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[1]Rotating Waves in the Laplace Domain for Angular Regions

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We introduce the rotating waves in a Laplace domain for formulating and solving wave problems with wedge shaped configurations. The method we develop is alternative to and possibly simpler than the one of Malyuzhinets. Applications of this method in this paper are concerned with the diffraction by isorefractive (or diaphaneous) wedges.

Keywords diffraction, Wiener-Hopf technique, wedge

Introduction

In a recent work [1,2,3] this author showed that the diffraction by an impenetrable wedge having arbitrary aperture angle always reduces to a standard Wiener-Hopf factorization. However, he encountered some difficulties in ascertaining the coincidence of Wiener-Hopf solutions with the ones obtained by the Malyuzhinets method. These difficulties are due to the use of two different spectral representations: the unilateral Fourier Transforms (or Laplace transforms) in the Wiener-Hopf technique and the Sommerfeld functions in the Malyuzhinets method. Moreover Sommerfeld integrals introduce the complex angular spectrum w, whereas the Fourier integrals introduce the complex wave numbers h. To simplify this comparison, it appeared more convenient to this author to formulate a Laplace approach in the angular spectrum w, **without using Sommerfeld integrals**. This can be accomplished with the introduction of the concept of the rotating waves. The main aim of the paper is the exposition of this theory. This author believes it is interesting for further understanding of the wave motion in angular regions. A second aim is to show the elegance of the rotating waves method. To this end the solution of the diffraction problem constituted by the diffraction of a plane wave by many isorefractive wedges is

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presented. Apparently Malyuzhinets' solutions for these penetrable wedges are not available in the literature.

Rotating spectral waves

It is well known that angular transmission lines are useful for studying waves in angular regions [4]. Of course, they imply the presence of clockwise and counterclockwise angular rotating waves. The Sommerfeld integrals allow the introduction of only one of these two opposite rotating waves. This simplification comes at a price: the presence in the integral representation of a complex and artificial integration path called Sommerfeld contour.

In addition the Sommerfeld integral constitutes an ansatz and has some limitations. For instance, looking at the link between the Sommerfeld and the Laplace representations [5], we may have waves that admit Laplace representations but not Sommerfeld representations. In fact, the Laplace representation always constitutes a natural and valid representation. With the purpose of avoiding the Sommerfeld contour and at the same time using only Laplace transforms, this author introduced the rotating waves in the Laplace domain. In order to obtain a precise definition of them, let us consider a two-dimensional electromagnetic field $E_z(r,j)$, $H_r(r,j)$ and $H_j(r,j)$ in the free-source angular region $0 \le r < \infty$, $j_1 \le j \le j_2$, (fig.1)

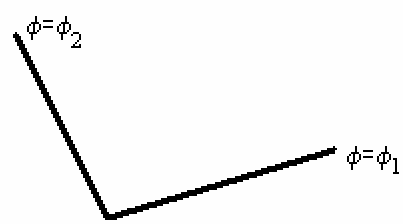

fig 1: angular region $0 \le \rho < \infty$, $\varphi_1 \le \varphi \le \varphi_2$

It is known [6], [7] that the function:

$$v(w,j) = k\sin(w+p)V_+(h,j)\Big|_{h=k\cos(w+p)} \tag{1}$$

where $V_+(h,j)$ is the radial Laplace transform and k the propagation constant of the medium:

$$V_+(h,j) = L[E_z(r,j)] = \int_0^\infty E_z(r,j)e^{-s\,r}dr\Big|_{s=-jh} \tag{2}$$

satisfies the equation:

$$\frac{\partial^2 v(w,j)}{\partial w^2} - \frac{\partial^2 v(w,j)}{\partial j^2} = 0 \tag{3}$$

The most general solution of eq. (3) is:

$$v(w,j) = v_1(w+j) + v_2(w-j) \tag{4}$$

We define $v_1(w)$ (clockwise) and $v_2(w)$ (counterclockwise) the rotating waves in the angular region.
Another important result [8] is that the Laplace transform of the radial component $H_r(r,j)$ of the magnetic field:

$$i(w,j) = k\, I_+(h,j) = k \int_0^\infty H_r(r,j) e^{jhr} dr \Big|_{h=k\cos(w+p)} \tag{5}$$

is expressed in terms of rotating waves in the form:

$$i(w,j) = Y_0 \left[v_1(w+j) - v_2(w-j) \right] \tag{6}$$

where Y_o is the admittance of the medium filling the angular region.
Eq.s (4) and (6) can be recast as transmission line equations with velocity 1, where the role of the time is assumed by the complex variable w:

$$\begin{aligned} \frac{\partial}{\partial j} v(w,j) &= Z_o \frac{\partial}{\partial w} i(w,j) \\ \frac{\partial}{\partial j} i(w,j) &= Y_o \frac{\partial}{\partial w} v(w,j) \end{aligned} \tag{7}$$

where $Y_o = 1/Z_o$

It follows that the rotating waves express the forward and the backward traveling waves of a uniform angular transmission line where the role of the time is assumed by the complex variable w.

Presence of incident plane waves

Let us consider an incident plane wave in the angular region:

$$E_z^i(r,j) = E_o e^{jk\, r\cos(j-j_o)} \quad , \quad H_r^i(r,j) = \frac{k}{w\, m} \sin(j - j_o) e^{jk\, r\cos(j-j_o)} E_o \tag{8}$$

Even though there is the presence of a source in the angular region, eqs (4) and (6) hold again since this source is far away and does not appear in the second member of the wave equation. The Laplace transforms (2), (5) and the eq.s (1), (4) and (6) evaluated for $j = 0$, yield the following rotating waves relevant to the plane wave:

$$\begin{cases} v_1^i[w] = \dfrac{j(\sin w + \sin j_o)}{2(\cos w - \cos j_o)} E_o \\ v_2^i[w] = \dfrac{j(\sin w - \sin j_o)}{2(\cos w - \cos j_o)} E_o \end{cases} \qquad (9)$$

In the following we will call $v_1^i(w)$ and $v_2^i(w)$ the incident rotating waves.

Properties of the scattered waves

The homogeneous region considered in fig.1 may be bounded by other homogeneous or non homogeneous regions. It generates a scattered field $E_z^s(r,j)$, and $H_r^s(r,j)$ in addition to the incident fields $E_z^i(r,j)$ and $H_r^i(r,j)$:

$$E_z^s(r,j) = E_z(r,j) - E_z^i(r,j), \qquad H_r^s(r,j) = H_r(r,j) - H_r^i(r,j)$$

The following theorem [8] has been shown: the Laplace transforms defined by:

$$V_+^s(h,j) = L[E_z(r,j) - E_z^i(r,j)] = \int_0^\infty [E_z(r,j) - E_z^i(r,j)]e^{-s\,r}\,dr\Big|_{s=-jh}$$

$$I_+^s(h,j) = L[H_r(r,j) - H_r^i(r,j)] = \int_0^\infty [H_r(r,j) - H_r^i(r,j)]e^{-s\,r}\,dr\Big|_{s=-jh}$$

are always regular in the $h-$upper half plane: $\mathrm{Im}[h] \geq \mathrm{Im}[-k]$ and in particular on the half-line $h = -ku, u \geq 0$. It follows that $V_+^s(-k\cos w, j)$ and $I_+^s(-k\cos w, j)$ are regular on the imaginary positive half-axis $w = jw''$, $w'' \geq 0$, that is the image of the half-line $h = -ku, u \geq 0$ in the w-plane. This regularity holds for all the values of $j : j_1 \leq j \leq j_2$ and induces the following fundamental property for the scattered rotating waves:

a) The scattered rotating waves $v_{1,2}^s(w)$ defined by:

$$v_{1,2}^s(w) = v_{1,2}(w) - v_{1,2}^i(w) \qquad (10)$$

are respectively regular in the strips $I_1\ \{j_1 \leq \mathrm{Re}[w] \leq j_2\}$ and $I_2\{-j_2 \leq \mathrm{Re}[w] \leq -j_1\}$ (see fig.2). To prove this property, let us observe that from eq.s (6) and (4) we have:

$$v_1^s(w+j) = \frac{-jk\sin w V_+^s(-k\cos w, j) + w\,m\,I_+^s(-k\cos w, j)}{2} \qquad (11a)$$

$$v_2^s(w-j) = \frac{-jk\sin w V_+^s(-k\cos w, j) - w\,m\,I_+^s(-k\cos w, j)}{2} \qquad (11b)$$

From these equations we ascertain that the regularity of $V_+^s(-k\cos w, j)$ and $I_+^s(-k\cos w, j)$ imply that also $v_1^s(jw''+j)$ and $v_2^s(jw''-j)$ are regular for all the values of $j : j_1 \leq j \leq j_2$ and w"≥0 . Consequently, putting $j = w'$, the analytical functions $v_1^s(w'+jw'')$ and $v_2^s(-w'+jw'')$ are holomorphic in the half-strips defined by:

$j_1 \le w' \le j_2, \quad w'' \ge 0$. In addition, the regularity of $V_+^s(-k\cos w, j)$ in w=0 $(h = -k)$ imposes that: $v^s(0, j) = v_1^s(j) + v_2^s(-j) = -k \sin 0\, V_+^s(-k\cos 0, j) = 0$,

i.e. $f(w) = v_1^s(w) + v_2^s(-w)$ is vanishing on the segment $j_1 \le \mathrm{Re}[w] \le j_2$, $\mathrm{Im}[w] = 0$. A process of analytical continuation [8] allows one to show the vanishing of f(w) in the whole complex plane w. The two opposite rotating waves are then related by the property b):

$$v_1^s(w) = -v_2^s(-w) \tag{12}$$

This property expresses a causality principle in the w-plane. Property a) follows immediately from the property b) and from the regularity of $v_1^s(w'+jw'')$ and $v_2^s(-w'+jw'')$ in the half-strip defined by: $j_1 \le w' \le j_2, \quad w'' \ge 0$.

On the interfaces $j = j_1$ and $j = j_2$ many kind of waves can be generated [4]. From a mathematical point of view these waves constitute poles or branch points in the h-plane. The mapping $h = k\cos(w+p)$ induces an infinite number of images of these points on the w-plane. When dealing with impenetrable wedges (see for example Fig.3), it is easy to show that the scattered rotating waves are meromorphic functions of w [8]. It yields only the presence of poles in the w-plane. Because of the property a) $v_1^s(w)$ and $v_2^s(w)$ have (infinite) poles that are respectively suited outside the strips I_1 $\{j_1 \le \mathrm{Re}[w] \le j_2\}$ and $I_2\{-j_2 \le \mathrm{Re}[w] \le -j_1\}$ (fig.2).

Other properties of the rotating waves in free-source angular regions.

Besides the properties a) and b), the rotating waves have the following properties:

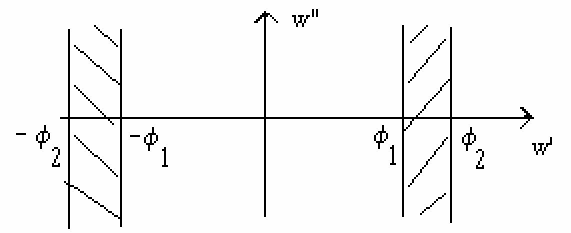

Fig.2. Regularity region of $v_1(w)$ (////) and $v_2(w)$ (\\\\\)

Property c): The functions $v_1(w+j)$ and $v_2(w-j)$ are bounded as $w \to \pm j\infty$ for every $j : j_1 \le j \le j_2$. These bounded values do not depend on j :

$$v_1(\pm j\infty + j) = v_1(\pm j\infty) \qquad v_2(\pm j\infty + j) = v_2(\pm j\infty). \tag{13}$$

To prove this property we first observe that the initial value theorem in the w- plane may be formulated in the form:

$$f(r)_{r=0} = \lim_{h \to j\infty}[-jh\, F_+(h)] = \lim_{w \to +j\infty}[\, jk\cos w F_+(-k\cos w)] \tag{14}$$

Taking into account the boundedness of E_z in $r = 0$ and that the Laplace transform $V_+(-ku, j)$ is bounded as $u \to +\infty$, it follows from eq.14 that $v(w, j)$ is bounded as $w \to +j\infty$. At the same time the boundedness of the Laplace transform $I_+(-ku, j)$ implies that also $i(w, j)$ is bounded as $w \to +j\infty$. Consequently by (11), (1) and (5) both $v_1(w+j)$ and $v_2(w-j)$ are bounded for $w \to +j\infty$ and assume the same value for every $j : j_1 \le j \le j_2$. Properties a) and b) extend these characteristics of $v_1(w+j)$ and $v_2(w-j)$ also in the range $w \to -j\infty$. This concludes the proof.

Property d): If the longitudinal field satisfies, near the edge $r = 0$, the condition $E_z(r, j) = O(r^c)$ with Re[c]>0, the rotating waves behave as $v_{1,2}(w) = O[\exp(-c|\mathrm{Im}[w]|)]$ as $w \to \pm j\infty$

Also the property d) follows from the behavior of the Laplace transforms for $w \to \pm j\infty$. Because of the property b), it is sufficient to study this behavior for $w \to j\infty$. If $E_z = O(r^{\,c})$ and $H_r = O(r^{\,c-1})$, it follows from the Watson lemma that:

$$\sin w\, L(E_z) = O(e^{|\mathrm{Im}.w|}\, e^{|\mathrm{Im}.w|(c-1)}) = O(e^{|\mathrm{Im}.w|c}), \qquad L(H_r) = O(e^{|\mathrm{Im}.w|c})$$

Substituting in eq.s (11) completes the proof.

Property e): When the Sommerfeld integral:

$$E_z(r, j) = \frac{1}{2pj}\left[\int_g s[w + j\,]e^{+jk\cos[w]r}\, dw\right] \tag{15a}$$

is valid, the Sommerfeld function s(w) is related to the clockwise wave v_1(w) through the equation:

$$v_1(w) = -j\, s(w) \tag{15b}$$

The link between the Laplace transform and the Sommerfeld integral has been studied by many authors including Malyuzhinets. For instance the fig.2 of [5], where the plane w is named a, shows that the Bromwich contour in the $h - plane$ has the image $\tilde{g}_+$ in the w or $a - plane$ that is not exactly the part g_+ of the Sommerfeld contour $g = g_+ \bigcup g_-$. When we can deform $\tilde{g}_+$ into g_+, the eq.(15b) does hold and the ansatz (15a) is valid. However no one can exclude the presence of infinite poles between these two contours. In such a case, even assuming on the contour g_+ high values of $\mathrm{Im}[a] = \mathrm{Im}[w]$, we cannot deform $\tilde{g}_+$ into g_+ and the ansatz (15a) is no longer valid.

Plane waves are distant sources in the direction j_o (Fig.3). For them the rotating waves are given by eqs.9. We observe the presence of the pole $w = j_o$ in the regularity strip I_1 of the clockwise waves and the presence of the pole $w = -j_o$ in the regularity strip I_2 of the counterclockwise waves. However, properties b) and c) hold again also for these rotating waves.

Solution of the transmission line equations

Some particular wedge problems require the solution of the transmission line equations (7) with suitable boundary conditions. To solve them, it often turns out to be convenient to introduce the following modified Fourier transforms and integrals [5]:

$$X(n) = -j\int_{-j\infty}^{j\infty} x(w)\exp[\,jn\ w]dw\,, \quad \mathrm{Re}[n] = 0 \tag{16a}$$

$$x(w) = -\frac{j}{2p}\int_{-j\infty}^{j\infty} X(n)\exp[-jn\ w]dn\,, \quad \mathrm{Re}[w] = 0 \tag{16b}$$

For Re.n =0 , $j_1 \le j \le j_2$, we have the following transforms of the rotating waves and the total field:

$$V_{1,2}[n, j] = -j\int_{-j\infty}^{j\infty} \left[v_{1,2}(w \pm j)\right]\exp[+jn\ w]dw\,, \tag{17a}$$

$$V(n, j) = V_1(n, j) + V_2(n, j)\,, \quad I(n, j) = Y_0[V_1(n, j) - V_2(n, j)] \tag{17b}$$

It should be observed that, in general, the rotating waves may be bounded but not vanishing for $w \to \pm j\infty$. Consequently we must define Fourier transforms and inverse Fourier transforms in the distribution space [8].
Evaluating the integrals (17a) for the incident rotating waves (9) yields [8]:

$$V_1^i[n, j] = -p\,\frac{\exp[+jn(j_o - p - j)]}{\sin(pn)}E_o$$
$$-2jp\,E_o\exp[\,jn\,(-j + j_o)][u(-j_o) + u(j - j_o)u(j_o)] \qquad j \ge 0\,, \tag{18a}$$

$$V_1^i[n, j] = -p\,\frac{\exp[+jn(j_o - p - j)]}{\sin(pn)}E_o$$
$$-2jp\,E_o\exp[\,jn\,(-j + j_o)][u(-j_o) - u(-j + j_o)u(-j_o)] \qquad j \le 0 \tag{18b}$$

$$V_2^i[n, j] = -V_1^i[-n, j] \tag{18c}$$

For what concerns the scattered waves, their regularity in the strips I_1 and I_2 allows the use of the following transport theorem [8]

$$V_{1,2}{}^s[n, j_b] = \exp[\mp jn(j_b - j_a)]V_{1,2}{}^s[n, j_a] \tag{19}$$

where $j_1 \le j_{a,b} \le j_2$

This theorem yields $\frac{\partial}{\partial w} \Rightarrow -jn$ and consequently the transmission lines eq.s (7) in the n −domain become:

$$-\frac{\partial}{\partial j} V^s(n,j) = jn\ Z_o\ I^s(n,j) \qquad (20a)$$

$$-\frac{\partial}{\partial j} I^s(n,j) = jn\ Y_o\ V^s(n,j) \qquad (20b)$$

Scattering of a plane wave solution by a perfectly conducting (PEC) wedge

In fig.3 the PEC wedge is immersed in a homogeneous isotropic medium defined by the

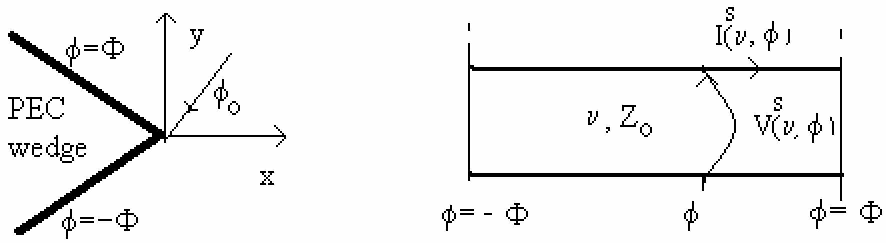

Fig. 3. Geometry and associated transmission line in the ν-domain for scattered waves

angular region $-\Phi \le j \le \Phi$. The source is an E-polarized plane wave propagating in the direction j_o (eq.s(8)). Taking into account the boundary conditions: $V(n,\pm\Phi)=0$, the equations of the transmission line yield:

$$\begin{vmatrix} e^{-jn\Phi} & e^{jn\Phi} \\ e^{jn\Phi} & e^{-jn\Phi} \end{vmatrix} \begin{vmatrix} V_1^s(n,0) \\ V_2^s(n,0) \end{vmatrix} = -\begin{vmatrix} V^i(n,\Phi) \\ V^i(n,-\Phi) \end{vmatrix} = -\begin{vmatrix} V_1^i(n,\Phi)+V_1^i(n,\Phi) \\ V_2^i(n,\Phi)+V_2^i(n,\Phi) \end{vmatrix} \qquad (21)$$

Substituting in the third member the eq.s (18) yield:

$$V_1^s(n,0) = \{p\,\frac{e^{jn\,(j_o-p)}}{\sin(p\ n)} + 2p\ j\,\frac{e^{-j\Phi n}\ \sin[(\Phi-j_o)n]}{\sin(2\Phi n)}\}E_o, \quad V_2^s(n,0) = -V_1^s(-n,0) \qquad (22)$$

Inverse transforming can be accomplishing in closed form, yielding

$$v_1(w) = v_1^i(w) + v_1^s(w) = -\frac{j}{n}\,\frac{\cos\frac{j_o}{n}}{\sin\frac{w}{n} - \sin\frac{j_o}{n}} E_o \qquad (23)$$

with $n = \frac{2\Phi}{p}$

Taking into account that the counterclockwise wave is $v_2(w)=-v_1(-w)$, the well known result follows:

$$E_z(r,j) = \frac{1}{2p}\int_{Br} V_+(h,j)e^{-jhr}\,dh = -\frac{1}{2p}\int_g v(w,j)e^{jkr\cos w}dw = \qquad (24)$$

$$= \frac{jE_o}{2pn}\cos\frac{j_o}{n}\int_g \left(\frac{1}{\sin\frac{w+j}{n}-\sin\frac{j_o}{n}}+\frac{1}{\sin\frac{w-j}{n}+\sin\frac{j_o}{n}}\right)e^{jkr\cos w}dw$$

where B_r is the Bromwich contour and g is its image in the complex plane w through the mapping $h=-k\cos w$.

Formulation of the Malyuzhinets problem in terms of rotating waves

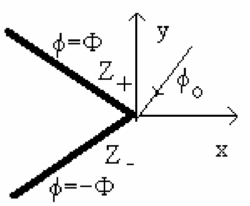

Fig. 4a Geometry

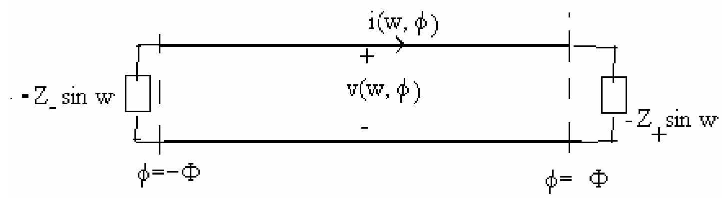

Fig. 4b: Equivalent transmission line in the w-domain

The boundary conditions on an impedance wedge are defined by: $E_z = \pm Z_\pm H_r$ on $j = \pm\Phi$ where the surface impedances $Z_\pm$ depends on the wedge material . Introducing the Laplace transforms, in the w-domain, the boundary conditions become:

$$-v(w,\pm\Phi) = k\sin wV_+(w,\pm\Phi) = \pm k\sin wZ_\pm I_+(w,\pm\Phi) = \pm\sin wZ_\pm i(w,\pm\Phi) \qquad (25)$$

Fig.4b shows the difficulty of solving the problem. In fact we are dealing with transmission lines loaded by w -variable impedances. If $Z_\pm$ were variables on r with the form $A_\pm r$ (4, p.674), the factor sin w would disappear in the last term of eq. (25) and we would be dealing again with transmission lines with constant loads.
In terms of rotating waves the eq.s (25) become:

$$v_1(w\pm\Phi)+v_2(w\mp\Phi) = \mp\frac{\sin w}{\sin q_\pm}\left[v_1(w\pm\Phi)-v_2(w\mp\Phi)\right] \qquad (26)$$

with $Z_{\pm} = Z_o / \sin q_{\pm}$ and $0 \le \mathrm{Re}[q_{\pm}] \le p/2$. Taking into account the property b), we eliminate $v_2(w)$ from (26) and obtain equations identical to those found by Malyuzhinets for the Sommerfeld functions s[w]. i.e.:

$$(\sin w \pm \sin q_{\pm}) v_1(w \pm \Phi) = (-\sin w \pm \sin q_{\pm}) v_1(-w \pm \Phi)$$

We can use the Malyuzhinets method for solving these equations [2], and obtain

$$v_1(w) = j \frac{E_o}{n} \frac{\cos[j_o / n]}{\sin[w/n] - \sin[j_o / n]} \frac{\Psi(w)}{\Psi(j_o)} \qquad (27)$$

where :

$$\Psi(w) = \Psi_{\Phi}(w + \Phi + \frac{p}{2} - q_{+})\, \Psi_{\Phi}(w + \Phi - \frac{p}{2} + q_{+})\, \Psi_{\Phi}(w - \Phi + \frac{p}{2} - q_{-})\, \Psi_{\Phi}(w - \Phi - \frac{p}{2} + q_{-})$$

and $\Psi_{\Phi}(w)$ is the Malyuzhinets function [5].

Scattering of a plane wave by isorefractive wedges

The problem of diffraction of electromagnetic waves by a penetrable wedge has produced a multitude of studies, and yet the problem is unsolved. In order to gain more insights into the behavior of penetrable wedges it is convenient to study the problem of the diffraction of isorefractive or diaphanous wedges. The importance of this problem is due to the fact that it constitutes a dynamical penetrable wedge problem that we can solve in closed form. The solution of isorefractive wedges has been accomplished in the past by using the Kontorovich-Lebedev transform [9] in the frequency domain and the Green function in the time domain [10]. Wiener-Hopf solutions for the right wedge are also available [11]. In this section we solve this problem by using the rotating waves method. This approach has two advantages. It can deal with an arbitrary number of isorefractive wedges and it gives the solution in a form that is more suitable for evaluating both the near field and the diffraction coefficients of the problem.
Fig.5 shows the geometry of the problem. We have isorefractive angular regions 1 and 2 excited by an E- plane wave (with intensity E_o=1) polarized in the z-direction. Let's introduce the rotating waves $v_1^q(w)$ and $v_2^q(w)$ where the superscript q=1,2 indicates the relevant region. By imposing the boundary conditions at the two interfaces $j = 0$ or $j = 2p$, $j = g$, the following system of linear difference equations is obtained:

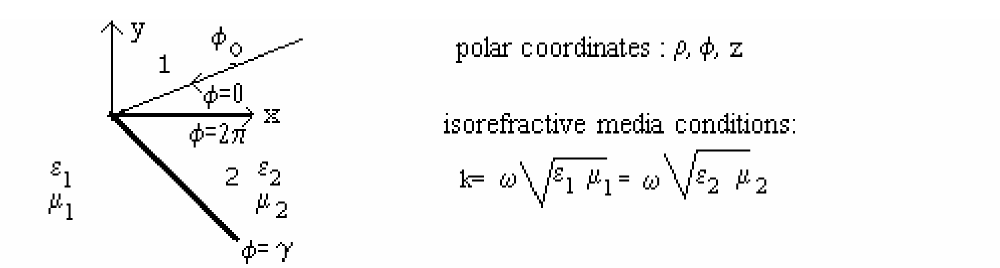

Fig.5 : scattering by an isorefractive wedge

-Interface $j = 0$ or $j = 2p$,
Electrical field matching:

$$v_1^1(w) + v_2^1(w) = v_1^2(w + 2p) + v_2^2(w - 2p) \qquad (28a)$$

Magnetic field matching

$$Y_1(v_1^1(w) - v_2^1(w)) = Y_2(v_1^2(w + 2p) - v_2^2(w - 2p)) \qquad (28b)$$

-Interface $j = g$
Electrical field matching:

$$v_1^1(w + g) + v_2^1(w - g) = v_1^2(w + g) + v_2^2(w - g) \qquad (28c)$$

Magnetic field matching

$$Y_1(v_1^1(w + g) - v_2^1(w - g)) = Y_2(v_1^2(w + g) - v_2^2(w - g)) \qquad (28d)$$

For solving eq.[28] we put:

$$v_1^1(w) = v_1(w) = v_1^s(w) + v_1^i(w)$$

$$v_2^1(w) = v_2(w) = v_2^s(w) + v_2^i(w)$$

and apply to them the Fourier transforms (17a). The transport theorem (19) and the Fourier transform of the incident rotating waves (18) yield algebraic equations involving the four unknowns $V_{1,2}^s(n), V_{1,2}^2(n,g)$. The same system holds by solving the circuit that involves the transmission lines in the $n - domain$ for the different media. The solution of the system has been obtained by using the program MATHEMATICA and is not reported here. The inverse transforming of $V_{1,2}^s(n), V_{1,2}^2(n,g)$ given by (16b) are in general complicated. They have been accomplished explicitly for certain cases by using the residue theorem [8]. For the right wedge $(g = 3p/2)$ we obtain [8]:

$$v_1(w) = \frac{p\, e^{-j(w + j_o - p)}}{\sin(w + j_o)}\Big[\frac{j\, e^{j[2cp + (1-c)(w + j_o - p)]}(Y_1^2 - Y_2^2)}{p[3Y_1^2 - 2(-3 + 4e^{jcp} + 2e^{j2cp})Y_1 Y_2 + 3Y_2^2]} +$$

$$+ \frac{j\, e^{j[-2cp + (1+c)(w + j_o - p)]}(Y_1^2 - Y_2^2)}{p[3Y_1^2 - 2(-3 + 4e^{-jcp} + 2e^{-j2cp})Y_1 Y_2 + 3Y_2^2]}\Big] + p\cot(w + j_o)\frac{Y_1^2 - Y_2^2}{jp(3Y_1 + Y_2)(Y_1 + 3Y_2)} +$$

$$+ \frac{p\, e^{j(w - j_o + p)}}{\sin(w - j_o)}\Big[\frac{e^{-j(-1+c)(w - j_o)}[4e^{jcp}Y_1 Y_2 - (Y_1 + Y_2)^2]}{jp[3Y_1^2 - 2(-3 + 4e^{jcp} + 2e^{j2cp})Y_1 Y_2 + 3Y_2^2]} +$$

$$+\frac{e^{j(1+c)(w-j_o)}[4e^{-jcp}Y_1\,Y_2\ -(Y_1\ +Y_2\ )^2\,]}{jp[3Y_1^2-2(-3+4e^{-jcp}\ +2e^{-j2cp}\ )Y_1\,Y_2\ +3Y_2^2]}]-p\ \cot(w-j_o)\frac{4Y_1\,Y_2\ +(Y_1\ +Y_2\ )^2}{jp[-4Y_1\,Y_2\ -3(Y_1\ +Y_2\ )^2\,]} \quad (29)$$

where

$$c=\frac{1}{p}\,Arc\ \tan[\frac{\sqrt{3Y_1^4+4Y_1^3Y_2\ -14Y_1^2Y_2^2+4Y_1\,Y_2^3+3Y_2^4}}{Y_1^2+6Y_1\,Y_2\ +Y_2^2}]\ , \qquad 0<\mathrm{Re}[c]<1$$

The asymptotic behavior for $w=\pm j\infty$ is given by:

$$v_1(w)=O\{\exp[-|\mathrm{Im}[w]|(1-c)]$$

It yields the near edge behavior:

$$E_z=O(r^{1-c}),\ H_r=O(r^{-c}),$$

This behavior is in agreement with the static behavior of a penetrable right wedge [11]. The analytical expression in Eq. (29) appears to be different from the analytical expression obtained by the Wiener-Hopf technique in [11]. The coincidence of the two solutions has been ascertained numerically by MATHEMATICA using different values of the parameters involved. Near and far field discussions have been reported in [11].

Three-dimensional case

Three-dimensional excitations imply the introduction of plane waves with skew incidence. We have the following longitudinal components relevant to the plane waves with skew incidence:

$$E_z^i=E_oe^{jt\,r\cos(j-j_o\}}e^{-ja\,z} \qquad H_z^i=H_oe^{jt\,r\cos(j-j_o)}e^{-ja\,z} \quad (30)$$

where, by indicating with q_o the angle between the edge and the direction of the plane wave, it is: $a=k\cos q_0,\quad t=k\sin q_o$

Consequently, in general, there is the presence of both the longitudinal components E_z and H_z for the total field. Again we introduce the rotating waves for the Laplace transform of the longitudinal components:

$$E_z\Rightarrow v(w,j)=-t\ \sin\,wV_{z+}(-t\ \cos w,j)=v_1(w+j)+v_2(w-j) \quad (31)$$

$$H_z\Rightarrow i(w,j)=-t\ \sin\,wI_{z+}(-t\ \cos w,j)=i_1(w+j)+i_2(w-j) \quad (32)$$

where we have introduced the Laplace transforms V_{z+} and I_{z+} for both the longitudinal electric and magnetic fields.

Starting from the eq.s (31) and (32) we deduced the following Laplace transform of the radial components [8]:

$$L\left[E_r\right]_{h=-t\cos w} = [-\frac{wm}{t^2}(i_1(w+j)-i_2(w-j)) - \frac{a}{t^2\tan w}(v_1(w+j)+v_2(w-j))]e^{-ja\,z} \tag{33}$$

$$L\left[H_r\right]_{h=-t\cos w} = [\frac{we}{t^2}(v_1(w+j)-v_2(w-j)) - \frac{a}{t^2\tan w}(i_1(w+j)+i_2(w-j))]e^{-ja\,z} \tag{34}$$

The previous equations allow us to formulate all the problems considered in the case of plane waves with skew incidence. Notice that for the isorefractive wedges the formulation always yields a system of difference linear equations with constant coefficients. This system can be solved with the same procedure described for the normal incidence. Conversely, the Malyuzhinets problems in general yield functional equations that are second order difference equations with non constant coefficients. Although some cases can be solved in closed form [12],[13], no general solution has been obtained up to now.

The role of the Kontorovich-Lebedev transforms

For obtaining the solution of wave problems in angular regions, sometimes we used the sequel:

$$r-\text{domain} \overset{\text{Laplace}}{\Rightarrow} h\text{-domain} \overset{\text{mapping } h=-k\cos w}{\Rightarrow} \text{w-domain} \overset{\text{Fourier}}{\Rightarrow} n\text{-domain}$$

It easy to ascertain that it is possible to pass directly from the $r-$domain to the n -domain by using the Kontorovich-Lebedev transform:

$$r-\text{domain} \overset{\text{Kontorovich-Lebedev}}{\Rightarrow} n\text{ - domain}$$

in fact we have (use the property: $\int_{-j\infty}^{j\infty} -\sin w e^{-jkr\cos w} e^{jn\,w} dw = \frac{p\,n}{kr} e^{-jnp/2} H_n^{(2)}(kr)$)

$$X(n) = -j\int_{-j\infty}^{j\infty}\left[x(w)\right]\exp[+jn\,w]dw = jk\int_{-j\infty}^{j\infty}\int_0^{\infty}\hat{x}(r)\exp[-jkr\cos w]\sin w\exp[+jn\,w]dr\,dw =$$

$$= -j\,e^{-jnp/2}\int_0^{\infty} H_n^{(2)}(kr)\frac{np}{r}\hat{x}(r)dr = -jn\,p\;e^{-jnp/2}KL[\hat{x}(r)] \tag{35}$$

where $KL[\hat{x}(r)]$ is the Kontorovich-Lebedev transform defined by (4, p.325):

$$KL[\hat{x}(r)] = \int_0^{\infty} H_n^{(2)}(kr)\frac{1}{r}\hat{x}(r)dr$$

Knowing $KL[\hat{x}(r)]$ we obtain $\hat{x}(r)$ by the inverse transform (4, p.325):

$$\hat{x}(r) = -\frac{1}{4}\int_{-j\infty}^{j\infty} n\, KL[\hat{x}(r)] H_n^{(2)}(kr)dn = \frac{1}{j4p}\int_{-j\infty}^{j\infty} e^{jn\,p/2} X(n) H_n^{(2)}(kr)dn \tag{36}$$

In some cases the Kontorovich-Lebedev transform may be useful . For instance, it has been used for the diffraction by an isorefractive wedge [9]. However, sometimes this transform does not exist. Moreover whereas the solutions in the h or w-plane allow one to use immediately powerful techniques for obtaining the far field (saddle point technique) and the near fields (Watson lemma), the solutions in the n - domain do not have these characteristics.

Conclusions

It can be shown that the theory of rotating waves may be successfully applied to all the problems approached by the Malyuzhinets method [8]. Comparing the two methods, we ascertain many deep-rooted analogies. However, in this author's opinion, the rotating waves present peculiar properties that simplify their use and make it reliable for studying wave problems in angular regions.

For instance this author claims the following facts must be appreciated:

-the rotating waves are based on the introduction of Laplace transforms and not on the ansatz constituted by the Sommerfeld integral which is sometimes invalid.

- the Malyuzhinets nullification theorem [5] has been avoided. Sometimes this theorem may be misinterpreted.

-We safely deal with the rotating waves in a larger number of angular regions. Instead we must be very careful if we use Malyuzhinets method in angular regions not defined in the strip $-\Phi \le \mathrm{Re}[w] \le \Phi$

-We can safely apply the Fourier technique ideated by Malyuzhinets for solving difference equations since the transport theorem is always valid for the scattered rotating waves. In addition, the distribution space for solving the system in the Fourier domain ensures the existence of the involved Fourier transforms and integrals.

-Many difficulties which are present in the Malyuzhinets method are overcome when we use the rotating waves. For instance, we have solved (again with many different angular regions) wave problems involving isorefractive regions. Up to now these problems have not been solved by the Malyuzhinets method.

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