Geophys. J. R. astr. Soc. (1979) **58**, 609–623

Rotational and elliptical splitting of the free oscillations of the Earth

F. A. Dahlen | *Department of Geological and Geophysical Sciences, Princeton University, Princeton, New Jersey 08540, USA*

R. V. Sailor* *Department of Geological Sciences, Harvard University, Cambridge, Massachusetts 02138, USA*

Received 1978 December 18; in original form 1978 March 27

Summary. We present a table of rotational and elliptical splitting parameters for earth model 1066A, including all terms through second order in rotation and first order in ellipticity. An algorithm for calculating the second-order Coriolis splitting by summing over all modes which are coupled to first order is given in detail. Coupling to secular (or zero frequency) modes, as well as the usual seismic modes, can provide significant contributions to these splitting parameters.

Introduction

The free oscillations of the Earth are split by rotation, ellipticity and lateral heterogeneity. Now that an error in the original formulation of normal mode perturbation theory has been corrected by Dziewonski & Sailor (1976), Woodhouse (1976), Dahlen (1976) and Woodhouse & Dahlen (1978), it is known that both the rotational and the elliptical splitting depend only weakly on the unperturbed spherical structure of the Earth. With this in mind, we present in this paper a tabulation of the rotational and elliptical splitting parameters for earth model 1066A of Gilbert & Dziewonski (1975). All terms through second order in the rotation and first order in the ellipticity are included. Every multiplet has been treated as if it were well isolated in the eigenfrequency spectrum. In at least one instance (${}_0S_{11}$ and ${}_0T_{12}$), the need for a quasi-degenerate calculation is clearly indicated, but no such calculations are presented here.

For any isolated multiplet, if rotation and ellipticity are the dominant perturbations, the observed splitting should be fairly well predicted by the parameters listed here, since both the radial structure of the Earth and the two perturbations have presumably been well represented. Any substantial deviation of the observed from the predicted splitting can therefore be ascribed to the Earth's lateral heterogeneity. It has been generally assumed that for the lowest frequency multiplets, rotation and ellipticity must be the dominant perturbations. This supposition has however never really been tested, in part because

* Present address: The Analytic Sciences Corporation, Reading, Massachusetts 01867, USA.

observations of splitting have so far been rather crude, and in part because of the error in previous determinations of the elliptical splitting. New instrumental networks (Agnew *et al.* 1976) and new methods of analysis (Buland & Gilbert 1978; Stein & Geller 1978) now offer the hope of improving dramatically the resolution and accuracy of splitting measurements. It is a matter of some interest to determine which of the Earth's low-frequency multiplets really are predominantly perturbed by rotation and ellipticity. If the splitting of one or more multiplets sensitive to lateral heterogeneity as well can be resolved, the results can be used to help constrain the heterogeneity.

Notation used in the table

Let r, θ, ϕ be a system of polar coordinates, with $\theta = 0$ aligned along the rotation axis of a hydrostatic ellipsoidal earth model. Let Ω be the rate of rotation, and let ϵ_h denote the hydrostatic ellipticity of the outer surface. We consider an isolated multiplet ${}_nS_l$ or ${}_nT_l$, with a degenerate eigenfrequency ω_0. Rotation and ellipticity remove the degeneracy completely. Furthermore, correct to zeroth order in both Ω/ω_0 and ϵ_h, each eigenfunction depends upon a single spherical harmonic Y_l^m. The $2l + 1$ zeroth-order eigenfunctions associated with a spheroidal multiplet ${}_nS_l$ are of the form

$$\mathbf{s}_m = \hat{\mathbf{r}} U Y_l^m + V \boldsymbol{\nabla}_1 Y_l^m, \tag{1}$$

while those associated with a toroidal multiplet ${}_nT_l$ are of the form

$$\mathbf{s}_m = W[-\hat{\mathbf{r}} \times \boldsymbol{\nabla}_1 Y_l^m]. \tag{2}$$

The scalars U, V and W depend only upon r, and $\boldsymbol{\nabla}_1 = \hat{\boldsymbol{\theta}} \partial_\theta + \hat{\boldsymbol{\phi}} (\sin\theta)^{-1} \partial_\phi$.

Correct to second order in Ω/ω_0 and first order in ϵ_h, the $2l + 1$ associated eigenfrequencies ω_m depend quadratically upon the index m, i.e.

$$\omega_m = \omega_0 [1 + a + mb + m^2 c]. \tag{3}$$

The term b arises from the first-order effect of the Coriolis force, and it is customarily written as

$$b = \beta(\Omega/\omega_0), \tag{4}$$

where β is the Coriolis splitting parameter first defined by Backus & Gilbert (1961). The terms a and c arise from the ellipticity and from the second-order effects of rotation. The first of these may be written in the form:

$$a = \alpha\epsilon_h + \tfrac{1}{3}[1 - l(l+1)\beta](\Omega/\omega_0)^2 + [\alpha' + \alpha_2](\Omega/\omega_0)^2 \tag{5}$$

and the second in the form

$$c = [-3/l(l+1)][\alpha\epsilon_h + \alpha'(\Omega/\omega_0)^2] + \gamma_2(\Omega/\omega_0)^2. \tag{6}$$

In equations (5) and (6), α is the ellipticity splitting parameter. As a check, we have computed α in two different ways, using the algorithms given by Dahlen (1976) and Woodhouse & Dahlen (1978). In general, the agreement between these two separate calculations was excellent. The terms $\tfrac{1}{3}[1 - l(l+1)\beta]$ and α' arise from the first-order effect of the rotational potential $\psi = -\tfrac{1}{3}\Omega^2 r^2 [1 - P_2(\cos\theta)]$; the spherical part $-\tfrac{1}{3}\Omega^2 r^2$ gives rise to $\tfrac{1}{3}[1 - l(l+1)\beta]$, and the aspherical part $\tfrac{1}{3}\Omega^2 r^2 P_2(\cos\theta)$ gives rise to α'. We have computed α' using the formula given by Woodhouse & Dahlen (1978). For a toroidal multiplet ${}_nT_l$, both $\tfrac{1}{3}[1 - l(l+1)\beta]$ and α' are identically zero. Finally, the terms α_2 and

γ_2 are due to the second order effect of the Coriolis force. The method we have used to calculate these quantities is described in the next section. Neither of the terms $\frac{1}{3}[1 - l(l+1)\beta]$ nor α_2 and γ_2 satisfy the diagonal sum rule (Gilbert 1971), the former because it is due to a spherical perturbation and the latter because they are of second order.

We have tabulated the quantities $\Omega/\omega_0, a, b, c, \alpha\epsilon_h, \alpha'(\Omega/\omega_0)^2, \alpha_2(\Omega/\omega_0)^2$ and $\gamma_2(\Omega/\omega_0)^2$ for all the usual seismic modes with $T = 2\pi/\omega_0$ greater than 500 s. Radau's approximation (Jeffreys 1970) has been used to find the hydrostatic ellipticity as a function of depth. Since model 1066A has the correct mass and moment of inertia, $\epsilon_h = 1/299.8$. All the entries in Table 1 are accurate to at least three figures.

Second-order Coriolis splitting

It is sufficient to consider the effect of the Coriolis force on the free oscillations of a spherical earth model. Terms of order $\epsilon_h\Omega/\omega_0$ will thereby be neglected. Let V be the volume occupied by the earth model, and let ρ_0 be its density. We shall say that two vector fields $\mathbf{u}$ and $\mathbf{v}$ in V are orthogonal ($\mathbf{u} \perp \mathbf{v}$) if

$$\int_V \rho_0 \mathbf{u} \cdot \mathbf{v}^* \, dV = 0, \tag{7}$$

where * denotes the complex conjugate.

Let H denote the Hermitian operator (with its associated natural boundary conditions) which characterizes the normal mode problem in the absence of rotation. We then seek to find eigenfrequencies ω and complex eigenfunctions $\mathbf{u}$ which satisfy the equation (Dahlen 1968)

$$H\mathbf{u} + 2i\omega\rho_0 \boldsymbol{\Omega} \times \mathbf{u} = \rho_0\omega^2\mathbf{u}. \tag{8}$$

In equation (8), $\boldsymbol{\Omega}$ is the angular velocity of rotation of the earth model. To take advantage of the symmetry, we have set $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$, so that $\theta = 0$ along the rotation axis. Following Backus & Gilbert (1961), let us look for solutions to equation (8) of the form

$$\mathbf{u} = \mathbf{u}_0 + (\Omega/\omega_0)\mathbf{u}_1 + (\Omega/\omega_0)^2\mathbf{u}_2 + \ldots \tag{9a}$$

$$\omega/\omega_0 = 1 + \sigma_1(\Omega/\omega_0) + \sigma_2(\Omega/\omega_0)^2 + \ldots \tag{9b}$$

where Ω/ω_0 is presumed to be a small parameter. Inserting equations (9) into equation (8) and equating powers of Ω/ω_0 leads to a set of perturbation equations, of which we write only the first three:

$$(H - \rho_0\omega_0^2)\mathbf{u}_0 = \mathbf{0} \tag{10a}$$

$$(H - \rho_0\omega_0^2)\mathbf{u}_1 = 2\rho_0\omega_0^2(\sigma_1\mathbf{u}_0 - i\hat{\mathbf{z}} \times \mathbf{u}_0) \equiv \boldsymbol{\xi}_1 \tag{10b}$$

$$\begin{aligned}(H - \rho_0\omega_0^2)\mathbf{u}_2 &= 2\rho_0\omega_0^2(\sigma_1\mathbf{u}_1 - i\hat{\mathbf{z}} \times \mathbf{u}_1)\\ &\quad + 2\rho_0\omega_0^2\sigma_1(\sigma_1\mathbf{u}_0 - i\hat{\mathbf{z}} \times \mathbf{u}_0)\\ &\quad + \rho_0\omega_0^2(2\sigma_2 - \sigma_1^2)\mathbf{u}_0 \equiv \boldsymbol{\xi}_2.\end{aligned} \tag{10c}$$

Equation (10a) simply states that ω_0 and $\mathbf{u}_0$ must be an eigenfrequency and associated eigenfunction of the unperturbed earth model. In the present problem, symmetry guarantees that when the degeneracy is removed (this will occur in the first order), the zeroth-order eigenfunctions $\mathbf{u}_0$ must be either of the form (1) or (2). Knowing this, we can proceed to

Table 1. Rotational and elliptical splitting parameters for model 1066A. Eigenfrequencies for each singlet are given by $\omega_m = \omega_0(1 + a + mb + m^2c)$. We have tabulated the perturbation parameter Ω/ω_0. The (sidereal) rotation rate Ω is $7.292115 \times 10^{-5}\ \mathrm{s}^{-1}$. The degenerate periods $T = 2\pi/\omega_0$ of all observed modes are tabulated in table 7 of Gilbert & Dziewonski (1975).

Mode	$\Omega/\omega_0(\times 10^{-3})$	$a(\times 10^{-3})$	$b(\times 10^{-3})$	$c(\times 10^{-3})$	$\alpha\epsilon_h(\times 10^{-3})$	$\alpha'(\Omega/\omega_0)^2$ $(\times 10^{-3})$	$\alpha_2(\Omega/\omega_0)^2$ $(\times 10^{-3})$	$\gamma_2(\Omega/\omega_0)^2$ $(\times 10^{-3})$	Comments
${}_0S_2$	37.514	0.376	14.905	−0.2671	0.102	0.238	0.686	−0.0973	
${}_0S_3$	24.773	0.463	4.621	−0.1179	0.152	0.074	0.490	−0.0614	
${}_0S_4$	17.941	0.544	1.834	−0.0751	0.284	0.031	0.341	−0.0278	
${}_0S_5$	13.816	0.452	0.841	−0.0472	0.366	0.015	0.124	−0.0092	
${}_0S_6$	11.185	0.391	0.407	−0.0331	0.423	0.006	−0.016	−0.0024	
${}_0S_7$	9.428	0.354	0.181	−0.0252	0.475	0.000	−0.119	0.0002	
${}_0S_8$	8.217	0.273	0.054	−0.0196	0.526	−0.004	−0.261	0.0022	
${}_0S_9$	7.360	0.015	−0.014	−0.0138	0.571	−0.006	−0.571	0.0050	
${}_0S_{10}$	6.728	−1.008	−0.040	−0.0033	0.601	−0.007	−1.627	0.0129	
${}_0S_{11}$	6.237	17.075	−0.047	−0.1284	0.618	−0.008	16.438	−0.1145	Close to ${}_0T_{12}$
${}_0S_{12}$	5.836	2.655	−0.045	−0.0241	0.627	−0.007	2.010	−0.0122	
${}_1S_1$	192.178	15.306	98.380	−0.5538	−6.317	7.503	14.413	1.2258	Unobserved
${}_1S_2$	17.064	1.177	4.173	−0.4283	0.642	0.074	0.507	−0.0704	
${}_1S_3$	12.342	0.922	2.633	−0.2151	0.757	0.051	0.194	−0.0132	
${}_1S_4$	9.889	0.795	1.948	−0.1219	0.747	0.040	0.103	−0.0038	
${}_1S_5$	8.465	0.696	1.437	−0.0750	0.702	0.031	0.061	−0.0017	
${}_1S_6$	7.625	0.618	0.873	−0.0490	0.631	0.021	0.041	−0.0025	
${}_1S_7$	7.013	0.561	0.564	−0.0329	0.572	0.016	0.030	−0.0014	
${}_1S_8$	6.455	0.500	0.427	−0.0228	0.518	0.015	0.020	−0.0006	
${}_1S_9$	5.915	0.446	0.349	−0.0165	0.472	0.014	0.010	−0.0003	
${}_2S_1$	28.743	2.094	15.074	0.1900	−0.779	0.195	2.692	−0.6865	Unobserved
${}_2S_2$	12.175	0.526	1.370	−0.2129	0.376	0.010	0.125	−0.0202	Unobserved
${}_2S_3$	9.352	0.662	0.668	−0.1534	0.633	0.005	0.020	0.0060	
${}_2S_4$	8.424	0.659	0.281	−0.0926	0.629	0.004	0.018	0.0023	
${}_2S_5$	7.668	0.681	0.159	−0.0635	0.640	0.006	0.027	0.0011	
${}_2S_6$	6.908	0.690	0.340	−0.0465	0.670	0.013	0.024	0.0022	
${}_2S_7$	6.224	0.673	0.410	−0.0357	0.679	0.016	0.013	0.0015	

${}_3S_1$	12.316	0.413	1.657	−0.3531	0.314	−0.034	0.097	0.0663	
${}_3S_2$	10.484	0.652	1.485	−0.2917	0.612	0.002	0.032	0.0154	
${}_3S_3$	8.057	0.455	0.460	−0.0978	0.349	0.003	0.097	−0.0100	Unobserved
${}_3S_4$	6.232	0.551	0.173	−0.0652	0.355	0.001	0.189	−0.0117	Unobserved
${}_4S_1$	8.214	0.469	3.082	−0.4616	0.256	0.041	0.167	−0.0166	Unobserved
${}_4S_2$	6.742	0.745	0.813	−0.3373	0.644	0.015	0.082	−0.0078	
${}_5S_1$	6.696	−0.988	2.712	1.5307	−1.028	0.000	0.037	−0.0102	Unobserved
${}_6S_1$	5.867	0.595	0.971	−0.8265	0.571	−0.009	0.025	0.0165	
${}_0S_0$	14.281	0.336	–	–	–	–	0.268	–	
${}_1S_0$	7.123	0.094	–	–	–	–	0.077	–	
${}_0T_2$	30.542	−1.335	5.090	−0.2314	2.013	–	−3.349	0.7752	Unobserved
${}_0T_3$	19.765	0.558	1.647	−0.2792	1.410	–	−0.852	0.0734	
${}_0T_4$	15.135	0.849	0.757	−0.1625	1.182	–	−0.333	0.0148	
${}_0T_5$	12.486	0.917	0.416	−0.1023	1.061	–	−0.144	0.0038	
${}_0T_6$	10.744	0.923	0.256	−0.0694	0.984	–	−0.061	0.0009	
${}_0T_7$	9.497	0.926	0.170	−0.0501	0.929	–	−0.002	−0.0004	
${}_0T_8$	8.549	0.961	0.119	−0.0384	0.887	–	0.074	−0.0014	
${}_0T_9$	7.799	1.069	0.087	−0.0313	0.854	–	0.214	−0.0028	
${}_0T_{10}$	7.186	1.358	0.065	−0.0280	0.828	–	0.530	−0.0054	
${}_0T_{11}$	6.675	2.398	0.051	−0.0315	0.805	–	1.593	−0.0132	
${}_0T_{12}$	6.241	−15.682	0.040	0.0992	0.787	–	−16.469	0.1143	Close to ${}_0S_{11}$
${}_0T_{13}$	5.865	−1.268	0.032	−0.0007	0.770	–	−2.038	0.0120	
${}_1T_1$	9.374	−0.604	4.687	0.8974	−0.562	–	−0.042	0.0546	
${}_1T_2$	8.779	0.250	1.463	−0.1563	0.391	–	−0.142	0.0394	
${}_1T_3$	8.054	0.517	0.671	−0.1283	0.521	–	−0.004	0.0020	
${}_1T_4$	7.311	0.576	0.366	−0.0839	0.556	–	0.019	−0.0005	
${}_1T_5$	6.622	0.599	0.221	−0.0579	0.572	–	0.027	−0.0007	Unobserved
${}_1T_6$	6.020	0.616	0.143	−0.0424	0.585	–	0.031	−0.0006	

employ an essentially non-degenerate version of perturbation theory. Suppose that $\mathbf{s}_0$ is some particular eigenfunction of the form (1) or (2), with an associated eigenfrequency ω_0. To determine the effect of the Coriolis force on ω_0 and $\mathbf{s}_0$, we simply set $\mathbf{u}_0 \equiv \mathbf{s}_0$ and use the remaining equations (10b), (10c) to find the perturbations σ_1, $\mathbf{u}_1$, σ_2, $\mathbf{u}_2$, etc. Since the normalization of $\mathbf{u}$ is arbitrary, we may without loss of generality require that $\mathbf{u}_1 \perp \mathbf{s}_0$, $\mathbf{u}_2 \perp \mathbf{s}_0$, etc.

We know from a theorem of Fredholm (Kolmogorov & Fomine 1974) that, since $H - \rho_0\omega_0^2$ is Hermitian, equation (10b) has a unique solution $\mathbf{u}_1 \perp \mathbf{s}_0$ if and only if $\boldsymbol{\xi}_1 \perp \mathbf{s}_0$, and equation (10c) has a unique solution $\mathbf{u}_2 \perp \mathbf{s}_0$ if and only if $\boldsymbol{\xi}_2 \perp \mathbf{s}_0$. The first of these conditions, rewritten as

$$\sigma_1 = \frac{\int_V \rho_0 \mathbf{s}_0 \cdot (i\hat{\mathbf{z}} \times \mathbf{s}_0)^* \, dV}{\int_V \rho_0 \mathbf{s}_0 \cdot \mathbf{s}_0^* \, dV}, \tag{11}$$

yields the first-order splitting parameter σ_1. Evaluation of equation (11) leads to the result $\sigma_1 = m\beta$ (Backus & Gilbert 1961), so that the degeneracy is completely removed in this order. The second condition leads to the result

$$2\sigma_2 - \sigma_1^2 = \frac{2\int_V \rho_0 \mathbf{u}_1 \cdot (i\hat{\mathbf{z}} \times \mathbf{s}_0)^* \, dV}{\int_V \rho_0 \mathbf{s}_0 \cdot \mathbf{s}_0^* \, dV}. \tag{12}$$

As usual, before we can find σ_2, we must first find $\mathbf{u}_1$. In the study of Dahlen (1968), $\mathbf{u}_1$ was calculated directly by numerical integration of the scalar version of equation (10b). The availability (Buland 1976) of a very complete table of eigenfrequencies and eigenfunctions for earth model 1066A has however prompted us to use a different method here.

We shall take it for granted that the set of all (unperturbed) eigenfunctions of the form (1), (2) is complete. Let us denote these eigenfunctions by $\mathbf{s}_0$, $\mathbf{s}_1$, $\mathbf{s}_2 \ldots$, and let us use $\omega_0, \omega_1, \omega_2 \ldots$ to denote their associated eigenfrequencies, so that

$$H\mathbf{s}_k = \rho_0\omega_k^2\mathbf{s}_k. \tag{13}$$

Since $\mathbf{u}_1 \perp \mathbf{s}_0$, it can be written in the form

$$\mathbf{u}_1 = \sum_{k=1}^{\infty} a_k \mathbf{s}_k. \tag{14}$$

To determine the coefficients a_k, we substitute equation (14) into equation (10b), and make use of equation (13) and the orthogonality relation $\mathbf{s}_j \perp \mathbf{s}_k$ if $j \neq k$. The result is

$$a_k = \frac{2\omega_0^2}{\omega_0^2 - \omega_k^2} \; \frac{\int_V \rho_0 \mathbf{s}_k^* \cdot (i\hat{\mathbf{z}} \times \mathbf{s}_0) \, dV}{\int_V \rho_0 \mathbf{s}_k \cdot \mathbf{s}_k^* \, dV}. \tag{15}$$

Insertion of equations (14) and (15) into equation (12) then leads to an explicit formal expression for $2\sigma_2 - \sigma_1^2$, namely

$$2\sigma_2 - \sigma_1^2 = 4 \sum_{k=1}^{\infty} \frac{\omega_0^2}{\omega_0^2 - \omega_k^2} \frac{\left| \int_V \rho_0 \mathbf{s}_k \cdot (i\hat{\mathbf{z}} \times \mathbf{s}_0)^* \, dV \right|^2}{\int_V \rho_0 \mathbf{s}_0 \cdot \mathbf{s}_0^* \, dV \int_V \rho_0 \mathbf{s}_k \cdot \mathbf{s}_k^* \, dV}. \tag{16}$$

Most of the coefficients a_k in equation (14) are zero, by virtue of the orthogonality of vector spherical harmonics. In fact, correct to this order, a toroidal mode ${}_nT_l$ is coupled only to the adjacent spheroidal modes ${}_{n'}S_{l\pm1}$, while a spheroidal mode ${}_nS_l$ is coupled to the adjacent toroidal modes ${}_{n'}T_{l\pm1}$ and also to the spheroidal modes ${}_{n'}S_l$, $n' \neq n$. The dominant contribution to the sums will come from those modes which are nearby in frequency. The factor $(\omega_0^2 - \omega_k^2)^{-1}$ in equations (15) and (16) is clearly a rough measure of the strength of the coupling. The above theory supposes that all the coupling is weak, i.e. that the multiplet is well isolated in the spectrum. If there is a mode in the sum with ω_k very near ω_0, the theory is no longer valid, and quasi-degenerate perturbation theory must be employed.

To insure completeness, we must be careful to incorporate into the above sums *absolutely all* the modes of the appropriate form. For an earth model like 1066A, with a solid inner core, a fluid outer core and a mantle, a complete catalogue of modes must include:

(i) The usual seismic toroidal modes of the mantle, designated ${}_1T_1$, ${}_0T_2$, etc.

(ii) The usual seismic spheroidal modes, designated ${}_0S_0$, ${}_1S_1$, ${}_0S_2$, etc. This category includes (Dziewonski & Gilbert 1973) Stoneley modes trapped at the boundaries of the inner and outer core, as well as shear-dominated modes confined largely to the solid inner core.

(iii) A set of toroidal modes of the inner core. The eigenfrequencies of these will coincide roughly with those of the spheroidal inner core modes mentioned above. The former correspond to *SH* waves in the inner core and the latter to *SV*. The toroidal inner core modes are completely decoupled from the rest of the Earth, whereas the spheroidal are in principle weakly visible at the surface.

(iv) A set of secular toroidal modes of the fluid outer core. These have $\omega_k = 0$, and $W = 0$ in the inner core and mantle; in the fluid W may be selected completely arbitrarily. These degenerate solutions exist because of the inability of the fluid core to resist any shear.

(v) A set of predominantly gravitational spheroidal modes confined largely to the fluid core (Dahlen 1974). The character of these modes is a sensitive function of the fluid density stratification; this is in turn best described by the squared Brunt–Väisälä frequency N^2. Only if a core model is everywhere stable ($N^2 > 0$) will all of these modes have $\omega_k^2 > 0$. The highest frequency modes will in that case have $\omega_k \approx N_{\max}$, and there will be an accumulation point at zero frequency. This is in contrast to the first three categories of modes, which have accumulation points at infinity. Model 1066A, it turns out, has regions of both stable and unstable stratification. In that case some, if not all, of these core gravity modes must have $\omega_k^2 < 0$. The central role played by these modes in static deformation problems has been elucidated by Dahlen & Fels (1978).

(vi) Finally, there are nine secular modes of degree $l = 1$ corresponding to rigid rotation of the mantle, rigid rotation of the inner core, and rigid translation of the Earth as a whole. The first three of these have $W = Cr$ (C is a constant) in the mantle, $W = 0$ elsewhere, and are occasionally designated ${}_0T_1$. The second three have $W = Cr$ in the inner core and $W = 0$

elsewhere. The remaining three have $U = V = C$ throughout the Earth, and are occasionally designated ${}_0S_1$. Since $\nabla_1 Y_0^0 = 0$, there are no toroidal modes of degree $l = 0$, and the spheroidal modes of degree $l = 0$ are all purely radial. Since radial motion necessarily involves compression, there are also no core gravity modes of degree $l = 0$, i.e. the only modes of degree $l = 0$ are the seismic radial modes ${}_0S_0$, ${}_1S_0$, etc. We remark finally that if the earth model under investigation has a well (say) an ocean, then the above catalogue must be amended in an obvious way. To avoid any confusion, we might also point out that nowhere in this paper is a prime used to denote differentiation; differentiation with respect to r will be denoted by ∂_r.

Without further ado, let us give an algorithm for computing σ_2. For both spheroidal and toroidal multiplets, σ_2 can be written in the form $\sigma_2 = \alpha_2 + m^2\gamma_2$. In what follows we shall use a, b and c to denote, respectively the radius of the Earth, the radius of the outer core and the radius of the inner core.

For a spheroidal mode ${}_nS_l$, the normalization integral N is

$$N = \int_0^a \rho_0[U^2 + l(l+1)V^2]r^2\,dr. \tag{17}$$

We define quantities A, B and C by

$$\begin{aligned} A = {} & \frac{1}{N}\frac{1}{l(l-1)}\int_c^b \rho_0[U + (l+1)V]^2 r^2\,dr \\ & + \sum_{n'T_{l-1}} \left(\frac{\omega_0^2}{\omega_0^2 - \omega_0'^2}\right)\left(\frac{1}{N\mathcal{N}_A'}\right)\left(\int_0^c \rho_0[U + (l+1)V]W' r^2\,dr\right)^2 \\ & + \sum_{n'T_{l-1}} \left(\frac{\omega_0^2}{\omega_0^2 - \omega_0'^2}\right)\left(\frac{1}{NN_A'}\right)\left(\int_b^a \rho_0[U + (l+1)V]W' r^2\,dr\right)^2 \end{aligned} \tag{18a}$$

$$\begin{aligned} B = {} & \frac{1}{N}\frac{1}{(l+1)(l+2)}\int_c^b \rho_0[U - lV]^2 r^2\,dr \\ & + \sum_{n'T_{l+1}} \left(\frac{\omega_0^2}{\omega_0^2 - \omega_0'^2}\right)\left(\frac{1}{N\mathcal{N}_B'}\right)\left(\int_0^c \rho_0[U - lV]W' r^2\,dr\right)^2 \\ & + \sum_{n'T_{l+1}} \left(\frac{\omega_0^2}{\omega_0^2 - \omega_0'^2}\right)\left(\frac{1}{NN_B'}\right)\left(\int_b^a \rho_0[U - lV]W' r^2\,dr\right)^2 \end{aligned} \tag{18b}$$

$$C = \sum_{\substack{n'S_l \\ n' \neq n}} \left(\frac{\omega_0^2}{\omega_0^2 - \omega_0'^2}\right)\left(\frac{1}{NN_C'}\right)\left(\int_0^a \rho_0[UV' + U'V + VV']r^2\,dr\right)^2. \tag{18c}$$

The quantity A represents the contribution of all the toroidal modes of degree $l - 1$, B represents the contribution of the toroidal modes of degree $l + 1$, and C represents the contribution of the spheroidal modes of degree l. In both equations (18a) and (18b), the first term is the contribution from the modes of the form (iv), the second from the modes of the form (iii) and the third from the modes of the form (i). In equation (18c) the sum should include core gravity modes (v) as well as the usual seismic spheroidal modes (ii).

The normalization integrals $\mathcal{N}_A'$, N_A', $\mathcal{N}_B'$, N_B' and N_C' are given by

$$\mathcal{N}_A' = l(l-1)\int_0^c \rho_0 W'^2 r^2\, dr \tag{19a}$$

$$N_A' = l(l-1)\int_b^a \rho_0 W'^2 r^2\, dr \tag{19b}$$

$$\mathcal{N}_B' = (l+1)(l+2)\int_0^c \rho_0 W'^2 r^2\, dr \tag{19c}$$

$$N_B' = (l+1)(l+2)\int_b^a \rho_0 W'^2 r^2\, dr \tag{19d}$$

$$N_C' = l(l+1)\int_0^a \rho_0 [U'^2 + l(l+1)V'^2] r^2\, dr. \tag{19e}$$

The splitting parameters α_2 and γ_2 for spheroidal modes ${}_nS_l$ are now given by

$$\alpha_2 = 2\,\frac{l^2(l-1)^2}{(2l+1)(2l-1)}\,A + 2\,\frac{(l+2)^2(l+1)^2}{(2l+3)(2l+1)}\,B \tag{20a}$$

$$\gamma_2 = \frac{1}{2}\beta^2 - 2\,\frac{(l-1)^2}{(2l+1)(2l-1)}\,A - 2\,\frac{(l+2)^2}{(2l+3)(2l+1)}\,B + 2C, \tag{20b}$$

where β is the first-order splitting parameter (Backus & Gilbert 1961),

$$\beta = \frac{1}{N}\int_0^a \rho_0 [V^2 + 2UV] r^2\, dr. \tag{21}$$

The cases $l = 0$, 1 and 2 must be treated separately. In those cases, equations (20a) and (20b) remain valid if we make the substitutions

$l = 0: A \to 0$

$$B \to B + \frac{1}{N}\left[\frac{\left(\int_0^c \rho_0 r U r^2\, dr\right)^2}{2\int_0^c \rho_0 r^4\, dr} + \frac{\left(\int_b^a \rho_0 r U r^2\, dr\right)^2}{2\int_b^a \rho_0 r^4\, dr}\right] \tag{22a}$$

$$l = 1: A \to 0 \tag{22b}$$

$$l = 2: A \to A + \frac{1}{N}\left[\frac{\left(\int_0^c \rho_0 r [U + 3V] r^2\, dr\right)^2}{2\int_0^c \rho_0 r^4\, dr} + \frac{\left(\int_b^a \rho_0 r [U + 3V] r^2\, dr\right)^2}{2\int_b^a \rho_0 r^4\, dr}\right]. \tag{22c}$$

The additional terms in equations (22a) and (22c) represent the contribution of the inner core and mantle rigid rotational modes. The rigid translational modes ${}_0S_1$ do not make any additional contribution to C in the case $l = 1$ by virtue of orthogonality.

For a toroidal mode (of the mantle) ${}_nT_l$, the normalization integral N is

$$N = l(l+1)\int_b^a \rho_0 W^2 r^2\, dr, \tag{23}$$

and we define A and B by

$$A = \sum_{n'S_{l-1}} \left(\frac{\omega_0^2}{\omega_0^2 - \omega_0'^2}\right)\left(\frac{1}{NN_A'}\right)\left(\int_b^a \rho_0 W[U' - (l-1)V']r^2\, dr\right)^2 \tag{24a}$$

$$B = \sum_{n'S_{l+1}} \left(\frac{\omega_0^2}{\omega_0^2 - \omega_0'^2}\right)\left(\frac{1}{NN_B'}\right)\left(\int_b^a \rho_0 W[U' + (l+2)V']r^2\, dr\right)^2 \tag{24b}$$

where N_A' and N_B' are given by

$$N_A' = \int_0^a \rho_0 [U'^2 + l(l-1)V'^2]r^2\, dr \tag{25a}$$

$$N_B' = \int_0^a \rho_0 [U'^2 + (l+1)(l+2)V'^2]r^2\, dr. \tag{25b}$$

In this case, A represents the contribution of all the spheroidal modes of degree $l-1$ and B the spheroidal modes of degree $l+1$. As with equation (18c), modes of the form (v) as well as (ii) must be included in the sums (24a) and (24b). The parameters α_2 and γ_2 are given by

$$\alpha_2 = 2\frac{l^2(l+1)^2}{(2l+1)(2l-1)}A + 2\frac{l^2(l+1)^2}{(2l+3)(2l+1)}B \tag{26a}$$

$$\gamma_2 = \frac{1}{2}\frac{1}{l^2(l+1)^2} - 2\frac{(l+1)^2}{(2l+1)(2l-1)}A - 2\frac{l^2}{(2l+3)(2l+1)}B. \tag{26b}$$

The rigid translational modes ${}_0S_1$ do not make any additional contribution to A in the case $l = 2$, because of the fact that $U = V$.

To calculate the contributions from the seismic modes (i) and (ii), we have made use of the compilation by Buland (1976), which is complete up to $T = 45$ s, including ${}_1S_1$, Stoneley modes and inner core spheroidal modes. We have supplemented this with a similar compilation of inner core toroidal modes (iii). The secular modes (iv) and (vi) were summed analytically, as described above. Taking proper account of the core gravity modes (v) is somewhat problematical. Since the parameter N^2 in the core is not well constrained in inversions, there is little reason to believe that the detailed structure of N^2 for model 1066A represents real features of the Earth. Since only a very minor adjustment of ρ_0 in the core suffices to make $N^2 = 0$ throughout, we have chosen to handle the core modes (v) as if they were the case. If the core is neutrally stable the modes of class (v) are all secular (zero frequency) and exactly confined to the core. Their contribution to A and B in equations (24a) and (24b) in that case vanishes, since the integration in those equations is carried out only over the mantle. There is, however, a contribution to C in equation (18c). We shall describe our method of evaluating that contribution in the next section.

Convergence of the sums over the modes (i), (ii) and (iii) was in all cases rapid, with the dominant contribution coming from a few modes nearby in frequency to ω_0. In every instance tabulated, we found the contribution of the toroidal inner core modes (iii) to be thoroughly negligible (we expected this, but decided it would not hurt to check). On the other hand, inclusion of the secular modes (iv) and (vi) is essential. For example, for ${}_0S_2$, roughly one-half of the contribution to α_2 comes from the toroidal core modes of degrees $l = 1$ and $l = 3$, and the other half comes from the $l = 1$ rigid rotational modes. Less than 1 per cent comes from the seismic modes ${}_0T_3$, ${}_1T_1$, ${}_1S_2$, etc. This is perhaps not too surprising, since ω_0^2 for ${}_0S_2$ is much nearer zero than it is to $\omega_0'^2$ for ${}_0T_3$, ${}_1T_1$ or ${}_1S_2$.

The parameters α_2 and γ_2 are very large for the modes ${}_0S_{11}$ and ${}_0T_{12}$. This is a result of their near degeneracy (the respective ω_0^2 differ by less than 0.15 per cent). Quasi-degenerate perturbation theory is clearly called for in this case.

The contribution from core gravity modes

Let κ be the bulk modulus in the fluid core, and let g_0 be the gravitational acceleration. The squared Brunt–Väisälä frequency N^2 is then given by $N^2 = -g_0\rho_0^{-1}\partial_r\rho_0 - g_0^2\rho_0\kappa^{-1}$. Let us denote the perturbations in fluid density, gravitational potential and fluid pressure associated with a spheroidal displacement of the form (1) by $\rho_1 Y_l^m$, $\phi_1 Y_l^m$ and $p_1 Y_l^m$ respectively. The normal mode equations in the fluid core can be written in the form

$$\rho_0\omega_0^2 U = \partial_r p_1 + \rho_0\partial_r\phi_1 + \rho_1 g_0 \tag{27a}$$

$$\rho_0\omega_0^2 rV = p_1 + \rho_0\phi_1 \tag{27b}$$

$$\rho_1 = -U\partial_r\rho_0 - \rho_0 X \tag{27c}$$

$$p_1 = \rho_0 g_0 U - \kappa X \tag{27d}$$

$$\nabla^2\phi_1 = 4\pi G\rho_1. \tag{27e}$$

The quantity X in equations (27c) and (27d) is the dilatation scalar, given by

$$X = \partial_r U + 2r^{-1}U - l(l+1)r^{-1}V, \tag{28}$$

and G is the gravitational constant. In equation (27e) and throughout this section, we have used the symbol ∇^2 to denote $\partial_r^2 + 2r^{-1}\partial_r - l(l+1)r^{-2}$. If $N^2 = 0$ throughout the core, all the core gravity modes have $\omega_0 = 0$, no displacement or any other perturbation within the inner core and mantle, and $\rho_1 = 0$, $\phi_1 = 0$ and $p_1 = 0$ within the fluid core. The scalars U and V must satisfy the two equivalent equations

$$U\partial_r\rho_0 + \rho_0 X = 0 \tag{29a}$$

$$\rho_0 g_0 U - \kappa X = 0 \tag{29b}$$

within the fluid core, and, in addition, U must vanish on both the inner core–outer core boundary and the outer core–mantle boundary. Any U and V which does satisfy those conditions constitutes the displacement eigenfunction of some core gravity mode, in the case $N^2 = 0$. Every such mode is secular, because every such deformation of a neutrally stable core leaves the elastic-gravitational potential energy of the whole earth model unchanged (in any core model which is not neutrally stratified, only toroidal deformations, which comprise the class (iv), have this property). Equations (29a) and (29b) can also be written in the form

$$\rho_0 l(l+1)V = r^{-1}\partial_r(\rho_0 r^2 U), \tag{30}$$

which is more useful for our purposes. A core gravity mode eigenfunction may be obtained, in the case $N^2 = 0$, by selecting U arbitrarily, subject to the constraint that $U = 0$ at $r = c$ and $r = b$, and then defining V by equation (30).

Having established the properties of the core gravity modes in the case $N^2 = 0$, we shall go on now to investigate the features of the usual seismic spheroidal modes in the same case. By combining equations (27a)–(27e), we may obtain the equation

$$\omega_0^2 \partial_r(rV) = (\omega_0^2 - N^2)U + (\rho_0 g_0)^{-1} N^2 p_1. \tag{31}$$

In the case $N^2 = 0$, this equation is satisfied automatically for a core gravity mode by virtue of the fact that $\omega_0 = 0$. For a seismic mode, $\omega_0 \neq 0$, and in the case $N^2 = 0$, equation (31) reduces to

$$U = \partial_r(rV). \tag{32}$$

This amazingly simple relation between U and V must prevail for every non-secular spheroidal mode throughout a neutrally stratified core. We have verified numerically that equation (32) is very well satisfied for all the spheroidal modes included in our table, even though the core of model 1066A is not exactly neutrally stratified. As expected, the worst agreement is for the mode ${}_1S_1$, which has $\omega_0^2 \sim |N^2|$.

Suppose now that s is an arbitrary spheroidal field of the form $\mathbf{s} = \hat{\mathbf{r}} U Y_l^m + V \nabla_1 Y_l^m$. Then $\nabla \cdot \mathbf{s}$ and $\nabla \times \mathbf{s}$ can be written in the form

$$\nabla \cdot \mathbf{s} = X Y_l^m \tag{33a}$$

$$\nabla \times \mathbf{s} = H[-\hat{\mathbf{r}} \times \nabla_1 Y_l^m] \tag{33b}$$

where X is given in equation (28) and where H is given by

$$H = r^{-1}[U - \partial_r(rV)]. \tag{34}$$

If s satisfies equation (32), we may write

$$X = \nabla^2(rV) \tag{35a}$$

$$H = 0. \tag{35b}$$

Equation (35b) states that the displacement associated with a non-secular spheroidal mode in a neutrally stratified core is irrotational. In that case, s can be written in the simple form $\mathbf{s} = \nabla(rVY_l^m)$.

To evaluate C in equation (18c), we break it into two parts, i.e. $C = C_s + C_g$, where C_s is the sum over the seismic modes ${}_{n'}S_l$, $n' \neq n$, and C_g is the corresponding core gravity mode contribution. The quantity C_s has been calculated in this study by direct summation, as have A and B. In the case $N^2 = 0$, since all the core gravity modes are secular, we can write C_g in the form

$$C_g = \frac{1}{N} \int_c^b \rho_0 (UV_g + U_g V + VV_g) r^2 \, dr, \tag{36}$$

where $\mathbf{s}_g = \hat{\mathbf{r}} U_g Y_l^m + V_g \nabla_1 Y_l^m$ is the projection of the vector field

$$\hat{\mathbf{r}}(V - \beta U) Y_l^m + [(U + V)/\{l(l+1)\} - \beta V] \nabla_1 Y_l^m \tag{37}$$

on to the space of core gravity modes of degree l. The quantities U_g and V_g satisfy equation (30) and, in addition, U_g must vanish at the fluid core boundaries $r = c$ and $r = b$. Using this information, together with equation (32), we can, after integrating by parts, convert equation (36) into the form

$$C_g = -\frac{1}{l(l+1)N}\int_c^b \rho_0 U_g r X r^2 dr. \tag{38}$$

To compute U_g, we employ the identity (valid for any spheroidal field)

$$\nabla^2(rU_g) = l(l+1)H_g + 2X_g + r\partial_r X_g, \tag{39}$$

where X_g and H_g are defined in terms of U_g and V_g by equations (28) and (34) respectively. From equation (29b), we know that X_g can be written in terms of U_g alone, in the form

$$X_g = \rho_0 g_0 \kappa^{-1} U_g. \tag{40}$$

Since the field (37) is the sum of its projection on to the core gravity modes of degree l and its projection on to the seismic modes of degree l, and since in the case $N^2 = 0$ the latter portion is irrotational, we know that we may also write

$$H_g = r^{-1}(\partial_r[\![r[(U+V)/\{l(l+1)\} - \beta V]]\!] - (V - \beta U)) = X/\{l(l+1)\}. \tag{41}$$

Upon substituting equations (40) and (41) into equation (39) and making use of equation (35a), we obtain an equation for U_g, namely

$$\nabla^2\{r(U_g - V)\} = r^{-1}\partial_r(r^2\rho_0 g_0 \kappa^{-1} U_g). \tag{42}$$

To find C_g, we have solved equation (42) numerically for U_g, subject to the boundary conditions that $U_g = 0$ at $r = c$ and $r = b$, and we have then evaluated the integral (38). Since $\rho_0 g_0 \kappa^{-1}$ is generally somewhat smaller than r^{-1} throughout the core, we have found it most convenient to solve equation (42) by iteration, using the solution with the right side equal to zero as the initial iterate.

As expected, we have found that C_g is generally insignificant compared to C_s except for those modes which have a substantial fraction of stored compressional energy in the fluid core. Mode ${}_6S_1$, for example, has more than 45 per cent of its total energy content in the form of compressional energy in the fluid core, and for that mode C_g is of the same order magnitude as C_s.

Conclusions

Perturbation theory as outlined above, which treats both Ω/ω_0 and ϵ_h as small parameters, is a valid procedure for calculating the influence of rotation and ellipticity on the seismic modes (i) and (ii) with the possible exception of ${}_1S_1$. For the mode ${}_1S_1$, the calculations of Smith (1976), which are first order in ϵ_h, but which make no assumption whatsoever about the magnitude of Ω/ω_0, should be superior to ours. Perturbation theory would in addition be valid for the toroidal inner core modes (iii), but they are of very little intrinsic interest. The remaining modes (iv), (v) and (vi) are also split and coupled by rotation and ellipticity. For all of these, perturbation theory fails completely since Ω/ω_0 is either very large or infinite. In addition, the coupling between them will be very strong, because they are all nearly degenerate.

The splitting of the rigid body modes (iv) is well understood (Dahlen & Smith 1975; Smith 1977). All but two of the resulting split modes still have a rigid body character, and are secular if viewed in an appropriate frame. The other two are more interesting; they are the Chandler wobble of the mantle and that of the inner core. In the former, the motion is confined largely to the mantle; the displacements in both the solid inner core and the fluid outer core are small. In the latter, the wobble of the inner core induces as well a rather unusual motion (governed by the Taylor–Proudman theorem) in the surrounding fluid outer core, but the torque exerted on the mantle is weak so that it participates very little in the motion (Kakuta, Okamoto & Sasao 1975).

The problem of determining the influence of rotation and ellipticity on the remaining fluid core modes (iv) and (v) is beginning to attract theoretical attenuation (Smith 1977; Johnson & Smylie 1977; Olson 1977). The implications for dynamo theory are obvious. In addition, if such modes could ever be observed, they would certainly shed light on the structure of the core.

Acknowledgments

We are especially grateful to Adam Dziewonski for his help in initiating this work, and to John Woodhouse for pointing out an error in an earlier version of this paper. Ray Buland provided tapes with the eigenfunctions of model 1066A. Grants from the National Science Foundation, EAR76-83914 (at Princeton) and EAR76-14839 (at Harvard), supported this study. This paper was written while one of us (FAD) was on leave at the Institut de Physique du Globe in Paris, partially supported by a grant from the Délégation Générale de Recherche Scientifique et Technique.

References

Agnew, D., Berger, J., Buland, R., Farrell, W. & Gilbert, F., 1976. International deployment of seismometers: a network for very long period seismology, *EOS*, **57**, 180–188.

Backus, G. & Gilbert, F., 1961. The rotational splitting of the free oscillations of the earth, *Proc. natn. Acad. Sci. U.S.A.*, 47, 362–371.

Buland, R., 1976. Retrieving the seismic moment tensor, *PhD thesis*, University of California, San Diego.

Buland, R. & Gilbert, F., 1978. Improved resolution of complex eigenfrequencies in analytically continued seismic spectra, *Geophys. J. R. astr. Soc.*, **52**, 457–470.

Dahlen, F. A., 1968. The normal modes of a rotating, elliptical earth, *Geophys. J. R. astr. Soc.*, **16**, 329–367.

Dahlen, F. A., 1974. On the static deformation of an earth model with a fluid core, *Geophys. J. R. astr. Soc.*, **36**, 461–485.

Dahlen, F. A., 1976. Reply to comments by A. M. Dziewonski and R. V. Sailor, *J. geophys. Res.*, **81**, 4951–4956.

Dahlen, F. A. & Fels, S. B., 1978. A physical explanation of the static core paradox, *Geophys. J. R. astr. Soc.*, **55**, 317.

Dahlen, F. A. & Smith, M. L., 1975. The influence of rotation on the free oscillations of the earth, *Phil. Trans. R. Soc. Lond. A*, **279**, 583–629.

Dziewonski, A. M. & Gilbert, F., 1973. Observations of normal modes from 84 recordings of the Alaskan earthquake of 1964 March 28 – II. Further remarks based on new spheroidal overtone data, *Geophys. J. R. astr. Soc.*, **35**, 401–437.

Dziewonski, A. M. & Sailor, R. V., 1976. Comments on a paper by F. A. Dahlen, *J. geophys. Res.*, **81**, 4947–4950.

Gilbert, F., 1971. The diagonal sum rule and averaged eigenfrequencies, *Geophys. J. R. astr. Soc.*, **23**, 119–123.

Gilbert, F. & Dziewonski, A. M., 1975. An application of normal mode theory to the retrieval of structural parameters and source mechanism from seismic spectra, *Phil. Trans. R. Soc. Lond. A*, **278**, 187–269.

Jeffreys, H., 1970. *The Earth*, 5th edn, pp. 183–192, Cambridge University Press.

Johnson, I. M. & Smylie, D. E., 1977. A variational approach to whole-earth dynamics, *Geophys. J. R. astr. Soc.*, **50**, 35–54.

Kakuta, C., Okamoto, I. & Sasao, T., 1975. Is the nutation of the solid inner core responsible for the 24-year libration of the pole?, *Publs astr. Soc. Japan*, **27**, 357–365.

Kolmogorov, A. & Fomine, S., 1974. *Eléments de la Théorie des Fonctions et de l'Analyse Fonctionelle*, pp. 463–467, Editions Mir, Moscow.

Olson, P., 1977. Internal waves in the Earth's core, *Geophys. J. R. astr. Soc.*, **51**, 183–215.

Smith, M. L., 1976. Translational inner core oscillations of a rotating slightly elliptical earth, *J. geophys. Res.*, **81**, 3055–3065.

Smith, M. L., 1977. Wobble and nutation of the Earth, *Geophys. J. R. astr. Soc.*, **50**, 103–140.

Stein, S. & Geller, R., 1978. Time domain observation and synthesis of split spheroidal and torsional free oscillations of the 1960 Chilean earthquake: preliminary results, *Bull. seism. Soc. Am.*, **68**, 325–332.

Woodhouse, J. H., 1976. On Rayleigh's principle, *Geophys. J. R. astr. Soc.*, **46**, 11–22.

Woodhouse, J. H. & Dahlen, F. A., 1978. The effect of a general aspherical perturbation on the free oscillations of the Earth, *Geophys. J. R. astr. Soc.*, **53**, 335–354.