

## Rotating Optical Soliton Clusters

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We introduce the concept of *soliton clusters*—multisoliton bound states in a homogeneous bulk optical medium—and reveal a key physical mechanism for their stabilization associated with a staircaselike phase distribution that induces a net angular momentum and leads to cluster rotation. The ringlike soliton clusters provide a nontrivial generalization of the concepts of two-soliton spiraling, optical vortex solitons, and necklace-type optical beams.

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Recent progress in generating spatial optical solitons in nonlinear bulk media opens the possibility to study truly two-dimensional self-trapping of light and interaction of multidimensional solitary waves [1]. The robust nature of spatial solitons that they display in interactions allows us to draw a formal analogy with atomic physics and treat spatial solitons as “atoms of light.” Our motivation here is to find out whether more complex objects, viewed as “atom clusters,” can be constructed from a certain number of simple solitons—“atoms.” In this Letter, we describe, for the first time to our knowledge, the basic principles for constructing the so-called “soliton clusters,” ringlike multisoliton bound states in a bulk media.

First, in order to discuss the formation of multisoliton bound states in a homogeneous bulk medium, we should recall the physics of the coherent interaction of two spatial solitons. It is well known [1] that such an interaction depends crucially on the relative soliton phase, say  $\theta$ , so that two solitons attract each other for  $\theta = 0$ , and they repel each other for  $\theta = \pi$ . For the intermediate values of the soliton phase,  $0 < \theta < \pi$ , the solitons undergo an energy exchange and display an inelastic interaction. As a result, *no stationary bound states of two coherently interacting solitons are possible in a bulk medium.*

*Soliton spiraling* was suggested theoretically [2] and observed experimentally [3] as a possible scenario for a *dynamical* two-soliton bound state formed when two solitons are launched with *initially twisted trajectories*. However, it was clarified later [4] that the experimental observation of the soliton spiraling was possible due to an effectively *vectorial* beam interaction. As a matter of fact, the soliton spiraling reported in Refs. [3,4] is associated with large-amplitude oscillations of a dipole-mode vector state [5] generated by the interaction of two initially mutually incoherent optical beams.

In spite of the fact that no bound states exist for two coherently interacting scalar solitons in a bulk medium, in this Letter we demonstrate that such bound states (or *ringlike clusters*) are indeed possible for a larger number of solitons, namely, for  $N \geq 4$ . The main reason for the ex-

istence of such multisoliton states can be explained with the help of simple physics. Indeed, let us analyze possible *stationary configurations* of  $N$  coherently interacting solitons in  $2 + 1$  dimensions. The only finite-energy structures that would balance out the phase-sensitive coherent interaction of the neighboring solitons should possess a ringlike geometry. However, a ringlike configuration of  $N$  solitons will be *radially unstable* due to an effective tension induced by bending of the soliton array. Thus, a ring of  $N$  solitons will collapse, if the mutual interaction between the neighboring solitons is attractive, or otherwise expand, resembling the expansion of the necklace beams [6]. Nevertheless, a simple physical mechanism will provide stabilization of the ringlike configuration of  $N$  solitons, if we introduce *an additional phase* on the scalar field that twists by  $2\pi m$  along the soliton ring. This phase introduces *an effective centrifugal force* that can balance out the tension effect and stabilize the ringlike soliton cluster. Because of a net angular momentum induced by such a phase distribution, such soliton clusters rotate with an angular velocity which depends on the number of solitons and phase charge  $m$ .

To build up the theory of the soliton clusters, we consider a coherent superposition of  $N$  solitons with the envelopes  $G_n(x, y, z)$ ,  $n = 1, 2, \dots, N$ , propagating in a self-focusing homogeneous bulk medium. The equation for the slowly varying field envelope  $E = \sum G_n$  can be written in the form of the nonlinear Schrödinger equation,

$$i \frac{\partial E}{\partial z} + \Delta_{\perp} E + f(I)E = 0, \quad (1)$$

where  $\Delta_{\perp}$  is the transverse Laplacian and  $z$  is the propagation distance measured in the units of the diffraction length. Function  $f(I)$  describes the nonlinear properties of an optical medium, and it is assumed to depend on the total beam intensity,  $I = |E|^2$ .

The general features of the dynamical system (1) are determined by its conservation laws, or *integrals of motion*: the *beam power*,  $P = \int |E|^2 d\mathbf{r}$ , the *linear momentum*,

$\mathbf{L} = \text{Im} \int E^* \nabla E \, d\mathbf{r}$ , and the angular momentum,  $M \mathbf{e}_z = \text{Im} \int E^* (\mathbf{r} \times \nabla E) \, d\mathbf{r}$ . For a ring of identical *weakly overlapping* solitons launched in parallel, we can calculate the integrals of motion employing a Gaussian ansatz for a single beam  $G_n$ ,

$$G_n = A \exp\left(-\frac{|\mathbf{r} - \mathbf{r}_n|^2}{2a^2} + i\alpha_n\right), \quad (2)$$

where  $\mathbf{r}_n = \{x_n; y_n\}$  defines the position of the soliton center, and  $\alpha_n$  is the phase of the  $n$ th beam. Then, the integrals of motion take the form

$$\begin{aligned} P &= \pi a^2 A^2 \sum_{n,k=1}^N e^{Y_{nk}} \cos \theta_{nk}, \\ \mathbf{L} &= \frac{\pi}{2} A^2 \sum_{n,k=1}^N e^{Y_{nk}} (\mathbf{r}_n - \mathbf{r}_k) \sin \theta_{nk}, \\ M &= \pi A^2 \sum_{n,k=1}^N e^{Y_{nk}} |\mathbf{r}_n \times \mathbf{r}_k| \sin \theta_{nk}, \end{aligned} \quad (3)$$

where  $Y_{nk} = -|\mathbf{r}_n - \mathbf{r}_k|^2/4a^2$  and  $\theta_{nk} = \alpha_n - \alpha_k$ . We assume that the beams are arranged in a *ring-shaped array*:  $\mathbf{r}_n = \{R \cos \varphi_n; R \sin \varphi_n\}$  with  $\varphi_n = 2\pi n/N$ , for which we find  $Y_{nk} = -(R/a)^2 \sin^2[\pi(n-k)/N]$ .

First of all, analyzing many-soliton clusters, we remove the center-of-the-mass motion and take  $\mathbf{L} = 0$ . Applying this constraint to Eqs. (3), we find the conditions for the soliton phases,  $\alpha_{i+n} - \alpha_i = \alpha_{k+n} - \alpha_k$ , which are easy to satisfy provided the phase  $\alpha_n$  has a linear dependence on  $n$ , i.e.,  $\alpha_n = \theta n$ , where  $\theta$  is the relative phase between two neighboring solitons in the ring. Then, we employ the phase periodicity condition taken in the form  $\alpha_{n+N} = \alpha_n + 2\pi m$  and find

$$\theta = \frac{2\pi m}{N}. \quad (4)$$

In terms of the classical fields, Eq. (4) gives the condition of the vanishing energy flow  $\mathbf{L} = 0$ , because the linear momentum  $\mathbf{L} = \int \mathbf{j} \, d\mathbf{r}$  can be presented through the local current  $\mathbf{j} = \text{Im}(E^* \nabla E)$ . Therefore, Eq. (4) determines a nontrivial phase distribution for the effectively elastic soliton interaction in the ring. In particular, for the well-known case of two solitons ( $N = 2$ ), this condition gives only two states with the zero energy exchange, when  $m$  is even ( $\theta = 0$ , mutual attraction) and when  $m$  is odd ( $\theta = \pi$ , mutual repulsion) [1].

For a given  $N > 2$ , the condition (4) predicts the existence of a discrete set of allowed stationary states corresponding to the values  $\theta = \theta^{(m)}$  with  $m = 0, \pm 1, \dots, \pm(N-1)$ . Here two states  $\theta^{(\pm|m|)}$  differ only by the sign (direction) of the angular momentum, similar to the case of vortex solitons. Moreover, for any positive (negative)  $m_+$  within the domain  $\pi < |\theta| < 2\pi$ , one can find the corresponding negative (positive) value  $m_-$  within the domain  $0 < |\theta| < \pi$ , so that both  $m_+$  and  $m_-$  describe the same cluster. For example, in the case  $N = 3$ , three states with zero energy exchange are

possible:  $\theta^{(0)} = 0$ ,  $\theta^{(1)} = 2\pi/3$ , and  $\theta^{(2)} = 4\pi/3$ , and the correspondence is  $\theta^{(\pm 1)} \leftrightarrow \theta^{(\mp 2)}$ . Therefore, it is useful to introduce the main value of  $\theta$  in the domain  $0 \leq \theta \leq \pi$ , keeping in mind that all allowed states inside the domain  $0 < \theta < \pi$  are degenerated with respect to the sign of the angular momentum. The absolute value of the angular momentum vanishes at both ends of this domain, when  $m = 0$ , for any  $N$ , and when  $m = N/2$ , for even  $N$ . The number  $m$  determines the full phase twist around the ring, and it plays a role of the topological charge of a phase dislocation associated with the ring.

In order to demonstrate the basic properties of the soliton clusters for a particular example, we select the well-known saturable nonlinear Kerr medium with  $F(I) = I(1 + sI)^{-1}$ , where  $s$  is a saturation parameter. This model supports stable  $(2 + 1)$ -dimensional solitons. First, we apply the variational technique to find the parameters of a single soliton described by the ansatz (2) and find  $A = 3.604$  and  $a = 1.623$  for  $s = 0.5$ . Then, substituting Eq. (2) into the system Hamiltonian, we calculate the effective interaction potential  $U(R) = H(R)/|H(\infty)|$ , where  $H$  is the system Hamiltonian,

$$H = \int \left\{ |\nabla E|^2 - \frac{1}{s} |E|^2 + \frac{1}{s^2} \ln(1 + s|E|^2) \right\} d\mathbf{r}.$$

As a result, for any  $N$  we find *three distinct types* of the interaction potential  $U(R)$ , shown in Fig. 1 for the particular case  $N = 5$ . Only one of them has a local minimum at finite  $R$  which indicates the cluster stabilization against collapse or expansion.

To verify the predictions of our effective-particle approach, we perform a series of simulations of different  $N$ -soliton rings, using the fast-Fourier-transform split-step numerical algorithm and monitoring the conservation of the integrals of motion. Alongside of nonstationary behavior, such as breathing and radiation emitting, we find the clusters dynamics in *excellent agreement with our analysis*. According to Fig. 1, the effective potential is always

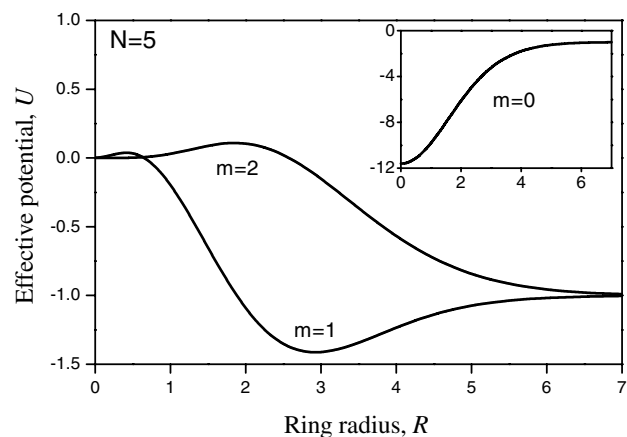


FIG. 1. Examples of the effective potential  $U(R)$  for a circular array of  $N = 5$  solitons. Corresponding values of the topological charge  $m$  are shown near the curves. The dynamically stable bound state is possible for  $m = 1$  only.

attractive for  $m = 0$ , and thus the ring of  $N = 5$  in-phase solitons should exhibit oscillations and, possibly, soliton fusion. Indeed, such a dynamics is observed in Fig. 2(a). Although the oscillations of the ring are well described by the effective potential  $U(R)$ , the ring dynamics is more complicated. Another scenario of the mutual soliton interaction corresponds to the repulsive potential (shown, e.g., in Fig. 1 for the case  $m = 2$  and  $\theta = 4\pi/5$ ). In the numerical simulations, the ringlike soliton array expands with the slowing down rotation, as is shown in Fig. 2(d).

Evolution of the stationary bound state that corresponds to a minimum of the effective potential  $U(R)$  in Fig. 1 (for  $m = 1$ ) is shown in Fig. 2(b). Here the repulsive centrifugal force balances out the soliton attraction. The effective potential predicts the stationary state at  $R_0 = 3$  with a good accuracy, and the cluster does not change its form while rotating during the propagation. To continue the analogy between the soliton cluster and a rigid body, we calculate the cluster's *moment of inertia*,  $I$ , and its *angular velocity*,  $\Omega$ :

$$I = \int |E|^2 \mathbf{r}^2 d\mathbf{r}, \quad \Omega = M/I. \quad (5)$$

For the case shown in Fig. 2(b), the numerically obtained value of the angular velocity is  $\Omega_{\text{num}} \approx \pi/20 = 0.157$ , while the formula (5) gives the value  $\Omega = 0.154$ . We also perform the numerical simulations of the "excited"

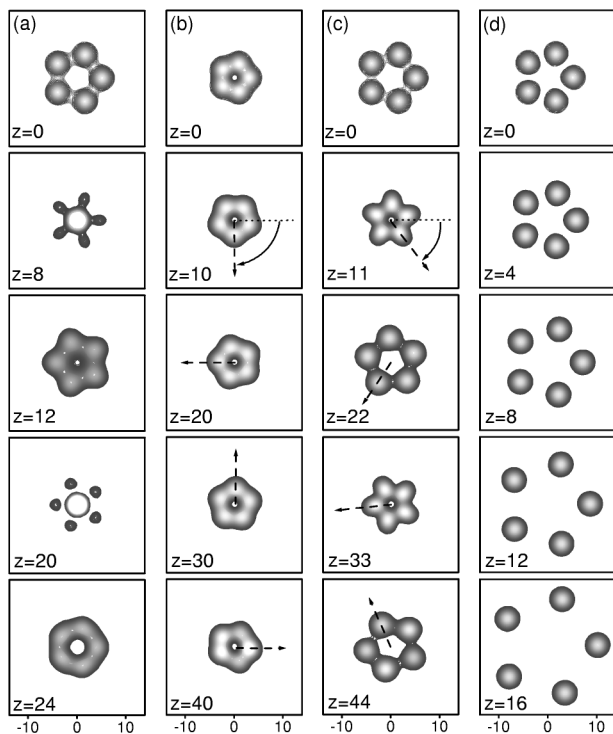


FIG. 2. Different regimes of the interaction of  $N = 5$  solitons: (a)  $m = 0$ , collapse and fusion through oscillations; (b)  $m = 1$ , a stationary bound state with  $R_0 = 3$ ; (c)  $m = 1$ , an excited bound state with the oscillation period  $z_{\text{period}} = 22$ ; (d)  $m = 2$ , the soliton repulsion. The initial radius of the ring in the cases (a), (c), and (d) is  $R = 5$ .

clusters, as shown in Fig. 2(c), and observed oscillations near the equilibrium state. Such a *vibrational* state of the " $N$ -soliton molecule" demonstrates the dynamical radial stability of the bound state in agreement with the effective-particle approximation.

Our analysis is valid for any  $N$ , and it allows us to classify all possible scenarios of the soliton interaction in terms of the phase jump  $\theta$  between the neighboring solitons in the array. Indeed, for  $\theta = 0$ , the ring of  $N$  solitons collapses through several oscillations. If the main value of  $\theta$  belongs to the segment  $0 < \theta \leq \pi/2$ , the interaction between solitons is *attractive*, the value of the induced angular momentum is finite, and there exists a rotating bound state of  $N$  solitons. However, if  $\theta$  belongs to the segment  $\pi/2 < \theta \leq \pi$ , the soliton interaction is *repulsive* and the soliton ring expands with or without ( $\theta = \pi$ ) rotation, similar to the necklace-type beams [6]. For example, in the case  $N = 3$  two stationary states are possible,  $\theta = 0$  and  $\theta = 2\pi/3$ , and there exist *no bound states*. For  $N = 4$  and  $m = 1$ , the value of  $\theta$  is  $\pi/2$  and a cluster is indeed possible, as is shown in Fig. 3.

Together with the intensity of the four-soliton cluster, in Fig. 3 we show the phase distribution for the distances up to 60 diffraction lengths. The initial staircase-like phase in the ring preserves its shape, and it is a nonlinear function of the polar angle  $\varphi$ , similar to the phase of the necklace-ring vector solitons with a fractional spin [7]. Note that the phase of such a state can be described as a *zeroth-order* term in the expansion of the vortex phase near the  $n$ th soliton center:

$$m\varphi \approx m \frac{2\pi n}{N} - m \frac{xy_n - yx_n}{R^2} \dots \quad (6)$$

Calculating the minimum of the potential  $U(R)$  that corresponds to the soliton cluster, we find that for given  $m$  and  $N \gg 1$ , the stationary cluster approaches a vortex soliton of the charge  $m$ . For example, the equilibrium radius  $R_0$  for the clusters with  $m = 1$  is  $R_0 = 3.8$  for  $N = 4$ , and for  $N \geq 4$  it approaches the corresponding vortex radius  $R_0 = 3$ . Furthermore, for  $m = 2$ , the soliton bound states are possible only if  $\theta = 4\pi/N \leq \pi/2$ . This gives the condition  $N \geq 8$ , and for  $N \geq 9$  we find  $R_0 = 5$  which

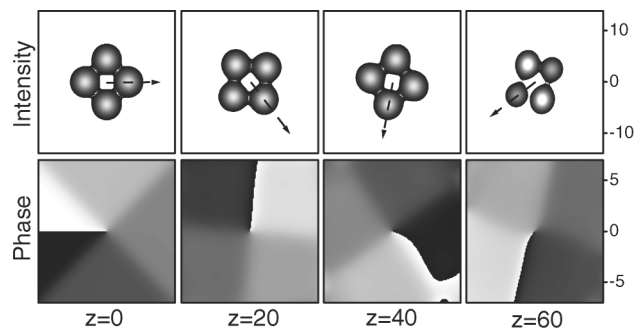


FIG. 3. Rotating cluster of  $N = 4$  solitons. The parameters are  $R_0 = 3.8$  and  $\Omega = 0.042$ . Phase images are scaled from  $-\pi$  (black) to  $+\pi$  (white).

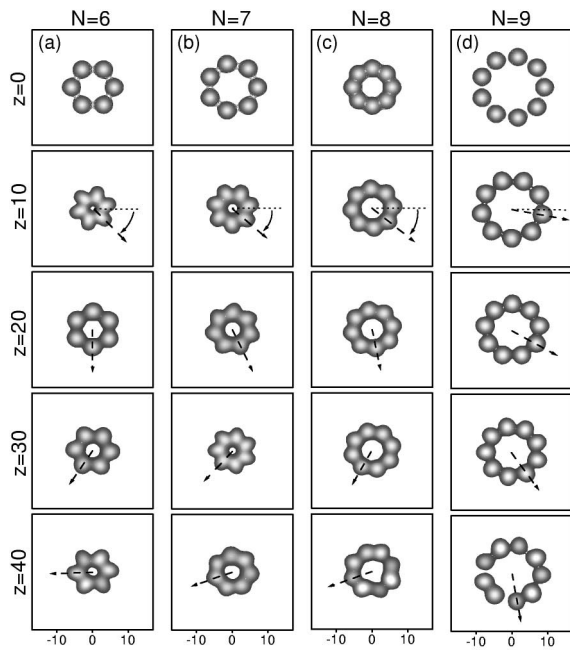


FIG. 4. Examples of the rotating soliton clusters: (a)  $N = 6$  and  $m = 1$ ; (b)  $N = 7$  and  $m = 1$ ; (c)  $N = 8$  and  $m = 2$ ; here we show the exact bound state with  $R_0 = 5.5$  and the angular velocity  $\Omega = 0.092$ ; (d)  $N = 9$  and  $m = 2$ .

is close to the radius of a double-charged vortex. Thus, the ringlike soliton cluster can be considered as a nontrivial “discrete” generalization of the optical vortex [8]. Clusters are generally metastable, and they experience the symmetry-breaking instability. However, they can propagate for many tenths of the diffraction length, being also asymptotically stable in some types of nonlinear media similar to the vortex solitons [9].

In Figs. 4(a), 4(b), and 4(d) we show some examples of the excited rotating clusters with a different number of solitons  $N$  and initial radius  $R = N$ , while in Fig. 4(c) we present the stationary rotating cluster of  $N = 8$  solitons.

We stress that the staircaselike phase distribution is a distinctive feature of the soliton cluster. The evolution of the soliton ring with the linear phases, i.e., those presented by the first-order terms of Eq. (6), can be associated with complex deformation of the vortex.

More interesting structures are found for the vector fields and incoherent interaction of solitons. For example,

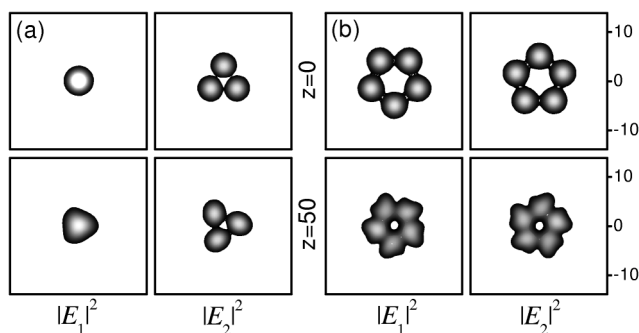


FIG. 5. Examples of the vector soliton clusters.

three coherently interacting solitons cannot form a bound state. However, adding a single “atom” to the incoherently coupled additional component  $E_1$  [see Fig. 5(a)] leads to mutual trapping of all beams. Here, the incoherent attraction balances out both the coherent repulsion, as in the case of multipole vector solitons [10], and the centrifugal force induced by a net angular momentum. This leads to the cluster rotation similar to the two-lobe rotating “propeller” soliton [11]. In Fig. 5(b), we show the five-lobe analog of the necklace-ring vector solitons recently discussed in Ref. [7]. Unlike the necklace-ring solitons, the vector cluster rotates and undergoes internal oscillations as it propagates.

In conclusion, we have revealed a key physical mechanism for stabilizing multisoliton bound states in a bulk medium in the form of rotating ringlike clusters. Such soliton clusters can be considered as a nontrivial generalization of the important concepts of the two-soliton spiraling, optical vortex solitons, and necklace scalar beams, and they provide an example of the next generation of multiple soliton-based systems operating entirely with light. We believe that the basic ideas presented in this Letter will be useful for other applications, such as the beam dynamics in plasmas [12] and the Skyrme model of a classical field theory [13], and they can be also generalized to the inhomogeneous systems (such as the Bose-Einstein condensates in a trap [14]).

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