \quad . \tag{37}$$

(*iv*). Correspondingly, solution (24) can be cast in the form:

$$\rho(x,p,z) = \frac{1}{2\pi B} \exp\left\{-\frac{1}{2B^2}\left[\sigma_p^2(z)x^2 - 2\sigma_{xp}(z)xp + \sigma_x^2(z)p^2\right]\right\} \quad . \tag{38}$$

Eq.n (38) is the phase-space distribution function of light rays associated with a Gaussian beam whose space density is

$$\Lambda_x(x,z) \equiv \int \rho(x,p,z)dp = \frac{1}{\sqrt{2\pi\sigma_x^2(z)}} \exp\left[-\frac{x^2}{2\sigma_x^2(z)}\right] \quad , \tag{39}$$

and whose ray-slope distribution is

$$\Lambda_p(p,z) \equiv \int \rho(x,p,z)dx = \frac{1}{\sqrt{2\pi\sigma_p^2(z)}} \exp\left[-\frac{x^2}{2\sigma_p^2(z)}\right] \quad . \tag{40}$$

It is worth to observe that, in case the beam is in a vacuum ($k = 0$), solution (24), i.e. (38), remains formally the same but provided that in Eq.s (25)-(27) is put $k = 0$.

Even if the distributions (39) and (40) do not depend explicitly on the constant B, to complete the present description we need to give for it a more precise physical meaning. To this end, we observe that the experimental observations show that, when we produce in vacuo the focusing of a light monochromatic beam of wavelength λ, a diffraction limit holds which can be written as [38]:

$$(\sigma_x \, \sigma_p)_{min} \simeq \frac{\lambda}{4\pi} \equiv \frac{\lambdabar}{2} \quad . \tag{41}$$

Consequently, it follows that

$$B = \frac{\lambdabar}{2} \quad . \tag{42}$$

Using this result in (35) and (38), we finally obtain, respectively

$$\sigma_x = \lambdabar b \quad , \quad \sigma_{xp} = \lambdabar a = \sigma_x\frac{d\sigma_x}{dz} \quad , \quad \sigma_p = \lambdabar c \quad , \tag{43}$$

and

$$\rho(x,p,z) = \frac{1}{\pi \, \lambdabar} \exp\left\{-\frac{2}{\lambdabar^2}\left[\sigma_p^2(z)x^2 - 2\sigma_{xp}(z)xp + \sigma_x^2(z)p^2\right]\right\} \quad . \tag{44}$$

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Remarkably, (41) and (44), show that, due to the diffraction limit, we cannot resolve among two or more light rays in phase-space regions of size the order of λ. If the limit $\lambda \to 0$ is not exactly taken, but nevertheless λ is considered however small, we are still in the context of geometrical optics. Thus, in the paraxial approximation Eq.n (23) still describes the phase-space evolution in a linear device. On the other hand, the diffraction limit introduced by non-zero λ introduces, by virtue of (41), an indistinguishability among the light rays.

In the next section we develop an *effective* phase-space description which takes into account this indistinguishability. We conclude the present section observing that $\Lambda_x(x,z)$ must also represent, according to the results of the previous section, the electromagnetic power density which is proportional to the modulus square of the electromagnetic field amplitude associated with the beam.

4 Deformed phase-space description

In this section we apply a deformation method used recently to transit from the classical phase-space ray equation to a quantum-like phase-space ray equation in semiclassical approximation [37]. We want to make here a similar *transition*, starting from the classical phase-space light ray equation (21). We still confine our attention to the 2-D case (the y-direction is ignored for simplicity) and make the same step as in Ref.[37].

(i). Let σ_0 be the minimum spot size that can be achieved in vacuo with an initial focusing condition, and let us define the parameter $\eta \equiv \lambda/(2\sigma_0)$. It is easy to see that, in paraxial approximation, this quantity is much smaller than 1. In fact, by denoting with σ_{p0} the r.m.s. of the ray-slopes corresponding to the above minimum spot size, from (41) is clear that:

$$\eta \equiv \frac{\lambda}{2\sigma_0} \simeq \sigma_{p0} \simeq \langle \left(\frac{dx}{dz}\right)^2 \rangle_{min}^{1/2} \ll 1 \quad . \tag{45}$$

(ii). The above 2-D phase-space light-ray equation for an arbitrary refractive index can be explicitly written as:

$$\frac{\partial \rho}{\partial z} + p\frac{\partial \rho}{\partial x} - \frac{\partial U}{\partial x}\frac{\partial \rho}{\partial p} = 0 \quad , \tag{46}$$

By introducing the dimensionless variables:

$$\overline{z} \equiv \frac{z}{2\sigma_0} \quad , \qquad \overline{x} \equiv \frac{x}{2\sigma_0} \quad , \tag{47}$$

Eq.n (46) assumes the form:

$$\frac{\partial \overline{\rho}}{\partial \overline{z}} + p\frac{\partial \overline{\rho}}{\partial \overline{x}} - \left(\frac{\partial \overline{U}}{\partial \overline{x}}\right)\frac{\partial \overline{\rho}}{\partial p} = 0 \quad , \tag{48}$$

where $\overline{\rho} \equiv \rho(x/2\sigma_0, p, z/2\sigma_0) \equiv \overline{\rho}(\overline{x}, p, \overline{z})$ and $\overline{U} \equiv U(x/2\sigma_0, z/2\sigma_0) \equiv \overline{U}(\overline{x}, \overline{z})$.

According to the previous results , the indistinguishability among two or more rays due to the paraxial diffraction is the order of $\eta \ll 1$. Thus, $\partial \overline{U}/\partial \overline{x}$ in (46) can be conveniently replaced by the following symmetrized Schwarz-like finite difference ratio:

$$\frac{\partial \overline{U}}{\partial x} \approx \frac{\overline{U}(\overline{x}+\eta/2) - \overline{U}(\overline{x}-\eta/2)}{\eta} \quad . \tag{49}$$

This way, (46) must be replaced by the following equation for an effective distribution, say $\overline{\rho}_w(\overline{x}, p, \overline{z}; \eta)$:

$$\frac{\partial \overline{\rho}_w}{\partial \overline{z}} + p\frac{\partial \overline{\rho}_w}{\partial \overline{x}} - \frac{\overline{U}(\overline{x} + \eta/2) - \overline{U}(\overline{x} - \eta/2)}{\eta}\frac{\partial \overline{\rho}_w}{\partial p} = 0 \quad . \tag{50}$$

Given the smallness of η, multiplying both numerator and denominator of the last term of the l.h.s. by the imaginary unit i, we have:

$$\frac{\overline{U}(\overline{x} + \eta/2) - \overline{U}(\overline{x} - \eta/2)}{i\eta} \, i\frac{\partial \overline{\rho}_w}{\partial p} \approx \frac{\overline{U}(\overline{x} + i\,(\eta/2)\,\partial/\partial p) - \overline{U}(\overline{x} - i\,(\eta/2)\,\partial/\partial p)}{i\eta}\overline{\rho}_w \quad , \tag{51}$$

Thus, going back to the old variables x and z, (50) assumes formally the look of a von Neumann equation [25, 39]:

$$\left\{\frac{\partial}{\partial z} + p\frac{\partial}{\partial x} + \frac{i}{\lambda}\left[U\left(x + i\frac{\lambda}{2}\frac{\partial}{\partial p}\right) - U\left(x - i\frac{\lambda}{2}\frac{\partial}{\partial p}\right)\right]\right\}\rho_w = 0 \quad , \tag{52}$$

where $\rho_w \equiv \overline{\rho}_w(2\sigma_0\overline{x}, p, 2\sigma_0\overline{z}; 2\sigma_0\eta) \equiv \rho_w(x, p, z; \lambda)$. Eq.n (52) shows that in the framework of this effective description, the phase-space evolution equation for light rays is a quantum-like phase-space equation where $\hbar$ and the time t are replaced by λ and the propagation coordinate z, respectively.
However, some considerations are in order.

- Approximation (49) is due both to the smallness of η and the fact that evaluation of $\overline{U}$-variation around the location $\overline{x}$ does not make sense within an interval of size η. This, in fact, corresponds to the intrinsic uncertainty produced among the rays by the paraxial diffraction. Thus, (50) represents a possible way to take into account the ray-mixing produced by the paraxial diffraction in this evaluation.

- Since $\overline{U}(\overline{x} + \frac{i\eta}{2}\frac{\partial}{\partial p}) - \overline{U}(\overline{x} - \frac{i\eta}{2}\frac{\partial}{\partial p}) = \frac{\partial \overline{U}}{\partial \overline{x}}\,i\eta\frac{\partial}{\partial p} + O\left(\eta^3\frac{\partial^3}{\partial p^3}\right)$, approximation (51) is equivalent to assume that terms $O\left(\eta^3\frac{\partial^3}{\partial p^3}\right)$ are small corrections compared to the lower-order ones, according to the paraxial approximation. Consequently, approximation (51) plays the role of *semi-classical approximation* [40].

- While the distribution $\rho(x, p, z)$ involved in (46) is introduced in a classical framework and it is positive definite, the function $\rho_w(x, p, z; \lambda)$ is introduced in a quantum-like framework and it is not positive definite. In fact, in this quantum-like context $\rho_w(x, p, z; \lambda)$ cannot be used to give information within the phase-space cells with size smaller than λ, due to the paraxial diffraction, i.e. due to the indistinguishability among the light rays. It is clear from the von Neumann-like equation (52) that ρ_w is a sort of *Wigner-like* function, which is not positive definite, due to the quantum-like uncertainty principle given in section 3. In analogy with quantum mechanics, $\rho_w(x, p, z; \lambda)$ can be defined a *quasidistribution*, even if its x-projection and p-projection are actually configuration-space distribution and momentum-space distribution, respectively. In particular, we assume that the probability $\Lambda_x(x, z; \lambda)$, introduced above, is:

$$\Lambda_x(x, z; \lambda) = \int \rho_w(x, p, z; \lambda)\, dp \quad , \tag{53}$$

provided that also ρ_w is normalized over the phase space.

Remarkably, from the above results it follows that it may exist a complex function, say $\Psi(x,z)$ such that

$$\Lambda_x(x,z;\chi) = \Psi(x,z)\Psi^*(x,z) \quad , \tag{54}$$

used also for description of pure quantum states, and the following *quantum-like density matrix*

$$G(x,x',z) = \Psi(x,z)\Psi^*(x',z) \quad , \tag{55}$$

used also for description of mixed quantum states, connected with ρ_w by means of the following Wigner-like transformation:

$$\rho_w(x,p,z;\chi) = \frac{1}{2\pi\,\chi}\int_{-\infty}^{\infty} G\left(x+\frac{y}{2}, x-\frac{y}{2}, z\right) \exp\left(i\frac{py}{\chi}\right)\, dy \quad , \tag{56}$$

or, for pure states

$$\rho_w(x,p,z;\chi) = \frac{1}{2\pi\,\chi}\int_{-\infty}^{\infty} \Psi^*\left(x+\frac{y}{2}, z\right) \Psi\left(x-\frac{y}{2}, z\right) \exp\left(i\frac{py}{\chi}\right)\, dy \quad . \tag{57}$$

Consequently, $\Psi(x,z)$ must obey to the following Schrödinger-like equation:

$$i\,\chi\,\frac{\partial\Psi}{\partial z} = -\,\frac{\chi^2}{2}\frac{\partial^2}{\partial x^2}\Psi + U(x,z)\Psi \quad , \tag{58}$$

which is exactly the Fock-Leontovich equation in the case of 2-D radiation beam (see Eq.n (10)). Note that (10) has been recovered with the present deformation method in semiclassical approximation only. Nevertheless, it is valid, in paraxial approximation, beyond the semiclassical approximation, as well.

5 Classical and semiclassical radiation fluids

In this section we consider the hierarchy of the moment equations generated by the von Neumann-like equation (52) up to the second order. This way we can give the picture that we could call *radiation fluid picture*. We distinguish the case of $\chi \to 0$ (*classical radiation fluid*) from the one of small wavelengths (*semiclassical radiation fluid*). To this end, one can calculate the set of moment equations associated with (52) respectively. Defining the following *Liouville operator*

$$\hat{\mathcal{L}} \equiv \frac{\partial}{\partial z} + p\frac{\partial}{\partial x} - \left(\frac{\partial U}{\partial x}\right)\frac{\partial}{\partial p} \quad , \tag{59}$$

(U being an arbitrary refractive index which can be expanded in Taylor series with respect to x), it is easy to see that (52) can be cast as:

$$\hat{\mathcal{L}}\rho_w = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{\chi}{2}\right)^{2k}\frac{\partial^{2k+1}U}{\partial x^{2k+1}}\frac{\partial^{2k+1}\rho_w}{\partial p^{2k+1}} \quad , \tag{60}$$

Note that (60) reduces to (46) when the sum at the r.h.s. is zero. Remarkably, this circumstance is verified not only in the limit $\chi \to 0$. In fact, it occurs also when, keeping non zero χ, the refractive index has a quadratic form in x: this is in full agreement with

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the results presented in Section 3. By introducing the ν–order (ν being a non-negative integer) moment of $\hat{\mathcal{L}}$ as

$$\mathcal{M}^{(\nu)}(x,z) \equiv \int_{-\infty}^{\infty} p^\nu \hat{\mathcal{L}} \rho_w \, dp \quad , \tag{61}$$

Eq.n (60) leads to: – the continuity equation, for $\nu = 0$

$$\frac{\partial \Lambda_x}{\partial z} + \frac{\partial}{\partial x}\left(\Lambda_x V\right) = 0 \quad ; \tag{62}$$

– the motion equation, for $\nu = 1$

$$\left(\frac{\partial}{\partial z} + V\frac{\partial}{\partial x}\right) V = -\frac{\partial U}{\partial x} - \frac{1}{\Lambda_x}\frac{\partial \Pi}{\partial x} \quad ; \tag{63}$$

– the energy equation, for $\nu = 2$

$$\frac{\partial u}{\partial z} + \frac{\partial}{\partial x}\left(uV\right) + \frac{\partial}{\partial x}\left(\Pi V\right) = -\left(\frac{\partial U}{\partial x}\right)\Lambda_x V - \frac{\partial Q}{\partial x} \quad ; \tag{64}$$

and so on, where

$$V(x,z) = \frac{1}{\Lambda_x}\int_{-\infty}^{\infty} p\rho_w \, dp \equiv \langle p \rangle_p \quad , \tag{65}$$

is the *current velocity*, which is experimentally the first order moment of ρ_w,

$$\Pi(x,z) \equiv \int_{-\infty}^{\infty} (p - V)^2 \rho_w \, dp \equiv \Lambda_x \langle (p - \langle p \rangle_p)^2 \rangle_p \quad , \tag{66}$$

which is essentially the radiation pressure or the second order moment of ρ_w,

$$u(x,z) \equiv \frac{1}{2}\Pi + \frac{1}{2}\Lambda_x V^2 \quad , \tag{67}$$

$$Q(x,z) \equiv \frac{1}{2}\int_{-\infty}^{\infty} (p - V)^3 \rho_w \, dp \equiv \Lambda_x \langle (p - \langle p \rangle_p)^3 \rangle_p \quad , \tag{68}$$

which is essentially the analog of the heat. Additionally, from (61) we obtain also:

$$\mathcal{M}^{(\nu)}(x,z) = -\sum_{k=1}^{k_{max} \leq (\nu-1)/2} (-1)^k \begin{pmatrix} \nu \\ 2k+1 \end{pmatrix} \left(\frac{\chi}{2}\right)^{2k} \frac{\partial^{2k+1} U}{\partial x^{2k+1}} \int_{-\infty}^{\infty} p^{\nu-2k-1}\rho_w \, dp \neq 0 \quad \forall \nu \geq 3 \quad . \tag{69}$$

The characteristic of these moment equations is that the one of ν–order is an evolution for the ν–order moment of ρ_w, but contains $(\nu + 1)$–order moment of this function. Provided that a *closure equation* is introduced, which relates $(\nu + 1)$–order moment with the lower-order ones, the truncated set of equations, consisting of moment equations up to the ν–order plus the closure equation, is fully equivalent or (60), respectively.

The fluid description is given when the truncation is introduced at $\nu = 1$ together with a closure relationship involving the second-order moment. Actually, the picture that we could get from the truncation involving (64) can be considered a fluid picture, as well. Furthermore, note that all the Eq.s (69) account for the quantum-like corrections beyond the semiclassical approximation.

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We can estimate the order of the paraxial diffraction introduced in Eq.n (63), assuming the form (44) for ρ_w and making use of (65) and (66). It results that

$$\Pi(x, z) = \frac{\chi^2}{4\sigma_x^2} \Lambda_x(x, z) \quad .$$

(70)

Since

$$\frac{\chi}{2\sigma_x} \leq \frac{\chi}{2\sigma_0} = \eta \ll 1 \quad ,$$

(71)

the last term in (63), viz.

$$\frac{1}{\Lambda_x} \frac{\partial \Pi}{\partial x} \sim \frac{\chi^2}{4\sigma_x^2} \frac{1}{\Lambda_x} \frac{\partial \Lambda_x}{\partial x} \quad ,$$

(72)

represents the semiclassical approximation of the paraxial diffraction at the level of the fluid description. Remarkably, truncating the hierarchy at the order ν more and more high, we get a mesoscopic description more and more deep. Taking all the infinite hierarchy, we will have the most deep mesoscopic description of the system (beyond the semiclassical approximation), which corresponds to a fluid scheme that we could call *Madelung's radiation fluid* (see Section 6).

5.1 Classical radiation fluid (diffraction-less beam)

For arbitrary refractive indexes U, the fluid description for a diffraction-less beam can be obtained from (62) and (63) in the limit $\chi \to 0$:

$$\frac{\partial \Lambda_x^{(0)}}{\partial z} + \frac{\partial}{\partial x} \left(\Lambda_x^{(0)} V^{(0)} \right) = 0 \quad ,$$

(73)

$$\left(\frac{\partial}{\partial z} + V^{(0)} \frac{\partial}{\partial x} \right) V^{(0)} = -\frac{\partial U}{\partial x} \quad ,$$

(74)

where the apexes (0) denotes that we are taking the above limit. In this limit we observe that

$$\rho_w(x, p, z; \chi \to 0) \to \Lambda_x^{(0)}(x, z) \, \delta\left(p - V^{(0)}(x, z) \right) \equiv \rho_0(x, p, z) \quad ,$$

(75)

and the local slope of the light rays, $p = dx/dz$, is determined only by the gradient of U. In particular, in vacuo ($U = 0$) a monochromatic beam has the phase-space density of the form $P_0 \, \delta(p - V_0)$, with P_0 and V_0 constants.

Note that system (73) and (74) is naturally closed. It has been used in radiation beam optics to solve a number of problems when the diffraction is negligible [41].

5.2 Semiclassical radiation fluid

Within the fluid description, we now take into account also the paraxial diffraction. Thus, the semiclassical fluid is described by the Eq.s (62) and (63) plus a suitable closure equation. The result shown by (70) for Gaussian beams suggests us to assume in general this relationship, at the present level of fluid description, as the required suitable closure equation. Consequently, Eq.s (62) and (63) become

$$\frac{\partial \Lambda_x^{(1)}}{\partial z} + \frac{\partial}{\partial x} \left(\Lambda_x^{(1)} V^{(1)} \right) = 0 \quad ,$$

(76)

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$$\left(\frac{\partial}{\partial z} + V^{(1)}\frac{\partial}{\partial x}\right) V^{(1)} = -\frac{\partial U}{\partial x} - \frac{\lambda^2}{4\sigma_x^2} \frac{1}{\Lambda_x^{(1)}} \frac{\partial \Lambda_x^{(1)}}{\partial x} \quad , \tag{77}$$

where the apexes (1) denote that the paraxial diffraction is now taken into account. This system is fully similar to the one that can be usually derived for the transverse motion of a dilute particle beam, assuming a fluid model with the ideal gas state equation [42]. In fact, in this analogy, the term $\frac{\lambda^2}{4\sigma_x^2} \frac{1}{\Lambda_x^{(1)}} \frac{\partial \Lambda_x^{(1)}}{\partial x}$ is replaced, for particle beams, by $\frac{\epsilon^2}{4\sigma_x^2} \frac{1}{n}\frac{\partial n}{\partial x} \simeq \frac{v_{th}^2}{c^2}\frac{1}{n}\frac{\partial n}{\partial x}$, where n is the beam number density, ϵ is the transverse particle beam emittance, v_{th} is the transverse thermal velocity, and where the following properties holds: $\epsilon/2\sigma_x \simeq v_{th}/c \ll 1$ (see Ref. [3]). The above ideal gas state equation assumed in this case is

$$\Pi = \frac{k_B T}{mc^2}n = \frac{\epsilon^2}{4\sigma_x^2}n \simeq \frac{v_{th}^2}{c^2}n \quad , \tag{78}$$

where here Π plays the role of the transverse kinetic pressure. On the other hand, radiation pressure is the effect that radiation produces on the surroundings (ponderomotive action) which is proportional to the square modulus of the electromagnetic field amplitude, E, i.e. $\Pi \propto |E|^2$. By taking into account the physical meaning of both the complex amplitude of Fock-Leontovich equation and the Gloge-Marcuse theory, and according to the results of the previous section, we note that $|E|^2$ and $\Lambda_x^{(1)}$ essentially coincide (apart from a normalization factor), i.e. $|E|^2 \propto \Lambda_x^{(1)}$. Consequently, we can provide for the following physical interpretation of the closure equation (70). We observe that since $\sigma_p \sim \frac{\lambda}{2\sigma_x}$, the mean transverse energy, due to the diffraction, associated with a single light ray (in vacuo) is $\mathcal{E}_0 \equiv \frac{1}{2}\sigma_p^2 \sim \frac{\lambda^2}{8\sigma_x^2}$. We remind that $\Lambda_x(x,z)$) is the probability to find a light ray at location (x,z). Thus, using arguments analogous to the ones used for particle systems (i.e. electronic-ray systems) we thus conclude that the transverse radiation pressure is given by

$$\Pi(x,z) = 2\Lambda_x^{(1)}(x,z)\mathcal{E}_0(z) \sim \frac{\lambda^2}{4\sigma_x^2(z)}\Lambda_x^{(1)}(x,z) \quad . \tag{79}$$

5.3 Coherent states in semiclassical fluid description

In this subsection we give a relevant example of the use of the results presented in the previous subsection. In particular, we show that Eq.s (76) and (77) are very suitable for describing in a very natural way coherent states associated with the radiation fluid motion.

Let us start by considering the case of $V(x,z)$ independent of x, viz.

$$V(x,z) \equiv p_0(z) \quad . \tag{80}$$

This way (77) can be easily integrated with respect to x, giving the normalized density:

$$\Lambda x^1(x,z) = \frac{\exp\left\{-(4\sigma_x^2(z)/\lambda^2)\left[U(x,z) + p_0'(z)x + g(z)\right]\right\}}{\int_{-\infty}^{\infty} \exp\left\{-(4\sigma_x(z)/\lambda^2)\left[U(x,z) + p_0'(z)x + g(z)\right]\right\}\,dx} \quad , \tag{81}$$

where $g(z)$ is an arbitrary function of z, and (76) becomes:

$$\frac{\partial \Lambda_x^1}{\partial z} = -p_0(z)\frac{\partial \Lambda_x^1}{\partial x} \quad . \tag{82}$$

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Note that the density is Gaussian if, and only if, U is quadratic in x. Thus, substituting (81) and (82), we get

$$\frac{\partial U}{\partial z} + p_0(z)\frac{\partial U}{\partial x} = -p_0^{''}(z)x - p_0(z)p_0^{'}(z) - g(z) \quad . \tag{83}$$

Let us define the center $x_0(z)$ of the transverse distribution $\Lambda_x(x,z)$, i.e. the mean value of x:

$$x_0(z) \equiv \int_{-\infty}^{\infty} x\Lambda_x(x,z) \, dx \quad . \tag{84}$$

In general, this quantity could not be zero. Taking into account this observation, we may define now $\sigma_x(z)$ as:

$$\sigma_x^2(z) \equiv \int_{-\infty}^{\infty} (x - x_0(z))^2 \, \Lambda_x(x,z) \, dx \quad . \tag{85}$$

We concentrate now the attention on the case in which the beam does not spread, namely:

$$\sigma_x(z) \equiv \sigma_{x0} = constant \quad . \tag{86}$$

Thus, by differentiating (85) with respect to z and taking into account (82) and (86) we obtain:

$$x_0^{'}(z) = p_0(z) \quad . \tag{87}$$

Let us concentrate our attention now only on the case in which U is independent of z. In this case (83) can be easily integrated with respect to x, obtaining:

$$U(x) = -\frac{1}{2}\frac{p_0^{''}(z)}{p_0(z)}x^2 - \frac{1}{p_0(z)}\left(\frac{1}{2}\frac{dp_0^2(z)}{dz} + g^{'}(z)\right)x + G \quad , \tag{88}$$

where G is an arbitrary constant which, without loss of generality, can be put equal to zero. Consequently, the only possible form of $U(x)$ compatible with (80) is to be quadratic with respect to x. For instance, by choosing:

$$U(x) = \frac{1}{2}kx^2 \, , \quad \text{with} \ \ k > 0 \quad , \tag{89}$$

from (88) we get:

$$p_0^{''} + kp_0 = 0 \quad , \tag{90}$$

and

$$g(z) = -\frac{p_0^2}{2} + g_0 \quad , \tag{91}$$

where g_0 is the an arbitrary constant. On the other hand, taking into account (83), (89), and (90), (81) can be cast in the form:

$$\Lambda_x(x,z) = \sqrt{\frac{k}{2\pi}}\frac{2\sigma_{x0}}{\chi}\exp\left[-\frac{4\sigma_{x0}k}{\chi^2}(x - x_0(z))^2\right] \quad , \tag{92}$$

with:

$$p_0^{'}(z) = -kx_0(z) \, , \quad \text{and} \ \ g(z) = \frac{1}{2}kx_0^2(z) \quad . \tag{93}$$

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Consequently, combining (87), (90), (91), and (93), we obtain

$$\frac{1}{2}p_0^2(z) + \frac{1}{2}kx_0^2(z) = g_0 = constant \quad , \tag{94}$$

and

$$x_0'' + kx_0 = 0 \quad . \tag{95}$$

Finally, by combining (85), (86), and (92), we obtain the condition which relates k, λ, and σ_{x0}:

$$k\sigma_{x0}^4 = \frac{\lambda^2}{4} \quad , \tag{96}$$

and Λ_x can be written as:

$$\Lambda_x(x,z) = \frac{1}{\sqrt{2\pi}\sigma_{x0}} \exp\left[\frac{-(x - x_0(z))^2}{2\sigma_{x0}^2}\right] \quad . \tag{97}$$

We thus can conclude that the distribution (97) with (86), (87), (90), (93), (95), and (96), describe a coherent state associated with the semiclassical radiation fluid. Its physical meaning is fully equivalent to the one given in the standard description [16]-[18]. We would like to point out that the quantum coherent states, which are described by the true Schrödinger equation are only analogs of the ones described by the Fock-Leontovich equation as in the Ref. [24]. The quantities $x_0(z)$ and $p_0(z)$ account for the real and the imaginary parts of the complex shift α which generates all the coherent states, starting from the ground state of both quantum [16]-[18] and quantum-like [31] harmonic oscillator for particle beams:

$$\alpha(z) = \frac{x_0(z)}{2\sigma_{x0}} + i\frac{\sigma_{x0}p_0(z)}{\lambda} \equiv \alpha_1(z) + i\alpha_2(z) \quad . \tag{98}$$

Still keeping U independent of z, we conclude this subsection considering the case of the equilibrium states (stationary states) associated with the semiclassical radiation fluid, which corresponds to the case of $x_0 = constant$. Thus (87) gives $p_0 = 0$ and from (83) we get $g = constant$. Consequently, (81) gives:

$$\Lambda_x(x) = \frac{\exp\left[-\frac{4\sigma_{x0}^2}{\lambda}U(x)\right]}{\int_{-\infty}^{\infty} \exp\left[-\frac{4\sigma_{x0}^2}{\lambda}U(x)\right] dx} \quad . \tag{99}$$

Note that (99) represents a stationary state of the radiation beam for an arbitrary refractive index $U(x)$ in semiclassical approximation.

6 Madelung's radiation fluids

In this section we give the full quantum-like description of the radiation beam beyond the semiclassical approximation developed in the previous section.

To this this end, let us start from the following eikonal representation of the complex electromagnetic field amplitude Ψ appearing in (58):

$$\Psi(x,z) = \Lambda_x^{1/2}(x,z)\exp\left[\frac{i}{\lambda}\Theta(x,z)\right] \quad . \tag{100}$$

Thus, substituting (100) in (58), we obtain the following system of equations:

$$\frac{\partial \Lambda_x}{\partial z} + \frac{\partial}{\partial x}\left(\Lambda_x v\right) = 0 \quad , \tag{101}$$

$$\left(\frac{\partial}{\partial z} + v\frac{\partial}{\partial x}\right) v = -\frac{\partial U}{\partial x} + \frac{\chi^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\Lambda_x^{1/2}}\frac{\partial^2 \Lambda_x^{1/2}}{\partial x^2}\right] \quad , \tag{102}$$

where the current velocity v is now given by

$$v(x,z) = \frac{\partial \Theta(x,z)}{\partial x} \quad . \tag{103}$$

Eq.s (101) and (102) has been widely used in literature [9]-[10] to describe the paraxial propagation of a radiation beam, especially in nonlinear media, where the refractive index depends on Λ_x (i.e. $|\Psi|^2$), being a functional of Λ_x. Moreover, Eq.s (101) and (102) constitute a closed system and are formally identical to the equation that describe the Madelung's fluid [43].

The last term of the r.h.s. of (102) accounts for the pressure term beyond the semi-classical approximation. If we take for Λ_x the form as the one given by (39), the pressure term of (102) coincides with the one shown in (72), and, thus, in this case, u coincides with V. In fact, the term $\frac{\chi^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\Lambda_x^{1/2}}\frac{\partial^2 \Lambda_x^{1/2}}{\partial x^2}\right]$ becomes $-\frac{\chi^2}{4\sigma_x^2}\frac{1}{\Lambda_x}\frac{\partial \Lambda_x}{\partial x}$. One important conse-quence of this result is that coherent states found in semiclassical approximation for the semiclassical radiation fluids (see Section 5) are exact solutions of the Madelung radiation fluid, as well. On the contrary, the stationary states found for non-quadratic refractive indexes (see section 5), are approximate solutions for the semiclassical radiation fluids only. In fact, when $v(x,z) = p_0(z) \equiv 0$, the (99) is not solution if U is not quadratic. For Madelung's fluid, stationary states must have a density Λ_x satisfying the following quantum-like eigenvalue problem associated with the Fock-Lentovich equation:

$$\frac{\chi^2}{2}\frac{\partial^2 \Lambda_x^{1/2}}{\partial x^2} + U(x)\Lambda_x^{1/2} = \mathcal{E}\Lambda_x^{1/2} \quad , \tag{104}$$

where $\mathcal{E}$ is a constant.

7 Conclusions, remarks, and future perspectives

In this paper we have proposed a deformation procedure, recently used to give the quantum-like semiclassical description of the electronic-ray optics [37], to describe, in a quantum-like context, the transition from geometrical optics to wave optics which is alternative to the one proposed by Gloge and Marcuse [13].

Starting from the light-ray equations provided by the Fermat's principle, we have given a phase-space description of the geometrical optics in terms of a classical probability den-sity distribution of the light rays for an arbitrary refractive index. This way, taking into account the quantum-like uncertainty relation (diffraction limit) between the r.m.s. trans-verse ray-position, σ_x, and the r.m.s. ray-slope, σ_p, the above deformation procedure has allowed us to transit to a von Neumann-like equation in semiclassical approximation which provides for a Wigner-like picture of the radiation beam optics in paraxial approximation.

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In turn, this picture has allowed us to recover, in semiclassical approximation, the Fock-Leontovich parabolic equation and its Gloge-Marcuse quantum-like interpretation. In this context, the possible negativity of the Wigner-like function has been correctly explained in terms of the above quantum-like uncertainty relation.

We have also determined the hierarchy of the moment equations associated with the von Neumann-like equation, and thus given both the classical and the semiclassical radiation fluid descriptions in paraxial approximation. In particular, the inclusion of the paraxial diffraction in the fluid context, that characterizes the semiclassical radiation fluid, has allowed us for naturally describing the coherent states associated with the radiation beam, whose fluid interpretation is in fully agreement with the standard one.

Finally, a comparison between the above radiation semiclassical fluid and the Madelung's fluids has been given.

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