Celest Mech Dyn Astr (2008) 101:69–103
DOI 10.1007/s10569-007-9116-3

ORIGINAL ARTICLE

Rotational dynamics of a solar system body under solar radiation torques

Daniel J. Scheeres · Sepidehsadat Mirrahimi

Received: 5 October 2007 / Revised: 17 December 2007 / Accepted: 26 December 2007 /
Published online: 6 February 2008

Abstract The rotational dynamics of a small solar system body subject to solar radiation torques is investigated. A set of averaged evolutionary equations are derived as an analytic function of a set of spherical harmonic coefficients that describe the torque acting on the body due to solar radiation. The analysis also includes the effect of thermal inertia. The resulting equations are studied and a set of possible dynamical outcomes for the rotation rate and obliquity of a small body are found and characterized.

Keywords YORP · Rotational dynamics · Asteroids · Asteroid rotation · Solar radiation torques

1 Introduction

Solar radiation has a significant role in the dynamical evolution of small solar system bodies such as asteroids. When operating on the orbital dynamics of such a body the effect is called the Yarkovsky effect. When operating on the rotational dynamics of a body this effect is called the YORP (Yarkovsky-O'Keefe-Raszievskii-Paddack) effect (Bottke et al. 2002). The recent detection of this effect (Kaasalainen et al. 2007, Lowry et al. 2007, Taylor et al. 2007) could possibly clarify many unexplained aspects about asteroids. For example, the YORP effect may be a possible contributor to the creation of binary asteroids, a reason for the presence of fast rotators in the NEA population, and may have a strong influence on the obliquities of these bodies.

The dynamics of the YORP effect has been studied previously. This physical effect was first introduced to the asteroid dynamics community by Rubincam (1995). Following this it was studied in detail with numerical models by Čapek and Vokrouhlický (2004) and Vokrouhlický and Čapek (2002). Recent analysis of this effect has been made by Scheeres (2007) where the use of Fourier coefficients was introduced to capture the

D. J. Scheeres (✉) · S. Mirrahimi
Aerospace Engineering Department, University of Michigan, Ann Arbor, MI, USA
e-mail: scheeres@unich.edu

functional form of the torques acting on the asteroid, and to enable a set of averaged equations to be defined. More recently, Nesvorný and Vokrouhlický (2007) have introduced a mapping from an object's spherical harmonic shape coefficients to the YORP torque coefficients that act on the rotation rate.

The current paper follows from the analysis presented in Scheeres (2007). In that paper a model of the torque acting on an asteroid at any given instant is given where the torque is decomposed into a Fourier series in terms of the solar longitude and is applied to an asteroid shape described using a polyhedral model. Following this, an analysis of the effect of YORP on the averaged rotational dynamics of a uniformly rotating asteroid was performed, supposing that the asteroid is rotating about its maximum moment of inertia. In Scheeres (2007) the Fourier coefficients in the torque expression are not constant, but depend on the obliquity of the asteroid and, therefore, in the dynamical equations the Fourier coefficients were averaged numerically. In the current paper we generalize this approach and express the solar torques using a full spherical harmonics field. This allows us to develop an explicit averaged set of evolutionary equations for the asteroid rotation state as a function of constant coefficients that only depend on the asteroid's shape. We also introduce a realistic model of the thermal inertia of an asteroid and incorporate that into the analysis. This leads us to some general results about the rotational dynamics of asteroids subject to YORP.

This paper is separated into the following sections. In Sect. 2, we provide some fundamental information. Particularly, we state some notation for the orientation of the asteroid relative to its heliocentric orbit, provide a model of the torque due to incident solar radiation and define its spherical harmonic expansion, and introduce a thermal inertia model. In Sect. 3, we give a representation of the rotational dynamics in terms of spherical harmonics coefficients and derive the secular rotational dynamics of an asteroid using the spherical harmonics expansion. Next, we average over one asteroid orbit to find a simple representation of the averaged dynamical system. In Sect. 4, we study the long-term dynamics of a rotating asteroid, using the results in Sect. 3. We also analyze the dynamics in presence and absence of thermal conductivity. Particularly, we discuss the generic dynamical evolution to the averaged rotational dynamics equations for some generic models of an asteroid's YORP spherical harmonic coefficients.

2 Rotational dynamics of asteroids

2.1 Terminology and determination of frames

To compute the interaction between the Sun and an asteroid, it is important to work in coordinate frames where the location of the Sun relative to the asteroid pole are stated in convenient coordinates. Hence, to facilitate our work we specify the asteroid's heliocentric orbit relative to it's rotation pole.

Let $\hat{\mathbf{X}}$ be the unit vector pointed along the asteroid orbit perihelion vector, $\hat{\mathbf{Z}}$ the unit vector pointed along the orbit angular momentum vector, and $\hat{\mathbf{Y}}$ the unit vector that completes the triad, and pointed along the perihelion velocity vector. This orbital coordinate system can be expressed in terms of an inertial frame (defined by the unit vectors $\hat{\mathbf{X}}_E$, $\hat{\mathbf{Y}}_E$, and $\hat{\mathbf{Z}}_E$) using classical orbit elements:

$$\hat{\mathbf{X}} = [\cos\varpi \ \cos\Omega - \sin\varpi \ \sin\Omega \ \cos i]\hat{\mathbf{X}}_E + [\cos\varpi \ \sin\Omega + \sin\varpi \ \cos\Omega \ \cos i]\hat{\mathbf{Y}}_E \\ + \sin\varpi \ \sin i \ \hat{\mathbf{Z}}_E$$

$$\hat{\mathbf{Y}} = -[\sin\varpi\ \cos\Omega + \cos\varpi\ \sin\Omega\ \cos i]\hat{\mathbf{X}}_E + [-\sin\varpi\ \sin\Omega + \cos\varpi\ \cos\Omega\ \cos i]\hat{\mathbf{Y}}_E$$
$$+ \cos\varpi\ \sin i\ \hat{\mathbf{Z}}_E$$
$$\hat{\mathbf{Z}} = \sin\Omega\ \sin i\ \hat{\mathbf{X}}_E - \cos\Omega\ \sin i\ \hat{\mathbf{Y}}_E + \cos i\ \hat{\mathbf{Z}}_E$$

where i is the inclination, Ω is the longitude of the ascending node, and ϖ is the argument of perihelion.

We can also specify the asteroid-fixed frame relative to the same inertial frame. We consider a body-fixed frame with unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, where the unit vector $\hat{\mathbf{z}}$ points along the maximum moment of inertia of the body (supposed to be the rotational axis). The $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ unit vectors lie in the body's equator and point at the minimum and intermediate moment of inertia axis, respectively.

$$\hat{\mathbf{x}} = -[\sin\alpha\ \cos\phi + \cos\alpha\ \sin\phi\ \sin\delta]\hat{\mathbf{X}}_E + [\cos\alpha\ \cos\phi - \sin\alpha\ \sin\phi\ \sin\delta]\hat{\mathbf{Y}}_E$$
$$+ \sin\phi\ \cos\delta\ \hat{\mathbf{Z}}_E$$

$$\hat{\mathbf{y}} = [\sin\alpha\ \sin\phi - \cos\alpha\ \cos\phi\ \sin\delta]\hat{\mathbf{X}}_E - [\cos\alpha\ \sin\phi + \sin\alpha\ \cos\phi\ \sin\delta]\hat{\mathbf{Y}}_E$$
$$+ \cos\phi\ \cos\delta\ \hat{\mathbf{Z}}_E$$

$$\hat{\mathbf{z}} = \cos\alpha\ \cos\delta\ \hat{\mathbf{X}}_E + \sin\alpha\ \cos\delta\ \hat{\mathbf{Y}}_E + \sin\delta\ \hat{\mathbf{Z}}_E$$

where α is the right ascension, δ is the declination, and ϕ is a rotation angle about the instantaneous rotation pole.

We use the body-fixed frame as the reference frame and determine the orbit's inclination, i_s, the longitude of the ascending node, Ω_s, and the argument of perihelion relative to this frame, ϖ_s:

$$\cos i_s = \hat{\mathbf{Z}} \cdot \hat{\mathbf{z}} = \sin\delta\ \cos i + \cos\delta\ \sin i\ \sin(\Omega - \alpha)$$

Let $\hat{\mathbf{n}}_{\Omega_s}$ be the node unit vector:

$$\hat{\mathbf{n}}_{\Omega_s} = \frac{\hat{\mathbf{z}} \times \hat{\mathbf{Z}}}{|\hat{\mathbf{z}} \times \hat{\mathbf{Z}}|} = \cos\Omega_s \hat{\mathbf{x}} + \sin\Omega_s \hat{\mathbf{y}}$$

Thus we can compute the longitude of the ascending node:

$$\tan\Omega_s = \frac{\hat{\mathbf{y}} \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{Z}})}{\hat{\mathbf{x}} \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{Z}})} = \frac{\hat{\mathbf{Z}} \cdot \hat{\mathbf{x}}}{-\hat{\mathbf{Z}} \cdot \hat{\mathbf{y}}} = \tan(\Omega_{s_0} - \phi)$$

$$\tan\Omega_{s_0} = \frac{\sin i\ \cos(\Omega - \alpha)}{\cos\delta\ \cos i - \sin\ \delta \sin i\ \sin(\Omega - \alpha)}$$

So we have $\Omega_s = \Omega_{s_0} - \phi$. Finally we determine the argument of perihelion. To do this we first compute the transverse vector $\hat{\mathbf{n}}_T = \hat{\mathbf{Z}} \times \hat{\mathbf{n}}_{\Omega_s}$, which lies in the orbit plane perpendicular to the node vector. So the argument of perihelion can be computed as:

$$\tan\varpi_s = \frac{\hat{\mathbf{X}} \cdot \hat{\mathbf{n}}_T}{\hat{\mathbf{X}} \cdot \hat{\mathbf{n}}_{\Omega_s}} = \frac{\hat{\mathbf{X}} \cdot \hat{\mathbf{z}}}{\hat{\mathbf{Y}} \cdot \hat{\mathbf{z}}} = \tan(\varpi_0 + \varpi)$$

$$\tan\varpi_0 = \frac{\cos\delta\ \cos(\Omega - \alpha)}{\sin\delta\ \sin i - \cos\delta \cos i\ \sin(\Omega - \alpha)}$$

Thus we have specified the orbital elements in the body-fixed frame. We can now compute the Sun's location in this frame, denoted by the unit vector $\hat{\mathbf{u}}$:

$$\hat{\mathbf{u}} = \cos(\varpi_s + \nu)\hat{\mathbf{n}}_{\Omega_s} + \sin(\varpi_s + \nu)\hat{\mathbf{n}}_T$$

where ν is the true anomaly. By defining $\nu' := \varpi_s + \nu$:

$$\hat{\mathbf{u}} = \cos(\nu')\hat{\mathbf{n}}_{\Omega_s} + \sin(\nu')\hat{\mathbf{n}}_T$$

And we specify this vector in the body-fixed frame:

$$\hat{\mathbf{u}} = \cos\delta_s \ \cos\lambda_s \,\hat{\mathbf{x}} + \cos\delta_s \ \sin\lambda_s \,\hat{\mathbf{y}} + \sin\delta_s \,\hat{\mathbf{z}}$$

We call δ_s the solar latitude and λ_s the solar longitude. They can be computed as:

$$\sin\delta_s = \sin i_s \ \sin(\varpi_s + \nu)$$
$$\lambda_s = \Omega_{s0} + \lambda_\nu - \phi$$

where

$$\tan\lambda_\nu = \cos i_s \ \tan(\varpi_s + \nu)$$

From this relationship we can also find expressions

$$\sin\lambda_\nu = \frac{\cos i_s \sin(\varpi_s + \nu)}{\cos\delta_s} \tag{1}$$

$$\cos\lambda_\nu = \frac{\cos(\varpi_s + \nu)}{\cos\delta_s} \tag{2}$$

The asteroid-Sun distance is assumed to vary with the true anomaly following Kepler's law:

$$R = \frac{a(1 - e^2)}{1 + e\cos\nu}$$

where a is the semi-major axis and e is the eccentricity.

Given these definitions we can specify the asteroid's rotation state in terms of the inclination and longitude of ascending node of the asteroid's solar orbit in the frame defined by its rotation pole. This is equivalent to specifying the obliquity and right ascension of the asteroid's rotation pole. In fact, the quantity i_s is precisely equal to the obliquity and the quantity Ω_s is precisely equal to the right ascension of the pole as measured relative to the asteroid's heliocentric orbit. In this study we use an asteroid-centric frame for describing the relative orientation of the asteroid and the Sun. This is motivated by our use of the Euler equations for describing the rotational dynamics of the body, as they are expressed in a body-fixed frame. Thus, in this frame, evolution of the asteroid's rotation pole can be viewed as the evolution of its heliocentric orbit elements relative to its rotation pole.

2.2 Forces and moments due to solar radiation

We assume that the asteroid consists of N facets, each of them being a flat plane with outward normal vector $\hat{\mathbf{n}}_i$ and with position vector $\mathbf{r}_i$ to their center. The combination of these facets defines the asteroid's surface. Assuming a zero thermal conductivity, we can determine the force acting on each surface element given the angular location of the Sun $\hat{\mathbf{u}}$ and the distance from the Sun to the asteroid R (McInnes 1999):

$$F(\mathbf{R}) = P(R)\mathbf{f}_i(\hat{\mathbf{u}})$$
$$\mathbf{f}_i(\hat{\mathbf{u}}) = -[\{\rho s(2\hat{\mathbf{n}}_i\hat{\mathbf{n}}_i - \mathbf{U}) + \mathbf{U}\} \cdot \hat{\mathbf{u}}\hat{\mathbf{u}} \cdot \hat{\mathbf{n}}_i + a_2\hat{\mathbf{n}}_i\hat{\mathbf{n}}_i \cdot \hat{\mathbf{u}}]H_i(\hat{\mathbf{u}})A_i$$

where $\mathbf{U}$ is the identity dyad, ρ is the reflectivity of the surface, s is the fraction of radiation with specular reflection, $a_2 = B(1 - s)\rho + (1 - \rho)B$, B being the Lambertian scattering coefficient of the surface, A_i is the surface area, and $H_i(\hat{\mathbf{u}})$ is the visibility function, which equals 1 when the Sun is above the horizon and 0 otherwise. $P(R)$ is the incident light pressure as a function of distance from the Sun and equals $P(R) = G_1/R^2$ where R is the distance between the asteroid and the Sun, $G_1 \sim 1 \times 10^8$ kg km^3 s^{-2} m^{-2}, and represents the light pressure at one astronomical unit. Here we assume that the surface properties are constant across the facet. So the total force and moment would be:

$$\frac{\mathbf{F}(R\hat{\mathbf{u}})}{P(R)} = \sum_{i=1}^{N} \mathbf{f}_i(\hat{\mathbf{u}})$$

$$\frac{\mathbf{M}(R\hat{\mathbf{u}})}{P(R)} = \sum_{i=1}^{N} \mathbf{m}_i(\hat{\mathbf{u}})$$

where $\mathbf{m}_i = \mathbf{r}_i \times \mathbf{f}_i$ and $\mathbf{r}_i$ is the position vector to the center of the facet.

In Scheeres (2007) the force and moment due to solar radiation were presented as a Fourier series decomposition in terms of longitude λ_s. Those Fourier coefficients were not constant, but depended on the solar latitude δ_s, as $\frac{\mathbf{M}}{P(R)}$ is a function of the solar latitude as well as the solar longitude. Thus, in that expression of the theory, to compute the average rates the Fourier coefficients for different latitudes had to be numerically averaged. This resulted in numerical formula for the average terms and not an analytical function. A natural idea is to write $\frac{\mathbf{M}}{P(R)}$ in a spherical harmonics decomposition as a function of solar latitude and longitude, instead of a Fourier decomposition just as a function of the longitude. Such a formulation can also be generalized more easily to the case where the asteroid rotates slowly and tumbles.

We note that $\frac{\mathbf{F}(R\hat{\mathbf{u}})}{P(R)}$, and $\frac{\mathbf{M}(R\hat{\mathbf{u}})}{P(R)}$ are piecewise continuous functions of δ_s and λ_s, where $0 \le \lambda_s \le 2\pi$, and $-\frac{\pi}{2} \le \delta_s \le \frac{\pi}{2}$. Therefore they can be represented as a series of spherical harmonics (MacRobert 1947):

$$\frac{\mathbf{M}(R, \delta_s, \lambda_s)}{P(R)} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} P_l^m(\sin(\delta_s))\{\mathbf{C}_{l,m}\cos(m\lambda_s) + \mathbf{D}_{l,m}\sin(m\lambda_s)\}$$

where $P_l^m(x)$ are the associated Legendre functions. We note that $\mathbf{C}_{l,m}$ and $\mathbf{D}_{l,m}$ are vectors. Computation of these coefficients for a given asteroid shape are discussed in the Appendix. A similar expression is available for the force.

2.3 The effect of finite thermal conductivity

A non-zero thermal conductivity delays the reemission of solar radiation and modifies the force acting on the asteroid, specifically the component due to the $(1 - \rho)B$ term in the a_2 coefficient, which represents reemission of absorbed solar radiation (McInnes 1999). In addition to decreasing the reemission, this delay can influence the dynamics of small asteroids by changing the inertial longitude where the photons are reradiated by an effective lag angle ϕ_{lag}. To model this we can modify the solar longitude term in the torque and force expression as $\lambda_s = \Omega_{s0} + \lambda_\nu - (\phi - \phi_{lag}) = \lambda_0 - \phi$, where $\lambda_0 = \Omega_{s0} + \lambda_\nu + \phi_{lag}$. We should expect this thermal lag to vary from point to point on the asteroid, but here we suppose that it is uniform.

We use the approximation of the lag angle given in Rubincam (1995):

$$\tan(\phi_{lag}) = \frac{\theta}{1+\theta}$$

$$\theta = \frac{\sqrt{\varrho c_p \kappa \omega}}{\sqrt{32}\varepsilon\sigma T_{eq}^3} \tag{3}$$

where ϱ is the density, c_p is the specific heat, κ the thermal conductivity, ε the emissivity, σ the Stefan–Boltzamnn constant, and T_{eq} is the equilibrium temperature, while $\sqrt{\frac{1}{2}\varrho c_p \kappa \omega}$ is the thermal inertia. The parameter θ is called the "Thermal Parameter" in Spencer et al. (1989). Defining $\mu = \frac{\sqrt{\varrho c_p \kappa}}{\sqrt{32}\varepsilon\sigma T_{eq}^3}$ we have the additional relations for ϕ_{lag}:

$$\cos(\phi_{lag}) = \frac{1+\mu\sqrt{\omega}}{\sqrt{1+2\mu\sqrt{\omega}+2\mu^2\omega}} \tag{4}$$

$$\sin(\phi_{lag}) = \frac{\mu\sqrt{\omega}}{\sqrt{1+2\mu\sqrt{\omega}+2\mu^2\omega}} \tag{5}$$

We note that this expression for thermal lag is only approximate and is based on a linearized theory. It is useful, however, as it provides us with a functional relationship between the asteroid rotation rate and the thermal reemission lag. This relation agrees with physical intuition, as the lag angle increases with an increased rotation rate or an increased thermal inertia. Also, we note that the lag angle using the linearized theory is capped at 45° for an arbitrarily fast rotation rate. The decrease in the reemission of the radiation is represented by a multiplicative factor $1/\sqrt{1+2\mu\sqrt{\omega}+2\mu^2\omega}$ in the linearized theory.

3 Rotational dynamics

Now, having an analytic description of the moments, we can study the dynamics of the asteroid. We first state the general form of the rotational dynamics equation. Under the assumption that the asteroid is close to principal axis rotation about its largest moment of inertia, we derive an approximate set of equations by introducing a linearization of these equations. We then explicitly solve these equations to find the secular rate of change of the asteroid's rotation state at a particular value of true anomaly. Then the resulting equations are averaged over one asteroid orbit to find the evolutionary equations for an asteroid's rotate state subject to the YORP effect. The derivation here completes the derivation given in Scheeres (2007) as it finds the evolutionary equations for a uniformly rotating asteroid in closed form as a function of its YORP torque coefficients.

3.1 General rotational dynamics equations

We use the Euler equations to describe the evolution of the asteroid angular velocity:

$$\mathbf{I}\cdot\dot{\mathbf{\Omega}} = -\mathbf{\Omega}\times\mathbf{I}\cdot\mathbf{\Omega} + \mathbf{M}$$

where $\mathbf{I}$ represents the inertia dyad of the asteroid, $\mathbf{\Omega}$ is the angular velocity vector in the body-fixed frame, and $\mathbf{M}$ is the moment vector in the body-fixed frame. Note that in this paper we use a dyadic notation for the inertia and the transformation matrix. This allows us to state these quantities without explicitly specifying the coordinate frame. They also allow for left and right dot products and cross products with these quantities to be well defined. Properties of dyads are summarized in Greenwood (2003).

The orientation of an asteroid can be represented by an axis of rotation $\hat{\mathbf{a}}$, and a rotation angle about this axis, ϕ. To derive a system of equations of motion without singularity we use the Euler parameters:

$$\boldsymbol{\epsilon} = \sin\left(\frac{\phi}{2}\right)\hat{\mathbf{a}}$$
$$\eta = \cos\left(\frac{\phi}{2}\right)$$

The equations of motion for these parameters (relative to the body fixed frame) are in the following form (Greenwood 2003).

$$\dot{\boldsymbol{\epsilon}} = -\frac{1}{2}\Omega \times \boldsymbol{\epsilon} + \frac{1}{2}\eta\Omega \tag{6}$$

$$\dot{\eta} = -\frac{1}{2}\Omega \cdot \boldsymbol{\epsilon} \tag{7}$$

Given the Euler parameters we can compute the transformation matrix $\mathbf{T}$, which takes a vector in the body fixed frame and expresses it in an inertial frame (Greenwood 2003, Scheeres 2007).

$$\mathbf{T} = \mathbf{U} + 2\tilde{\boldsymbol{\epsilon}} \cdot \tilde{\boldsymbol{\epsilon}} + 2\eta\tilde{\boldsymbol{\epsilon}} \tag{8}$$

Thus, if the asteroid's initial rotation pole is $\hat{\mathbf{z}}_o$, to find the asteroid's rotate pole at a later time in inertial space we compute $\mathbf{T} \cdot \hat{\mathbf{z}}_o$. The quantity $\mathbf{U}$ is the identity dyad and the quantity $\tilde{\mathbf{n}}$ can be called the cross product dyad and, given $\mathbf{n} = n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}$:

$$\tilde{\mathbf{n}} = n_x\left(\hat{\mathbf{z}}\hat{\mathbf{y}} - \hat{\mathbf{y}}\hat{\mathbf{z}}\right) + n_y\left(\hat{\mathbf{x}}\hat{\mathbf{z}} - \hat{\mathbf{z}}\hat{\mathbf{x}}\right) + n_z\left(\hat{\mathbf{y}}\hat{\mathbf{x}} - \hat{\mathbf{x}}\hat{\mathbf{y}}\right) \tag{9}$$

Given the cross product $\mathbf{a} \times \mathbf{b}$, using the dyad formulation we note the following equivalences: $\mathbf{a} \times \mathbf{b} = \tilde{\mathbf{a}} \cdot \mathbf{b} = \mathbf{a} \cdot \tilde{\mathbf{b}}$

3.2 Linearization of the general equations

We make an assumption that the asteroid is rotating about its maximum moment of inertia and that the radiation torques are very small. This allows us to introduce a linearization approach to our analysis. We assume that the angular velocity vector can be described as $\Omega = \omega_o\hat{\mathbf{Z}} + \boldsymbol{\omega}$ where $|\boldsymbol{\omega}| << \omega_o$. We consider only the case where the asteroid is rotating uniformly and fast enough such that the rotation angle $\phi = \omega_o t$ varies more rapidly than the true anomaly ν. For an asteroid with a semi-major axis of 1AU, a rotation period of 3.5 days, which is long compared to the average rotation period of asteroids, will still provide two orders of magnitude difference between the rotation rate of the asteroid and the mean motion of the asteroid.

Linearizing Euler's equation about the nominal $\omega_o\hat{\mathbf{z}}$ solution we have:

$$\mathbf{I} \cdot \dot{\boldsymbol{\omega}} = -\omega_o\hat{\mathbf{z}} \times \mathbf{I} \cdot \boldsymbol{\omega} - \omega_o\boldsymbol{\omega} \times \mathbf{I} \cdot \hat{\mathbf{z}} + \mathbf{M} \tag{10}$$

Or:

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \end{bmatrix} = \mathbf{A}_\sigma \begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} + \begin{bmatrix} \frac{M_x}{I_x} \\ \frac{M_y}{I_y} \end{bmatrix} \tag{11}$$

$$\dot{\omega}_z = \frac{M_z}{I_z} \tag{12}$$

where

$$\mathbf{A}_\sigma = \begin{bmatrix} 0 & -\omega_o \sigma_x \\ \omega_o \sigma_y & 0 \end{bmatrix}$$

$$\sigma_x = \frac{I_z - I_y}{I_x} \quad (13)$$

$$\sigma_y = \frac{I_z - I_x}{I_y} \quad (14)$$

$$\sigma^2 = \sigma_x \sigma_y \quad (15)$$

and I_x, I_y and I_z are the minimum, intermediate and maximum moments of inertia, respectively.

We can rewrite the differential equation for ω_x and ω_y in the following form (Scheeres 2007):

$$\dot{\boldsymbol{\omega}}_\perp = \mathbf{A}_\sigma \boldsymbol{\omega}_\perp + \mathbf{I}_\perp^{-1} \mathbf{M}_\perp(t) \quad (16)$$

where $\boldsymbol{\omega}_\perp = [\omega_x; \omega_y]$, $\mathbf{M}_\perp = [M_x; M_y]$, $\mathbf{I}_\perp$ is the 2×2 diagonal matrix containing I_x and I_y, and the matrix $\mathbf{A}_\sigma$ is constant.

Now we investigate the dynamics of asteroid orientation by deriving the linearized evolution of the Euler parameters. We define $\hat{\mathbf{a}}_0$ to be the initial spin axis, choosing it equal to $\hat{\mathbf{z}}$, and we denote $\mathbf{a}_\perp$ to be the first order deviation from the spin axis, where $\hat{\mathbf{a}}_0 \cdot \mathbf{a}_\perp = 0$. We note that if $|\mathbf{a}_\perp| \ll 1$ then $\hat{\mathbf{a}} = \hat{\mathbf{a}}_0 + \mathbf{a}_\perp$ will still be of unit magnitude. Thus we have the following equation for the rotation angle:

$$\dot{\phi} = \hat{\mathbf{a}} \cdot \Omega = (\hat{\mathbf{a}}_0 + \mathbf{a}_\perp) \cdot ((\omega_o + \omega_z)\hat{\mathbf{z}} + \boldsymbol{\omega}_\perp) = \omega_o + \omega_z + \cdots \quad (17)$$

Given our decomposition of $\boldsymbol{\Omega} = (\omega_o + \omega_z)\hat{\mathbf{z}} + \boldsymbol{\omega}_\perp$, we rewrite the differential equation for the vector $\boldsymbol{\epsilon}$ (Eq. 6) as:

$$\dot{\boldsymbol{\epsilon}} = -\frac{\omega_o + \omega_z}{2} \left[\tilde{\hat{\mathbf{z}}} \cdot \boldsymbol{\epsilon} - \eta \hat{\mathbf{z}} \right] - \frac{1}{2} \tilde{\boldsymbol{\omega}}_\perp \cdot \boldsymbol{\epsilon} + \frac{1}{2} \eta \boldsymbol{\omega}_\perp \quad (18)$$

Next, we can decompose the Euler parameter vector as $\boldsymbol{\epsilon} = \sin(\phi/2) \left[\hat{\mathbf{a}} + \mathbf{a}_\perp \right] = \boldsymbol{\epsilon}_o + \boldsymbol{\epsilon}_\perp$. We note that $\boldsymbol{\omega}_\perp$ and $\boldsymbol{\epsilon}_\perp$ are mutually orthogonal to $\hat{\mathbf{z}}$ but are not necessarily parallel to each other, however their mutual dot product can be ignored as a second order term. Inserting and keeping terms of the appropriate order, we find:

$$\dot{\boldsymbol{\epsilon}}_\perp = -\frac{\omega_o}{2} \tilde{\hat{\mathbf{z}}} \cdot \boldsymbol{\epsilon}_\perp + \frac{1}{2} \tilde{\boldsymbol{\epsilon}}_o \cdot \boldsymbol{\omega}_\perp + \frac{\eta}{2} \boldsymbol{\omega}_\perp \quad (19)$$

where the differential equation describing the evolution of the nominal spin axis, $\dot{\boldsymbol{\epsilon}}_o = \frac{1}{2} \omega_o \eta \hat{\mathbf{z}}$, drops out. In the following we can rewrite this vector equation in a two-dimensional form as the deviation of the spin axis along the z-direction can be ignored. In conjunction with this we introduce the following notation: $\tilde{\hat{\mathbf{z}}} = -\mathbf{J}$, where

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (20)$$

Rewriting this equation into a two-dimensional form gives us:

$$\begin{bmatrix} \dot{\epsilon}_x \\ \dot{\epsilon}_y \end{bmatrix} = \frac{\omega_o}{2}\mathbf{J}\begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix} + \frac{1}{2}e^{-\mathbf{J}\phi/2}\begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} \tag{21}$$

$$e^{-\mathbf{J}\phi/2} = \begin{bmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{bmatrix} \tag{22}$$

where we note that the combination $[\cos(\phi/2)\mathbf{U} - \sin(\phi/2)\mathbf{J}] = e^{-\mathbf{J}\phi/2}$, and $\mathbf{U}$ is the 2 by 2 identity matrix.

To compute the transformation matrix from the body-fixed frame to the inertial frame, first express the Euler parameter vector as $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_o + \boldsymbol{\epsilon}_\perp$. Then insert this into Eq. 8 and ignore second order terms to find:

$$\mathbf{T} = \mathbf{U} + 2\tilde{\boldsymbol{\epsilon}}_o \cdot \tilde{\boldsymbol{\epsilon}}_o + 2\eta\tilde{\boldsymbol{\epsilon}}_o + 2\left[\tilde{\boldsymbol{\epsilon}}_o \cdot \tilde{\boldsymbol{\epsilon}}_\perp + \tilde{\boldsymbol{\epsilon}}_\perp \cdot \tilde{\boldsymbol{\epsilon}}_o + \eta\tilde{\boldsymbol{\epsilon}}_\perp\right] + \ldots \tag{23}$$

Specifically, we want to find the inertial attitude of the spin pole and its deviation from the original pole direction. To find this we compute the dot product of the body-fixed spin pole $\hat{\mathbf{z}}_o$ with $\mathbf{T}$. Given that $\tilde{\boldsymbol{\epsilon}}_o = \sin(\phi/2)\tilde{\hat{\mathbf{z}}}_\mathbf{o}$, this results in a great simplification given that $\tilde{\mathbf{z}} \cdot \hat{\mathbf{z}} = 0$. The evolved orientation of the spin pole in inertial space is denoted as $\hat{\mathbf{h}} = \mathbf{T} \cdot \hat{\mathbf{z}}_o$ and found as:

$$\hat{\mathbf{h}} = \hat{\mathbf{z}}_o + 2\left[\tilde{\boldsymbol{\epsilon}}_o \cdot \tilde{\boldsymbol{\epsilon}}_\perp + \eta\tilde{\boldsymbol{\epsilon}}_\perp\right] \cdot \hat{\mathbf{z}}_o \tag{24}$$

3.3 Solutions of the linearized equations and secular rates

As all of the equations have been reduced to a linear form, their explicit solution can be found. We first state their general solution and then extract the non-periodic, or secular, terms of these equations. Once these are found, we can evaluate the secular rate of change of the parameter over one rotation period by directly differentiating the secular solution.

In all of these solutions we note the spherical harmonic series representation of $\mathbf{M}$:

$$\frac{\mathbf{M}(\delta_s, \lambda_s)}{P(R)} = \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_l^m(\sin(\delta_s)\{\mathbf{C}_{l,m}\cos(m\lambda_s) + \mathbf{D}_{l,m}\sin(m\lambda_s)\})$$

where $\lambda_s = \lambda_0 - \omega_o t$, and $\lambda_0 = \Omega_{s_o} + \lambda_\nu + \phi_{lag}$. Under the linearized assumption, and at a fixed true anomaly, the only term in the moment expression that changes in time is the angle $\omega_o t$, thus it is useful to rewrite the spherical harmonic expansion explicitly as a function of time:

$$\frac{\mathbf{M}(\delta_s, \lambda_s)}{P(R)} = \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_l^m(\sin(\delta_s)\{\mathbf{C}^0_{l,m}\cos(m\omega_o t) + \mathbf{D}^0_{l,m}\sin(m\omega_o t)\})$$

where

$$\mathbf{C}^0_{l,m} = \mathbf{C}_{l,m}\cos(m\lambda_0) + \mathbf{D}_{l,m}\sin(m\lambda_0)$$

$$\mathbf{D}^0_{l,m} = \mathbf{C}_{l,m}\sin(m\lambda_0) - \mathbf{D}_{l,m}\cos(m\lambda_0)$$

where we specifically note that $\mathbf{C}^0_{l,0} = \mathbf{C}_{l,0}$.

Now we state the solutions to the linear equations. It is easy to solve the equation for ω_z:

$$\omega_z = \omega_o + \frac{1}{I_z}\int_0^t M_z d\tau \tag{25}$$

We note that the variations in the rotation rate will be periodic except for the secular terms:

$$\omega_z^s = \omega_o + t\frac{P(R)}{I_z}\sum_{l=0}^{\infty} P_l^0(\sin(\delta_s))C_{l,0,z} \tag{26}$$

where the superscript "s" denotes the non-periodic, or secular, component of the solution. We can compute the rate of change in ω_z^s by taking the time derivative of this secular term to find:

$$\dot{\omega}_z^s = \frac{P(R)}{I_z}C_{0,z} \tag{27}$$

where:

$$\mathbf{C}_0 = \sum_{l=0}^{\infty} \mathbf{C}_{l,0} P_l^0(\sin(\delta_s))$$

We note that the secular change in rotation rate at a given distance from the Sun is independent of λ_0 and thus is independent of the thermal lag parameter.

Next, we compute the off-axis component of angular velocity. We will see that we can neglect this for the secular evolution of the angular velocity, however they become important for the evolution of the asteroid rotation pole. Eq. 16 has the following solution:

$$\boldsymbol{\omega}_\perp = e^{\mathbf{A}_\sigma t}\boldsymbol{\omega}_{\perp,0} + \int_0^t e^{\mathbf{A}_\sigma (t-\tau)}\mathbf{I}_\perp^{-1}\mathbf{M}_\perp(\tau)d\tau \tag{28}$$

where $\omega_{\perp,0}$ is the initial condition for the angular velocity and $e^{\mathbf{A}_\sigma t}$ is the exponential matrix:

$$e^{\mathbf{A}_\sigma t} = \begin{bmatrix} \cos(\omega_o \sigma t) & -\frac{\sigma_x}{\sigma}\sin(\omega_o\sigma t) \\ -\frac{\sigma_y}{\sigma}\sin(\omega_o\sigma t) & \cos(\omega_o\sigma t) \end{bmatrix} \tag{29}$$

Noting that $0 < \sigma < 1$ and using the spherical harmonics expansion of $\mathbf{M}$, we can verify that all of the terms in the integral will be periodic and there are no small divisors. Thus there is no secular term present and the off-axis rotational dynamics of the asteroid can be neglected at this stage as being small and quasi-periodic with frequencies of the form $(n \pm \sigma)$ and n.

Next we move on to the orientation equations of the asteroid. First consider the solution to Eq. 17. If we assume that the secular rate of change of the angular velocity keeps constant, we can derive the following expression for ϕ:

$$\phi^s = \phi_0 + \omega_o(t - t_0) + \frac{(t-t_0)^2}{2}\dot{\omega}_z^s + \cdots \tag{30}$$

We note that the rate of acceleration is small enough to allow us to treat the angle as linearly increasing in time over short time periods. Using this expression we can compute $|\boldsymbol{\epsilon}|$ and η.

The equation for the evolution of $\boldsymbol{\epsilon}_\perp$, Eq. 19, falls into the same canonical form as Eq. 16 for the angular velocities, but now the resulting matrix exponential is of the form:

$$e^{\mathbf{J}\phi/2} = \begin{bmatrix} \cos(\phi/2) & \sin(\phi/2) \\ -\sin(\phi/2) & \cos(\phi/2) \end{bmatrix} \tag{31}$$

where we use $\omega_o t = \phi$. Combining terms and taking the angle ϕ as our independent parameter for the angular velocity evolution we find the solution to be of the form:

$$\boldsymbol{\epsilon}_\perp = e^{\mathbf{J}\phi/2}\boldsymbol{\epsilon}_{\perp_o} + \frac{e^{\mathbf{J}\phi/2}}{2\omega_o}\int_0^\phi e^{-\mathbf{J}\phi'}\boldsymbol{\omega}_\perp(\phi'/\omega_o)d\phi' \tag{32}$$

where we note that $e^{-\mathbf{J}\phi}$ has entries $\cos\phi$ and $\sin\phi$. From our previous discussion the solution for $\boldsymbol{\omega}_\perp$ has combinations of frequencies of the form $(n \pm \sigma)$ and n and thus there is now a resonance within the integral for $n = 1$, causing a secular drift in the Euler parameters. Specifically, the $\cos\phi$ and $\sin\phi$ terms in the $e^{-\mathbf{J}\phi}$ matrix will combine with the $n = 1$ terms in the $\boldsymbol{\omega}_\perp$ solution, giving rise to a linear drift in the rotation axis with time.

The first term in Eq. 32 is always periodic and hence we neglect it as small in general. In the second term, we substitute the general solution for the angular velocity, Eq. 28. To have a more specific focus, we consider the general nth order Fourier coefficient terms to find:

$$P(R)\frac{e^{\mathbf{J}\phi/2}}{2\omega_o^2}\int_0^\phi\int_0^{\phi'} e^{-\mathbf{J}\phi'}\begin{bmatrix}\cos\sigma(\phi''-\phi') & \frac{\sigma_x}{\sigma}\sin\sigma(\phi''-\phi') \\ -\frac{\sigma_y}{\sigma}\sin\sigma(\phi''-\phi') & \cos\sigma(\phi''-\phi')\end{bmatrix}$$
$$\times\left[\cos(n\phi'')\mathbf{I}_\perp^{-1}\mathbf{C}_{\perp,n}^0 + \sin(n\phi'')\mathbf{I}_\perp^{-1}\mathbf{D}_{\perp,n}^0\right]d\phi''d\phi' \tag{33}$$

where

$$\mathbf{C}_n = \sum_{l=1}^{\infty}\mathbf{C}_{l,n}P_l^n(\sin(\delta_s))$$

$$\mathbf{D}_n = \sum_{l=1}^{\infty}\mathbf{D}_{l,n}P_l^n(\sin(\delta_s))$$

and the $\perp$ sub-script signifies that only the x and y components are used.

When considering all the possible contributions from terms of this type, we find that only when $n = 1$ do we have a secular term. Specifically, only terms of the form $\int_0^\phi\int_0^{\phi'}\mathrm{cs}(\phi''-\phi')\mathrm{sc}\sigma(\phi''-\phi')d\phi''d\phi'$, yield a secular term of the form $\sigma\phi/(1-\sigma^2)$ or $-\phi/(1-\sigma^2)$, corresponding to the combination $\cos\phi\sin\sigma\phi$ and $\sin\phi\cos\sigma\phi$, respectively. All other contributions are sinusoidal in nature and will only contribute to an oscillation in the spin pole but not a secular drift. Combining all the terms, and re-introducing the Fourier coefficients, we have the explicit solution for the secular component of the vector $\boldsymbol{\epsilon}_\perp$, now recast in a 3-vector form:

$$\boldsymbol{\epsilon}_\perp^s = \frac{\phi P(R)}{4\omega_o^2(1-\sigma^2)}e^{\mathbf{J}\phi/2}\begin{bmatrix}-\frac{1-\sigma_x}{I_y}C_{1,y}^0 & -\frac{1-\sigma_y}{I_x}D_{1,x}^0 \\ -\frac{1-\sigma_x}{I_y}D_{1,y}^0 & +\frac{1-\sigma_y}{I_x}C_{1,x}^0\end{bmatrix} \tag{34}$$

Using the definitions for σ_x, σ_y and σ^2 from Eqs. 13 to 15 we note the following identities:

$$\frac{1-\sigma_x}{1-\sigma^2}\frac{1}{I_y} = \frac{1-\sigma_y}{1-\sigma^2}\frac{1}{I_x} = \frac{1}{I_z} \tag{35}$$

Thus we can simplify the expression further to:

$$\boldsymbol{\epsilon}_\perp^s = \frac{\phi P(R)}{4\omega_o^2 I_z}e^{\mathbf{J}\phi/2}\begin{bmatrix}-C_{1,y}^0 & -D_{1,x}^0 \\ -D_{1,y}^0 & +C_{1,x}^0\end{bmatrix} \tag{36}$$

We use this result to compute the transformation matrix that defines the movement of the asteroid's angular momentum.

$$\hat{\mathbf{h}}^s = \hat{\mathbf{z}}_o + 2\left[\cos(\phi/2)\mathbf{U} - \sin(\phi/2)\mathbf{J}\right] \cdot \tilde{\boldsymbol{\epsilon}}^s_\perp \cdot \hat{\mathbf{z}}_o \tag{37}$$

where $\mathbf{U}$ is again the 2 by 2 identity matrix. Applying the cross product identities to the above $\tilde{\boldsymbol{\epsilon}}^s_\perp \cdot \hat{\mathbf{z}}_o = -\tilde{\hat{\mathbf{z}}}_o \cdot \boldsymbol{\epsilon}^s_\perp = \mathbf{J} \cdot \boldsymbol{\epsilon}^s_\perp$, given our earlier definition $\tilde{\hat{\mathbf{z}}}_o = -\mathbf{J}$. Thus we find the general expression for the spin pole in the inertial frame at a specific value of true anomaly:

$$\hat{\mathbf{h}}^s = \hat{\mathbf{z}}_o + 2e^{-\mathbf{J}\phi/2} \cdot \mathbf{J} \cdot \boldsymbol{\epsilon}^s_\perp \tag{38}$$

$$= \hat{\mathbf{z}}_o + \frac{\phi P(R)}{2\omega_o^2 I_z} e^{-\mathbf{J}\phi/2} \cdot \mathbf{J} \cdot e^{\mathbf{J}\phi/2} \begin{bmatrix} -C^0_{1,y} & -D^0_{1,x} \\ -D^0_{1,y} & +C^0_{1,x} \end{bmatrix} \tag{39}$$

An additional simplification can be made, noting the identity $e^{-\mathbf{J}\phi/2} \cdot \mathbf{J} \cdot e^{\mathbf{J}\phi/2} = \mathbf{J}$. This results in the final, explicit solution for the secular evolution of the rotation pole with respect to the inertial frame:

$$\hat{\mathbf{h}}^s = \hat{\mathbf{z}}_o + \frac{\phi P(R)}{2\omega_o^2 I_z} \mathbf{J} \cdot \begin{bmatrix} -C^0_{1,y} & -D^0_{1,x} \\ -D^0_{1,y} & +C^0_{1,x} \end{bmatrix} \tag{40}$$

Of particular interest is the secular time rate of change of the rotation pole. This can be found by formally differentiating Eq. 40 with respect to time. The only time-varying quantity (assuming a fixed true anomaly) is the angle ϕ^s, with $\dot{\phi}^s = \omega_o$. Thus, the secular rate of change in $\hat{\mathbf{h}}^s$ is:

$$\dot{\hat{\mathbf{h}}}^s = \frac{P(R)}{2\omega_o I_z} \mathbf{J} \cdot \begin{bmatrix} -C^0_{1,y} & -D^0_{1,x} \\ -D^0_{1,y} & +C^0_{1,x} \end{bmatrix} \tag{41}$$

This is linear in the Fourier coefficients, which allows us to add the different components of radiation forces together to find the net rate of change in this quantity.

At this point we substitute for C^0_n and D^0_n into Eq. 41 for the current case of $n = 1$ to give the secular rate of change in the rotation pole in terms of the body-fixed Fourier coefficients. Substituting these relations yields:

$$\dot{\hat{\mathbf{h}}}^s = \frac{P(R)}{2\omega_o I_z} \begin{bmatrix} (C_{1,x} + D_{1,y}) & (D_{1,x} - C_{1,y}) \\ -(D_{1,x} - C_{1,y}) & (C_{1,x} + D_{1,y}) \end{bmatrix} \begin{bmatrix} \cos\lambda_o \\ \sin\lambda_o \end{bmatrix} \tag{42}$$

3.4 Secular variation of the rotation state

Given the secular solutions for $\hat{\mathbf{h}}$ and $\dot{\hat{\mathbf{h}}}$ (note, we will drop the "s" superscript now), we derive how they affect the obliquity and right ascension of the asteroid rotation pole, or in terms of our formulation how they change the solar inclination, i_s, and longitude, Ω_s. To derive equations for the secular rate of change of these quantities we can refer back to their definitions in Sect. 2. Specifically, note that if we take the cross product of the current asteroid rotation pole, $\hat{\mathbf{h}}$, and the orbit pole, $\hat{\mathbf{Z}}$, that we form the current node vector multiplied by $\sin i_s$:

$$\hat{\mathbf{h}} \times \hat{\mathbf{Z}} = \sin i_s \hat{\mathbf{n}}_{\Omega_s} \tag{43}$$

$$\hat{\mathbf{n}}_{\Omega_s} = \cos\Omega_s \hat{\mathbf{x}} + \sin\Omega_s \hat{\mathbf{y}} \tag{44}$$

Now consider the time derivative of this, noting that the orbit pole $\hat{\mathbf{Z}}$ does not move relative to inertial space:

$$\dot{\mathbf{h}} \times \hat{\mathbf{Z}} = \cos i_s \hat{\mathbf{n}}_{\Omega_s} \dot{i}_s + \sin i_s \frac{\partial \hat{\mathbf{n}}_{\Omega_s}}{\partial \Omega_s} \dot{\Omega}_s \tag{45}$$

$$\frac{\partial \hat{\mathbf{n}}_{\Omega_s}}{\partial \Omega_s} = -\sin \Omega_s \hat{\mathbf{x}} + \cos \Omega_s \hat{\mathbf{y}} \tag{46}$$

We can extract unique relationships for $\dot{i}_s$ and $\dot{\Omega}_s$ from this equation as they are each multiplied by a mutually orthogonal vector.

First, take the dot product of this equation with respect to $\hat{\mathbf{n}}_{\Omega_s}$. The second term containing $\dot{\Omega}_s$ will vanish and the leading term on the right-hand side just equals $\cos i_s \dot{i}_s$. Remaining to resolve is the term $\hat{\mathbf{n}}_{\Omega_s} \cdot \left(\dot{\mathbf{h}} \times \hat{\mathbf{Z}}\right) = \dot{\mathbf{h}} \cdot \left(\hat{\mathbf{Z}} \times \hat{\mathbf{n}}_{\Omega_s}\right)$. But the cross product of $\hat{\mathbf{Z}}$ and $\hat{\mathbf{n}}_{\Omega_s}$ equals the transverse vector $\hat{\mathbf{n}}_T$, or

$$\hat{\mathbf{Z}} \times \hat{\mathbf{n}}_{\Omega_s} = -\cos i_s \sin \Omega_{s_o} \hat{\mathbf{x}} + \cos i_s \cos \Omega_{s_o} \hat{\mathbf{y}} + \sin i_s \hat{\mathbf{z}} \tag{47}$$

The dot product of $\dot{\mathbf{h}}$ with this will not pick up the $\hat{\mathbf{z}}$ term, and can be rewritten as:

$$\cos i_s \dot{i}_s = \cos i_s \hat{\mathbf{n}}_{\Omega_{s_o}} \mathbf{J} \dot{\mathbf{h}} \tag{48}$$

The $\cos i_s$ terms cancel each other, unless $i_s = \pi/2$. However, in this case an alternate derivation finds the same secular evolution equation for i_s. Carrying out the multiplications and simplifications yields:

$$\dot{i}_s = \frac{P(R)}{2\omega_o I_z} \left[-\left(D_{1,x} - C_{1,y}\right) \cos(\lambda_o - \Omega_{s_o}) + \left(C_{1,x} + D_{1,y}\right) \sin(\lambda_o - \Omega_{s_o})\right] \tag{49}$$

Now, recall the general expressions for λ_o:

$$\lambda_o = \Omega_{s_o} + \lambda_\nu + \phi_{lag} \tag{50}$$

Separating the λ_ν and ϕ terms we find:

$$\begin{aligned} \dot{i}_s = \frac{P(R)}{2\omega_o I_z} &\left\{\left[-\left(D_{1,x} - C_{1,y}\right) \cos(\phi_{lag}) + \left(C_{1,x} + D_{1,y}\right) \sin(\phi_{lag})\right] \cos(\lambda_\nu)\right. \\ &\left.+ \left[\left(C_{1,x} + D_{1,y}\right) \cos(\phi_{lag}) + \left(D_{1,x} - C_{1,y}\right) \sin(\phi_{lag})\right] \sin(\lambda_\nu)\right\} \end{aligned} \tag{51}$$

A similar derivation can be made for $\dot{\Omega}_s$, now taking the dot product of Eq. 45 with $\partial \hat{\mathbf{n}}_{\Omega_s} / \partial \Omega_s$ to find:

$$\sin i_s \dot{\Omega}_s = \dot{\mathbf{h}} \cdot \left(\hat{\mathbf{Z}} \times \frac{\partial \hat{\mathbf{n}}_{\Omega_s}}{\partial \Omega_s}\right) \tag{52}$$

$$= -\cos i_s \dot{\mathbf{h}} \cdot \hat{\mathbf{n}}_{\Omega_s} \tag{53}$$

which leads to:

$$\begin{aligned} \dot{\Omega}_s = -\cot(i_s) \frac{P(R)}{2\omega_o I_z} &\left[\left(C_{1,x} + D_{1,y}\right) \cos(\lambda_o - \Omega_{s_o})\right. \\ &\left.+ \left(D_{1,x} - C_{1,y}\right) \sin(\lambda_o - \Omega_{s_o})\right] \end{aligned} \tag{54}$$

Making final substitutions and rewriting in terms of the longitude λ_ν we find:

$$\begin{aligned}\dot{\Omega}_s = &-\cot(i_s)\frac{P(R)}{2\omega_o I_z}\\ &\times\left\{\left[\left(C_{1,x}+D_{1,y}\right)\cos(\phi_{lag})+\left(D_{1,x}-C_{1,y}\right)\sin(\phi_{lag})\right]\cos(\lambda_\nu)\right.\\ &\left.-\left[-\left(D_{1,x}-C_{1,y}\right)\cos(\phi_{lag})+\left(C_{1,x}+D_{1,y}\right)\sin(\phi_{lag})\right]\sin(\lambda_\nu)\right\}\end{aligned} \tag{55}$$

We recall once more the definition of these coefficients:

$$\mathbf{C}_1 = \sum_{l=1}^{\infty}\mathbf{C}_{l,1}P_l^1(\sin(\delta_s))$$

$$\mathbf{D}_1 = \sum_{l=1}^{\infty}\mathbf{D}_{l,1}P_l^1(\sin(\delta_s))$$

We note that not all the coefficients will have the same thermal lag. In general we need to separate out the insolation and specular and diffuse reflection terms and only apply the thermal lag to the reemission term. However, previous studies have noted that the reemission term should dominate the YORP torques, implying that only these coefficients need be accounted for (Rubincam 1995, Vokrouhlický and Čapek 2002).

Thus, we find the explicit evolutionary equations for the rotation rate and the rotation pole at a fixed true anomaly relative to the Sun, represented by Eqs. 27, 51, and 55. We note that the Fourier coefficients $\mathbf{C}_1$ and $\mathbf{D}_1$ are currently a function of solar latitude which in turn is a function of true anomaly, and thus the next step in our analysis is to perform an averaging of these equations over the asteroid true anomaly relative to the Sun.

3.5 Orbit averaging

To evaluate the net effect of YORP over one asteroid year we average the preceding equations over the mean anomaly of the heliocentric orbit. This is a valid approach to derive the mean evolutionary equations of the asteroid spin state given the small change in rotation rate and asteroid rotation pole over a single orbit about the Sun. In this integration we consider the changing distance of the asteroid to the Sun, as well as the changing solar latitude and longitude. The relation between solar latitude, solar longitude and true anomaly is stated again below:

$$\sin(\delta_s) = \sin(i_s)\sin(\varpi_s+\nu) \tag{56}$$

$$\sin(\lambda_s) = \frac{\cos(i_s)\sin(\varpi_s+\nu)}{\cos\delta_s} \tag{57}$$

$$\cos(\lambda_s) = \frac{\cos(\varpi_s+\nu)}{\cos\delta_s} \tag{58}$$

The time rate of change of the dynamical variables we study here are of the form, $\dot{x} = P(R)f(x,\varpi_s+\nu)$, where $P(R)$ is the solar pressure term, R is the asteroid heliocentric orbit radius, and $f(x,\varpi_s+\nu)$ is a function of the state x, the argument of the perihelion ϖ_s and the true anomaly ν. The magnitude of $\dot{x}$ being very small we average $\dot{x}$ over one year to obtain the net rate of change in x:

$$\dot{\bar{x}} = \frac{1}{2\pi}\int_0^{2\pi}\frac{G_1}{R^2}f(x,\varpi_s+\nu)dM \tag{59}$$

where M is the mean anomaly of the asteroid in its orbit. We use $dM = \frac{R^2}{a^2\sqrt{1-e^2}}d\nu$ to transform the independent parameter to true anomaly.

$$\dot{\bar{x}} = \frac{1}{2\pi}\frac{G_1}{a^2\sqrt{1-e^2}}\int_0^{2\pi} f(x,\nu')d\nu' \tag{60}$$

where $\nu' = \varpi_s + \nu$. Note that this change of integration variable eliminates the varying radius in the solar pressure term. We should note that in performing this average we ignore any possible variation in the equilibrium temperature of the asteroid as a function of its location in orbit, which in turn could modify the phase lag in a systematic way. However, it should not alter the basic functional form of the phase lag we are assuming.

We consider the equations in turn, first concentrating on $\dot{\omega}_z$.

$$\dot{\bar{\omega}}_z = \frac{G_1}{I_z a^2\sqrt{1-e^2}}\bar{C}_{0,z} \tag{61}$$

where

$$\begin{aligned}\bar{C}_{0,z} &= \frac{1}{2\pi}\int_0^{2\pi} C_{0,z}d\nu' = \frac{1}{2\pi}\int_0^{2\pi}\sum_{l=0}^{\infty} C_{l,0,z}P_l^0(\sin(\delta_s))d\nu' \\ &= \frac{1}{2\pi}\sum_{l=0}^{\infty} C_{l,0,z}\int_0^{2\pi} P_l^0(\beta\sin(\nu'))d\nu'\end{aligned}$$

We replace the Legendre function by its polynomial representation (Kaula 2000):

$$P_l^0(x) = \frac{1}{2^l}\sum_{k=0}^{\lfloor\frac{l}{2}\rfloor}(-1)^k\binom{l}{k}\binom{2l-2k}{l}x^{l-2k}$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ and $\lfloor m\rfloor$ is the integer part of m. So

$$\bar{C}_{0,z} = \frac{1}{2\pi}\sum_{l=0}^{\infty} C_{l,0,z}\left[\frac{1}{2^l}\sum_{k=0}^{\lfloor\frac{l}{2}\rfloor}(-1)^k\binom{l}{k}\binom{2l-2k}{l}\beta^{l-2k}\int_0^{2\pi}\sin^{l-2k}(\nu')d\nu'\right]$$

So we have to compute $\int_0^{2\pi}\sin^{l-2k}(\nu')d\nu'$. To do this we make the following observations:

If l is odd $l-2k$ is odd too, and as sine is an odd function this integral is zero. If l is even, $l = 2t$ and we use the following formula that is a simple integration by parts:

$$\int_0^{2\pi}\sin^n(x)dx = \frac{-\sin^{n-1}x\cos x}{n}|_0^{2\pi} + \frac{n-1}{n}\int_0^{2\pi}\sin^{n-2}xdx = \frac{n-1}{n}\int_0^{2\pi}\sin^{n-2}xdx$$

so

$$\begin{aligned}\int_0^{2\pi}\sin^{2t-2k}(\nu')d\nu' &= \frac{2(t-k)-1}{2(t-k)}\cdot\frac{2(t-k)-3}{2(t-k)-2}\cdots\frac{1}{2}\cdot 2\pi \\ &= \frac{(2t-2k)!}{(t-k)!^2\cdot 2^{2(t-k)}}\cdot 2\pi\end{aligned}$$

Substituting this we find:

$$\bar{C}_{0,z} = \sum_{t=0}^{\infty} C_{2t,0,z}\left[\frac{1}{2^{2t}}\sum_{k=0}^{t}(-1)^k\frac{(4t-2k)!}{k!(2t-k)!(t-k)!^2}\left(\frac{\beta}{2}\right)^{2t-2k}\right] \tag{62}$$

To carry out the averaging for i_s and Ω_s we must compute the following:

$$\begin{aligned}\bar{\mathbf{C}}_1 &= \frac{1}{2\pi}\int_0^{2\pi} \sin\lambda_\nu \mathbf{C}_1 d\nu' = \frac{1}{2\pi}\int_0^{2\pi} \sin\lambda_\nu \left[\sum_{l=1}^{\infty} \mathbf{C}_{l,1} P_l^1(\sin(\delta_s))\right] d\nu' \\ &= \frac{1}{2\pi}\sum_{l=1}^{\infty} \mathbf{C}_{l,1}\int_0^{2\pi} \sin\lambda_\nu P_l^1(\sin(\delta_s)) d\nu'\end{aligned}$$

and similarly

$$\begin{aligned}\bar{\bar{\mathbf{C}}}_1 &= \frac{1}{2\pi}\sum_{l=1}^{\infty} \mathbf{C}_{l,1}\int_0^{2\pi} \cos\lambda_\nu P_l^1(\sin(\delta_s)) d\nu' \\ \bar{\mathbf{D}}_1 &= \frac{1}{2\pi}\sum_{l=1}^{\infty} \mathbf{D}_{l,1}\int_0^{2\pi} \sin\lambda_\nu P_l^1(\sin(\delta_s)) d\nu' \\ \bar{\bar{\mathbf{D}}}_1 &= \frac{1}{2\pi}\sum_{l=1}^{\infty} \mathbf{D}_{l,1}\int_0^{2\pi} \cos\lambda_\nu P_l^1(\sin(\delta_s)) d\nu'\end{aligned}$$

The fundamental computations to be made are:

$$\frac{1}{2\pi}\int_0^{2\pi} P_l^1(\sin(\delta_s)) \sin\lambda_\nu d\nu' \tag{63}$$

$$\frac{1}{2\pi}\int_0^{2\pi} P_l^1(\sin(\delta_s)) \cos\lambda_\nu d\nu' \tag{64}$$

where we have

$$\begin{aligned}P_l^1(x) &= (-1)(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_l^0(x) \\ &= \frac{-1}{2^l}(1-x^2)^{\frac{1}{2}} \sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor} (-1)^k \binom{l}{k}\binom{2l-2k}{l}(l-2k)x^{l-2k-1}\end{aligned}$$

With $x = \sin(\delta_s) = \beta\sin(\nu')$:

$$P_l^1(\sin(\delta_s)) = \frac{-1}{2^l}\cos(\delta_s) \sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor} (-1)^k \binom{l}{k}\binom{2l-2k}{l}(l-2k)(\beta\sin(\nu'))^{l-2k-1}$$

So

$$\begin{aligned}\bar{\mathbf{C}}_1 &= \frac{1}{2\pi}\sum_{l=1}^{\infty} \mathbf{C}_{l,1}\int_0^{2\pi} \sin\lambda_\nu P_l^1(\sin(\delta_s)) d\nu' \\ &= \frac{1}{2\pi}\sum_{l=1}^{\infty} \mathbf{C}_{l,1}\int_0^{2\pi} \frac{\cos(i_s)\sin(\nu')}{\cos(\delta_s)} \frac{-1}{2^l}\cos(\delta_s) \\ &\quad\times \sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor} (-1)^k \binom{l}{k}\binom{2l-2k}{l}(l-2k)(\beta\sin(\nu'))^{l-2k-1} d\nu' \\ &= \frac{-\cos(i_s)}{2\pi}\sum_{l=1}^{\infty} \mathbf{C}_{l,1}\frac{1}{2^l}\sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor} (-1)^k \binom{l}{k}\binom{2l-2k}{l}(l-2k)(\beta)^{l-2k-1}\int_0^{2\pi}\sin^{l-2k}(\nu') d\nu'\end{aligned}$$

Conversely:

$$\bar{\bar{\mathbf{C}}}_1 = \frac{1}{2\pi}\sum_{l=1}^{\infty}\mathbf{C}_{l,1}\int_0^{2\pi}\cos\lambda_\nu P_l^1(\sin(\delta_s))d\nu'$$
$$= \frac{1}{2\pi}\sum_{l=1}^{\infty}\mathbf{C}_{l,1}\int_0^{2\pi}\frac{\cos(\nu')}{\cos(\delta_s)}\frac{-1}{2^l}\cos(\delta_s)$$
$$\times\sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor}(-1)^k\binom{l}{k}\binom{2l-2k}{l}(l-2k)(\beta\sin(\nu'))^{l-2k-1}d\nu'$$
$$= \frac{-\cos(i_s)}{2\pi}\sum_{l=1}^{\infty}\mathbf{C}_{l,1}\frac{1}{2^l}\sum_{k=0}^{\lfloor\frac{l-1}{2}\rfloor}(-1)^k\binom{l}{k}\binom{2l-2k}{l}(l-2k)(\beta)^{l-2k-1}$$
$$\times\int_0^{2\pi}\cos(\nu')\sin^{l-2k-1}(\nu')d\nu'$$

We first note that $\int_0^{2\pi}\cos(\nu')\sin^{l-2k-1}(\nu')d\nu' = 0$ for all $l-2k-1 \geq 0$. We have already computed $\int_0^{2\pi}\sin^{l-2k}(\nu')d\nu'$ where we find it equals zero for l odd and for l even, $l = 2t$, it equals:

$$\int_0^{2\pi}\sin^{2t-2k}(\nu')d\nu' = \frac{(2t-2k)!}{(t-k)!^2\cdot 2^{2(t-k)}}\cdot 2\pi$$

Hence we have $\bar{\bar{\mathbf{C}}}_1 = \bar{\bar{\mathbf{D}}}_1 = 0$ and:

$$\bar{\mathbf{C}}_1 = -\cos(i_s)\sum_{t=1}^{\infty}\mathbf{C}_{2t,1}\frac{1}{2^{2t}}\left[\sum_{k=0}^{t-1}(-1)^k\frac{(4t-2k)!(t-k)}{k!(2t-k)!(t-k)!^2}\left(\frac{\beta}{2}\right)^{2t-2k-1}\right] \quad (65)$$

$$\bar{\mathbf{D}}_1 = -\cos(i_s)\sum_{t=1}^{\infty}\mathbf{D}_{2t,1}\frac{1}{2^{2t}}\left[\sum_{k=0}^{t-1}(-1)^k\frac{(4t-2k)!(t-k)}{k!(2t-k)!(t-k)!^2}\left(\frac{\beta}{2}\right)^{2t-2k-1}\right] \quad (66)$$

Thus we find the averaged evolutionary equations for the obliquity and the right ascension to be:

$$\dot{i}_s = \frac{G_1}{\omega_0 I_z a^2\sqrt{1-e^2}}\left[\left(\bar{C}_{1,x}+\bar{D}_{1,y}\right)\cos(\phi_{lag})+\left(\bar{D}_{1,x}-\bar{C}_{1,y}\right)\sin(\phi_{lag})\right] \quad (67)$$

$$\dot{\Omega}_s = -\frac{\cot(i_s)G_1}{\omega_0 I_z a^2\sqrt{1-e^2}}\left[-\left(\bar{D}_{1,x}-\bar{C}_{1,y}\right)\cos(\phi_{lag})+\left(\bar{C}_{1,x}+\bar{D}_{1,y}\right)\sin(\phi_{lag})\right] \quad (68)$$

Thus to completely describe the dynamical equations of 61, 67, and 68 it remains only to compute the spherical harmonic coefficients $\mathbf{C}_{l,m}$ and $\mathbf{D}_{l,m}$ for $m = 0, 1$. The computation of these harmonic coefficients is discussed in the Appendix for a given asteroid shape model.

3.6 Symmetry properties

The averaged coefficients that describe the rotational evolution of an asteroid have a number of symmetry properties as a function of the obliquity and as a function of in which sense the asteroid rotates.

3.6.1 Symmetry in obliquity

The simplest symmetry to observe in the coefficients is in the obliquity, i_s, about a value of $\pi/2$. We first note that the sine function is symmetric about $\pi/2$, $\sin(i_s) = \sin(\pi - i_s)$, and that the cosine function is odd about $\pi/2$, $\cos(i_s) = -\cos(\pi - i_s)$. We immediately note that the $\bar{\mathbf{C}}_0$ coefficients are only a function of $\sin i_s$ and thus are symmetric about this line. Next, we note that the coefficients $\bar{\mathbf{C}}_1$ and $\bar{\mathbf{D}}_1$ are functions of $\sin i_s$ and are both multiplied by $\cos i_s$. Thus, these coefficients are anti-symmetric in obliquity. Thus:

$$\bar{\mathbf{C}}_0(i_s) = \bar{\mathbf{C}}_0(\pi - i_s) \tag{69}$$

$$\bar{\mathbf{C}}_1(i_s) = -\bar{\mathbf{C}}_1(\pi - i_s) \tag{70}$$

$$\bar{\mathbf{D}}_1(i_s) = -\bar{\mathbf{D}}_1(\pi - i_s) \tag{71}$$

3.6.2 Symmetry in rotation sense

The computation of these coefficients assumes that the asteroid uniformly rotates about its maximum moment of inertia, and implicitly assumes one of two possible directions for the asteroid to rotate. Under long-term evolution we will see in the next section that, under the current model assumptions, it is possible for an asteroid's rotation rate to approach zero, and should that happen it is feasible that the body would commence rotating in the opposite direction. The actual dynamics are more complex and may involve perturbations from solar gravitational torques as well as non-uniform rotation. Nonetheless, it is of interest to recompute the relevant coefficients under the assumption that the asteroid rotates in the opposite sense. Doing so reveals some interesting symmetry properties of the coefficients.

To model this, we consider "flipping" the asteroid by 180 degrees about the x or y axis—for definiteness we consider a rotation about the x axis. Upon consideration, it is clear that the resulting moment equation should equal the original one, but with the change in spherical coordinates $\delta \rightarrow -\delta$ and $\lambda \rightarrow -\lambda$ and with the coefficient vectors being subject to the change $()_x \rightarrow ()_x$, $()_y \rightarrow -()_y$, and $()_z \rightarrow -()_z$, which can be accomplished by pre-multiplication by the rotation matrix:

$$T_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{72}$$

As the shape distribution of the asteroid does not change, the absolute values of the coefficients should likewise be unchanged. If we denote the Fourier coefficients for the flipped mass distribution as $\mathbf{C}'_{l,m}$ and $\mathbf{D}'_{l,m}$, we then have the identity:

$$\sum_{l=0}^{\infty}\sum_{m=0}^{l} P_l^m(\sin(\delta_s))\{\mathbf{C}'_{l,m}\cos(m\lambda_s) + \mathbf{D}'_{l,m}\sin(m\lambda_s)\}$$
$$= \sum_{l=0}^{\infty}\sum_{m=0}^{l} P_l^m(\sin(-\delta_s))\{T_x\mathbf{C}_{l,m}\cos(-m\lambda_s) + T_x\mathbf{D}_{l,m}\sin(-m\lambda_s)\}$$

The associated Legendre functions have the symmetry property $P_l^m(-x) = (-1)^{l-m}P_l^m(x)$ and the sine and cosine functions have the symmetry property $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$. Thus, for a body rotating in the opposite sense the solar torque coefficients can be defined to be:

$$\mathbf{C}'_{l,m} = (-1)^{l-m} T_x \mathbf{C}_{l,m} \tag{73}$$
$$\mathbf{D}'_{l,m} = -(-1)^{l-m} T_x \mathbf{D}_{l,m} \tag{74}$$

For the coefficients of specific interest to us we find:

$$C'_{2t,0,z} = -C_{2t,0,z} \tag{75}$$
$$C'_{2t,1,x} = -C_{2t,1,x} \tag{76}$$
$$C'_{2t,1,y} = C_{2t,1,y} \tag{77}$$
$$D'_{2t,1,x} = D_{2t,1,x} \tag{78}$$
$$D'_{2t,1,y} = -D_{2t,1,y} \tag{79}$$

In terms of the spin state, we find the following symmetry changes when the asteroid rotates in the opposite sense. The longitude of the ascending node shifts by π, or $\Omega'_s = \Omega_s + \pi$, and the solar inclination is replaced by its supplement, $i'_s = \pi - i_s$. We see that the value of $\sin i'_s = \sin i_s$, and thus the secular equations do not change their form, just their coefficients. In terms of the original Fourier coefficients we thus find the secular evolution equations for the spin state of the asteroid rotating in the opposite sense to be:

$$\dot{\omega}'_z = -\frac{G_1}{I_z a^2 \sqrt{1-e^2}} \bar{C}_{0,z} \tag{80}$$

$$\dot{i}'_s = \frac{G_1}{\omega_0 I_z a^2 \sqrt{1-e^2}} \left[-\left(\bar{C}_{1,x} + \bar{D}_{1,y}\right) \cos(\phi_{lag}) + \left(\bar{D}_{1,x} - \bar{C}_{1,y}\right) \sin(\phi_{lag}) \right] \tag{81}$$

$$\dot{\Omega}'_s = \frac{\cot(i_s) G_1}{\omega_0 I_z a^2 \sqrt{1-e^2}} \left[-\left(\bar{D}_{1,x} - \bar{C}_{1,y}\right) \cos(\phi_{lag}) - \left(\bar{C}_{1,x} + \bar{D}_{1,y}\right) \sin(\phi_{lag}) \right] \tag{82}$$

Thus we note that the overall evolutionary equations for the inclination and longitude are not symmetric, but that the effect of thermal lag can change the magnitude and direction of the evolution. The spin rate evolution just has a change in sign, and changes from deceleration to acceleration, or vice-versa. The obliquity change is non-symmetric if the thermal inertia is non-zero. Thus, when an asteroid rotates in the opposite sense its obliquity dynamics may be quite different.

4 A study of the rotational dynamics equations

From our analysis we can state the averaged dynamics equations for i_s and ω. In the following we do not consider the dynamics of Ω_s as it does not affect the other states.

4.1 Normalized form of the equations

The general form of the dynamical equations can be stated as:

$$\dot{\omega} = \frac{G_1}{I_z a^2 \sqrt{1-e^2}} \sum_{t=0}^{\infty} C_{2t,0,z} \left[\frac{1}{2^{2t}} \sum_{k=0}^{t} (-1)^k \frac{(4t-2k)!}{k!(2t-k)!(t-k)!^2} \left(\frac{\sin i_s}{2} \right)^{2t-2k} \right] \tag{83}$$

$$\dot{i_s} = \frac{-G_1 \cos(i_s)}{\omega I_z a^2 \sqrt{1-e^2}} \sum_{t=1}^{\infty} \frac{1}{2^{2t}} \times [(C_{2t,1\,x} + D_{2t,1\,y}) \cos(\phi_{lag}) + (D_{2t,1\,x} - C_{2t,1\,y}) \sin(\phi_{lag})] \times \left[\sum_{k=0}^{t-1} (-1)^k \frac{(4t-2k)!(t-k)}{k!(2t-k)!(t-k)!^2} \left(\frac{\sin i_s}{2} \right)^{2t-2k-1} \right] \quad (84)$$

Following the definition in Scheeres (2007), we introduce a normalization to the spherical harmonic coefficients to define them in a dimensionless form. The normalizing factor is rI_z/M, where r is the mean radius of the body, I_z is its maximum moment of inertia, and M is the total mass of the body. Thus, the normalizing factor scales as r^3. We divide the coefficients by this factor and multiply the leading terms by it. We may also rearrange the series summations to extract the term $\sin i_s$ to find a simple form for the equations:

$$\dot{\omega} = g \sum_{k=0}^{\infty} a_k \sin^{2k}(i_s)$$

$$\dot{i_s} = g \frac{\cos(i_s)}{\omega} \sum_{k=0}^{\infty} \sin^{2k+1}(i_s)[b_k \cos(\phi_{lag}) + c_k \sin(\phi_{lag})]$$

where

$$g = \frac{G_1 r}{M a^2 \sqrt{1-e^2}} \quad (85)$$

and the coefficients a_k, b_k and c_k are defined as:

$$a_k = \sum_{k \le t} \tilde{C}_{2t,0,z} \frac{(-1)^{t-k}}{2^{2t+2k}} \cdot \frac{(2t+2k)!}{(t-k)!(t+k)!k!^2}$$

$$b_k = \sum_{1+k \le t} (\tilde{C}_{2t,1,x} + \tilde{D}_{2t,1,y}) \frac{(-1)^{t-k}}{2^{2t+2k+1}} \cdot \frac{(2t+2k+2)!(k+1)}{(t-k-1)!(t+k+1)!(k+1)!^2}$$

$$c_k = \sum_{1+k \le t} (\tilde{D}_{2t,1,x} - \tilde{C}_{2t,1,y}) \frac{(-1)^{t-k}}{2^{2t+2k+1}} \cdot \frac{(2t+2k+2)!(k+1)}{(t-k-1)!(t+k+1)!(k+1)!^2}$$

and define an additional term to be used later, $d_k = b_k + c_k$,

$$d_k = \sum_{1+k \le t} (\tilde{C}_{2t,1,x} + \tilde{D}_{2t,1,y} + \tilde{D}_{2t,1,x} - \tilde{C}_{2t,1,y}) \times \frac{(-1)^{t-k}}{2^{2t+2k+1}} \cdot \frac{(2t+2k+2)!(k+1)}{(t-k-1)!(t+k+1)!(k+1)!^2}$$

where $\tilde{C}_{lm} = C_{lm}/(rI_z/M)$ and $\tilde{D}_{lm} = D_{lm}/(rI_z/M)$. The coefficients a_k, b_k, c_k and d_k are pure functions of the geometry of the asteroid while g depends on its mass, size and heliocentric orbit, and controls how fast the evolution of the system is.

If we consider rotation in the opposite sense we note the symmetry transformations:

$$a_k' = -a_k \tag{86}$$
$$b_k' = -b_k \tag{87}$$
$$c_k' = c_k \tag{88}$$
$$d_k' = -b_k + c_k \tag{89}$$

and thus we note again the non-symmetric change in the obliquity evolution.

The thermal lag angle can be approximated using Eqs. 4 and 5:

$$\cos(\phi_{lag}) = \frac{1+\mu\sqrt{\omega}}{\sqrt{1+2\mu\sqrt{\omega}+2\mu^2\omega}}$$
$$\sin(\phi_{lag}) = \frac{\mu\sqrt{\omega}}{\sqrt{1+2\mu\sqrt{\omega}+2\mu^2\omega}}$$

In addition, we can apply the reemission factor if we only consider the diffuse emission term:

$$\frac{1}{\sqrt{1+2\mu\sqrt{\omega}+2\mu^2\omega}}$$

Then the dynamics equations have the following form:

$$\dot{\omega} = \frac{g}{\sqrt{1+2\mu\sqrt{\omega}+2\mu^2\omega}} \sum_{k=0}^{\infty} a_k \sin^{2k}(i_s)$$

$$\dot{i_s} = \frac{g\cos(i_s)}{\omega\left(1+2\mu\sqrt{\omega}+2\mu^2\omega\right)} \sum_{k=0}^{\infty} \sin^{2k+1}(i_s)\left[b_k\,(1+\mu\sqrt{\omega}) + c_k\,(\mu\sqrt{\omega})\right]$$

Defining $d_k = b_k + c_k$ we have:

$$\dot{i_s} = \frac{g\,\cos(i_s)}{\omega\left(1+2\mu\sqrt{\omega}+2\mu^2\omega\right)} \sum_{k=0}^{\infty} \sin^{2k+1}(i_s)\left[b_k + d_k\,\mu\sqrt{\omega}\right]$$

For $(i_s, \omega) \in [0, \pi] \times (\mathbb{R} - \{0\})$, these equations are regular. We see that if $i_s = \frac{k\pi}{2}$, for k an integer, $\dot{i_s} = 0$ for any angular velocity not equal to zero.

If we define $\beta = \sin i_s$ we have the following system of equations for ω and β:

$$\dot{\omega} = \frac{g}{\sqrt{1+2\mu\sqrt{\omega}+2\mu^2\omega}} \sum_{k=0}^{\infty} a_k\,\beta^{2k} \tag{90}$$

$$\dot{\beta} = \frac{g(1-\beta^2)}{\omega\left(1+2\mu\sqrt{\omega}+2\mu^2\omega\right)} \sum_{k=0}^{\infty} \beta^{2k+1}\left[b_k + \mu d_k\sqrt{\omega}\right]$$

where $0 \le \beta \le 1$. We note that the evolution of β contains in it the symmetric evolution of i_s over the intervals $[0, \pi/2]$ and $[\pi, \pi/2]$. As we have seen before $\beta = 0, 1$ are asymptotic solutions, and if $\beta \to 0$ or 1 then $\dot{\beta} \to 0$. Thus, we also note that β is trapped to lie in the interval [0, 1] in accordance with its definition.

To simplify the following discussion we define the following functions:

$$A(\beta) = \sum_{k=0}^{\infty} a_k \beta^{2k} \tag{91}$$

$$B(\beta) = \sum_{k=0}^{\infty} b_k \beta^{2k} \tag{92}$$

$$D(\beta) = \sum_{k=0}^{\infty} d_k \beta^{2k} \tag{93}$$

We note that some asteroids investigated to date have some basic characteristics in terms of the number of zeros these functions have over the internal $\beta \in [0, 1]$. Namely, $A(\beta)$ generally has one zero in this interval, often at a value close to 60°, or $\beta \sim \sqrt{3}/2$ (Nesvorný and Vokrouhlický 2007). Conversely, the functions $B(\beta)$ and $D(\beta)$ tend to have no roots in this interval, although again some asteroid shapes show exceptions to this (Scheeres 2007). This is discussed in more detail in (Vokrouhlický and Čapek 2002 and Nesvorný and Vokrouhlický 2007). In the following we will take these results as the "generic" case and discuss the dynamical evolution under these hypotheses. A more general discussion can also be made of these dynamics for different situations Mirrahimi (2007).

More precisely, we define our generic functions A^g, B^g and D^g to have the following properties:

$$A^g(\beta_o) = 0 \tag{94}$$

$$A^g(\beta) \neq 0 \quad \forall \beta \neq \beta_o \tag{95}$$

$$B^g(\beta) D^g(\beta) \neq 0 \quad \forall \beta \in [0, 1] \tag{96}$$

An example case for asteroid Castalia that satisfies these properties is shown in Fig. 1. An example which does not satisfy these properties, asteroid 1998 KW4, is shown in Fig. 2. Also of interest is asteroid 1999 KY26, Fig. 3, which exhibits the generic case for its nominal rotation sense but has a drastically changed $D(\beta)$ for rotation in the opposite sense. These particular asteroids are used as there are precise shape models in existence for them. Shape models for the asteroids Apollo and YORP, for which YORP has been detected, are not known as precisely.

To characterize the dynamics for the generic form of the functions it suffices to develop a notation to track their signs. Specifically, we use the "sign" function $\text{sgn}(x)$ (equal to 1 for $x > 0$, -1 for $x < 0$, and equal to zero for $x = 0$) to define the following:

$$s_A = \text{sgn}(A^g(0)) \tag{97}$$

$$s_B = \text{sgn}(B^g(\beta)) \tag{98}$$

$$s_D = \text{sgn}(D^g(\beta)) \tag{99}$$

Thus we note that if $s_A = 1$ that $\text{sgn}(A^g(1)) = -1$. Also, from Vokrouhlický and Čapek (2002) we note that $s_A > 0$ is called Type I while $s_A < 0$ is Type II. We do not assume any correlation between the signs s_A, s_B and s_D, although such correlations may exist (Nesvorný and Vokrouhlický 2007).

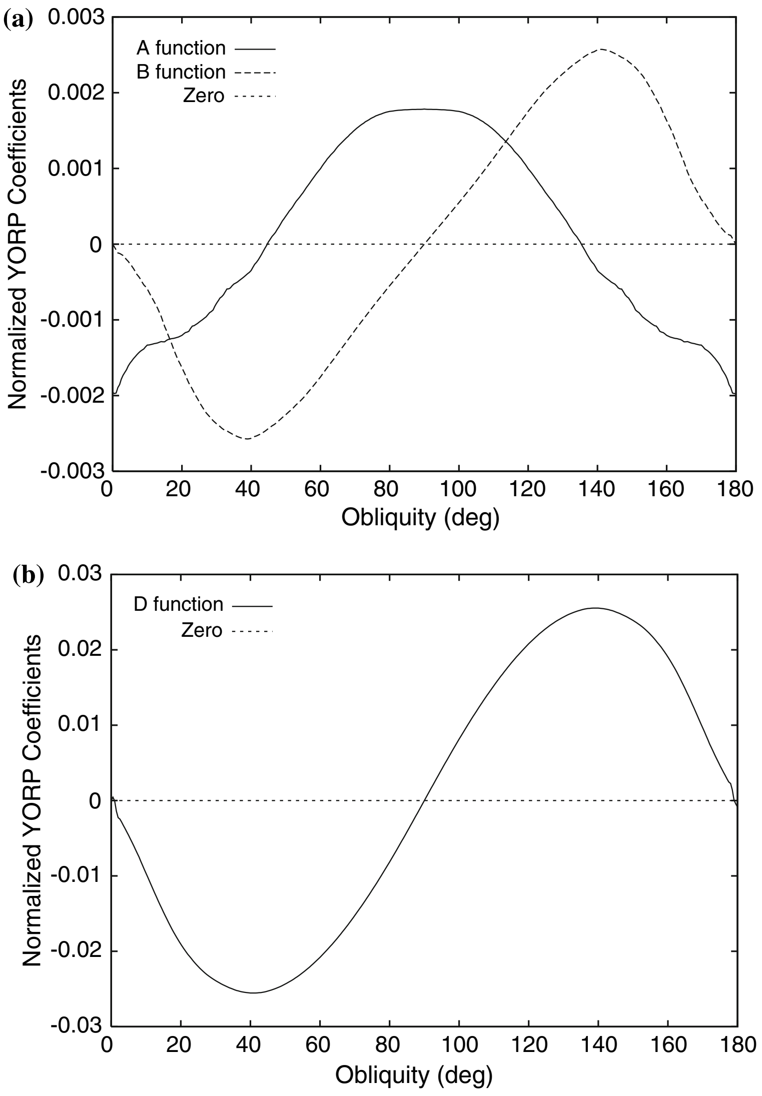

Fig. 1 Functions A, $\beta(1-\beta^2)B$ and $\beta(1-\beta^2)D$ for the asteroid Castalia (Hudson and Ostro 1994)

4.2 Dynamics with zero thermal conductivity

We first study the case where there is no thermal conductivity, or $\mu = 0$.

$$\dot{\omega} = g\,A(\beta)$$

$$\dot{\beta} = g\frac{\beta(1-\beta^2)}{\omega}B(\beta)$$

Depending on the signs of A and B we will get different asymptotic solutions. There are a total of 4 different possibilities, outlined in Table 1.

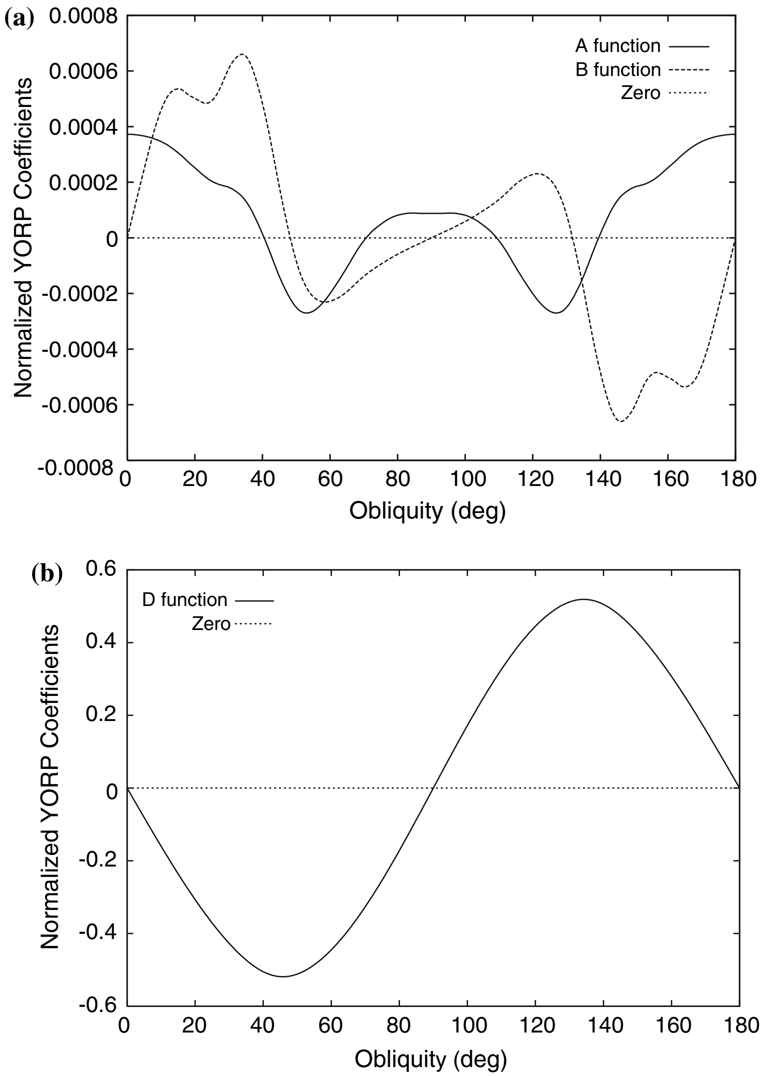

Fig. 2 Functions A, $\beta(1-\beta^2)B$ and $\beta(1-\beta^2)D$ for the asteroid KW4, note that A has multiple roots (Scheeres et al. 2006)

A generic depiction of these different solutions, both forwards and backwards in time, is given in Fig. 4. We note that under the spin-reversal symmetry, the solutions with $s_A s_B > 0$ transform into each other as do the solutions with $s_A s_B < 0$. Thus, if an asteroid spins to a zero rotation rate and starts to spin in the opposite direction, its path will merely retrace the path it initially took, and eventually end up with a zero rotation rate again. Also, it is interesting to note that shapes with $s_A s_B > 0$ have a characteristic maximum spin rate, which occurs at β_o, while shapes with $s_A s_B < 0$ have a minimum spin rate, which again occurs at β_o.

If there exists a zero for $B(\beta)$, called β^*, in the interval $\beta \in [0, 1]$, we note that the obliquity dynamics will then be trapped in one of the intervals $[0, \beta^*)$ and $(\beta^*, 1]$. One of these intervals will contain β_o, and dynamics in that interval will be similar to that in Table 1.

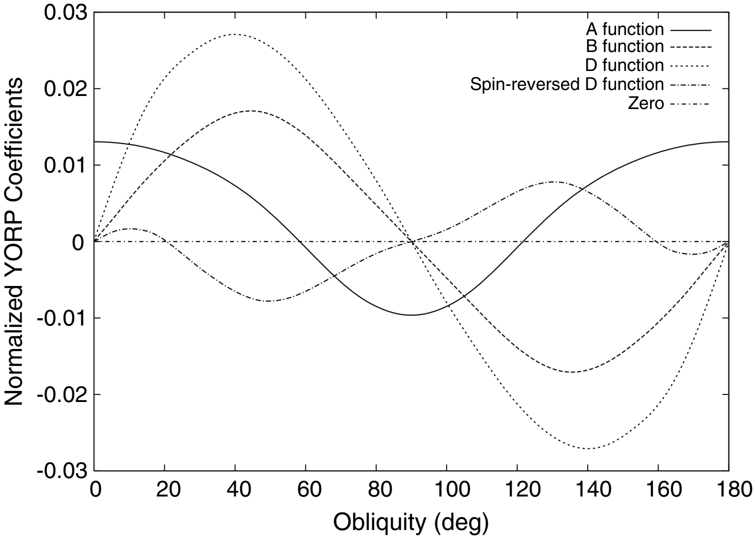

Fig. 3 Functions A, B and D for the asteroid 1999 KY26, showing both the nominal configuration and the D function if spun in the opposite direction (Ostro et al. 1999)

Table 1 Asymptotic dynamics for the case with zero thermal conductivity

Type	s_A	s_B	$t \to \infty$		$t \to -\infty$	
			ω	β	ω	β
I_0^+	1	1	0	1	0	0
II_0^+	−1	−1	0	0	0	1
I_0^-	1	−1	∞	0	∞	1
II_0^-	−1	1	∞	1	∞	0

The Type I/II designation is taken from Vokrouhlický and Čapek (2002), the "0" represents zero thermal conductivity, and the $\pm$ represents the product $s_A s_B$

The other interval will have trajectories that travel from 0 to ∞ in ω, either forwards or backwards in time, as the obliquity travels between the interval limits. An additional zero in $A(\beta)$ will cause a more complex evolution of the spin rate, with the possibility of there being local maximum and minimum spin rates.

Regardless of the number of zeroes of the functions A and B, the solution to these differential equations can be expressed in closed form. Define an interval of β, (β_1, β_2), such that $B(\beta) \neq 0 \; \forall \beta \in (\beta_1, \beta_2)$. Then, regardless of the number of zeros $A(\beta)$ may have in this interval, the rotation rate can be explicitly expressed as:

$$\omega(\beta) = \omega_o e^{G(\beta)} \tag{100}$$

$$G(\beta) = \int_{\beta_0}^{\beta} \frac{A(\beta')}{B(\beta')\beta'(1-\beta'^2)} d\beta' \tag{101}$$

$$\forall \beta_0, \quad \beta \in (\beta_1, \beta_2) \tag{102}$$

Rubincam (1995) presents a similar analytical solution for an asteroid subjected to the YORP effect.

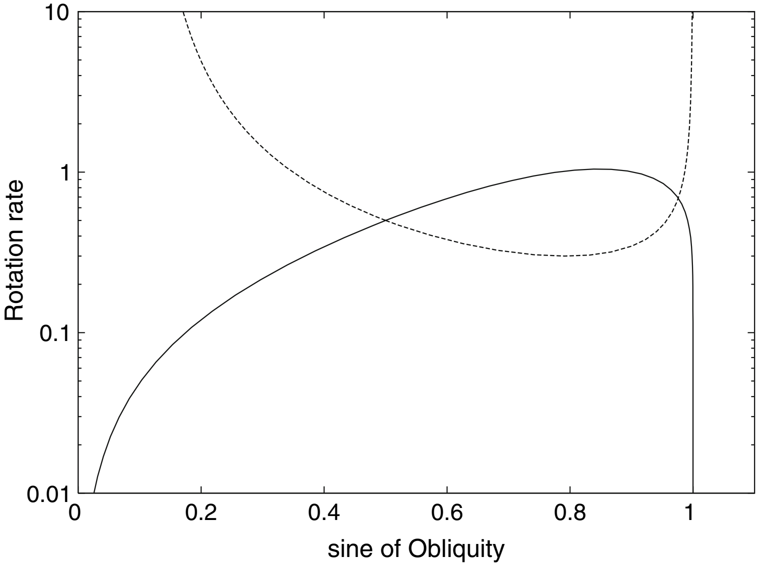

Fig. 4 Generic rotational dynamics phase plane possibilities for the case of zero thermal inertia. Solutions can traverse these curves in either direction, depending on their Type

4.3 Dynamics with nonzero thermal conductivity

These previous results assume no thermal conductivity. One of the effects of thermal conductivity is to modify the time between insolation and reemission of solar photons leading to a lag angle. As derived previously, we see that this has a significant effect on the obliquity dynamics, but does not modify the rotational acceleration.

Now we consider rotational dynamics in the general case with a nonzero thermal conductivity:

$$\dot{\omega} = \frac{g}{\sqrt{1 + 2\mu\sqrt{\omega} + 2\mu^2\omega}} A(\beta)$$

$$\dot{\beta} = \frac{g\beta(1-\beta^2)}{\omega\left(1 + 2\mu\sqrt{\omega} + 2\mu^2\omega\right)} \left[B(\beta) + \mu\sqrt{\omega}D(\beta)\right]$$

Again, for our generic case we assume a single root for $A^g(\beta)$ and no roots for the functions $B^g(\beta)$ and $D^g(\beta)$. We first discuss the dynamics of this "generic" case, which can be done in terms of the signs of $A(0)$, B and D. We introduce the new notation $s_D = \text{sgn}(D(\beta))$. Again, we note that $\dot{\beta} = 0$ at $\beta = 0$ and $\beta = 1$, and thus these can serve as asymptotic solutions. Now, however, the stability of these solutions is more complex and depends on the signs of both $B(\beta)$ and $D(\beta)$.

The simplest case occurs if B and D have the same sign, either positive or negative, or if $s_B s_D > 0$. Under our generic assumption the entire interval of $\dot{\beta}$ will be non-zero, and we have the equivalent situation as in the zero thermal inertia case. Specifically, we have four different possible situations for the future and past of the asteroid rotation state as stated in Table 1. Now, however, we note that under reversal of spin direction that $D' \neq -D$, and hence the nature of the obliquity dynamics can change if a reversal in spin direction occurs.

Table 2 Asymptotic dynamics for the generic case with non-zero thermal conductivity

Type	s_A	s_B	s_D	$t \to \infty$		$t \to -\infty$	
				ω	β	ω	β
I^+	1	1	1	0	1	0	0
II^+	−1	−1	−1	0	0	0	1
I^-	1	−1	−1	∞	0	∞	1
II^-	−1	1	1	∞	1	∞	0
I^H	1	1	−1	∞	0	0	0
				0	1	∞	1
II^H	−1	−1	1	0	0	∞	0
				∞	1	0	1
I^C	1	−1	1	Circulation			
II^C	−1	1	−1	Circulation			

We note that the first four cases are identical with those for zero thermal conductivity. For the more complex cases we can now have multiple possible asymptotic solutions, depending on the starting conditions. In some cases there can be circulation of the solution that either converges to an equilibrium or that does not. The Type *I* and *II* definitions are again taken from Vokrouhlický and Čapek (2002), and now the *H* and *C* super-scripts stand for the equilibrium point existing and being either hyperbolic or circulatory

A more complex situation occurs if B and D have opposite signs, or $s_B s_D < 0$, as now a stationary value in the obliquity rate can occur other than at $\beta = 0, 1$. Specifically, we see that $\dot{\beta} = 0$ if $\mu\sqrt{\omega} = -B(\beta)/D(\beta)$. Thus, if the rotation rate passes through this value the flux direction of β will change sign. We can evaluate the nature of the solutions at $\beta = 0, 1$ by considering the different cases of s_A, s_B, and s_D again, given in Table 2.

For the last four cases in Table 2 we note that the asymptotic dynamics become more complex, and indeed that for the last two cases there are no solutions asymptotic to $\beta = 0, 1$. This complication arises from the presence of a equilibrium point in the dynamics, located at β_o and $\mu\sqrt{\omega_o} = -B^g(\beta_o)/D^g(\beta_o)$ for the generic system (note that this ω_o is distinct from that defined earlier). The properties of the motion can be better understood by studying the stability properties of this equilibrium solution, as this organizes the flow of the asteroid's rotation state. To analyze its stability we consider small deviations from this relative equilibrium point, $\omega = \omega_o + \delta\omega$ and $\beta = \beta_o + \delta\beta$, and form the linearized dynamical equations:

$$\begin{bmatrix} \delta\dot{\omega} \\ \delta\dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & \dot{\omega}_\beta\big|_o \\ \dot{\beta}_\omega\big|_o & \dot{\beta}_\beta\big|_o \end{bmatrix} \begin{bmatrix} \delta\omega \\ \delta\beta \end{bmatrix} \tag{103}$$

where the subscript notation denotes partial differentiation with respect to that variable and

$$\dot{\omega}_\beta\big|_o = A_\beta\big|_o \frac{g}{\sqrt{1 + 2\mu\sqrt{\omega_o} + 2\mu^2\omega_o}} \tag{104}$$

$$\dot{\beta}_\beta\big|_o = \frac{g\beta_o(1-\beta_o^2)}{\omega_o\left(1 + 2\mu\sqrt{\omega_o} + 2\mu^2\omega_o\right)} \tag{105}$$

$$\dot{\beta}_\omega\big|_o = \frac{g\beta_o(1-\beta_o^2)}{2\omega_o^{3/2}\left(1 + 2\mu\sqrt{\omega_o} + 2\mu^2\omega_o\right)} \tag{106}$$

The characteristic equation for this system is found to be:

$$\lambda^2 - \dot{\beta}_\beta\lambda - \dot{\beta}_\omega\dot{\omega}_\beta = 0 \tag{107}$$

with solutions

$$\lambda_{\pm} = \frac{1}{2}\left[\dot{\beta}_{\beta} \pm \sqrt{\dot{\beta}_{\beta}^2 + 4\dot{\beta}_{\omega}\dot{\omega}_{\beta}}\right] \tag{108}$$

If the real part of either $\lambda_{\pm}$ is positive the equilibrium point is unstable and has a well-defined unstable manifold along which the solution would depart from the equilibrium point. If the real part of either root is negative, there also exists a stable manifold which approaches the equilibrium. If the roots have an imaginary term, there is also oscillatory motion as the solution follows the stable or unstable manifolds. Determining the stability of the equilibrium can be reduced to determining the signs of the partial derivatives $\dot{\beta}_{\beta}$, $\dot{\beta}_{\omega}$, and $\dot{\omega}_{\beta}$.

For our generic models we find the following results, used in our discussion. First, if $s_A > 0$ then $\dot{\omega}_{\beta} < 0$ and vice-versa. Next, if $s_D > 0$ then $\dot{\beta}_{\omega} > 0$ and vice-versa. The sign of $\dot{\beta}_{\beta}$ is more difficult to determine as it depends on the slopes of $B(\beta)$ and $D(\beta)$ at β_o, which are not constrained to any particular sign and magnitude.

If $s_A s_D < 0$ then $\dot{\omega}_{\beta}\dot{\beta}_{\omega} > 0$ and we note that the equilibrium point has a hyperbolic structure independent of the sign of $\dot{\beta}_{\beta}$, with a positive real eigenvalue and a negative real eigenvalue. The stable and unstable manifolds associated with the equilibrium point then continue into the solutions asymptotic to $\beta = 0, 1$ forwards and backwards in time. This corresponds to Types I^H and II^H in Table 2, and a generic example is shown in Fig. 5. We note that the rotation state dynamics are isolated in their respective quadrants, which are separated by the stable and unstable manifolds to the equilibrium point. Non-averaged dynamics can be more complex, of course, and crossing of the manifolds may be possible for actual rotation state evolution. The study of these more realistic evolutions are of interest for the future. For the case of non-zero thermal inertia, if the spin direction of the asteroid is switched the phase space of the dynamics is no longer self-similar, specifically the sign of D may not change even though the sign of A changes, thus potentially changing the qualitative nature of the dynamics.

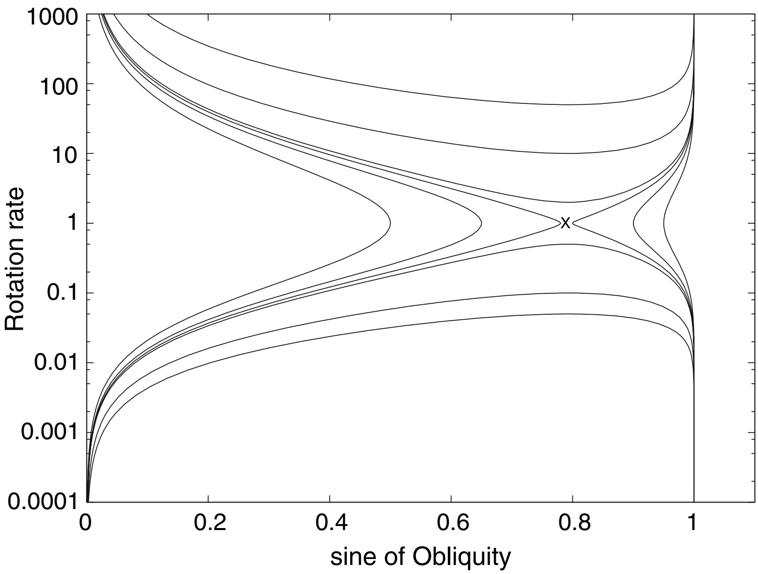

Fig. 5 Generic rotational dynamics phase plane for the case of a hyperbolic root for the equilibrium point

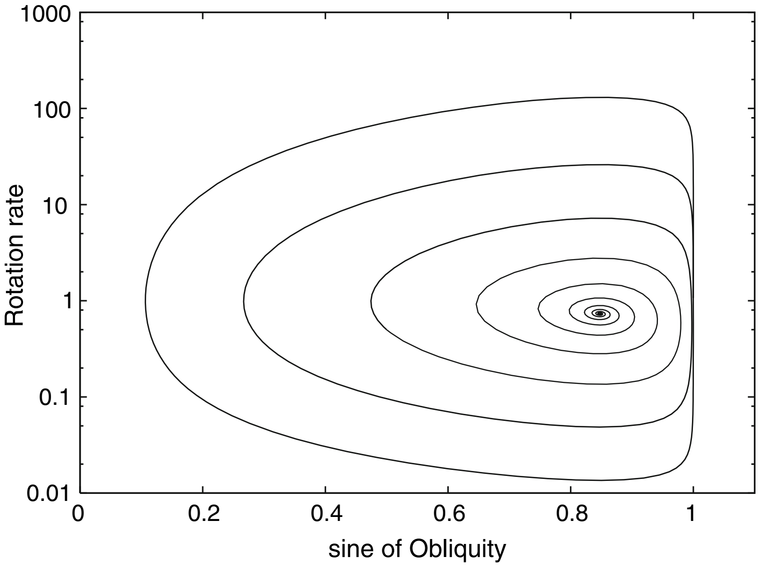

Fig. 6 Generic rotational dynamics phase plane for the case of a stable equilibrium point with oscillatory behavior

If $s_A s_D > 0$ then $\dot{\omega}_\beta \dot{\beta}_\omega < 0$ and the square root quantity is less in magnitude than $\dot{\beta}_\beta$. For this case, Types I^C and II^C in Table 2, we note that there are no asymptotic solutions to $\beta = 0, 1$, and the only possible asymptotic solution is the equilibrium point, if it is stable. If $\dot{\beta}_\beta < 0$ then the equilibrium point is stable, and solutions will converge upon the point forwards in time, and spiral away from it backwards in time. A generic case is shown in Fig. 6. Conversely, if $\dot{\beta}_\beta > 0$ the equilibrium point is unstable and the situation is reversed. In this case the solution has no limiting value and continues to evolve indefinitely. Practically, the solution spends most of its time either along the $\beta = 0$ of 1 axes, but never stays there forever. For the special case of $\dot{\beta}_\beta = 0$ the equilibrium point is purely oscillatory, and we see that the dynamics have a similar oscillatory behavior for non-linear motion (see Fig. 7).

In the non-generic case the situation can become considerably more complex. However, as the rotational dynamics have been reduced to a two-dimensional dynamical system, they can be studied and understood using basic techniques of dynamical systems.

5 Conclusion

In this paper an analysis of the dynamics of a uniformly rotating asteroid subject to the YORP effect is given. By solving the equations of rotational motion we have derived the secular evolution of rotation rate and solar inclination of the asteroid. The equations derived for the dynamics of this system have a relatively simple form which leads to a standardized discussion of the dynamics. The effect of thermal conductivity and the thermal inertia is included in the study and modify the equations. Using the theoretical results in this report, a study of dynamical evolution of asteroid rotation states due to YORP can be performed easily. Also

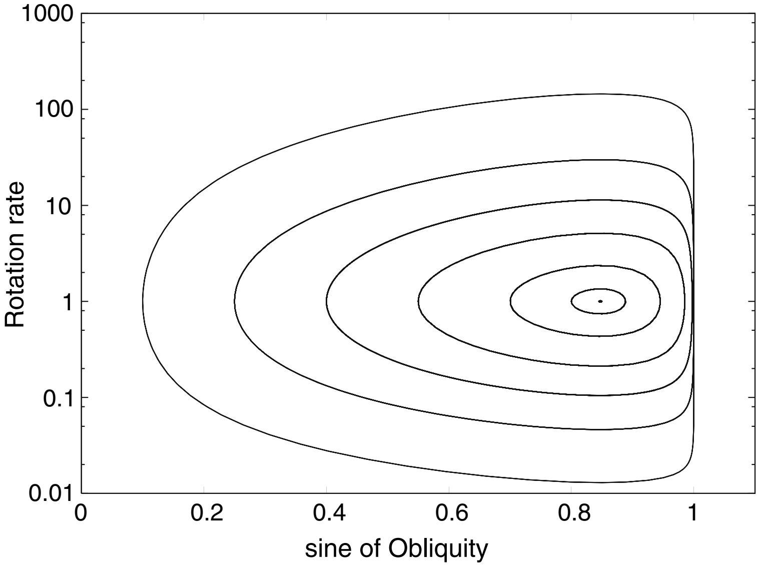

Fig. 7 Generic rotational dynamics phase plane for the special case of pure oscillatory behavior

given in this paper is an explicit model of the YORP torques acting on an asteroid in terms of spherical harmonic coefficients. The definition of this model enables the averaging analysis to be performed entirely analytically.

Acknowledgements DJS acknowledges support from NASA's Planetary Geology and Geophysics program. The authors thank the anonymous reviewers for their helpful comments and corrections.

Appendix A

Spherical harmonic coefficients

The spherical harmonics defined as below form a complete set of orthonormal functions and thus form a vector space analogous to unit basis vectors (MacRobert 1947):

$$Y_l^m(\delta, \lambda) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\sin\delta) e^{im\lambda}$$

These functions are orthonormal, thus:

$$\int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\delta) Y_l^m(\delta, \lambda) Y_{l'}^{m'*}(\delta, \lambda) d\delta d\lambda = \delta_{ll'} \delta_{mm'}$$

where

$$\delta_{l,m} = \begin{cases} 1 & \text{if } l = m \\ 0 & \text{if } l \neq m \end{cases}$$

So if a function f is written as a series of spherical harmonics:

$$f(\delta,\lambda)=\sum_{l=0}^{\infty}\sum_{m=0}^{l}a_{l,m}Y_l^m(\delta,\lambda)$$

We can compute the coefficients by following formulas:

$$a_{l,m}=\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(\delta)f(\delta,\lambda)Y_l^{m*}d\delta d\lambda \tag{A.1}$$

To find the real parts we have:

$$\begin{aligned}\frac{1}{2}(Y_l^m+(-1)^mY_l^{-m})&=\frac{1}{2}\sqrt{\frac{2l+1}{4\pi}}\left[\sqrt{\frac{(l-m)!}{(l+m)!}}P_l^m(\sin(\delta))e^{im\lambda}\right.\\&\quad\left.+(-1)^m\sqrt{\frac{(l+m)!}{(l-m)!}}P_l^{-m}(\sin(\delta))e^{-im\lambda}\right]\\&=\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\sin\delta)\left[\frac{e^{im\lambda}+(-1)^{2m}e^{-im\lambda}}{2}\right]\\&=\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\sin\delta)\cos(m\lambda)\end{aligned}$$

So

$$\frac{1}{2}(Y_l^m+(-1)^mY_l^{-m})=\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\sin\delta)\cos(m\lambda) \tag{A.2}$$

And

$$\frac{1}{2i}(Y_l^m-(-1)^mY_l^{-m})=\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\sin\delta)\sin(m\lambda) \tag{A.3}$$

Using (A.1), (A.2), and (A.3) we can compute the spherical harmonic coefficients of $f(\delta,\lambda)=\frac{M}{P(R)}$, i.e. $C_{l,m}$ and $D_{l,m}$:

$$\frac{C_{l,m}}{2N_{l,m}}+\frac{D_{l,m}}{2N_{l,m}i}=a_{l,m}$$

$$\frac{C_{l,m}}{2N_{l,m}}-\frac{D_{l,m}}{2N_{l,m}i}=(-1)^ma_{l,-m}$$

or

$$C_{l,m}=N_{l,m}[a_{l,m}+(-1)^ma_{l,-m}] \tag{A.4}$$

$$D_{l,m}=iN_{l,m}[a_{l,m}-(-1)^ma_{l,-m}] \tag{A.5}$$

where $N_{l,m}=\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}$. We can rewrite the values of $C_{l,m}$ and $D_{l,m}$ in the following form, using A.1:

$$C_{l,m}=2N_{l,m}^2\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(\delta)f(\delta,\lambda)P_l^m(\sin(\delta))\cos(m\lambda)d\delta d\lambda \tag{A.6}$$

$$D_{l,m} = 2N_{l,m}^2 \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\delta) f(\delta, \lambda) P_l^m(\sin(\delta)) \sin(m\lambda) d\delta d\lambda \quad \text{(A.7)}$$

To compute completely all the terms of the average dynamic equations of 61, 67, 68 we need to know $C_{0,z}$, $C_{1,x}$, $C_{1,y}$, $D_{1,x}$, $D_{1,y}$. We have the expression of $\frac{\mathbf{M}}{P(R)}$, (Čapek and Vokrouhlický 2004):

$$\frac{\mathbf{M}}{P(R)} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{f}_i(\hat{\mathbf{u}})$$

where $\mathbf{f}_i(\hat{\mathbf{u}}) = -[\{\rho s(2\hat{\mathbf{n}}_i\hat{\mathbf{n}}_i - \mathbf{U}) + \mathbf{U}\} \cdot \hat{\mathbf{u}}\hat{\mathbf{u}} \cdot \hat{\mathbf{n}}_i + a_2\hat{\mathbf{n}}_i\hat{\mathbf{n}}_\mathbf{i} \cdot \hat{\mathbf{u}}]H_i(\hat{\mathbf{u}})A_i$.
Here we have supposed that the asteroid consists of N facets, each of them being a flat plane. $\hat{n}_i$ is the normal vector to the facet i. $\mathbf{U}$ is the identity matrix. $\hat{\mathbf{u}}$ is the solar direction, and $H_i(\hat{\mathbf{u}})$ is the visibility function for the facet i, which equals 1 when the Sun is above the horizon and zero otherwise.
So to compute the coefficients, we have to compute the following integrals:

$$\mathbf{v}_{2t,m}^i = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\delta_s)\hat{\mathbf{u}}H_i(\hat{\mathbf{u}})P_{2t}^m(\sin(\delta)) \cos(m\lambda) d\delta d\lambda$$

$$\mathbf{v}_{2t,m}^{\prime i} = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\delta_s)\hat{\mathbf{u}}H_i(\hat{\mathbf{u}})P_{2t}^m(\sin(\delta)) \sin(m\lambda) d\delta d\lambda$$

$$\mathbf{w}_{2t,m}^i = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\delta_s)\hat{\mathbf{u}}\hat{\mathbf{u}}H_i(\hat{\mathbf{u}})P_{2t}^m(\sin(\delta)) \cos(m\lambda) d\delta d\lambda$$

$$\mathbf{w}_{2t,m}^{\prime i} = \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\delta_s)\hat{\mathbf{u}}\hat{\mathbf{u}}H_i(\hat{\mathbf{u}})P_{2t}^m(\sin(\delta)) \sin(m\lambda) d\delta d\lambda$$

for $m = 0, 1$.

The visibility function $H_i(\hat{u})$ equals 1 when the Sun is above the horizon and equals 0 otherwise. In general, for each facet it will be defined by two solar longitudes, the one when it rises λ_{r_i}, and the one when it sets λ_{s_i}. Note that these longitudes are a function of δ in general and are computed from $\lambda_{r,s} = \lambda_i \pm \arccos(-\tan\delta_s \tan\delta_i)$ where δ_i is the solar latitude of the surface element. Thus, in general only the integral in longitude can be computed in closed form, the other being left in integral form (note, a closed form integration over these limits is discussed in Scheeres (2007)). Shadowing can also be accommodated in performing this integral, being represented in a unique way by the shadowing function H_i at each facet. A method for incorporating this into the computation is given in Scheeres (2007).

$$\mathbf{v}_{2t,m}^i = \int_{\lambda_{r_i}}^{\lambda_{s_i}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\delta_s)\hat{\mathbf{u}}P_{2t}^m(\sin(\delta)) \cos(m\lambda) d\delta d\lambda$$

$$\mathbf{v}_{2t,m}^{\prime i} = \int_{\lambda_{r_i}}^{\lambda_{s_i}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\delta_s)\hat{\mathbf{u}}P_{2t}^m(\sin(\delta)) \sin(m\lambda) d\delta d\lambda$$

$$\mathbf{w}_{2t,m}^i = \int_{\lambda_{r_i}}^{\lambda_{s_i}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\delta_s)\hat{\mathbf{u}}\hat{\mathbf{u}}P_{2t}^m(\sin(\delta)) \cos(m\lambda) d\delta d\lambda$$

$$\mathbf{w}'^{i}_{2t,m} = \int_{\lambda_{r_i}}^{\lambda_{s_i}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\delta_s)\hat{\mathbf{u}}\hat{\mathbf{u}} P_{2t}^{m}(\sin(\delta)) \sin(m\lambda) d\delta d\lambda$$

where

$$\hat{\mathbf{u}} = \begin{pmatrix} \cos(\delta_s)\cos(\lambda_s) \\ \cos(\delta_s)\sin(\lambda_s) \\ \sin(\delta_s) \end{pmatrix}$$

and

$$\hat{\mathbf{u}}\hat{\mathbf{u}} = \begin{pmatrix} \cos^2(\delta_s)\cos^2(\lambda_s) & \cos^2(\delta_s)\cos(\lambda_s)\sin(\lambda_s) & \cos(\delta_s)\sin(\delta_s)\cos(\lambda_s) \\ \cos^2(\delta_s)\cos(\lambda_s)\sin(\lambda_s) & \cos^2(\delta_s)\sin^2(\lambda_s) & \cos(\delta_s)\sin(\delta_s)\sin(\lambda_s) \\ \cos(\delta_s)\sin(\delta_s)\cos(\lambda_s) & \cos(\delta_s)\sin(\delta_s)\sin(\lambda_s) & \sin^2(\delta_s) \end{pmatrix}.$$

Thus we find

$$\mathbf{v}^{i}_{2t,m} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{I}^{1}_{m,i} \cos(\delta_s) P_{2t}^{m}(\sin(\delta)) H_i(\delta) d\delta \tag{A.8}$$

$$\mathbf{v}'^{i}_{2t,m} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{I}'^{1}_{m,i} \cos(\delta_s) P_{2t}^{m}(\sin(\delta)) H_i(\delta) d\delta \tag{A.9}$$

$$\mathbf{w}^{i}_{2t,m} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{I}^{2}_{m,i} \cos(\delta_s) P_{2t}^{m}(\sin(\delta)) H_i(\delta) d\delta \tag{A.10}$$

$$\mathbf{w}'^{i}_{2t,m} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathbf{I}'^{2}_{m,i} \cos(\delta_s) P_{2t}^{m}(\sin(\delta)) H_i(\delta) d\delta \tag{A.11}$$

where

$$\mathbf{I}^{1}_{0,i} = \int_{\lambda_s}^{\lambda_r} \hat{\mathbf{u}} d\lambda \tag{A.12}$$

$$= \begin{bmatrix} \cos(\delta_s)\,(\sin(\lambda_r) - \sin(\lambda_s)) \\ -\cos(\delta_s)\,(\cos(\lambda_r) - \cos(\lambda_s)) \\ \sin(\delta_s)\Delta\lambda \end{bmatrix} \tag{A.13}$$

$$\mathbf{I}^{2}_{0,i} = \int_{\lambda_s}^{\lambda_r} \cos(\lambda)\hat{\mathbf{u}}\hat{\mathbf{u}} d\lambda \tag{A.14}$$

$$\text{Entry }(1,1) = \frac{1}{2}\cos^2(\delta_s)\left[\Delta\lambda + \frac{1}{2}(\sin 2\lambda_r - \sin 2\lambda_s)\right] \tag{A.15}$$

$$\text{Entry }(1,2) = -\frac{1}{4}\cos^2(\delta_s)\,[\cos 2\lambda_r - \cos 2\lambda_s] \tag{A.16}$$

$$\text{Entry }(1,3) = \sin(\delta_s)\cos(\delta_s)\,(\sin(\lambda_r) - \sin(\lambda_s)) \tag{A.17}$$

$$\text{Entry }(2,2) = \frac{1}{2}\cos^2(\delta_s)\left[\Delta\lambda - \frac{1}{2}(\sin 2\lambda_r - \sin 2\lambda_s)\right] \tag{A.18}$$

$$\text{Entry }(2,3) = -\sin(\delta_s)\cos(\delta_s)\,(\cos(\lambda_r) - \cos(\lambda_s)) \tag{A.19}$$

$$\text{Entry }(3,3) = \sin^2(\delta_s)\Delta\lambda \tag{A.20}$$

where $\Delta\lambda = \lambda_r - \lambda_s$.

For the case $m = 1$ there are two sets of integrals, for the cosine and for the sine term.

Cosine $m = 1$ term

$$\mathbf{I}^1_{1,i} = \int_{\lambda_s}^{\lambda_r} \cos(\lambda)\hat{\mathbf{u}} d\lambda \tag{A.21}$$

$$= \begin{bmatrix} \frac{1}{2}\cos(\delta_s)\left[\Delta\lambda + \frac{1}{2}(\sin 2\lambda_r - \sin 2\lambda_s)\right] \\ -\frac{1}{4}\cos(\delta_s)\left[\cos 2\lambda_r - \cos 2\lambda_s\right] \\ \sin(\delta_s)\left(\sin(\lambda_r) - \sin(\lambda_s)\right) \end{bmatrix} \tag{A.22}$$

$$\mathbf{I}^2_{1,i} = \int_{\lambda_s}^{\lambda_r} \cos(\lambda)\hat{\mathbf{u}}\hat{\mathbf{u}} d\lambda \tag{A.23}$$

$$\text{Entry } (1,1) = \frac{1}{3}\cos^2(\delta_s) \times \left[\sin\lambda_r\left(\cos^2\lambda_r + 2\right) - \sin\lambda_s\left(\cos^2\lambda_s + 2\right)\right] \tag{A.24}$$

$$\text{Entry } (1,2) = \frac{-1}{3}\cos^2(\delta_s)\left[\cos^3\lambda_r - \cos^3\lambda_s\right] \tag{A.25}$$

$$\text{Entry } (1,3) = \frac{1}{2}\sin(\delta_s)\cos(\delta_s)\left[\Delta\lambda + \frac{1}{2}(\sin 2\lambda_r - \sin 2\lambda_s)\right] \tag{A.26}$$

$$\text{Entry } (2,2) = \frac{1}{3}\cos^2(\delta_s)\left[\sin^3\lambda_r - \sin^3\lambda_s\right] \tag{A.27}$$

$$\text{Entry } (2,3) = -\frac{1}{4}\sin(\delta_s)\cos(\delta_s)\left[\cos 2\lambda_r - \cos 2\lambda_s\right] \tag{A.28}$$

$$\text{Entry } (3,3) = \sin^2(\delta_s)\left(\sin(\lambda_r) - \sin(\lambda_s)\right) \tag{A.29}$$

Sine $m = 1$ term

$$\mathbf{I}'^1_{1,i} = \int_{\lambda_s}^{\lambda_r} \sin(\lambda)\hat{\mathbf{u}} d\lambda \tag{A.30}$$

$$= \begin{bmatrix} -\frac{1}{4}\cos(\delta_s)\left[\cos 2\lambda_r - \cos 2\lambda_s\right] \\ \frac{1}{2}\cos(\delta_s)\left[\Delta\lambda - \frac{1}{2}(\sin 2\lambda_r - \sin 2\lambda_s)\right] \\ -\sin(\delta_s)\left(\cos(\lambda_r) - \cos(\lambda_s)\right) \end{bmatrix} \tag{A.31}$$

$$\mathbf{I}'^2_{1,i} = \int_{\lambda_s}^{\lambda_r} \sin(\lambda)\hat{\mathbf{u}}\hat{\mathbf{u}} d\lambda \tag{A.32}$$

$$\text{Entry } (1,1) = \frac{-1}{3}\cos^2(\delta_s)\left[\cos^3\lambda_r - \cos^3\lambda_s\right] \tag{A.33}$$

$$\text{Entry } (1,2) = \frac{1}{3}\cos^2(\delta_s)\left[\sin^3\lambda_r - \sin^3\lambda_s\right] \tag{A.34}$$

$$\text{Entry } (1,3) = -\frac{1}{4}\sin(\delta_s)\cos(\delta_s)\left[\cos 2\lambda_r - \cos 2\lambda_s\right] \tag{A.35}$$

$$\text{Entry } (2,2) = \frac{-1}{3}\cos^2(\delta_s) \times \left[\cos\lambda_r\left(\sin^2\lambda_r + 2\right) - \cos\lambda_s\left(\sin^2\lambda_s + 2\right)\right] \tag{A.36}$$

$$\text{Entry } (2,3) = \frac{1}{2}\sin(\delta_s)\cos(\delta_s)\left[\Delta\lambda - \frac{1}{2}(\sin 2\lambda_r - \sin 2\lambda_s)\right] \tag{A.37}$$

$$\text{Entry } (3,3) = -\sin^2(\delta_s)\left(\cos(\lambda_r) - \cos(\lambda_s)\right) \tag{A.38}$$

Once we have computed these values we can derive the coefficients by the following formulas:

$$\mathbf{C}_{l,m} = -2N_{l,m}^2 \sum_{i=1}^{N} \boldsymbol{r_i} \times [\{\rho s(2\hat{\boldsymbol{n}}_{\boldsymbol{i}}\hat{\boldsymbol{n}}_{\boldsymbol{i}} - \boldsymbol{U}) + \boldsymbol{U}\} \cdot \mathbf{w}_{l,m^i} \cdot \hat{\boldsymbol{n}}_{\boldsymbol{i}} + a_2 \hat{\boldsymbol{n}}_{\boldsymbol{i}}\hat{\boldsymbol{n}}_{\boldsymbol{i}} \cdot \mathbf{v}_{l,m^i}]A_i$$

$$\mathbf{D}_{l,m} = -2N_{l,m}^2 \sum_{i=1}^{N} \mathbf{r_i} \times [\{\rho s(2\hat{\boldsymbol{n}}_{\boldsymbol{i}}\hat{\boldsymbol{n}}_{\boldsymbol{i}} - \mathbf{U}) + \mathbf{U}\} \cdot \mathbf{w}_{l,m'^i} \cdot \hat{\boldsymbol{n}}_{\boldsymbol{i}} + a_2 \hat{\boldsymbol{n}}_{\boldsymbol{i}}\hat{\boldsymbol{n}}_{\boldsymbol{i}} \cdot \mathbf{v}_{l,m'^i}]A_i$$

Note, we have not stated the closed form of the integrals given in Eqs. A.8–A.11, although they are available for some special cases in terms of elliptic integrals (Scheeres 2007). For convenience, these can be computed numerically once a shape is given.

References

Bottke, W.F., Vokrouhlický, D., Rubincam, D.P., Brož, M.: The effect of Yarkovsky thermal forces on the dynamical evolution of asteroids and meteoroids. In: Bottke, et al. (eds.) Asteroids III, pp. 405 (2002)

Čapek, D., Vokrouhlický, D.: The YORP effect with finite thermal conductivity. Icarus **172**, 526–536 (2004)

Greenwood, D.T.: Advanced Dynamics. Cambridge University Press (2003)

Hudson, R.S., Ostro, S.J.: Shape of asteroid 4769 Castalia (1989 PB) from inversion of radar images. Science **263**, 940 (1994)

Kaasalainen, M. et al.: Acceleration of the rotation of asteroid 1862 Apollo by radiation torques. Nature **446**, 420–422 (2007)

Kaula, W.M.: Theory of Satellite Geodesy: Applications of Satellites to Geodesy. Dover Publications, Inc., Mineola, New York (2000)

Lowry, S. et al.: Direct detection of the Asteroidal YORP effect. Science **316**, 272–274 (2007)

MacRobert, T.M.: Spherical Harmonics—An Elementary Treatise on Harmonic Functions. Methuen & Co. Ltd. (1947)

McInnes, C.R.: Solar Sailing: Technology, Dynamics and Mission Applications. Springer-Praxis, Chichester, UK (1999)

Mirrahimi, S.: Rotational Dynamics of a Solar System Body Under Solar Radiation Torques. Internship research report, The University of Michigan (2007)

Nesvorný, D., Vokrouhlický, D.: Analytic theory of the YORP effect for near-spherical objects. Astronom J **134**, 1750–1768 (2007)

Ostro, S.J. et al.: Radar and Optical Observations of Asteroid 1998 KY26. Science **285**, 557–559 (1999)

Rubincam, D.P.: Asteroid orbit evolution due to thermal lag. J. Geoph. Res. **100**(E1), 1585–1594 (1995)

Scheeres, D.J. et al.: Dynamical configuration of binary near-earth asteroid (66391) 1999 KW4. Science **314**, 1280–1283 (2006)

Scheeres, D.J.: The dynamical evolution of uniformly rotating asteroids subject to YORP. Icarus **188**, 430–450 (2007)

Spencer, D. et al.: Systematic biases in radiometric diameter determinations. Icarus **78**, 337–354 (1989)

Taylor, P. et al.: Spin rate of asteroid (54509) 2000 PH5 increasing due to the YORP effect. Science **316**, 274–277 (2007)

Vokrouhlický, D., Čapek, D.: YORP-Induced long-term evolution of the spin state of small asteroids and meteoroids: Rubincam's approximation. Icarus **159**, 449–467 (2002)