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ROOT STRINGS WITH THREE OR FOUR REAL ROOTS IN KAC-MOODY ROOT SYSTEMS

Dedicated to Professor Eiichi Abe on his sixtieth birthday

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0. Introduction. A characterization and a presentation of a (universal) Kac-Moody group over a field (of any characteristic) have been given by Tits [6]. Such a presentation, which is a natural generalization of Steinberg's one for a (simply connected) split semisimple algebraic group over a field (cf. [5]), is conjectured by E. Abe and established by J. Tits. The most interesting part of the presentation is the so-called "commutation relation", which is deeply related to the root strings and whose explicit description is given in [4]. In this paper, we will discuss certain root strings in Kac-Moody root systems, and give some direct applications to the associated Kac-Moody groups. Our main result is as follows.

Let $A=(a_{ij})$ be an $n\times n$ generalized Cartan matrix, Δ the associated root system, and Δ^{re} the set of real roots. Put $r(\alpha;\beta)=\sharp|\{\beta+k\alpha\,|\,k\in\boldsymbol{Z}\}\cap\Delta^{re}|$ for $(\alpha,\beta)\in\Delta^{re}\times\Delta$. Then the following two conditions are equivalent.

(1) $r(\alpha;\beta)=3$ or 4 for some $(\alpha,\beta)\in\Delta^{re}\times\Delta$.

(2) $a_{ij}=-1$ and $a_{ji}<-1$ for some i, j $(1\leqq i, j\leqq n)$.

As a corollary, we can simplify the Steinberg-Tits presentation of the associated Kac-Moody group in the case when A has a certain property.

1. Notation and lemmas. Let $A=(a_{ij})_{i,j\in I}$ be an $n\times n$ generalized Cartan matrix, $(\mathfrak{h}, \Pi, \Pi^\vee)$ a realization of A, and $\mathfrak{g}(A)$ the Kac-Moody Lie algebra (over $\boldsymbol{C}$ associated with A), where $I=\{1,2,\cdots,n\}$, $\Pi=\{\alpha_1,\cdots,\alpha_n\}$, $\Pi^\vee=\{h_1,\cdots,h_n\}$ and $\alpha_i(h_j)=a_{ji}$ (cf. [1]). We denote by W the Weyl group with simple reflections $w_1,\cdots,w_n$. Let Δ be the root system of $\mathfrak{g}(A)$ with Π as simple roots, $\Delta^{re}=\{w(\alpha)\,|\,w\in W, \alpha\in\Pi\}$ the set of real roots, Δ_+ the set of positive roots, and Δ_+^{re} the set of positive real roots. For each $\alpha\in\Delta^{re}$, let $h_\alpha\in\mathfrak{h}$ be the dual root of α. Then both $\alpha(h_\beta)$ and $\beta(h_\alpha)$ have the same sign (one of $+, 0, -$) for all $\alpha,\beta\in\Delta^{re}$ (cf. [3]). Put $\mathrm{ht}(\alpha)=\sum_{k=1}^n c_k$, called the height of α, if $\alpha=\sum_{k=1}^n c_k\alpha_k\in\Delta$. Let $S(\alpha;\beta)=\{\beta+k\alpha\,|\,k\in\boldsymbol{Z}\}\cap\Delta$ for $(\alpha,\beta)\in\Delta^{re}\times\Delta$. This $S(\alpha;\beta)$ is called the α-string through β. Let $r(\alpha; \beta) = \#|S(\alpha; \beta) \cap \Delta^{\mathrm{re}}|$ for each $(\alpha, \beta) \in \Delta^{\mathrm{re}} \times \Delta$. Then one sees $r(\alpha; \beta) = 0, 1, 2, 3$ or 4. Our interest in this paper (in view of Steinberg-Tits presentation) is when $r(\alpha; \beta)$ is 3 or 4 for some $(\alpha, \beta) \in \Delta^{\mathrm{re}} \times \Delta$. Set $R = \{(\alpha, \beta) \in \Delta^{\mathrm{re}} \times \Delta^{\mathrm{re}} \mid \alpha - \beta \notin \Delta,\ r(\alpha; \beta) = 3$ or $4\}$ and $R_+ = R \cap (\Delta_+^{\mathrm{re}} \times \Delta_+^{\mathrm{re}})$. Then $(\alpha, \beta) \in R$ implies that $\alpha(h_\beta) = -1$ and $\beta(h_\alpha) < -1$.

LEMMA 1. *Let $i, j \in I$, and $\alpha = \sum_{k=1}^{n} c_k \alpha_k \in \Delta_+$. Suppose $\alpha_i(h_j) = \alpha_j(h_i) = -2$.*

(1) *In general, $\alpha(h_i + h_j) \leqq 0$.*

(2) *If $\alpha(h_i + h_j) = 0$, then $\alpha(h_i) = -\alpha(h_j) \equiv 0 \pmod 2$.*

PROOF. Put $\alpha' = \sum_{k \neq i,j} c_k \alpha_k$. Since $\alpha'(h_i) \leqq 0$, $\alpha'(h_j) \leqq 0$ and $(c_i\alpha_i + c_j\alpha_j)(h_i + h_j) = 0$, we obtain $\alpha(h_i + h_j) \leqq 0$. Suppose $\alpha(h_i + h_j) = 0$. Then $\alpha'(h_i) = \alpha'(h_j) = 0$. Therefore $\alpha(h_i) = (c_i\alpha_i + c_j\alpha_j)(h_i) = 2(c_i - c_j) \equiv 0$ (mod 2). □

LEMMA 2. *Let $i, j \in I$, and $\alpha = \sum_{k=1}^{n} c_k \alpha_k \in \Delta_+$. Suppose $\alpha_i(h_j) = -4$ and $\alpha_j(h_i) = -1$.*

(1) *In general, $\alpha(2h_i + h_j) \leqq 0$.*

(2) *If $\alpha(h_i) = -1$ and $\alpha(h_j) = 2$, then $\alpha = \alpha_j + m\xi$, where $m \in \boldsymbol{Z}_{\geqq 0}$ and $\xi = \alpha_i + 2\alpha_j$.*

PROOF. By the same reason as in Lemma 1(1), we see $\alpha(2h_i + h_j) \leqq 0$. Suppose $\alpha(h_i) = -1$ and $\alpha(h_j) = 2$. Then $\alpha' = \sum_{k \neq i,j} c_k \alpha_k$ must be zero and $\alpha = c_i\alpha_i + c_j\alpha_j$, since $\alpha'(h_i) = \alpha'(h_j) = 0$. If $\mathrm{ht}(\alpha) = 1$, then $\alpha = \alpha_i$ or α_j, hence $\alpha = \alpha_j$ by the condition. Suppose $\mathrm{ht}(\alpha) > 1$. Then $c_i > 0$ and $c_j > 0$, and $(\alpha - \alpha_j)(h_i) = (\alpha - \alpha_j)(h_j) = 0$. Therefore $\alpha - \alpha_j = m\xi$ with $m \in \boldsymbol{Z}_{>0}$. □

LEMMA 3. *Let $i, j \in I$, and suppose $\alpha_i(h_j) \cdot \alpha_j(h_i) > 4$. Put $V = \bigoplus_{k=1}^{n} \boldsymbol{R}\alpha_k$ and $V' = \{\lambda \in V \mid \lambda(h_i) = \lambda(h_j) = 0\}$.*

(1) *$V = \boldsymbol{R}\alpha_i \oplus \boldsymbol{R}\alpha_j \oplus V'$.*

(2) *If $\mu = b_i\alpha_i + b_j\alpha_j + \mu' \in V$ $(b_i, b_j \in \boldsymbol{R},\ \mu' \in V')$ with $\mu(h_i) \leqq 0$ and $\mu(h_j) \leqq 0$, then $b_i \geqq 0$ and $b_j \geqq 0$.*

(3) *If $\mu \in \Delta_+$ and $\mu(h_i) \geqq m$ for some $m \in \boldsymbol{Z}_{>0}$, then $(w_j\mu)(h_i) \leqq -(m+1)$.*

PROOF. For $\mu \in V$, put

$$b_i = \frac{2\mu(h_i) - \alpha_j(h_i)\mu(h_j)}{4 - \alpha_i(h_j)\alpha_j(h_i)}\,, \quad b_j = \frac{2\mu(h_j) - \alpha_i(h_j)\mu(h_i)}{4 - \alpha_i(h_j)\alpha_j(h_i)}\,,$$

and $\mu' = \mu - b_i\alpha_i - b_j\alpha_j$. Then $\mu = b_i\alpha_i + b_j\alpha_j + \mu'$ and $\mu' \in V'$. If $\mu \in (\boldsymbol{R}\alpha_i \oplus \boldsymbol{R}\alpha_j) \cap V'$, then $\mu = 0$ since $\alpha_i(h_j) \cdot \alpha_j(h_i) > 4$. Hence $V = \boldsymbol{R}\alpha_i \oplus$

$\boldsymbol{R}\alpha_j \oplus V'$. If $\mu(h_i) \leqq 0$ and $\mu(h_j) \leqq 0$, then $b_i \geqq 0$ and $b_j \geqq 0$. Next suppose $\mu = \sum_{k=1}^{n} c_k\alpha_k \in \Delta_+$ and $\mu(h_i) \geqq m$ for some $m \in \boldsymbol{Z}_{>0}$. Put $\mu_0 = \sum_{k\neq i,j} c_k\alpha_k$. Then $\mu_0(h_i) \leqq 0$ and $\mu_0(h_j) \leqq 0$. Therefore, by (2), we can write $\mu_0 = b_i\alpha_i + b_j\alpha_j + \mu_0'$ ($b_i, b_j \geqq 0$, $\mu_0' \in V'$). Then $\mu = d_i\alpha_i + d_j\alpha_j + \mu_0'$, where $d_i = b_i + c_i > 0$ and $d_j = b_j + c_j \geqq 0$. Hence

$$\begin{aligned}(w_j\mu)(h_i) &= (\mu - \mu(h_j)\alpha_j)(h_i) = \mu(h_i) - \mu(h_j)\alpha_j(h_i)\\ &= (d_i\alpha_i + d_j\alpha_j)(h_i) - (d_i\alpha_i + d_j\alpha_j)(h_j)\alpha_j(h_i)\\ &= 2d_i + d_j\alpha_j(h_i) - d_i\alpha_i(h_j)\alpha_j(h_i) - 2d_j\alpha_j(h_i)\\ &= (2 - \alpha_i(h_j)\alpha_j(h_i))d_i - d_j\alpha_j(h_i) < -2d_i - d_j\alpha_j(h_i)\\ &= -(2d_i + d_j\alpha_j(h_i)) = -\mu(h_i) \leqq -m\,.\end{aligned}$$

Therefore, $(w_j\mu)(h_i) \leqq -(m + 1)$. □

2. Main result. In this section, we will establish the following theorem.

THEOREM. *Notation is as in Section* 1. *Then the following conditions are equivalent.*

(1) $r(\alpha; \beta) = 3$ *or* 4 *for some* $(\alpha, \beta) \in \Delta^{\mathrm{re}} \times \Delta$.

(2) $a_{ij} = -1$ *and* $a_{ji} < -1$ *for some* $i, j \in I$.

COROLLARY. *The following conditions are equivalent.*

(1) $a_{ij} = -1$ *if and only if* $a_{ji} = -1$ $(i, j \in I)$.

(2) $r(\alpha; \beta) = 0$, 1 *or* 2 *for all* $(\alpha, \beta) \in \Delta^{\mathrm{re}} \times \Delta$.

PROOF OF THEOREM. The condition (2) implies $r(\alpha_j; \alpha_i) = 3$ or 4 and, hence, the condition (1). Therefore it is required to show the converse. Suppose $r(\alpha; \beta) = 3$ or 4 for some $(\alpha, \beta) \in \Delta^{\mathrm{re}} \times \Delta$. Then we can assume $(\alpha, \beta) \in R_+$. Let $Q = R_+ \cap W\cdot(\alpha, \beta)$. Then we can also assume $\mathrm{ht}(\alpha + \beta)$ is minimal in Q. Since $\alpha + \beta \in \Delta^{\mathrm{re}}$ and $\mathrm{ht}(\alpha + \beta) > 1$, there is $\alpha_i \in \Pi$ such that $(\alpha + \beta)(h_i) > 0$. Then $\alpha \neq \alpha_i$ for $(\alpha + \beta)(h_\alpha) \leqq 0$. If $\beta \neq \alpha_i$, then $(w_i\alpha, w_i\beta) \in Q$ and $\mathrm{ht}(w_i\alpha + w_i\beta) < \mathrm{ht}(\alpha + \beta)$, which is a contradiction. Therefore $\beta = \alpha_i$. Since $\alpha \in \Delta_+^{\mathrm{re}}$, there are $\alpha_{i_0} \in \Pi$ and $i_1, i_2, \cdots, i_l \in I$ $(l \geqq 0)$ such that $\alpha = w_{i_l}w_{i_{l-1}} \cdots w_{i_1}\alpha_{i_0}$ and $\beta_{s-1}(h_{i_s}) < 0$ $(1 \leqq s \leqq l)$, where $\beta_0 = \alpha_{i_0}$, $\beta_s = w_{i_s}w_{i_{s-1}} \cdots w_{i_1}\alpha_{i_0}$ $(1 \leqq s \leqq l)$, and $\beta_l = \alpha$. Let $j = i_l$. Then we claim $a_{ij} = -1$ and $a_{ji} < -1$, which is our goal. If $l = 0$, then $\alpha = \alpha_{i_0} = \alpha_j$. Since $(\alpha_j, \alpha_i) \in R_+$, one sees $a_{ij} = \alpha_j(h_i) = -1$ and $a_{ji} = \alpha_i(h_j) < -1$. Therefore we suppose, from now on, $l > 0$, hence $\mathrm{ht}(\alpha) > 1$. Then $j \neq i$ since $\alpha(h_i) = -1$ and $\alpha(h_j) > 0$. Put $\alpha' = \beta_{l-1}$. If $\alpha_i(h_j) = 0$, then $(\alpha', \alpha_i) = w_j(\alpha, \alpha_i) \in Q$ and $\mathrm{ht}(\alpha' + \alpha_i) < \mathrm{ht}(\alpha + \alpha_i)$, which is a contradiction. Thus, $\alpha_i(h_j) < 0$ and $\alpha_j(h_i) < 0$. If $\alpha'(h_i) < 0$, then $\alpha(h_i) = (w_j\alpha')(h_i) = (\alpha' - \alpha'(h_j)\alpha_j)(h_i) = \alpha'(h_i) - \alpha'(h_j)\alpha_j(h_i) \leqq -2$. Hence $\alpha'(h_i) \geqq 0$, since $\alpha(h_i) = -1$.

Case 1: $\alpha'(h_i) = 0$. In this case, we obtain $-1 = \alpha(h_i) = (w_j\alpha')(h_i) = \alpha'(h_i) - \alpha'(h_j)\alpha_j(h_i) = -\alpha'(h_j)\alpha_j(h_i)$ and $\alpha'(h_j) = \alpha_j(h_i) = -1$. If $\alpha_i(h_j) = -1$, then $(\alpha', \alpha_j) = w_iw_j(\alpha, \alpha_i) \in Q$ and $\mathrm{ht}(\alpha' + \alpha_j) < \mathrm{ht}(\alpha + \alpha_i)$, a contradiction. Hence $\alpha_i(h_j) < -1$, so $a_{ij} = -1$ and $a_{ji} < -1$.

Case 2: $\alpha'(h_i) > 0$. We proceed in several steps.

Step 1. Suppose $\alpha_i(h_j) = \alpha_j(h_i) = -2$. Then $\alpha(h_i + h_j) \leqq 0$ by Lemma 1(1). Since $\alpha(h_i) = -1$ and $\alpha(h_j) > 0$, one sees $-1 < \alpha(h_i) + \alpha(h_j) \leqq 0$, hence $\alpha(h_i + h_j) = 0$. By Lemma 1(2), we obtain a contradiction: $-1 = \alpha(h_i) \equiv 0 \pmod 2$.

Step 2. Suppose $\alpha_i(h_j)\cdot\alpha_j(h_i) > 4$. Then $\alpha' \in \Delta_+$ and $\alpha'(h_i) > 0$ imply a contradiction: $\alpha(h_i) = (w_j\alpha')(h_i) < -1$ by Lemma 3(3).

Step 3. We have just got $\{\alpha_i(h_j), \alpha_j(h_i)\} = \{-1, -1\}$, $\{-1, -2\}$, $\{-1, -3\}$ or $\{-1, -4\}$. If $w_iw_j(\alpha) \in \Delta_-^{\mathrm{re}}$, then $\alpha' = w_j(\alpha) = \alpha_i$, hence $\alpha = \alpha_i - \alpha_i(h_j)\alpha_j$ and $-1 = \alpha(h_i) = 2 - \alpha_i(h_j)\alpha_j(h_i)$, so $\alpha_i(h_j)\alpha_j(h_i) = 3$. If $\alpha_i(h_j) = -1$ and $\alpha_j(h_i) = -3$, then $\alpha = w_j(\alpha_i) = \alpha_i + \alpha_j$ and $(\alpha, \alpha_i) \notin R$, a contradiction. If $\alpha_i(h_j) = -3$ and $\alpha_j(h_i) = -1$, then $\alpha = w_j(\alpha_i) = \alpha_i + 3\alpha_j$ and $(\alpha, \alpha_i) \notin R$, also a contradiction. Therefore $w_iw_j(\alpha) \in \Delta_+^{\mathrm{re}}$ and $(w_iw_j\alpha, w_iw_j\alpha_i) \in Q$.

Step 4. Our hypothesis, the minimality of $\mathrm{ht}(\alpha + \beta)$ in Q, leads to

$$\begin{aligned}&\mathrm{ht}(w_iw_j(\alpha + \alpha_i)) - \mathrm{ht}(\alpha + \alpha_i)\\ &\quad = -(\alpha + \alpha_i)(h_i) - (\alpha + \alpha_i)(h_j) + (\alpha + \alpha_i)(h_j)\alpha_j(h_i)\\ &\quad = -(\alpha + \alpha_i)(h_j)[1 - \alpha_j(h_i)] - 1 \geqq 0\,,\end{aligned}$$

which implies $(\alpha + \alpha_i)(h_j) < 0$ and $\alpha_i(h_j) < -1$. Therefore $\alpha_j(h_i) = -1$ and $\alpha_i(h_j) = -2, -3, -4$. Hence our theorem has been established. We, however, want to continue in order to obtain a stronger result.

Step 5. Suppose $\alpha_j(h_i) = -1$ and $\alpha_i(h_j) = -2$. Then Step 4 says $\alpha(h_j) = 1$ and $\alpha'(h_i) = (\alpha - \alpha_j)(h_i) = 0$, a contradiction.

Step 6. Suppose $\alpha_j(h_i) = -1$ and $\alpha_i(h_j) = -3$. Then Step 4 says $\alpha(h_j) = 1$ or 2, and $\alpha'(h_i) = \alpha(h_i) - \alpha(h_j)\alpha_j(h_i) = -1 + \alpha(h_j)$. Therefore $\alpha(h_j) = 2$ since $\alpha'(h_i) > 0$. Hence $\alpha'(h_i) = 1$. Put $w_0 = w_jw_iw_jw_iw_j \in W$. Then $w_0(\alpha, \alpha_i) = (\alpha - \alpha_i - 2\alpha_j, \alpha_i) \in Q$ and $\mathrm{ht}(w_0(\alpha + \alpha_i)) < \mathrm{ht}(\alpha + \alpha_i)$, a contradiction.

Step 7. Suppose $\alpha_j(h_i) = -1$ and $\alpha_i(h_j) = -4$. Then Step 4 says $\alpha(h_j) = 1, 2$ or 3, and $\alpha'(h_i) = -1 + \alpha(h_j)$. Therefore $\alpha(h_j) = 2$ or 3 since $\alpha'(h_i) > 0$. Suppose $\alpha(h_j) = 3$. We inductively define γ_t ($t \in \boldsymbol{Z}_{\geqq 0}$) by $\gamma_0 = \alpha$, $\gamma_{2m+1} = w_j(\gamma_{2m})$ and $\gamma_{2m+2} = w_i(\gamma_{2m+1})$ for $m \in \boldsymbol{Z}_{\geqq 0}$. Then one can easily check that $\gamma_{2m}(h_j) = 2m + 3 > 0$ and $\gamma_{2m+1}(h_i) = m + 2 > 0$. This means that α must be of the form $c_i\alpha_i + c_j\alpha_j \in \Delta_+^{\mathrm{re}}$, since $\mathrm{ht}(\gamma_t) < 0$ for some (sufficiently large) t. Then $0 \geqq \alpha(2h_i+h_j) = 2\alpha(h_i)+\alpha(h_j) = -2+3 = 1$, a contradiction. Therefore $\alpha(h_j) = 2$ and $\alpha(h_i) = -1$. By Lemma 2(2), we obtain $\alpha = \alpha_j + m\xi$, where $m \in \boldsymbol{Z}_{\geqq 0}$ and $\xi = \alpha_i + 2\alpha_j$.

Step 8. In particular, we have established that $\alpha'(h_i) > 0$ implies $a_{ij} = -1$ and $a_{ji} = -4$. □

3. Relations in Kac-Moody groups. (1) *Steinberg-Tits presentation.* Let A be a generalized Cartan matrix and $G(A)$ the associated (universal) Kac-Moody group over a field K. Then $G(A)$ has the following presentation (cf. Tits [6]):

generators

$$x_\alpha(t) \quad \text{for all} \quad \alpha \in \Delta^{\mathrm{re}} \text{ and } t \in K\,,$$

relations

(A) $x_\alpha(s)\cdot x_\alpha(t) = x_\alpha(s+t)$,

(B) $[x_\alpha(s), x_\beta(t)] = \prod_{i\alpha+j\beta \in \Delta^{\mathrm{re}}; i,j>0} x_{i\alpha+j\beta}(c_{\alpha\beta ij}s^i t^j)$ if $(\boldsymbol{Z}_{>0}\alpha + \boldsymbol{Z}_{>0}\beta) \cap \Delta^{\mathrm{im}} = \varnothing$,

(B′) $w_\alpha(u)\cdot x_\beta(t)\cdot w_\alpha(-u) = x_{\beta'}(u't)$,

(C) $h_\alpha(u)\cdot h_\alpha(v) = h_\alpha(uv)$

for all $\alpha, \beta \in \Delta^{\mathrm{re}}$, $s, t \in K$ and $u, v \in K^\times$, where $c_{\alpha\beta ij}$ is a certain integer, $\beta' = \beta - \beta(h_\alpha)\alpha$, $u' = \pm u^{-\beta(h_\alpha)}t$, $w_\alpha(u) = x_\alpha(u)\cdot x_{-\alpha}(-u^{-1})\cdot x_\alpha(u)$ and $h_\alpha(u) = w_\alpha(u)\cdot w_\alpha(-1)$. An explicit description of the right-hand side in (B) has been calculated (cf. [4]). We must notice that the coefficients $c_{\alpha\beta ij}$ are deeply related to the root strings in the rank two subsystem generated by α and β.

(2) *Symmetry of* -1. Suppose that $A = (a_{ij})_{i,j\in I}$ has the property that $a_{ij} = -1$ if and only if $a_{ji} = -1$ ($i, j \in I$). Then the above relation (B) can be simplified as follows:

$$\text{(B)} \qquad [x_\alpha(s), x_\beta(t)] = \begin{cases} 1 & \text{if} \quad \alpha+\beta \notin \Delta\,, \\ x_{\alpha+\beta}(\pm st) & \text{if} \quad \alpha+\beta \in \Delta^{\mathrm{re}}\,. \end{cases}$$

The other type relations for (B) (cf. [4]) do not happen here. This comes from our theorem (or its corollary). Then we should compare this to the corresponding relation for SL_n.

(3) *A_2-subsystems.* As a direct consequence of Kac-Peterson conjugacy theorems (cf. [2]), we obtain the equivalence of the following two conditions.

(i) There exist $\alpha, \beta \in \Delta^{\mathrm{re}}$ such that α and β generate an A_2-subsystem of Δ.

(ii) There are some $i, j \in I$ such that $a_{ij}\cdot a_{ji} = 1$ or 3.

(4) *No entry of* -1. If A has no -1 as an entry, then from (2) and (3) we see that the relation (B) is just

(B) $$[x_\alpha(s), x_\beta(t)] = 1 \quad \text{if} \quad \alpha + \beta \notin \Delta .$$

(5) *The set* $P(A)$. Let $P(A)$ be the set of all the prime numbers p having the property that p divides $|a_{ij}|$ for some $i, j \in I$ with $a_{ji} = -1$. If char K does not belong to $P(A)$, then the following two conditions are equivalent.

(i) $[x_\alpha(s), x_\beta(t)] = 1$.

(ii) $\alpha + \beta \notin \Delta$.

Here $\alpha, \beta \in \Delta^{\mathrm{re}}$ and $s, t \in K^\times$. This equivalence is due to [4], [6] and the proof of Theorem. For example, $P(B_n) = \{2\}$, $P(G_2) = \{3\}$, $P(A_1^{(1)}) = \varnothing$, and $P\left(\begin{pmatrix} 2 & -6 \\ -1 & 2 \end{pmatrix}\right) = \{2, 3\}$.

(6) *Example*. Let $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ with $ab \geqq 4$, and $U(A)$ the subgroup of $G(A)$ generated by $x_\alpha(t)$ for all $\alpha \in \Delta_+^{\mathrm{re}}$ and $t \in K$. Put $\Phi_i = \{\alpha \in \Delta_+^{\mathrm{re}} \mid \alpha(h_i) > 0\}$ for each $i = 1, 2$. Then $\Delta_+^{\mathrm{re}} = \Phi_1 \cup \Phi_2$. Let U_i be the subgroup of $U(A)$ generated by $x_\alpha(t)$ for all $\alpha \in \Phi_i$ and $t \in K$ $(i = 1, 2)$. If char $K = 0$, then we see $U(A) \simeq U_1 * U_2$, the free product of U_1 and U_2 (cf. [6], (1)). If $a > 1$ and $b > 1$, then each U_i is abelian by Theorem. Suppose $a = 1$ (, hence $b \geqq 4$). If char K belongs to $P(A)$, then each U_i is abelian. Otherwise each U_i is meta-abelian (not abelian).

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