Role of the doubly stochastic Neyman type-A and Thomas counting distributions in photon detection

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The Neyman type-A and Thomas counting distributions provide a useful description for a broad variety of phenomena from the distribution of larvas on small plots of land to the distribution of galaxies in space. They turn out to provide a good description for the counting of photons generated by multiplied Poisson processes, as long as the time course of the multiplication is short compared with the counting time. Analytic expressions are presented for the probability distributions, moment generating functions, moments, and variance-to-mean ratios. Sums of Neyman type-A and Thomas random variables are shown to retain their form under the constraint of constant multiplication parameter. Conditions under which the Neyman type-A and Thomas converge in distribution to the fixed multiplicative Poisson and to the Gaussian are presented. This latter result is most important for it provides a ready solution to likelihood-ratio detection, estimation, and discrimination problems in the presence of many kinds of signal and noise. The doubly stochastic Neyman type-A, Thomas, and fixed multiplicative Poisson distributions are also considered. A number of explicit applications are presented. These include (1) the photon counting scintillation detection of nuclear particles, when the particle flux is low, (2) the photon counting detection of weak optical signals in the presence of ionizing radiation, (3) the design of a star-scanner spacecraft guidance system for the hostile environment of space, (4) the neural pulse counting distribution in the cat retinal ganglion cell at low light levels, and (5) the transfer of visual signal to the cortex in a classical psychophysics experiment. A number of more complex contagious distributions arising from multiplicative processes are also discussed, with particular emphasis on photon counting and direct-detection optical communications.

I. Introduction

It is just over 40 years since Neyman introduced a family of straightforward but intriguing generalizations of the binomial and Poisson distributions.[1] He called the simplest and perhaps most useful of these "the contagious distribution of Type-A with two parameters." Neyman's work was motivated by a variety of experimental observations in entomology and bacteriology with which calculations based on the ordinary Poisson failed to agree. He reasoned that the effects of contagion were important in these studies, and he succeeded in introducing this property in a remarkably elegant yet simple way.

The description contagious implies that each favorable event increases (or decreases) the probability of succeeding favorable events. Feller[2] and others have argued that there are essentially two kinds of contagion: true contagion as described above and apparent contagion, where there is an inhomogeneity of the population. It has long been known that certain distributions, such as the negative binomial and the Neyman type-A, could be derived in terms of both types of contagion.[2] Indeed the negative binomial distribution was obtained by Greenwood and Yule[3] in terms of apparent contagion and subsequently by Eggenberger and Polya in an independent study in terms of true contagion.[4–6] It is generally assumed that a detailed study of the correlation between various time intervals is required to distinguish between the two types.[7]

Distributions exhibiting true contagion are special cases of cluster (self-exciting) point processes, which have found application in many disciplines including ecology, economics, entomology, oncology, bacteriology, neurophysiology, epidemiology, forestry, cosmology, operations research, traffic studies, reliability, geophysics, and detection theory. In the cluster case, a mother (or primary) process (often a Poisson point process) generates at each point, with a particular time course, a sequence of daughter (or subsidiary) events. (The multiplicative nature of the process is therefore clear.) The mother process may, for example, be high-energy ionizing particles and the subsidiary or cluster events optical photons. There are two impor-

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tant special cases[8]: the Bartlett-Lewis cluster process in which the subsidiary events are generated cumulatively as a finite renewal process[9,10], and the Neyman-Scott cluster process[11,12] in which the subsidiary events are generated additively, each of the random number of events being independently displaced from its primary generating event. The resultant subsidiary process is, in general, a nonstationary nonrenewal point process. A wealth of information relating to cluster point processes is contained in a remarkable volume edited by Lewis in 1972. Various aspects of the cluster point process are treated in chapters by Lawrance,[13] Daley and Vere-Jones,[14] Fisher,[15] Çinlar,[16] and Neyman and Scott,[12] all in this volume. Snyder has more recently provided an excellent description of self-exciting point processes.[17]

Other contagious distributions originally considered by Neyman[1] are type-A with three or more parameters and type-B and type-C, which are essentially linear combinations of type-A. Beall and Rescia[18] have considered an infinite sequence of two-parameter contagious distributions for arbitrary β, where β is not necessarily an integer. ($\beta = 0, 1, 2$ correspond to the two-parameter type-A, -B, and -C distributions, respectively.) Gurland[19] has shown that the limiting case ($\beta = \infty$) is equivalent to the Polya-Aeppli distribution. The major distinction between this latter distribution and that generated by lower values of β is that any multimodality in the region of low count numbers is diminished as β increases. The Polya-Aeppli superficially resembles the negative binomial; the latter can never be multimodal, however. Following Feller,[2] Gurland[19] has systematically converted the fixed parameters in Neyman's original formulation to random variables, thereby creating an even more general class of contagious distributions. A number of other generalizations have also been put forward.[8]

A class of point processes in which apparent contagion is the key element was first studied by Cox.[20] It was given the appellation doubly stochastic point process (DSPP) to emphasize that in this case two kinds of randomness take place: randomness associated with the point process itself and an independent randomness associated with its rate. Much of the recent development of the properties of the DSPP has been in the context of optics,[21–27] and several excellent reference books are available.[17,28,29] A special case of the DSPP obtains when the stochastic rate is shot noise; it is therefore convenient to call this the shot-noise-driven doubly stochastic Poisson point process (SNDP).[13,30] An important result, first shown by Bartlett,[30] is that the SNDP is a particular Neyman-Scott cluster process.[13] We have recently studied the SNDP in detail, obtaining the singlefold and multifold counting and time statistics[31] as well as time statistics in the presence of dead time and sick time.[32] One important result that emerges from our study is that in the limit of counting times long in comparison with the fluctuation time of the shot noise, a unique SNDP counting distribution emerges, and it is the Neyman type-A.[33]

We are therefore led to an interesting conclusion. The SNDP is a DSPP exhibiting apparent contagion, and it is also a Neyman-Scott cluster process exhibiting true contagion. Evidently there can be no distinction between true and apparent contagion in connection with the SNDP and with the Neyman type-A. More generally, Snyder has shown that a DSPP can be expressed in terms of an equivalent self-exciting point process (see Ref. 17, pp. 292–293).

Another closely related counting distribution will prove useful in our study. In 1949, Thomas[34] introduced a two-parameter counting distribution distinct from the Neyman type-A only in that mother pulses appeared in the final process along with daughter pulses. Although she originally referred to this as the double-Poisson distribution, it has since come to be called the Thomas distribution. Like the Neyman type-A, it is also obtainable from the Neyman-Scott cluster process and will become the limiting counting distribution in a number of important applications.

In the following we develop various properties and limits of the simple and doubly stochastic versions of the two-parameter Neyman type-A, Thomas, and fixed multiplicative Poisson distributions.[35] We will deal with applications to photon, particle, and pulse counting in optics and vision and touch on the performance of systems containing such counting detectors. Finally, we will examine a number of related contagious distributions pertinent to photon counting and optical communications.

II. Properties of the Simple Neyman Type-A and Thomas Counting Distributions

Following Neyman,[1] Feller,[2] and McGill,[36] we consider a mother (primary) distribution describable by the probability law $p(m|W)$. This represents the probability of obtaining m clusters (in time or space) with a given driving rate parameter $W(>0)$. The number of daughters (subsidiaries) per cluster n is assumed to vary independently from one cluster to another according to a probability law $p(n|m)$ that is the same for all clusters. When only daughter pulses appear in the final process, and when mother and daughter distributions are both Poisson, the conditional probability equation yields the Neyman type-A counting distribution[37]

$$p(n|W) = \sum_{m=0}^{\infty} p(n|m)p(m|W)$$

$$= \sum_{m=0}^{\infty} \frac{(am)^n e^{-am}}{n!} \frac{W^m e^{-W}}{m!}, \tag{1a}$$

$$p(0|W) = \exp[-W(1 - e^{-a})]. \tag{1b}$$

The quantity $a(>0)$ is called the Neyman type-A (or multiplication) parameter and provides a measure of the average number of daughters per mother event, assuming that all daughters are included in the counting interval. The moment generating function $Q_n(t) = \langle e^{tn} \rangle$ can be readily calculated from Eq. (1)[38]:

$$Q_n(t) = \exp(W\{\exp[a(e^t - 1)] - 1\}). \tag{2}$$

The mean, variance, and variance-to-mean ratio are

$$\langle n \rangle = aW, \tag{3a}$$

$$\langle (\Delta n)^2 \rangle = (1 + a)aW, \tag{3b}$$

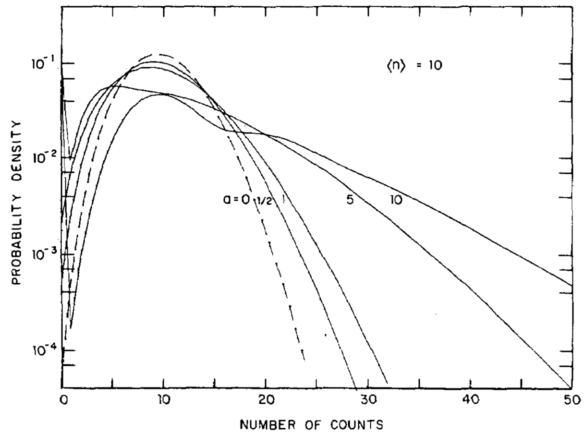

Fig. 1. Semilogarithmic plot of the Neyman type-A counting distribution $p(n|W)$ vs the count number n. The overall mean $\langle n \rangle$ = 10. Curves are labeled according to the value of the parameter a. In the limit as $a \to 0$, the Neyman type-A approaches the Poisson (dashed curve). The multimodal character of the distribution is evident.

$$\langle(\Delta n)^2\rangle/\langle n \rangle = 1 + a. \tag{3c}$$

It is apparent from Eq. (1a) that $W = \langle m \rangle$. Some years ago Grimm[39] compiled tables of the cumulative density function for various values of $\langle n \rangle$ and a. The behavior of this distribution is best illustrated graphically, however. In Fig. 1, we present a plot of the Neyman type-A counting distribution [$\log p(n|W)$ vs n] for fixed overall mean $\langle n \rangle = 10$, with the parameter a varying between 0 and 10. Its multimodal character is apparent.

The Thomas distribution[34] is similar, but mother pulses appear in the final process along with daughter pulses. It is written as the finite sum

$$p(n|W) = \sum_{m=1}^{n} \frac{(\alpha m)^{n-m} e^{-\alpha m}}{(n-m)!} \frac{W^m e^{-W}}{m!}, \tag{4a}$$

$$p(0|W) = \exp[-W]. \tag{4b}$$

The Thomas parameter $\alpha(\geq 0)$ denotes the average number of daughters per mother and plays a role analogous to the a parameter in the Neyman type-A. Equation (4a) can be most readily understood by considering the upper and lower limits on the summation index m: for $m = n$, all n counts arise from mother pulses ($n - m = 0$), whereas for $m = 1$ only one count is associated with a mother pulse, the remainder arising from daughter pulses ($n - m = n - 1$). Equation (4b) expresses the requirement that recording 0 pulses required 0 mother pulses. The moment generating function $Q_n(t)$ is

$$Q_n(t) = \exp(W\{e^t \exp[\alpha(e^t - 1)] - 1\}), \tag{5}$$

from which the mean, variance, and variance-to-mean ratio are calculated to be

$$\langle n \rangle = (1 + \alpha)W, \tag{6a}$$

$$\langle(\Delta n)^2\rangle = [(1+\alpha)^2 + \alpha]W = (1 + 3\alpha + \alpha^2)W, \tag{6b}$$

$$\langle(\Delta n)^2\rangle/\langle n \rangle = \left[(1+\alpha) + \left(\frac{\alpha}{1+\alpha}\right)\right]. \tag{6c}$$

Again, from Eq. (4a) it is clear that $W = \langle m \rangle$. The appearance of the mother pulses in the Thomas distribution regularizes the pulse train. Thus the variance-to-mean ratio is lower for a Thomas than for a Neyman type-A of equal mean.

Because of the way in which both the Neyman type-A and the Thomas distributions are generated, they should become indistinguishable as $a, \alpha \to \infty$. In that limit, for $\langle m \rangle$ fixed, Eqs. (3) and (6) become

$$\langle n \rangle = AW, \tag{7a}$$

$$\langle(\Delta n)^2\rangle = A^2 W, \tag{7b}$$

$$\langle(\Delta n)^2\rangle/\langle n \rangle = A, \tag{7c}$$

where we have set $a = \alpha = A$. These are, of course, the often-used moments for the fixed multiplicative Poisson distribution with parameter A (≥ 1):

$$p(n|W) = \sum_{m=0}^{\infty} p(n|m)p(m|W) = \sum_{m=0}^{\infty} \delta\left(\frac{n}{A} - m\right)\frac{W^m e^{-W}}{m!}$$
$$= \frac{W^{n/A} e^{-W}}{(n/A)!} \qquad n/A = 1,2,3,\ldots, \tag{8a}$$

$$p(0|W) = \exp[-W]. \tag{8b}$$

In this limit, the width of the daughter distribution $p(n|m)$ is negligible, and the distribution usually takes on a scalloped appearance. The moment generating function for the fixed multiplicative Poisson distribution is easily calculated to be

$$Q_n(t) = \exp[W(e^{At} - 1)]. \tag{9}$$

At the opposite extreme, the Neyman type-A approaches the Poisson when $a \to 0$ with $\langle n \rangle$ finite (see Fig. 1), and the Thomas becomes the Poisson when $\alpha = 0$. The fixed multiplicative Poisson is identical with the Poisson for $A = 1$. We will subsequently demonstrate that both the Neyman type-A and the Thomas converge in distribution to the Gaussian for a, α fixed and finite, when the mean of the driving distribution $W = \langle m \rangle$ increases without limit (see Sec. IV).

Finally we note that the characteristic function is obtained from Eqs. (2) and (5) by the simple substitution $t \to iu$, whereas the probability generating function is obtained from Eqs. (2) and (5) by the substitution $e^t \to v$.[40]

III. Sums of Neyman Type-A and Thomas Random Variables

Let z_s represent the sum of s mutually independent random variables n_i,

$$z_s = \sum_{i=1}^{s} n_i \tag{10}$$

with moment generating functions

$$Q_{n_i}(t) = \langle e^{tn_i} \rangle, \tag{11}$$

whence

$$Q_{z_s}(t) = \prod_{i=1}^{s} Q_{n_i}(t). \tag{12}$$

For Neyman type-A random variables with identical a parameters, Eq. (2) provides

$$Q_{n_i}(t) = \exp(W_i\{\exp[a(e^t - 1)] - 1\}), \qquad (13)$$

which, when combined with Eq. (12), leads to

$$Q_{z_s}(t) = \exp\left(\left\{\sum_{i=1}^{s} W_i\right\}\{\exp[a(e^t - 1)] - 1\}\right). \qquad (14)$$

Equation (14) represents the moment generating function for a Neyman type-A distribution with an a parameter identical with that of each constituent distribution, and with mean

$$\langle n \rangle = aW = a \sum_{i=1}^{s} W_i \qquad (15a)$$

and variance

$$\langle(\Delta n)^2\rangle = (1 + a)aW = (1 + a)a \sum_{i=1}^{s} W_i. \qquad (15b)$$

For Thomas random variables with identical α parameters, Eq. (5) gives

$$Q_{n_i}(t) = \exp(W_i\{e^t \exp[\alpha(e^t - 1)] - 1\}), \qquad (16)$$

which, when combined with Eq. (12), yields

$$Q_{z_s}(t) = \exp\left(\left\{\sum_{i=1}^{s} W_i\right\}\{e^t \exp[\alpha(e^t - 1)] - 1\}\right). \qquad (17)$$

Equation (17) represents the moment generating function for a Thomas distribution with an α parameter identical with that of each constituent distribution and with mean

$$\langle n \rangle = (1 + \alpha)W = (1 + \alpha) \sum_{i=1}^{s} W_i \qquad (18a)$$

and variance

$$\langle(\Delta n)^2\rangle = (1 + 3\alpha + \alpha^2)W = (1 + 3\alpha + \alpha^2) \sum_{i=1}^{s} W_i. \qquad (18b)$$

We conclude that the sum of an arbitrary number of Neyman type-A (Thomas) random variables will remain Neyman type-A (Thomas) provided that the $a(\alpha)$ parameter associated with all the random variables is identical. The mean and variance of the summated random variable are represented by the sums of the component means and variances, respectively. This property of the Neyman type-A and Thomas distributions makes them similar to the simple Poisson and the fixed multiplicative Poisson [see Eq. (9)]; indeed all are infinitely divisible distributions.[41]

It is of interest to note that a simple result of this form will not emerge if the a, α, or A parameters differ, even if we consider only two random variables, nor will it for the summation of a Neyman type-A with a Thomas or Poisson random variable.

We will demonstrate in the next section, however, that the Neyman type-A and Thomas converge in distribution to the Gaussian as $W \to \infty$. In that limit, the summation of s Neyman type-A random variables, with individual means $\langle n_i \rangle$ and parameters a_i, will result in a Gaussian random variable with mean

$$\langle n \rangle = \sum_{i=1}^{s} \langle n_i \rangle = \sum_{i=1}^{s} a_i W_i \qquad (19a)$$

and variance

$$\langle(\Delta n)^2\rangle = \sum_{i=1}^{s} \langle(\Delta n_i)^2\rangle = \sum_{i=1}^{s} (1 + a_i)a_i W_i. \qquad (19b)$$

Similarly, the summation of s Thomas random variables, with individual means $\langle n_i \rangle$ and parameters α_i, will result in a Gaussian random variable with mean

$$\langle n \rangle = \sum_{i=1}^{s} \langle n_i \rangle = \sum_{i=1}^{s} (1 + \alpha_i) W_i \qquad (20a)$$

and variance

$$\langle(\Delta n)^2\rangle = \sum_{i=1}^{s} \langle(\Delta n_i)^2\rangle = \sum_{i=1}^{s} (1 + 3\alpha_i + \alpha_i^2) W_i. \qquad (20b)$$

It is clear that, in this same limit, the Neyman type-A (Thomas) random variable may be added together with any number of arbitrary independent random variables that converge in distribution to the Gaussian (e.g., Poisson, binomial, Neyman type-A, Thomas) to yield an overall Gaussian random variable with mean $\langle n \rangle = \Sigma_i \langle n_i \rangle$ and variance $\langle(\Delta n)^2\rangle = \Sigma_i \langle(\Delta n_i)^2\rangle$. This result is extremely useful in a broad variety of practical applications (see Sec. VI).

IV. Convergence in Distribution of Neyman Type-A and Thomas to Gaussian

We demonstrate that the Neyman type-A and Thomas counting distributions converge to the Gaussian by examining the moment generating functions $Q_n(t)$. In the Appendix, we explicitly show that the random variable

$$k = \frac{n - \langle n \rangle}{\langle(\Delta n)^2\rangle^{1/2}} \qquad (21)$$

converges in distribution to the standard normal random variable when the mean of the driving distribution W increases without limit.

It is also evident from the calculations presented in the Appendix that for fixed W and $a \to \infty$, finite terms other than $t^2/2$ appear in the moment generating functions. Indeed, in that limit the Neyman type-A and Thomas approach the fixed multiplicative Poisson rather than the Gaussian distribution. We have already mentioned that the former has a scalloped appearance, whereas the latter, of course, is a smooth bell-shaped function. Martin and Katti[42] have shown that for small W, the Neyman type-A can be approximated by a Poisson distribution with additional zeros.

For the Neyman type-A, behavior representing the limits described above is apparent in Fig. 1 ($\langle n \rangle = 10$), where the $a \to 0$ curve is Poisson, and where the ($a = 1/2$, $W = 20$) and ($a = 1$, $W = 10$) curves are smooth and bell-like, whereas the ($a = 5$, $W = 2$) and ($a = 10$, $W = 1$) curves have begun to scallop. A quantitative measure of the goodness of the Gaussian approximation has been presented by Martin and Katti[42] for limited ranges of the parameters W and a.

V. Properties of the Doubly Stochastic Neyman Type-A and Thomas Distributions

In an attempt to explain the relative frequency of multiple occurrences of accidents in a factory population, Greenwood and Yule[3] used the notion of an inhomogeneous rate parameter to generalize the Poisson distribution. These authors assumed that although the probability of accident for a given worker (in a British munitions factory) followed the simple Poisson law, variation in individual proneness to accident caused the accident rate to vary from individual to individual in the population. They then calculated the overall probability of multiple accidents using certain plausible density functions for this individual variation of rate parameter. The result is the doubly stochastic (in this case mixed) Poisson counting distribution (which exhibits apparent contagion).

In this section, we introduce an inhomogeneity of the rate parameter into the Neyman type-A, Thomas, and fixed multiplicative Poisson distributions in the manner described above. We thereby generate what we call the doubly stochastic Neyman type-A, doubly stochastic Thomas, and doubly stochastic fixed multiplicative Poisson distributions.

We begin with the Neyman type-A kernel represented in Eq. (1). Define $P(W)$ as the density function for the driving rate parameter W, which is now stochastic. The counting statistics emerge from removal of the conditioning on W expressed in Eq. (1):

$$p(n) = \int_0^\infty p(n|W)P(W)dW, \tag{22a}$$

$$= \int_0^\infty \sum_{m=0}^\infty p(n|m)p(m|W)P(W)dW, \tag{22b}$$

$$= \sum_{m=0}^\infty p(n|m)p(m). \tag{22c}$$

Equation (22a) may be interpreted as the Neyman type-A distribution with a smeared mean; Eq. (22c), on the other hand, which is its equivalent, is readily interpreted as a doubly stochastic Poisson mother distribution $p(m)$ giving rise to clusters of daughters that are Poisson.

The moment generating function $Q_n^*(t)$ is easily calculated from its definition

$$Q_n^*(t) = \sum_{n=0}^\infty e^{tn}p(n) = \sum_{n=0}^\infty e^{tn} \int_0^\infty p(n|W)P(W)dW$$

$$= \int_0^\infty P(W)dW \sum_{n=0}^\infty e^{tn}p(n|W)$$

$$= \int_0^\infty P(W)dWQ_n(t), \tag{23}$$

where $Q_n(t)$ represents the moment generating function for the simple Neyman type-A. Substituting Eq. (2) into Eq. (23), we obtain

$$Q_n^*(t) = \int_0^\infty \exp(W\{\exp[a(e^t - 1)] - 1\})P(W)dW$$

$$= Q_W (\exp[a(e^t - 1)] - 1). \tag{24}$$

Because of the exponential form of Eq. (2), the calculation of the moment generating function for the doubly stochastic Neyman type-A is a trivial enterprise. Simply write down the moment generating function for the stochastic rate $Q_W(s)$ and substitute $s = (\exp[a(e^t - 1)] - 1)$. The exponential form for the moment generating functions of the Thomas and fixed multiplicative Poisson simplify our calculations in those cases as well.

The mean and variance are easily obtained from Eq. (24), but we provide a more direct route to these results by using well-known formulas for the conditional expectation.[41]

Using the notation $E[\cdot]$ and $\mathrm{var}[\cdot]$ to represent expectation and variance with respect to W, we obtain

$$\langle n \rangle = E[\langle n|W\rangle] = E[aW] = a\langle W\rangle \tag{25a}$$

with the help of Eq. (3a), and

$$\begin{aligned}\langle(\Delta n)^2\rangle &= E[\langle(\Delta n|W)^2\rangle] + \mathrm{var}[\langle n|W\rangle] \\ &= E[(1+a)aW] + \mathrm{var}[aW] \\ &= (1+a)a\langle W\rangle + a^2\langle(\Delta W)^2\rangle\end{aligned} \tag{25b}$$

with the help of Eqs. (3a) and (3b). Equation (25b) may be expressed in words by saying that the unconditional variance is equal to the mean of the conditional variance plus the variance of the conditional mean. From Eqs. (25a) and (25b), the variance-to-mean ratio is

$$\frac{\langle(\Delta n)^2\rangle}{\langle n\rangle} = (1+a) + a\frac{\langle(\Delta W)^2\rangle}{\langle W\rangle}. \tag{25c}$$

The limits of Eq. (25c) are seen to be proper. For W fixed, $\langle(\Delta W)^2\rangle = 0$, and the ratio reduces to $(1 + a)$, which is appropriate for the simple Neyman type-A; for $a \rightarrow 0$ with $a\langle W\rangle$ finite, $\langle(\Delta n)^2\rangle/\langle n\rangle \rightarrow 1 + \langle(\Delta aW)^2\rangle/\langle aW\rangle$, which is precisely that for the doubly stochastic Poisson[43]; and, for a increasing without limit, the ratio approaches $a(1 + \langle(\Delta W)^2\rangle/\langle W\rangle)$, which is the result for the doubly stochastic fixed multiplicative Poisson as we will see subsequently.

Little imagination is required to generate parallel results for the doubly stochastic Thomas distribution. Combining Eqs. (4) and (22) gives us the unconditional probability distribution $p(n)$. The moment generating function $Q_n^*(t)$ is calculated by combining Eqs. (5) and (23) to provide

$$Q_n^*(t) = Q_W(e^t \exp[\alpha(e^t - 1)] - 1). \tag{26}$$

Using the conditional expectation relations provided earlier and making use of Eq. (6), the mean, variance, and variance-to-mean ratios are calculated to be

$$\langle n\rangle = (1+\alpha)\langle W\rangle, \tag{27a}$$

$$\langle(\Delta n)^2\rangle = [(1+\alpha)^2 + \alpha]\langle W\rangle + (1+\alpha)^2\langle(\Delta W)^2\rangle, \tag{27b}$$

$$\frac{\langle(\Delta n)^2\rangle}{\langle n\rangle} = (1+\alpha) + \frac{\alpha}{1+\alpha} + (1+\alpha)\frac{\langle(\Delta W)^2\rangle}{\langle W\rangle}. \tag{27c}$$

The limits of Eq. (27c) are also proper. For W fixed, $\langle(\Delta n)^2\rangle/\langle n\rangle$ reduces to $(1+\alpha) + \alpha/(1+\alpha)$, which is the simple Thomas result; for $\alpha = 0$, the ratio becomes $1 + \langle(\Delta W)^2\rangle/\langle W\rangle$, which is the doubly stochastic Poisson result[43]; and of course for $\alpha \rightarrow \infty$, Eq. (27c) becomes $\alpha(1 + \langle(\Delta W)^2\rangle/\langle W\rangle)$, which is the result for the doubly stochastic Neyman type-A and doubly stochastic fixed multiplicative Poisson, as expected.

The counting distribution for the doubly stochastic fixed multiplicative Poisson is obtained by inserting Eq. (8) into Eq. (22). Combining Eqs. (9) and (23) provides the moment generating function

$$Q_n^*(t) = Q_W(e^{At} - 1). \tag{28}$$

Finally, using Eq. (7) and the conditional expectation formulas cited earlier yields

$$\langle n \rangle = A\langle W \rangle, \tag{29a}$$

$$\langle(\Delta n)^2\rangle = A^2\langle W \rangle + A^2\langle(\Delta W)^2\rangle, \tag{29b}$$

$$\frac{\langle(\Delta n)^2\rangle}{\langle n \rangle} = A + A\frac{\langle(\Delta W)^2\rangle}{\langle W \rangle}, \tag{29c}$$

verifying the promised conjunction of the doubly stochastic Neyman type-A, Thomas and fixed multiplicative Poisson distributions as a, $\alpha \to \infty$.

It is clear from Eqs. (22a), (22c), (25c), (27c), and (29c) that the doubly stochastic Neyman type-A, Thomas, and fixed multiplicative Poisson distributions are broader than their respective underlying kernels. They are also broader than the equivalent [same $P(W)$] doubly stochastic Poisson distributions.[43]

VI. Applications in Pulse, Particle, and Photon Counting

Although the distributions and mathematical properties discussed to this point enjoy application in a broad variety of disciplines, we restrict our attention to their use in optics and vision. We consider specific examples in retinal neural pulse counting, scintillation counting, and photon counting in the presence of ionizing radiation. These models will apply when the counting time is large compared with the characteristic decay time of the multiplication process.

In a neurophysiological study of the responses of cat retinal ganglion cells to light, Barlow *et al.*[44] found that single quantal absorptions stimulated multiple neural impulses. A study of the experimental mean and variance of the pulse-counting distributions produced results in accord with Eqs. (3) and (7), with $a, A \simeq 2$. Although Barlow *et al.* did not explicitly refer to the fixed multiplicative Poisson and the Neyman type-A by name, these are indeed the distributions they used to model the statistical behavior of the discharge in the cat's retinal ganglion cell at low light levels and in darkness. We have since provided a model for the maintained discharge time statistics in terms of the SNDP.[32] Another example of the use of the Neyman type-A in neural counting was provided by McGill,[36] who hypothesized that it plays a vital role in visual psychophysics. He supposed that the distribution $p(m|W)$ [see Eq. (1)] represented the Poisson flow of photons from an incandescent light source used as the stimulus, whereas $p(n|m)$ reflected the Poisson distribution of neural impulses, induced by m photons, at some central counting center. McGill argued that the smearing together of many neural paths at a hypothetical counting center in the chain to the visual cortex will produce a Poisson-like central noise process under a broad range of conditions. There is ample support for such a convergence to the Poisson from a mathematical point of view.[16] We have recently performed a series of psychophysical experiments at the threshold of seeing in humans that are consistent with such an interpretation.[45,46]

We now consider an example in the nuclear counting domain where we often detect ionizing radiation optically through a radiation–matter interaction in which a single high-energy particle produces a shower of lower-energy particles. A case in point is the scintillation detector, which is a combination of a scintillation crystal (e.g., NaI:Tl, plastic) with a photomultiplier tube.[47] When the incident high-energy particles (e.g., β, p, γ) are Poisson distributed, and when each particle produces a Poisson distribution of luminescence photons with a specified efficiency, such as occurs in fluorescence and phosphorescence, the resulting photon counting distribution representing the signal is the Neyman type-A.[31,33] If Čerenkov radiation is also present, mother pulses will be registered, and the outcome is the Thomas, assuming again that we are performing photon counting.[48] It is usually assumed in the literature, generally tacitly, that the fixed multiplicative Poisson distribution describes the luminescence photon statistics, and indeed when the counting time is long and a is large, this is a good approximation. But if photon counting is used, and the luminescence is weak, it will be necessary to use the more accurate forms described above. Similar statistics will apply to the detection of cathodoluminescence and photoluminescence.

In certain applications where we wish to count the photons in an optical signal (e.g., astronomy), it sometimes happens that the distributions discussed in the previous paragraph are characteristic of the noise rather than of the signal. Viehmann and Eubanks[49,50] have discussed sources of noise in photomultiplier tubes in the radiation environment of space. Such noise may arise from several mechanisms such as luminescence and Čerenkov radiation in the photomultiplier window; secondary electron emission from the window, photocathode, and dynodes; Bremsstrahlung in turn causing such secondary electron emission; cosmic-ray bursts; and, of course, thermionic emission dark current. These effects clearly degrade both the dynamic range and the photometric accuracy of low-light-level measurements and therefore must be clearly understood. Photon counting can be particularly advantageous in such situations: Even if a large number of photoelectrons are produced by the Čerenkov radiation arising from a single charged particle (which constitutes noise in this case), they will be counted as only a single pulse, since the Čerenkov radiation emission time is much shorter than the transit time in the photomultiplier. We will shortly report on a series of photon counting experiments carried out in the presence of ionizing radiation that demonstrate the usefulness of the Neyman type-A and Thomas distributions.[33,51] We mention the image intensifier as another photon counting application of the distributions discussed in Sec. II.

The mathematical properties of the Neyman type-A, Thomas, and fixed multiplicative Poisson distributions discussed in Secs. III and IV are very useful for the study of the performance of systems. The permanence under convolution of distributions with identical parameters a, α, or A facilitates their use in statistical detection and estimation problems for signal and noise distributions of arbitrary means.[52] But it is the convergence in distribution of the Neyman type-A and Thomas to the Gaussian, demonstrated in Sec. IV, that can simplify calculations enormously in many neural, nuclear, and photon counting applications. In that limit (a, α finite, $W \to \infty$), as outlined at the end of Sec. III, the Neyman type-A (Thomas) random variable with arbitrary $a(\alpha)$ may be added together with any number of arbitrary random variables that converge in distribution to the Gaussian (e.g., Poisson, binomial, Neyman type-A, Thomas) to yield an overall Gaussian random variable with mean $\langle n \rangle = \Sigma_i \langle n_i \rangle$ and variance $\langle (\Delta n)^2 \rangle = \Sigma_i \langle (\Delta n_i)^2 \rangle$.

This is a powerful result because it means that the entire statistical decision theory and estimation theory literature relating to Gaussian random variables[53] can be brought to bear on the problem. Some results that follow immediately are single-threshold detection,[54] simple forms for expressing system performance and system SNR,[52–55] well-known solutions for estimation problems,[52] and a simple form for the just-noticeable difference or detection law.[56]

As an interesting example of the convergence to the Gaussian, we consider the design of a photon counting photomultiplier star scanner for the guidance system of a spacecraft exposed to intense ionizing radiation. Quantities important to characterize the detection performance and attitude accuracy for this kind of system are error probabilities and the system SNR. To carry our example further, we assume the absence of atmospheric turbulence that would corrupt our starlight signal, and we assume a counting time much greater than the coherence time of the starlight. In this case the signal will produce Poisson photon counts (mean $\langle n_s \rangle$). Drawing on our earlier discussion of noise in photomultiplier tubes in the radiation environment of space, we may expect the (independent) dominant sources of noise to be Neyman type-A counts from γ-induced fluorescence (driving mean $\langle m_\gamma \rangle$, parameter $\overline{a}$), Thomas counts from charged-particle-induced Čerenkov radiation and fluorescence (driving mean $\langle m_\beta \rangle$, parameter $\overline{\alpha}$), and thermionic emission Poisson dark counts (mean $\langle n_d \rangle$). Using the usual definition of the SNR for Gaussian random variables[41,55] and designating σ_n as the standard deviation of the noise distribution, we obtain

$$\mathrm{SNR} \simeq \frac{\langle n_s \rangle}{\sqrt{2}\sigma_n} = \frac{\langle n_s \rangle}{\sqrt{2}[(1+\overline{a})\overline{a}\langle m_\gamma \rangle + (1 + 3\overline{\alpha} + \overline{\alpha}^2)\langle m_\beta \rangle + \langle n_d \rangle]^{1/2}}. \tag{30}$$

For a, $\alpha \gg 1$, and negligible dark counts, under the same constraint of Gaussian signal and noise, Eq. (30) provides the very simple result

$$\mathrm{SNR} \simeq \frac{\langle n_s \rangle}{(2\overline{a}^2\langle m_\gamma \rangle + 2\overline{\alpha}^2\langle m_\beta \rangle)^{1/2}}. \tag{31}$$

A calculation of the exact error probabilities, especially when small, may be more difficult because they are particularly sensitive to the validity of the Gaussian approximation in the tails of the distribution.[57] The results obtained by Martin and Katti[42] and the use of Chernov bounds and Monte Carlo simulations in the manner of Personick *et al.*[57] may be useful in this connection. Aside from reporting on photon counting experiments, we will carry out specific estimation calculations for a system similar to the star-scanner guidance system envisioned for the NASA/JPL Galileo mission scheduled to orbit Jupiter in 1988.[58]

The doubly stochastic Neyman type-A, Thomas, and fixed multiplicative Poisson distributions will be useful in photon, particle, and pulse counting when there is a modulation of the driving-rate parameter. Three examples readily present themselves: (1) We have extended the neural counting model for visual psychophysics proposed by McGill[36] to the case of intensity modulated radiation, leading to the doubly stochastic Neyman type-A distribution.[45,46] Experimental results are consistent with this model.[45,46] (2) Fluorescence and Čerenkov radiation produced by the interaction of a modulated relativistic electron beam with a material will lead to doubly stochastic Thomas photon counts. (3) The photon counting distribution at the output of an image intensifier will be doubly stochastic Neyman type-A if the incident optical radiation is modulated and the counting time is long compared with the time scale of the multiplication.

VII. Related Contagious Distributions Useful in Photon Counting and Optical Communications

It has been pointed out in Sec. I that a broad variety of contagious distributions may be constructed, although to this point we have limited our discussion to the simple and doubly stochastic Neyman type-A, Thomas, and fixed multiplicative Poisson distributions. In this final section, we explicitly mention a number of more complex distributions particularly related to photon, particle, and pulse counting. The reader is cautioned, however, that whereas the simple Neyman type-A, Thomas, and fixed multiplicative Poisson distributions are described by two parameters, many of these more complex distributions require more than two parameters for their specification.

Feller[2] and Rogers[40] discuss the Poisson/binomial distribution, which may be useful when not all daughter events are included in a sampling interval. It has been shown by Pielou[59] and Gleeson and Douglas[60] that quadrat size (or sampling interval) and cluster spread affect the estimation of parameters for these distributions. In general, the driving rate parameter is overestimated, whereas the multiplication parameter is underestimated, as is intuitively expected. Clearly the SNDP will provide a more realistic model here.[31,32]

Another case of interest in particle and pulse counting occurs when the mother-pulse distribution $p(m \mid W)$ is a dead-time-modified Poisson[61,62] rather than a Poisson. The primary process then exhibits self-excitation in its own right. If the dead time is triggered by the daughter pulses, however, the situation is more difficult, although we have recently obtained results for the time-interval probability density function[32] and for the count mean and variance.[51]

Finally, we explicitly consider two rather complex contagious distributions useful in the design of practical photon counting and optical communication systems.

Personick[63] considered the performance of a simple binary direct-detection fiber-optic communication link making use of a unilateral gain (single-carrier ionization) avalanche photodiode. He assumed Gaussian thermal noise statistics, zero dark count, and signal statistics that are Poisson/shifted-Bose-Einstein which may be approximated by the Poisson/exponential for large gain. This distribution arises in the limit of a sequence of Thomas distributions driving Thomas distributions (successive compounding of the Thomas). This kind of nesting may be referred to as higher-order clustering; in this framework, the simple Neyman type-A and Thomas are first-order clustered. Personick made use of the Chernov technique with noise alone to upper bound the false-alarm probability and with signal plus noise to upper bound the miss probability. Fixing both of these probabilities at 10^{-9} (maximum-likelihood criterion), he calculated the *maximum mean number* of detected photons per light pulse required to achieve this performance. He independently used a model consisting of Gaussian noise statistics, Poisson dark counts, and fixed multiplicative Poisson signal statistics to obtain a lower bound to the required mean number of photons per pulse. In a generalization of this first paper,[64] he considered upper-bound results for two-carrier unequal ionization probability avalanche photodiodes (e.g., Si), but the calculations are considerably more complex. The signal statistics can, nevertheless, be represented as a Poisson photon-induced carrier distribution driving a random gain-multiplication distribution substantially more complicated than a shifted Bose-Einstein or exponential.[57,65] Personick[64] also performed an analysis for the twin-channel receiver (orthogonal signal format) in the presence of dark counts and compared it with the single-channel receiver when the incident photon statistics are negative binomial. In a subsequent paper[57] he and his collaborators compared system performance calculated on the basis of four distinct approaches: exact computer calculation; Monte Carlo simulation, Chernov bounds; and the Gaussian approximation, demonstrating the usefulness of each.

Lachs[66] carried out a similar study for a binary direct-detection optical communication system but assumed that the detector was a nonideal photomultiplier tube rather than an avalanche photodiode. He assumed that the primary source of noise arose from interfering chaotic radiation (Lorentzian spectral shape) and also ignored dark counts. In this model, the input photon distribution drives a cascade of stages, each described by the Polya distribution. This leads to a recursion relation; the lowest-order (one-stage) result is described by a distribution very similar to the negative-binomial/Polya. Lachs presented performance curves in graphical form for various system parameters, assuming maximum-likelihood detection. Although the high-gain photomultiplier tube is the instrument of choice for photon-counting applications,[48] it should be mentioned that under very specialized operating conditions, avalanche photodiodes (e.g., Si just below avalanche breakdown) can also be used for this purpose, as demonstrated by Cummings and Lachs.[67] The presence of intense ionizing radiation can lead to extensive damage in semiconductor devices, however.

As a closing note relating to radiation damage, it is marvelous to consider that Neyman himself continues to deal with important problems that make use of the basic character of contagious distributions like the Neyman type-A. He has, most recently with Puri,[68] been exploring models that describe the damage to living cells resulting from primary radiation particles which generate clusters of secondaries that, in turn, produce damage in the cell. In light of the discussion in Sec. VI, this is a familiar application indeed.

Appendix: Explicit Calculation Demonstrating Convergence in Distribution of Neyman Type-A and Thomas to Gaussian

We begin with the Neyman type-A, for which

$$k = \frac{n - aW}{[(1+a)aW]^{1/2}}. \tag{A1}$$

Since k is of the form

$$k = bn + c, \tag{A2}$$

it is clear that

$$Q_k(t) = \exp(ct)Q_n(bt) \tag{A3}$$

so that

$$Q_k(t) = \exp\left[-\left(\frac{aW}{1+a}\right)^{1/2} t\right] Q_n\left(\frac{t}{[(1+a)aW]^{1/2}}\right), \tag{A4}$$

which, when combined with Eq. (2), yields

$$Q_k(t) = \exp\left[-\left(\frac{aW}{1+a}\right)^{1/2} t\right] \times \exp(W\{\exp[a(e^{t/[(1+a)aW]^{1/2}} - 1)] - 1\}). \tag{A5}$$

A Taylor expansion of the quantity $[a(e^{t/[(1+a)aW]^{1/2}} - 1)]$ provides

$$[a(e^{t/[(1+a)aW]^{1/2}} - 1] = \frac{at}{[(1+a)aW]^{1/2}} + \frac{at^2}{2(1+a)aW} + \frac{at^3}{6[(1+a)aW]^{3/2}} + \ldots, \tag{A6}$$

so that

$$\exp[a(e^{t/[(1+a)aW]^{1/2}} - 1)] = \exp\left[\frac{at}{[(1+a)aW]^{1/2}}\right] \cdot \exp\left[\frac{at^2}{2(1+a)aW}\right] \cdot \exp\left[\frac{at^3}{6[(1+a)aW]^{3/2}}\right] \ldots. \tag{A7}$$

A further Taylor expansion of each factor above leads to

$$\exp[a(e^{t/[(1+a)aW]^{1/2}}-1)] = \left(1+\frac{at}{[(1+a)aW]^{1/2}}+\frac{a^2t^2}{2(1+a)aW}+\frac{a^3t^3}{6[(1+a)aW]^{3/2}}+\ldots\right) \cdot\left(1+\frac{at^2}{2(1+a)aW}+\frac{a^2t^4}{8[(1+a)aW]^2}+\ldots\right) \cdot\left(1+\frac{at^3}{6[(1+a)aW]^{3/2}}+\frac{a^2t^6}{72[(1+a)aW]^3}+\ldots\right) \ldots . \quad \text{(A8)}$$

Retaining all terms to order $W^{3/2}$ in the denominator, we obtain

$$\exp[a(e^{t/[(1+a)aW]^{1/2}}-1)] = 1+\frac{at}{[(1+a)aW]^{1/2}}+\frac{at^2}{2(1+a)aW} +\frac{a^2t^2}{2(1+a)aW} +\frac{(1+3a+a^2)t^3}{6(1+a)^{3/2}a^{1/2}W^{3/2}}+\ldots, \quad \text{(A9)}$$

so that

$$(W\{\exp[a(e^{t/[(1+a)aW]^{1/2}}-1)]-1\}) = \left(\frac{aW}{1+a}\right)^{1/2}t+\frac{t^2}{2}+\frac{(1+3a+a^2)t^3}{6(1+a)^{3/2}a^{1/2}W^{1/2}}+\ldots . \quad \text{(A10)}$$

Hence

$$Q_n\left(\frac{t}{[(1+a)aW]^{1/2}}\right) = \exp\left[\left(\frac{aW}{1+a}\right)^{1/2}t+\frac{t^2}{2} +\frac{(1+3a+a^2)t^3}{6(1+a)^{3/2}a^{1/2}W^{1/2}}+\ldots\right], \quad \text{(A11)}$$

which, when substituted into Eq. (A4), yields

$$Q_k(t) = \exp\left[\frac{t^2}{2}+\frac{(1+3a+a^2)t^3}{6(1+a)^{3/2}a^{1/2}W^{1/2}}+\ldots\right]. \quad \text{(A12)}$$

In the limit as $a \to 0$, but $\langle n\rangle = aW$ remains finite, the moment generating function becomes

$$\lim_{\substack{a\to 0\\ 0<\langle n\rangle<\infty}} Q_k(t) = \exp\left[\frac{t^2}{2}+\frac{t^3}{6\langle n\rangle^{1/2}}+\ldots\right], \quad \text{(A13)}$$

which is identical to the Poisson result, as expected.

On the other hand, if a is fixed at any finite value and the mean of the driving distribution $W = \langle m\rangle$ increases without limit, the moment generating function is identical to that for the standard normal random variable:

$$\lim_{W\to\infty} Q_k(t) = \exp[t^2/2]. \quad \text{(A14)}$$

This result demonstrates that the Neyman type-A converges in distribution to the Gaussian with mean aW and standard deviation $[(1+a)aW]^{1/2}$.

The treatment for the Thomas distribution is similar. Using Eqs. (6) and (21), we consider the random variable

$$k = \frac{n-(1+\alpha)W}{[(1+3\alpha+\alpha^2)W]^{1/2}}. \quad \text{(A15)}$$

Then, using Eqs. (5), (A2), and (A3), the moment generating function is calculated to be

$$Q_k(t) = \exp\left[-\left(\frac{(1+\alpha)^2W}{1+3\alpha+\alpha^2}\right)^{1/2}t\right] \cdot\exp\left(W\left\{\exp\left[\frac{t}{[(1+3\alpha+\alpha^2)W]^{1/2}}\right] \times\exp[\alpha(e^{t/[(1+3\alpha+\alpha^2)W]^{1/2}}-1)]-1\right\}\right). \quad \text{(A16)}$$

In analogy with the development for the Neyman type-A, it is easily shown that [see Eq. (A9)]

$$\exp[\alpha(e^{t/[(1+3\alpha+\alpha^2)W]^{1/2}}-1)] = 1+\frac{\alpha t}{[(1+3\alpha+\alpha^2)W]^{1/2}} +\frac{(1+\alpha)\alpha t^2}{2(1+3\alpha+\alpha^2)W} +\frac{(1+3\alpha+\alpha^2)\alpha t^3}{6[(1+3\alpha+\alpha^2)W]^{3/2}}+\ldots . \quad \text{(A17)}$$

The calculation for the Thomas is a bit more involved than that for the Neyman type-A because of the additional exponential factor in the generating function [compare Eqs. (A5) and (A16)]. Expanding it, we obtain

$$\exp\left[\frac{t}{[(1+3\alpha+\alpha^2)W]^{1/2}}\right] = 1+\frac{t}{[(1+3\alpha+\alpha^2)W]^{1/2}} +\frac{t^2}{2(1+3\alpha+\alpha^2)W} +\frac{t^3}{6[(1+3\alpha+\alpha^2)W]^{3/2}}+\ldots, \quad \text{(A18)}$$

so that

$$\left(W\left\{\exp\left[\frac{t}{[(1+3\alpha+\alpha^2)W]^{1/2}}\right]\cdot\exp[\alpha(e^{t/[(1+3\alpha+\alpha^2)W]^{1/2}}-1)]-1\right\}\right) = \left(\frac{(1+\alpha)^2W}{1+3\alpha+\alpha^2}\right)^{1/2}t+\frac{t^2}{2}+\frac{(1+7\alpha+6\alpha^2+\alpha^3)t^3}{6[(1+3\alpha+\alpha^2)]^{3/2}W^{1/2}}+\ldots . \quad \text{(A19)}$$

Thus

$$Q_n\left(\frac{t}{[(1+3\alpha+\alpha^2)W]^{1/2}}\right) = \exp\left[\left(\frac{(1+\alpha)^2W}{1+3\alpha+\alpha^2}\right)^{1/2}t+\frac{t^2}{2} +\frac{(1+7\alpha+6\alpha^2+\alpha^3)t^3}{6[(1+3\alpha+\alpha^2)]^{3/2}W^{1/2}}+\ldots\right], \quad \text{(A20)}$$

which, when combined with Eq. (A16), yields

$$Q_k(t) = \exp\left[\frac{t^2}{2}+\frac{(1+7\alpha+6\alpha^2+\alpha^3)t^3}{6[(1+3\alpha+\alpha^2)]^{3/2}W^{1/2}}+\ldots\right], \quad \text{(A21)}$$

in analogy with Eq. (A12) for the Neyman type-A.

For $\alpha = 0$, the moment generating function becomes

$$\lim_{\alpha\to 0} Q_k(t) = \exp\left[\frac{t^2}{2} + \frac{t^3}{6W^{1/2}} + \ldots\right], \quad \text{(A22)}$$

which is the Poisson form for the driving distribution. This is as it should be, since there are only mother pulses in the process in that case, and these have been assumed at the outset to be Poisson distributed.

The more interesting case occurs when α is fixed at any finite value, and the mean of the driving distribution $W = \langle m \rangle$ increases without limit; here the moment generating function is again identical to that for the standard normal random variable [see Eq (A14)]. Thus the Thomas converges in distribution to the Gaussian with mean $(1 + \alpha)W$ and standard deviation $[(1 + 3\alpha + \alpha^2)W]^{1/2}$.

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