Rotation Equivariant Minkowski Valuations

Rolf Schneider and Franz E. Schuster

Dedicated to Professor Peter Gruber
on the occasion of his sixty-fifth birthday

Abstract

The projection body operator Π, which associates with every convex body in Euclidean space $\mathbb{R}^n$ its projection body, is a continuous valuation, it is invariant under translations and equivariant under rotations. It is also well known that Π maps the set of polytopes in $\mathbb{R}^n$ into itself. We show that Π is the only non-trivial operator with these properties.

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1 Introduction and Main Results

Let $\mathcal{K}^n$ denote the space of convex bodies (non-empty, compact, convex sets) in n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$), endowed with the Hausdorff metric. A convex body $K \in \mathcal{K}^n$ is determined by its support function $h(K, \cdot)$, defined on $\mathbb{R}^n$ by $h(K, x) = \max\{\langle x, y\rangle : y \in K\}$, where $\langle \cdot, \cdot \rangle$ is the scalar product of $\mathbb{R}^n$. The *projection body* ΠK of K is defined by

$$h(\Pi K, u) = V_{n-1}(K|u^{\perp}) \qquad \text{for } u \in S^{n-1}.$$

Here, $K|u^{\perp}$ denotes the image of K under orthogonal projection to the $(n-1)$-dimensional subspace orthogonal to u, and S^{n-1} is the unit sphere of $\mathbb{R}^n$. Generally, we denote by $V_k(M)$ the k-dimensional volume of a k-dimensional convex body M.

The projection body operator was already introduced by Minkowski [22]. In recent years it has attracted increased attention due to its numerous applications in different areas, see [3, 4, 5, 7, 13, 31]. Projection bodies of convex bodies are centered convex bodies called *zonoids*. For their role in geometry, we refer to the surveys [28, 6]. Projection bodies of convex polytopes, called *zonotopes*, have appeared in optimization, computational geometry, and other areas, see [32].

In this paper, the emphasis is on the fact that the projection body operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$ is a *Minkowski valuation*, i.e., a valuation with respect to Minkowski addition on $\mathcal{K}^n$. In general, a mapping $\varphi : \mathcal{K}^n \to A$ into an abelian semigroup $(A, +)$ is called a *valuation* if

$$\varphi(K \cup M) + \varphi(K \cap M) = \varphi(K) + \varphi(M)$$

whenever $K, M, K \cup M \in \mathcal{K}^n$. Valuations on convex bodies are a classical concept. Probably the most famous result in this area is Hadwiger's classification of rigid motion invariant real valued continuous valuations, see [8, 12] and the surveys [21], [20]. In recent years, many new results on real and body valued valuations have been obtained, see [1, 2, 10, 11, 13, 14, 15, 16, 17, 27].

An immediate consequence of a result obtained by M. Ludwig in [15, Corollary 2.2], extending a previous result from [13], is the following characterization of the projection body operator.

Theorem. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, translation invariant valuation with the property that, for all $K \in \mathcal{K}^n$ and every $\alpha \in SL(n)$,*

$$\Phi\alpha K = \alpha^{-T}\Phi K.$$

Then there is a constant $c \geq 0$ such that $\Phi = c\Pi$.

Thus, among all continuous, translation invariant valuations from $\mathcal{K}^n$ to $\mathcal{K}^n$, the projection body operator is characterized, up to a factor, by its $SL(n)$ contravariance. It was also shown in [15] that the assumption of continuity can be omitted when $\mathcal{K}^n$ as the domain of Φ is replaced by $\mathcal{P}^n$, the set of convex polytopes in $\mathbb{R}^n$.

In the following, we will consider continuous, translation invariant valuations $\Phi : \mathcal{K}^n \to \mathcal{K}^n$, but we will replace the strong assumption of $SL(n)$ contravariance, which belongs to affine geometry, by the Euclidean condition of rotation equivariance, i.e., the property that, for all $K \in \mathcal{K}^n$ and every ϑ in the rotation group $SO(n)$ of $\mathbb{R}^n$,

$$\Phi\vartheta K = \vartheta\Phi K.$$

The projection body operator is no longer characterized by these properties. Simple further examples are the trivial maps $\mathcal{I}$ and $-\mathcal{I}$ given by

$$\mathcal{I}(K) = K - s(K) \quad \text{and} \quad (-\mathcal{I})(K) = -K + s(K) \qquad \text{for } K \in \mathcal{K}^n.$$

Here, $s : \mathcal{K}^n \to \mathbb{R}^n$ denotes the Steiner point map, defined by

$$s(K) = n \int_{S^{n-1}} h(K, u)u \, \mathrm{d}u, \tag{1}$$

where the integration is with respect to the rotation invariant probability measure on the sphere. The Steiner point map is the unique vector valued, rigid motion equivariant and continuous valuation on $\mathcal{K}^n$, see [23, Satz 2].

A large class of non-trivial examples is provided by translation invariant Minkowski endomorphisms. This class of operators was introduced and investigated by the first author, see [24, 25], and more recently studied by Kiderlen [9]. They are precisely the continuous valuations from $\mathcal{K}^n$ to $\mathcal{K}^n$, invariant under translations and equivariant under rotations, that are homogeneous of degree one. Here a function φ from $\mathcal{K}^n$ to $\mathbb{R}$ or $\mathcal{K}^n$ is called *homogeneous of degree* j if $\varphi(\lambda K) = \lambda^j \varphi(K)$ for $K \in \mathcal{K}^n$ and $\lambda \geq 0$. The case of valuations homogeneous of degree $n-1$, called Blaschke Minkowski homomorphisms, was investigated recently by the second author in [29, 30].

The main object of this paper is to find an additional assumption which suffices to single out among this large class of valuations the combinations of the projection body operator Π and the mappings $\mathcal{I}$ and $-\mathcal{I}$. This additional assumption will be the property that polytopes are mapped to polytopes.

Theorem 1. *Let $n \geq 3$. Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, translation invariant and rotation equivariant valuation. If Φ maps polytopes to polytopes, then*

$$\Phi = c_1 \Pi + c_2 \mathcal{I} + c_3(-\mathcal{I})$$

with constants $c_1, c_2, c_3 \geq 0$.

Remarks. The assumption that Φ maps polytopes to polytopes is convenient to formulate, but stronger than necessary. As the proof shows, only the following is needed. To every $j \in \{0, \ldots, n\}$, there exists a convex body K of dimension j such that $\Phi m K$ is a polytope, for $n+1$ different values of m.

In the plane, where the rotation group is abelian, the assertion has to be modified. Let $\Phi : \mathcal{K}^2 \to \mathcal{K}^2$ be a continuous, translation invariant and rotation equivariant valuation. If the image of Φ contains some polygon with more than one point, then there are rotations $\vartheta_1, \ldots, \vartheta_r$ of $\mathbb{R}^2$ and positive numbers $\lambda_1, \ldots, \lambda_r$ such that

$$\Phi K = \lambda_1 \vartheta_1 [K - s(K)] + \ldots + \lambda_r \vartheta_r [K - s(K)]$$

for all $K \in \mathcal{K}^2$. This was proved in [25, Satz 3].

Under the additional assumption of homogeneity of degree one, the combinations of $\mathcal{I}$ and $-\mathcal{I}$ were characterized by the first author in [24, Corollary 1.12].

Among a subclass of the Blaschke Minkowski homomorphisms (which includes the even ones), the projection body operator was characterized (up to a factor) by the second author in [29, Theorem 5.3], by the assumption that it maps some n-dimensional convex body to a polytope. The wish to generalize this characterization has led us to the following result.

Theorem 2. *Let $n \geq 3$. Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, translation invariant and rotation equivariant valuation. If Φ maps bodies of dimension $n-2$ to $\{0\}$ and maps some n-dimensional convex body to a polytope, then $\Phi = c\Pi$ with some constant $c \geq 0$.*

Without the assumption that Φ maps some n-dimensional convex body to a polytope, every Blaschke Minkowski homomorphism would satisfy the conditions of Theorem 2. Thus, in particular, the multiples of the projection body operator are the only Blaschke Minkowski homomorphisms that map some n-dimensional convex body to a polytope, see Corollary 1.

2 Results on homogeneous valuations

In this section we collect further material on convex bodies and some well-known results from the theory of real valued valuations. General references are the books by Schneider [26] and by Klain and Rota [12]. At the end of this section we will prove the main tool for the proofs of Theorems 1 and 2.

A convex body $K \in \mathcal{K}^n$ is uniquely determined by its support function $h(K, \cdot)$, which is positively homogeneous of degree one and sublinear. Conversely, every function with these properties is the support function of a convex body. From the definition of $h(K, \cdot)$ it is easily seen that $h(\vartheta K, u) = h(K, \vartheta^{-1}u)$ for every $u \in \mathbb{R}^n$ and every $\vartheta \in SO(n)$. The support function $h(K, \cdot)$ of a convex body $K \in \mathcal{K}^n$ is piecewise linear if and only if K is a polytope.

A convex body $K \in \mathcal{K}^n$ with non-empty interior is also determined up to translation by its surface area measure $S_{n-1}(K, \cdot)$. For a Borel set $\omega \subseteq S^{n-1}$, the value $S_{n-1}(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all boundary points of K at which there exists a normal vector of K belonging to ω. The relation $S_{n-1}(\lambda K, \cdot) = \lambda^{n-1} S_{n-1}(K, \cdot)$ holds for all $K \in \mathcal{K}^n$ and $\lambda \geq 0$. For $\vartheta \in SO(n)$, we have $S_{n-1}(\vartheta K, \cdot) = \vartheta S_{n-1}(K, \cdot)$, where $\vartheta S_{n-1}(K, \cdot)$ is the image measure of $S_{n-1}(K, \cdot)$ under the rotation ϑ. By Minkowski's existence theorem, a non-negative measure μ on S^{n-1} is the surface area measure of a convex body if and only if μ has its center of mass at the origin and is not concentrated on any great subsphere.

We collect some auxiliary results on translation invariant real valued valuations, which will be employed repeatedly.

Lemma 1 (Hadwiger [8, p. 79]). *If $\varphi : \mathcal{K}^n \to \mathbb{R}$ is a continuous, translation invariant valuation, homogeneous of degree n, then $\varphi = cV_n$ with a constant c.*

Lemma 2 (McMullen [18]). *Every continuous, translation invariant valuation $\varphi : \mathcal{K}^n \to \mathbb{R}$ has a unique representation*

$$\varphi = \varphi_0 + \ldots + \varphi_n,$$

where $\varphi_j : \mathcal{K}^n \to \mathbb{R}$ is a continuous, translation invariant valuation which is homogeneous of degree j.

A valuation φ on $\mathcal{K}^n$ is called *simple* if $\varphi(K) = 0$ whenever $\dim K < n$. A function φ from $\mathcal{K}^n$ to $\mathbb{R}$ or $\mathcal{K}^n$ is called *even* (resp. *odd*) if $\varphi(-K) = \varphi(K)$ (resp. $\varphi(-K) = -\varphi(K)$) for all $K \in \mathcal{K}^n$. The following classification of translation invariant, continuous and simple valuations, due to Klain [10] (for even valuations) and the first author [27], will be useful.

Lemma 3. *If $\varphi : \mathcal{K}^n \to \mathbb{R}$ is a continuous, translation invariant and simple valuation, then*

$$\varphi(K) = cV_n(K) + \int_{S^{n-1}} g(u) \, \mathrm{d}S_{n-1}(K, u) \qquad \text{for } K \in \mathcal{K}^n,$$

where c is a constant and g is an odd, continuous real function on S^{n-1}.

From Lemma 3 we deduce a result on homogeneous valuations which are not necessarily simple. For a subspace E of $\mathbb{R}^n$, we denote by $\mathcal{K}(E)$ the set of convex bodies contained in E, and by $SO(E)$ the subgroup of rotations in $SO(n)$ mapping E into itself. The map $\pi_E : \mathbb{R}^n \to E$ is the orthogonal projection.

Lemma 4. *Let $\varphi : \mathcal{K}^n \to \mathbb{R}$ be a continuous, translation invariant valuation which is homogeneous of degree j, for a given $j \in \{0, 1, \dots, n-1\}$.*

(a) *If φ is even and if $\varphi(K) = 0$ whenever $\dim K = j$, then $\varphi = 0$.*

(b) *If $\varphi(K) = 0$ whenever $\dim K = j + 1$, then $\varphi = 0$.*

Proof. Assertion (a) was proved by Klain [11, Corollary 3.2]. In order to prove assertion (b), we can assume $j \in \{0, 1, \dots, n-2\}$. Let E be a $(j+2)$-dimensional linear subspace of $\mathbb{R}^n$. By the continuity of φ, we have $\varphi(K) = 0$ if $\dim K \leq j + 1$, thus the restriction of the valuation φ to $\mathcal{K}(E)$ is simple. It is continuous and invariant under translations of E into itself, therefore we deduce from Lemma 3, applied in E, that it is a linear combination of valuations that are homogeneous of degrees $j+2$ and $j+1$, respectively. Since φ is homogeneous of degree j, we get $\varphi = 0$ on $\mathcal{K}(E)$. Since E was an arbitrary $(j + 2)$-dimensional subspace, we have $\varphi(K) = 0$ whenever $\dim K \leq j + 2$. Now we can repeat the argument with a $(j + 3)$-dimensional subspace, and so on, to conclude finally that $\varphi(K) = 0$ holds for all convex bodies $K \in \mathcal{K}^n$. $\square$

If φ satisfies all assumptions of Lemma 4 (a) except the evenness, then it follows that the valuation defined by $K \mapsto \varphi(K) + \varphi(-K)$ is identically zero. In particular, one can deduce that $\varphi(K) = 0$ for all centrally symmetric convex bodies K.

Lemma 4 leads to the following auxiliary result on valuations taking their values in the space of convex bodies. The equation $\Phi = \{0\}$ means that Φ maps every convex body to the one-pointed set containing only the origin of $\mathbb{R}^n$.

Lemma 5. *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, translation invariant and rotation equivariant valuation which is homogeneous of degree j, for a given $j \in \{0, 1, \dots, n-1\}$. If $\Phi K = \{0\}$ whenever $\dim K = j$, then $\Phi = \{0\}$.*

Proof. If the assumptions are satisfied, we deduce from Lemma 4 (a) (applying it to $h(\Phi K + \Phi(-K), u)$ with $u \in \mathbb{R}^n$) that Φ is odd and that $\Phi K = \{0\}$ holds for all centrally symmetric bodies $K \in \mathcal{K}^n$.

To extend the latter result to general convex bodies, we use an argument employed by Klain [10]. Let E be a $(j + 1)$-dimensional linear subspace of $\mathbb{R}^n$. Let Δ be a simplex in E, say $\Delta = \mathrm{conv}\{0, v_1, \dots, v_{j+1}\}$, without loss of generality. Let $v := v_1 + \ldots + v_{j+1}$ and $\Delta' := \mathrm{conv}\{v, v - v_1, \dots, v - v_{j+1}\}$. The parallelepiped, P, that is spanned by $v_1, \dots, v_{j+1}$, is the union of Δ, Δ' and a centrally symmetric polytope Q, where $\dim(\Delta \cap Q) = \dim(\Delta' \cap Q) = j$ and $\Delta \cup Q$ is convex. By assumption, the valuation $h(\Phi(\cdot), u)$, for given $u \in \mathbb{R}^n$, vanishes on convex bodies of dimension smaller than $j + 1$. Therefore, the restriction of $h(\Phi(\cdot), u)$ to $\mathcal{K}(E)$ is a simple valuation. It follows that $h(\Phi\Delta, u) + h(\Phi Q, u) + h(\Phi\Delta', u) = h(\Phi P, u)$. Since $u \in \mathbb{R}^n$ was arbitrary and $\Phi Q = \{0\}$, this yields $\Phi\Delta + \Phi\Delta' = \Phi P = \{0\}$. Since the summands on the left-hand side are convex bodies, this is only possible if $\Phi\Delta$ is one-pointed. Now we use a standard argument: every polytope can be decomposed into simplices, and a simple valuation on polytopes has an additive extension to the finite unions of polytopes. Together with the continuity of Φ, this yields that ΦK is one-pointed for all $K \in \mathcal{K}(E)$, say $\Phi K = \{t^K\}$. Consider the odd map $t : \mathcal{K}(E) \to E$ defined by $t(K) := \pi_E t^K$. Then $t + s : \mathcal{K}(E) \to E$ is a continuous valuation, equivariant under the rigid motions of E. By the uniqueness property of the Steiner point map mentioned after definition (1), this yields $t(K) = 0$ for all $K \in \mathcal{K}(E)$. Thus, t^K is contained in $E^\perp$ for all $K \in \mathcal{K}(E)$. If $\dim E \leq n-2$, the vector t^K is invariant under every rotation of $\mathbb{R}^n$ that leaves E pointwise fixed, thus $t^K = 0$. If $\dim E = n-1$, we have $t^K = \varphi(K)e$, where e is a unit normal vector of E and $\varphi : \mathcal{K}(E) \to \mathbb{R}$ is an odd, translation invariant, continuous and simple valuation. For every $\vartheta \in SO(E)$ fixing e we get

$$\varphi(\vartheta K) = h(\Phi\vartheta K, e) = h(\vartheta\Phi K, e) = h(\vartheta\Phi K, \vartheta e) = h(\Phi K, e) = \varphi(K).$$

Thus, φ is also invariant with respect to rotations of E into itself. The valuation φ can be represented as in Lemma 3 (applied in E); here $c = 0$ since φ is odd, and by the rotation invariance, the odd function g is constant and hence vanishes. This gives $\varphi = 0$ and thus $\Phi K = \{0\}$, for all $K \in \mathcal{K}(E)$.

Since E was an arbitrary $(j+1)$-dimensional subspace, we have $\Phi K = \{0\}$ for all convex bodies K with $\dim K \leq j+1$. Therefore, by Lemma 4 (b), the valuation $h(\Phi(\cdot), u)$, for given $u \in \mathbb{R}^n$, vanishes identically. Since u was arbitrary, this yields $\Phi = \{0\}$. $\square$

In [24], the first author started an investigation of continuous maps $\Psi : \mathcal{K}^n \to \mathcal{K}^n$, called (Minkowski) endomorphisms, with the following properties:

(a) Ψ is Minkowski additive, i.e., $\Psi(K + L) = \Psi K + \Psi L$ for $K, L \in \mathcal{K}^n$.

(b) $\Psi(\vartheta K + t) = \vartheta\Psi K + t$ for $K \in \mathcal{K}^n, \vartheta \in SO(n)$ and $t \in \mathbb{R}^n$.

Note that continuity and Minkowski additivity imply that Ψ is homogeneous of degree one. Moreover, since $(K \cup L) + (K \cap L) = K + L$ whenever $K, L, K \cup L \in \mathcal{K}^n$, the map Ψ is a Minkowski valuation. Thus, it follows from the equivariance properties of the Steiner point map s that the map $\Psi - s$ is a translation invariant, continuous Minkowski valuation that is equivariant with respect to rotations and homogeneous of degree one.

Apart from constructing a large class of non-trivial examples, the main purpose of [24] was to find reasonable additional assumptions to single out suitable combinations of dilatations and reflections among the class of Minkowski endomorphisms. One of the obtained results was the following, see [24, Theorem 1.8 (b)], which will be used below.

Theorem 3. *Let $\Psi : \mathcal{K}^n \to \mathcal{K}^n$ be an endomorphism. If the image under Ψ of some convex body is a segment, then there are constants $c_2, c_3 \geq 0$ such that*

$$\Psi K = c_2[K - s(K)] + c_3[-K + s(K)] + s(K) \qquad \text{for } K \in \mathcal{K}^n.$$

Motivated by the work of Schneider, the second author [29] recently investigated continuous operators $\Psi : \mathcal{K}^n \to \mathcal{K}^n$, called Blaschke Minkowski homomorphisms, with the following properties:

(a) Ψ is Blaschke Minkowski additive, i.e., $\Psi(K \# L) = \Psi K + \Psi L$ for $K, L \in \mathcal{K}^n$.

(b) Ψ is translation invariant and equivariant with respect to rotations.

Here, the Blaschke sum $K \# L$ of the convex bodies $K, L \in \mathcal{K}^n$ is the convex body with $S_{n-1}(K \# L, \cdot) = S_{n-1}(K, \cdot) + S_{n-1}(L, \cdot)$ and, say, the Steiner point at the origin. Property (a) and the continuity of Ψ imply that Ψ is homogeneous of degree $n-1$. Moreover, since $(K \cup L) \# (K \cap L) = K \# L$ whenever $K, L, K \cup L \in \mathcal{K}^n$, the map Ψ is a Minkowski valuation. A result of McMullen [19] on continuous, translation invariant real valued valuations implies (compare the proof of Theorem 1.2 in [29]) that Blaschke Minkowski homomorphisms are precisely the continuous, translation invariant valuations, homogeneous of degree $n-1$, that are equivariant with respect to rotations.

Among other results, the following, essentially unique, representation of Blaschke Minkowski homomorphisms Ψ was obtained in [29, Theorem 1.2 and Lemma 4.6]:

$$h(\Psi K, u) = \int_{S^{n-1}} [p(\langle u, v\rangle) + q(\langle u, v\rangle)] \, \mathrm{d}S_{n-1}(K, v), \qquad u \in S^{n-1}, \tag{2}$$

for $K \in \mathcal{K}^n$, where p, q are continuous functions on $[-1, 1]$, p is even, q is odd, and $p(\langle \cdot, v\rangle)$ is the restriction of a support function to S^{n-1}. The following result is a version of Theorem 5.3 in [29], it corresponds to Theorem 3 for Blaschke Minkowski homomorphisms.

Theorem 4. *Let $\Psi : \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke Minkowski homomorphism. If the image under Ψ of some convex body M of dimension at least $n-1$ is a polytope, then $\Psi K = c_1 \Pi K$ for each centrally symmetric $K \in \mathcal{K}^n$, where $c_1 \geq 0$ is a real constant.*

We note here that Theorem 5.3 in [29] was formulated for an n-dimensional body M and a certain class of Blaschke Minkowski homomorphisms, but the proof needs only minor modifications to give the result stated as Theorem 4; the only requirement is that the support of $S_{n-1}(M, \cdot)$ is not empty, which is satisfied if $\dim M = n-1$.

Continuous, translation invariant and rotation equivariant valuations $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ which are homogeneous of some degree $j \in \{2, \ldots, n-2\}$, have not been much investigated. Examples are the mappings Π_j defined by

$$h(\Pi_j K, u) = \frac{1}{2} \int_{S^{n-1}} |\langle u, v\rangle| \, \mathrm{d}S_j(K, v) \qquad \text{for } u \in S^{n-1},$$

where $S_j(K, \cdot)$ is the area measure of order j of the convex body K. The body $\Pi_j K$ is known as the projection body of order j of the convex body K; see [5, p. 161]. These examples can be generalized considerably.

The proofs of Theorems 1 and 2 will make use of the following generalization of Theorems 3 and 4, concerning homogeneous valuations of arbitrary degrees.

Theorem 5. *Let $n \geq 3$. Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, translation invariant and rotation equivariant valuation which is homogeneous of degree j and maps some convex body of dimension j to a polytope.*

(a) *If $j = n$, then $\Phi = \{0\}$.*

(b) *If $j = n - 1$, then $\Phi = c_1 \Pi$ with a constant $c_1 \geq 0$.*

(c) *If $j \in \{2, \dots, n-2\}$, then $\Phi = \{0\}$.*

(d) *If $j = 1$, then $\Phi = c_2 \mathcal{I} + c_3(-\mathcal{I})$ with constants $c_2, c_3 \geq 0$.*

(e) *If $j = 0$, then $\Phi = \{0\}$.*

Proof. (a) Let $j = n$. For $u \in \mathbb{R}^n$, the function φ defined by $\varphi(K) = h(\Phi K, u)$ satisfies the assumptions of Lemma 1. It follows that $h(\Phi K, u) = f(u) V_n(K)$ for $K \in \mathcal{K}^n$. This defines a function f on $\mathbb{R}^n$. For $\vartheta \in SO(n)$ we have

$$f(\vartheta u) V_n(K) = f(\vartheta u) V_n(\vartheta K) = h(\Phi \vartheta K, \vartheta u) = h(\vartheta \Phi K, \vartheta u) = h(\Phi K, u) = f(u) V_n(K),$$

hence $f(u) = a\|u\|$ and thus $\Phi K = a V_n(K) B^n$ with a constant a, where B^n denotes the unit ball. Inserting for K an n-dimensional convex body for which ΦK is a polytope, we get that aB^n is a polytope. This is only possible if $a = 0$ and hence $f = 0$. This proves part (a).

(b) Let $j = n-1$. Since Φ is homogeneous of degree $n-1$, it is a Blaschke Minkowski homomorphism. Since Φ maps some convex body M of dimension $n-1$ to a polytope, Theorem 4 yields that $\Phi K = c\Pi K$ holds for every centrally symmetric convex body K, where $c \geq 0$ is a constant.

Let $K \in \mathcal{K}^n$, and let $K \# (-K)$ be the Blaschke sum of K and $-K$. Using (2), it is easy to see that Φ commutes with the reflection in the origin. Thus, it follows from the fact that $K \# (-K)$ is centrally symmetric that

$$\Phi K + (-\Phi K) = \Phi K + \Phi(-K) = \Phi(K \# (-K)) = c\Pi(K \# (-K)).$$

Suppose, first, that $c = 0$. Then ΦK is one-pointed, say $\Phi K = \{t(K)\}$. The map $t + s : \mathcal{K}^n \to \mathbb{R}^n$, where s is the Steiner point map, is a continuous valuation which is equivariant under translations and rotations. From the characterization of the Steiner point mentioned after (1), we obtain that $t = 0$, hence $\Phi K = \{0\} = c\Pi K$ for $K \in \mathcal{K}^n$.

Let $c > 0$. Let B be an n-dimensional polytope with the property that any three of the outer unit normal vectors of its facets are linearly independent. Writing $\Phi B =: Q$, we get $Q + (-Q) = c\Pi(B \# (-B)) =: Z$ and, clearly, $c\Pi B = \frac{1}{2} Z$. Then Q is a summand of the polytope Z and is, therefore, itself a polytope. Let F be a two-dimensional face of Q, and let $u \in S^{n-1}$ be such that $F = F(Q, u)$, the face of Q with outer normal vector u. Then $F(Q, u)$ is a summand of $F(Z, u)$. Since the normal cone of Q at F has dimension $n - 2$ and hence cannot be covered by normal cones of Z at faces of dimensions larger than 2, we can choose u in such a way that $F(Z, u)$ is a two-face of Z. Due to the way how the directions of the edges of Z are determined by the facet normals of B, the assumption that any three normal vectors of the facets of B are linearly independent implies that $F(Z, u)$ is a parallelogram. Every summand of a parallelogram is either a parallelogram or a segment or a singleton. In any case, we deduce that $F(Q,u)$ is centrally symmetric. Since F was an arbitrary two-face of Q, the polytope Q is a zonotope ([26, Theorem 3.5.1]). In particular, Q is centrally symmetric. (This holds also if Q has no two-faces.) Therefore, $Q = \frac{1}{2}Z + t(B)$, and thus $\Phi B = c\Pi B + t(B)$, with some translation vector $t(B)$. Using (1), (2) and the fact that the center of mass of $S_{n-1}(B,\cdot)$ is the origin, we obtain $s(\Phi B) = 0 = s(c\Pi B)$ and thus $t(B) = 0$. Since any convex body can be approximated by polytopes B satisfying the assumption on the facet normals, and since Φ and Π are continuous, we deduce that $\Phi K = c\Pi K$ for all convex bodies $K \in \mathcal{K}^n$. This completes the proof of part (b).

(c) Let $j \in \{2,\dots,n-2\}$. We choose linear subspaces $E \subset U \subset \mathbb{R}^n$ with $\dim E = j$ and $\dim U = j+1$. Let $\pi_U : \mathbb{R}^n \to U$ denote the orthogonal projection, and define a map $\Psi : \mathcal{K}(U) \to \mathcal{K}(U)$ by

$$\Psi K := \pi_U \Phi K \qquad \text{for } K \in \mathcal{K}(U).$$

Then Ψ is a continuous valuation on $\mathcal{K}(U)$, it is invariant under the translations of U into itself and equivariant under the rotations in $SO(U)$. By assumption, Φ maps some j-dimensional convex body to a polytope. By the translation invariance and rotation equivariance of Φ, there also exists such a body that is contained in U. Now we can apply Part (b) of Theorem 5, with $\mathbb{R}^n$ replaced by U. It follows that $\Psi = c\Pi^U$, where Π^U denotes the projection body operator in U, and $c \geq 0$ is a constant (depending on U). Let $K \subset E$ be any j-dimensional convex body. Then $\Pi^U K =: S$ is a (non-degenerate) segment in U, centered at 0 and orthogonal to E, thus

$$\pi_U \Phi K = cS.$$

In particular, the orthogonal projection of ΦK to E is equal to $\{0\}$. Therefore, ΦK is contained in the orthogonal complement $E^\perp$ (with respect to $\mathbb{R}^n$) of E. Every rotation of $\mathbb{R}^n$ that leaves E pointwise fixed maps ΦK to itself, hence ΦK is a ball with center 0 and dimension $n - j \geq 2$ or dimension zero. But for a suitable j-dimensional body $M \subset E$, the set ΦM is also a polytope. It follows that $\Phi M = \{0\}$. This implies that $c = 0$. In particular, we have $\Phi K = \{0\}$ for every $K \in \mathcal{K}(E)$. Here E can be any j-dimensional subspace, hence $\Phi K = \{0\}$ holds for all $K \in \mathcal{K}^n$ with $\dim K \leq j$. An application of Lemma 5 now completes the proof of part (c).

(d) Let $j = 1$. The continuous, translation invariant valuation Φ is homogeneous of degree one and hence Minkowski additive (see [8] or [18]). The map defined by $K \mapsto \Phi K + s(K)$ is an endomorphism of $\mathcal{K}^n$ in the sense of [24]. A one-dimensional convex body, that is, a segment, is a polytope and hence is mapped by Φ to a polytope. Since the image has rotational symmetry and $n \geq 3$, it can only be a segment. From Theorem 3 we can now conclude that

$$\Phi K = c_2[K - s(K)] + c_3[-K + s(K)] \qquad \text{for } K \in \mathcal{K}^n,$$

with constants $c_2, c_3 \geq 0$.

(e) Let $j = 0$. A continuous translation invariant valuation which is homogeneous of degree zero is constant, hence for any $u \in \mathbb{R}^n$ we get $h(\Phi K, u) = f(u)$. As in the proof of case (a), we obtain $\Phi K^n = aB^n$ with a constant $a \geq 0$. Choosing for K a one-pointed set for which ΦK is a polytope, we get $a = 0$ and hence $\Phi = \{0\}$. This completes the proof of Theorem 5. $\square$

Note that the proof of Theorem 5 (b) also holds under the modified assumption that some n-dimensional convex body is mapped to a polytope, which leads to the following generalization of Theorem 4:

Corollary 1. *Let $\Psi : \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke Minkowski homomorphism. If the image under Ψ of some convex body M of dimension at least $n-1$ is a polytope, then $\Psi K = c\Pi K$, where $c \geq 0$ is a real constant.*

3 Proof of Theorem 1

We assume that $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ satisfies the assumptions of Theorem 1. Let $u \in \mathbb{R}^n$. By Lemma 2, the real valued valuation $K \mapsto h(\Phi K, u)$ has a decomposition

$$h(\Phi K, u) = \sum_{i=0}^{n} f_i(K, u), \qquad K \in \mathcal{K}^n, \tag{3}$$

where $f_i(\cdot, u)$ is a continuous translation invariant valuation that is homogeneous of degree i. In (3), we replace K by mK for $m = 1, 2, \ldots, n+1$. The resulting system of linear equations,

$$h(\Phi mK, u) = \sum_{i=0}^{n} m^i f_i(K, u), \qquad m = 1, \ldots, n+1,$$

can be solved to give representations

$$f_j(K, u) = \sum_{m=1}^{n+1} a_{jm} h(\Phi mK, u), \qquad j = 0, \ldots, n,$$

with coefficients a_{jm} depending only on j and m. From this representation we read off the following:

(a) For each rotation $\vartheta \in SO(n)$ we have $f_j(\vartheta K, u) = f_j(K, \vartheta^{-1} u)$.

(b) The function $f_j(K, \cdot)$ is positively homogeneous.

(c) If K is a polytope, then the function $f_j(K, \cdot)$ is piecewise linear.

We do not know, at this point, whether each function $f_i(K, \cdot)$ is a support function; only the following can be shown.

Lemma 6. *Suppose that the convex body $K \in \mathcal{K}^n$ satisfies*

$$h(\Phi \lambda K, \cdot) = \sum_{i=k}^{l} f_i(\lambda K, \cdot) \tag{4}$$

for $\lambda > 0$, with some $k, l \in \{0, \dots, n\}$, $k \le l$. Then there exist convex bodies $\Phi_k K, \Phi_l K$ such that $h(\Phi_k K, \cdot) = f_k(K, \cdot)$ and $h(\Phi_l K, \cdot) = f_l(K, \cdot)$. If $\Phi\lambda K$ is a polytope for $\lambda > 0$, then $\Phi_k K$ and $\Phi_l K$ are polytopes.

If $E \subseteq \mathbb{R}^n$ is a linear subspace and (4) holds for all $K \in \mathcal{K}(E)$, then the maps $\Phi_k, \Phi_l : \mathcal{K}(E) \to \mathcal{K}^n$ defined in this way are continuous valuations, invariant under translations and equivariant under rotations of E into itself, and homogeneous of degrees k and l, respectively.

Proof. Let $u_1, u_2 \in \mathbb{R}^n$ and $\lambda > 0$. Since $h(\Phi\lambda K, \cdot)$ is sublinear, (4) yields

$$\begin{aligned} 0 &\ge h(\Phi\lambda K, u_1 + u_2) - h(\Phi\lambda K, u_1) - h(\Phi\lambda K, u_2) \\ &= \sum_{i=k}^{l} \lambda^i [f_i(K, u_1 + u_2) - f_i(K, u_1) - f_i(K, u_2)]. \end{aligned}$$

Dividing by λ^k and letting λ tend to zero, we see that the function $f_k(K, \cdot)$ is sublinear. Being positively homogeneous, it is a support function, hence there exists a convex body $\Phi_k K$ with $f_k(K, \cdot) = h(\Phi_k K, \cdot)$. If all bodies $\Phi\lambda K$ are polytopes, then $\Phi_k K$ is a polytope, since $h(\Phi_k K, \cdot)$ is piecewise linear.

Similarly, dividing by λ^l and letting λ tend to infinity, we obtain that $f_l(K, \cdot)$ is sublinear and hence $f_l(K, \cdot) = h(\Phi_l K, \cdot)$ with a convex body $\Phi_l K$. The remaining assertions are clear. $\square$

First we apply Lemma 6 with $k = 0$ and $l = n$ (and $E = \mathbb{R}^n$). Theorem 5 (a) and (e) implies that $\Phi_0 = \{0\}$ and $\Phi_n = \{0\}$, hence

$$h(\Phi K, \cdot) = \sum_{i=1}^{n-1} f_i(K, \cdot) \qquad \text{for } K \in \mathcal{K}^n. \tag{5}$$

Now Lemma 6 with $l = n - 1$ yields the existence of a map $\Phi_{n-1} : \mathcal{K}^n \to \mathcal{K}^n$ which is a continuous, translation invariant and rotation equivariant valuation, homogeneous of degree $n - 1$, and satisfying $h(\Phi_{n-1} K, \cdot) = f_{n-1}(K, \cdot)$. If K is a polytope, then $\Phi_{n-1} K$ is a polytope. Theorem 5 (b) shows that $\Phi_{n-1} = c_1 \Pi$ with a constant $c_1 \ge 0$.

Similarly, we conclude from (5) that $f_1(K, \cdot) = h(\Phi_1 K, \cdot)$ with a continuous, translation invariant and rotation equivariant valuation $\Phi_1 : \mathcal{K}^n \to \mathcal{K}^n$ which is homogeneous of degree one, and that $\Phi_1 K$ is a polytope if K is a segment. From Theorem 5 (d) we obtain that $\Phi_1 = c_2 \mathcal{I} + c_3(-\mathcal{I})$ with constants $c_2, c_3 \ge 0$. Therefore, (5) can be replaced by

$$h(\Phi K, \cdot) = c_1 h(\Pi K, \cdot) + \sum_{i=2}^{n-2} f_i(K, \cdot) + c_2 h(\mathcal{I} K, \cdot) + c_3 h(-\mathcal{I} K, \cdot). \tag{6}$$

This finishes the proof if $n = 3$. We assume, therefore, that $n \ge 4$. We have to show that the remaining functions $f_i(K, \cdot)$ are zero.

Let $j \in \{2, \dots, n-2\}$. We choose a j-dimensional linear subspace $E \subset \mathbb{R}^n$. Let $K \in \mathcal{K}(E)$. Since a continuous, translation invariant valuation that is homogeneous of degree i vanishes on convex bodies of dimension smaller than i, we have

$$h(\Phi\lambda K, \cdot) = \sum_{i=1}^{j} f_i(\lambda K, \cdot)$$

for $\lambda > 0$. By Lemma 6, there is a convex body $\Phi_j K$ with $h(\Phi_j K, \cdot) = f_j(K, \cdot)$, and if K is a polytope, then $\Phi_j K$ is a polytope. For $u \in \mathbb{R}^n$, Lemma 1 gives $h(\Phi_j K, u) = f(u)V_j(K)$, with a function f on $\mathbb{R}^n$. Taking for K a j-dimensional polytope in E, we see that f is the support function of a polytope, say P. By the rotation equivariance, P is invariant under the rotations mapping E into itself and keeping $E^\perp$ pointwise fixed. Therefore, the projection $\pi_E P$ is a ball in E, centered at 0. It can only have radius zero, hence P is contained in $E^\perp$. Every rotation of $\mathbb{R}^n$ that leaves E pointwise fixed maps P to itself, hence P is a centered ball of dimension $n - j \geq 2$ or of dimension zero. We deduce that $P = \{0\}$.

We have shown that

$$f_j(K, \cdot) = 0 \text{ whenever } \dim K = j,\ j = 2, \dots, n-2. \tag{7}$$

From Lemma 4 (a) we conclude that

$$f_j(K, \cdot) + f_j(-K, \cdot) = 0 \qquad \text{for all } K \in \mathcal{K}^n,\ j = 2, \dots, n-2. \tag{8}$$

Now let $E \subset \mathbb{R}^n$ be an $(n-1)$-dimensional linear subspace. Define $\Psi : \mathcal{K}(E) \to \mathcal{K}(E)$ by $\Psi K = \pi_E \Phi K$ for $K \in \mathcal{K}(E)$. Then Ψ is a continuous valuation, invariant under the translations of E into itself and equivariant under the rotations of $SO(E)$. It maps polytopes to polytopes. Let $K \in \mathcal{K}^n$. The support function, on E, of ΨK is the restriction of $h(\Phi K, \cdot)$ to E. Hence, it follows from (5) that

$$h(\Psi K, u) = \sum_{i=1}^{n-1} f_i(K, u) \qquad \text{for } u \in E.$$

On the other hand, from the result (6), applied in E, we have

$$h(\Psi K, u) = c_E h(\Pi^E K, u) + \sum_{i=1}^{n-3} g_i(K, u) \qquad \text{for } u \in E,$$

where Π^E is the projection body operator in E and $g_i(\cdot, u)$ is homogeneous of degree i. By homogeneity, we must have

$$c_E h(\Pi^E K, u) = f_{n-2}(K, u) \qquad \text{for } u \in E.$$

Let $K \subset E$ be an $(n-1)$-dimensional centrally symmetric body. Since $\Pi^E K \neq \{0\}$, we deduce from (8) that $c_E = 0$. This yields

$$f_{n-2}(K, u) = 0 \qquad \text{for all } K \in \mathcal{K}(E),\ u \in E. \tag{9}$$

Let e be one of the unit normal vectors of E, and let $K \in \mathcal{K}(E)$. Then (6) can be written in the form

$$h(\Phi K, \cdot) = c_1 V_{n-1}(K)|\langle e, \cdot\rangle| + \sum_{i=1}^{n-2} f_i(K, \cdot). \tag{10}$$

Let $H_e^+ := \{u \in \mathbb{R}^n : \langle e, u\rangle \geq 0\}$ and $H_e^- = -H_e^+$. For $u_1, u_2 \in H_e^+$ and $\lambda > 0$, we have

$$\begin{aligned} 0 &\geq h(\Phi\lambda K, u_1 + u_2) - h(\Phi\lambda K, u_1) - h(\Phi\lambda K, u_2) \\ &= \sum_{i=1}^{n-2} \lambda^i [f_i(K, u_1 + u_2) - f_i(K, u_1) - f_i(K, u_2)], \end{aligned}$$

from which we obtain

$$f_{n-2}(K, u_1 + u_2) \leq f_{n-2}(K, u_1) + f_{n-2}(K, u_2).$$

We replace K by $-K$ and use (8). Together with the preceding inequality, this yields

$$f_{n-2}(K, u_1 + u_2) = f_{n-2}(K, u_1) + f_{n-2}(K, u_2).$$

Since this holds for all $u_1, u_2 \in H_e^+$ and since $f_{n-2}(K, \cdot)$ is positively homogeneous, we conclude that $f_{n-2}(K, \cdot)$ is linear on H_e^+, thus there is a vector x_K such that

$$f_{n-2}(K, u) = \langle x_K, u\rangle \qquad \text{for } u \in H_e^+.$$

Similarly, there is a vector y_K such that

$$f_{n-2}(K, u) = \langle y_K, u\rangle \qquad \text{for } u \in H_e^-.$$

By (9), for any vector $v \perp e$ we have $\langle x_K, v\rangle = f_{n-2}(K, v) = 0$, and analogously $\langle y_K, v\rangle = 0$, hence x_K and y_K are parallel to e. Thus,

$$f_{n-2}(K, u) = \langle e, u\rangle \varphi_{n-2}^{\pm}(K) \qquad \text{for } u \in H_e^{\pm},$$

where $\varphi_{n-2}^+, \varphi_{n-2}^- : \mathcal{K}(E) \to \mathbb{R}$ are translation invariant, continuous valuations, homogeneous of degree $n-2$, and by (7) they are simple. Moreover, for every rotation $\vartheta \in SO(E)$ fixing e we have

$$\varphi_{n-2}^+(\vartheta K) = f(\vartheta K, e) = f(K, e) = \varphi_{n-2}^+(K).$$

Thus, φ_{n-2}^+ is also invariant with respect to rotations of E into itself. By Lemma 3, applied in E, this is only possible if $\varphi_{n-2}^+ = 0$. Similarly, we obtain $\varphi_{n-2}^- = 0$ and thus $f_{n-2}(K, \cdot) = 0$ for all $K \in \mathcal{K}(E)$. Since E was an arbitrary $(n-1)$-dimensional subspace, we have $f_{n-2}(K, \cdot) = 0$ for arbitrary convex bodies K with $\dim K \leq n-1$. Now Lemma 4 (b) yields $f_{n-2}(K, \cdot) = 0$ for all convex bodies $K \in \mathcal{K}^n$.

Let $j \in \{2, \ldots, n-3\}$ and suppose it has already been proved that

$$f_i(K, \cdot) = 0 \qquad \text{for all } K \in \mathcal{K}^n,\ i = j+1, \ldots, n-2.$$

We choose a $(j+1)$-dimensional linear subspace $E \subset \mathbb{R}^n$. For all $K \in \mathcal{K}(E)$ we have $\Pi K = \{0\}$ and hence, by (6),

$$h(\Phi K, \cdot) = \sum_{i=1}^{j} f_i(K, \cdot).$$

By Lemma 6, the valuation $\Phi_j : \mathcal{K}(E) \to \mathcal{K}^n$ with $h(\Phi_j K, \cdot) = f_j(K, \cdot)$ is defined. By (7), it satisfies $\Phi_j K = 0$ if $\dim K = j$. The proof of Lemma 5 shows that $\Phi_j = \{0\}$. Thus, $f_j(K, \cdot) = 0$ whenever $\dim K \leq j+1$. Now Lemma 4 (b) yields

$$f_j(K, \cdot) = 0 \qquad \text{for all } K \in \mathcal{K}^n.$$

In this way we continue, until we obtain $f_2(K, \cdot) = 0$ for all $K \in \mathcal{K}^n$. This completes the proof of Theorem 1. □

4 Proof of Theorem 2

We assume that the assumptions of Theorem 2 are satisfied. As in the proof of Theorem 1, relation (3) holds (but $f_j(K, \cdot)$ need not have property (c) listed there), thus

$$h(\Phi \lambda K, \cdot) = \sum_{j=0}^{n} \lambda^j f_j(K, \cdot) \qquad \text{for } K \in \mathcal{K}^n,$$

for all $\lambda > 0$. If $\dim K \leq n-2$, then $\Phi \lambda K = \{0\}$ by assumption and continuity, and we deduce that $f_j(K, \cdot) = 0$ for $j = 0, \dots, n$.

We assert that $f_j(K, \cdot) = 0$ for $K \in \mathcal{K}^n$ and for $j = 0, \dots, n-2$. For $j = 0$, this follows from $f_0(K, u) = a\|u\|$ (obtained as before), by inserting a convex body K for which $f_0(K, \cdot) = 0$. Suppose that $j \in \{1, \dots, n-2\}$ and that $f_i(K, \cdot) = 0$ for all $K \in \mathcal{K}^n$ has been proved for $i < j$. Then

$$h(\Phi K, \cdot) = \sum_{i=j}^{n} f_i(K, \cdot) \qquad \text{for } K \in \mathcal{K}^n.$$

By Lemma 6, the map $\Phi_j : \mathcal{K}^n \to \mathcal{K}^n$ is defined. It maps bodies of dimension smaller than $n-1$ to $\{0\}$; in particular, $\Phi_j K = 0$ if $\dim K = j$. Lemma 5 shows that $\Phi_j = \{0\}$. Thus, (3) reduces to

$$h(\Phi K, \cdot) = f_n(K, \cdot) + f_{n-1}(K, \cdot) = h(\Phi_n K, \cdot) + h(\Phi_{n-1} K, \cdot)$$

for $K \in \mathcal{K}^n$, where we have already inserted the homogeneous valuations Φ_n, Φ_{n-1} that exist by Lemma 6.

By Lemma 1, $h(\Phi_n K, u) = g(u) V_n(K)$ with some function g. By the rotation equivariance of Φ, we obtain $\Phi_n K = a V_n(K) B^n$, with a constant $a \geq 0$. By assumption, there exists an n-dimensional convex body M such that ΦM is a polytope P. This gives $P = a V_n(M) B^n + \Phi_{n-1} M$, where B^n denotes the unit ball. Since the polytope P cannot have a ball with positive radius as a summand, this yields $a = 0$. Therefore, $\Phi = \Phi_{n-1}$. Now Corollary 1 completes the proof of Theorem 2. □

5 Open Problems

A positive answer to the following problem would simplify the proof of Theorem 1 considerably:

Problem 1. *Let $n \geq 3$. Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, translation invariant and rotation equivariant valuation. Is there a (unique) representation of Φ of the form*

$$\Phi = \Phi_0 + \ldots + \Phi_n,$$

where $\Phi_j : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, translation invariant and rotation equivariant valuation which is homogeneous of degree j?

As remarked in the introduction, the assumptions of Theorem 1 are stronger than necessary. Theorems 2 and 5 lead to the following question.

Problem 2. *Let $n \geq 3$. Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, translation invariant and rotation equivariant valuation. Assume that Φ maps some n-dimensional convex body to a polytope. Are there constants $c_1, c_2, c_3 \geq 0$ such that*

$$\Phi = c_1\Pi + c_2\mathcal{I} + c_3(-\mathcal{I})?$$

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Rolf Schneider: Mathematisches Institut,Universität Freiburg, Eckerstraße 1,
79104 Freiburg i. Br., Germany
E-mail address: rolf.schneider@math.uni-freiburg.de

Franz E. Schuster: Institut für Diskrete Mathematik und Geometrie,
Technische Universität Wien, Wiedner Hauptstraße 8-10/104, 1040 Wien, Austria
E-mail address: fschuster@osiris.tuwien.ac.at