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ORIGINAL PAPER

Rough approximation models via graphs based on neighborhood systems

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Abstract

Neighborhood systems are used to approximate graphs as finite topological structures. Throughout this article, we construct new types of eight neighborhoods for vertices of an arbitrary graph, say, j-adhesion neighborhoods. Both notions of Allam et al. and Yao will be extended via j-adhesion neighborhoods. We investigate new types of j-lower approximations and j-upper approximations for any subgraph of a given graph. Then, the accuracy of these approximations will be calculated. Moreover, a comparison between accuracy measures and boundary regions for different kinds of approximations will be discussed. To generate j-adhesion neighborhoods and rough sets on graphs, some algorithms will be introduced. Finally, a sample of a chemical example for Walczak will be introduced to illustrate our proposed methods.

Keywords Neighborhood system · Rough sets · Lower approximations · Upper approximations · Graphs · j-Accuracy measure

1 Introduction

Motivated by many analyzes requiring rough sets, the present paper aims for a new approach to the study of rough sets from the points of view of both neighborhood systems and graphs. Neighborhood systems on graphs based on rough sets are a generalization of Pawlak's rough set model.

Rough set theory was initially developed (Pawlak 1981) as a new mathematical methodology to deal with the vagueness and uncertainty in information systems. Many proposals made for generalizing and interpreting rough sets (Orlowska and Pawlak 1984; Pomykala 1987; Skowron and Stepaniuk 1996; Yao and Line 1996; Zirako 1994). Some applicable examples of real-life fields of the rough set method can be cited such as in Process Control, Economics, Medical Diagnosis, Biochemistry, Environmental Science, Biology, Chemistry, Psychology, Conflict Analysis, Pharmacology, Banking, Market Research, Engineering, Speech Recognition, Material Science, Information Analysis, Data Analysis, Data Mining, Control and Linguistics and many other fields [See, (Allam et al. 2005; Benouini et al. 2020; Dong et al. 2004; Jensen and Shen 2004; Leung et al. 2006; Pal and Mitra 2004; Yao and Chen 2005; Zhao and Liu 2011; Zhan et al. 2019)]. In 1999, Yao (1999) introduced generalized rough sets through a binary relation; while, these approximations are not satisfied with Pawlak's properties that were applied on an equivalence relation. For this reason, Zhu (2007) studied rough approximations that depend on general relations. These approximations help to prove some properties that were not easy to prove in the classical case. From this time onwards, many types of approximations are investigated. In 2008, Abu-Donia (2008) discussed three types of lower approximations and upper approximations with respect to any binary relation based on the right neighborhoods. This generalization of approximations converted into two ways via a finite number of binary relations. In 2014, Abd El-Monsef et al. (2015) presented the main ideas about the concept of j-neighborhood systems and studied eight approaches for approximating rough sets. Many researchers studied the j-neighborhood systems on different spaces such as Abbas et al. (2016); Amer et al. (2017); Atef et al.

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(2020); Hosny (2018); Huang and Li (2018); Kozae et al. (2019).

Graph theory (Chartrand et al. 2016) is an important mathematical tool in diverse subjects. A graph $G = (V, E)$, is an ordered pair of different sets (V, E), where V is a nonempty set and E is a subset of unordered pairs of V. The vertices and edges of a graph G are the elements $V = V(G)$ and $E = E(G)$, respectively. A graph G is finite (respectively, infinite) if the set $V(G)$ is finite (respectively, infinite). The degree of a vertex $u \in V(G)$ is the number of edges containing u. If there is no edge in a graph G but contains a vertex u, then u is called an isolated point, and so the degree of u is zero. An edge that has the same vertex to end is called a loop, and the edge with a distinct end is called a link. A graph is simple if it has no loop and no pair of its links join the same pair of vertices. A graph that has no edge is called a null graph. A directed graph is a graph in which edges have a certain way. In addition, an undirected graph is a graph in which edges have no way. Many scholars work on the theory of graphs and applied it in many fields, see (Akram and Zafar 2018; Atef et al. 2020; Liu et al. 2020; Qin et al. 2018; Malik et al. 2018; Malik and Akram 2018; Mandal and Ranadive 2019; William-West and Singh 2018). In 2018, Nada et al. (2018) initiated the study on topological structures via graphs based on the right neighborhoods. Recently, the neighborhood systems, rough sets on graphs are used to represent structures such as self-similar fractals (El Atik and Nasef 2020) and human heart (El Atik and Nasef 2020) which are useful in physics and medicine, respectively.

As a continuation of the development in the use of general relations, we construct in the present paper new types of j-adhesion neighborhoods from adjacent vertices of general graphs. Based on j-adhesion neighborhoods, we define j-lower approximations and j-upper approximations and the comparison between them and some other types of lower approximations and upper approximations will be discussed. In Sect. 2, we present the fundamental concepts and properties of that used in this paper. In Sect. 3, we introduce the new concepts of j-adhesion neighborhoods and study their basic properties and examples. The goal of Sect. 4 is to generalize some of Pawlak's properties. A comparison between the proposed method and the previous one is shown in Sect. 5. Finally, we apply the results on a sample that is deduced from a reduction by similarity (El Atik 2020) for Walczak's example in chemistry.

2 Basic concepts and properties

In this section, some basic notions of rough sets, graph theory, and a j-neighborhood system will be presented.

Definition 1 (Yao 1999) Let R be a binary relation on a nonempty set U and $A \subseteq U$. Lower approximations and upper approximations of A are defined by

$\underline{R}(A) = \{x \in U : xR \subseteq A\}$, and
$\overline{R}(A) = \{x \in U : xR \cap A \neq \phi\}$, where
$xR = \{y \in U : xRy\}$.

The following properties of lower approximations and upper approximations for Pawlak (Pawlak 1982; Pawlak and Skowron 1994; Pawlak 1997) will be stated.

(L1) $\underline{R}(X) \subseteq X$.
(L2) $\underline{R}(\phi) = \phi$.
(L3) $\underline{R}(U) = U$.
(L4) $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$.
(L5) If $X \subseteq Y$, then $\underline{R}(X) \subseteq \underline{R}(Y)$.
(L6) $\underline{R}(X) \cup \underline{R}(Y) \subseteq \underline{R}(X \cup Y)$.
(L7) $\underline{R}(X^c) = (\overline{R}(X))^c$.
(L8) $\underline{R}(\underline{R}(X)) = \underline{R}(X)$.
(L9) $\underline{R}((\underline{R}(X))^c) = (\underline{R}(X))^c$.
(U1) $X \subseteq \overline{R}(X)$.
(U2) $\overline{R}(\phi) = \phi$.
(U3) $\overline{R}(U) = U$.
(U4) $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$.
(U5) If $X \subseteq Y$, then $\overline{R}(X) \subseteq \overline{R}(Y)$.
(U6) $\overline{R}(X) \cap \overline{R}(Y) \supseteq \overline{R}(X \cap Y)$.
(U7) $\overline{R}(X^c) = (\underline{R}(X))^c$.
(U8) $\overline{R}(\overline{R}(X)) = \overline{R}(X)$.
(U9) $\overline{R}((\overline{R}(X))^c) = (\overline{R}(X))^c$.

Definition 2 (Abu-Donia and Salama 2012) Let R be a binary relation on U and $A \subseteq U$. Then, the following properties are held.

(1) Roughly R-definable, if $\underline{R}(A) \neq \phi$ and $\overline{R}(A) \neq X$;
(2) Internally R-undefinable, if $\underline{R}(A) = \phi$ and $\overline{R}(A) \neq X$;
(3) Externally R-undefinable, if $\underline{R}(A) \neq \phi$ and $\overline{R}(A) = X$;
(4) Totally R-undefinable, if $\underline{R}(A) = \phi$ and $\overline{R}(A) = X$.

Definition 3 (Nada et al. 2018) Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. Lower approximations and upper approximations of $V(H)$ are defined by

$\underline{\mathscr{R}}(V(H)) = \{x \in V(G) : xR \subseteq V(H)\}$, and
$\overline{\mathscr{R}}(V(H)) = \{x \in V(G) : xR \cap V(H) \neq \phi\}$.

Definition 4 (Yao 1999 and Allam et al. 2005) Let $G = (V(G), E(G))$ be a graph, for each $x \in V(G)$. The j-neighborhood systems for x, $\forall\ j \in \{r, l, <r>, <l>, u, i, <u>, <i>\}$ are defined by

(1) $N_r(x) = \{y \in V(G) : xRy\}$.

(2) $N_l(x) = \{y \in V(G) : yRx\}$.
(3) $N_{<r>}(x) = \bigcap_{x \in N_r(y)} N_r(y)$.
(4) $N_{<l>}(x) = \bigcap_{x \in N_l(y)} N_l(y)$.
(5) $N_u(x) = N_r(x) \cup N_l(x)$.
(6) $N_i(x) = N_r(x) \cap N_l(x)$.
(7) $N_{<u>}(x) = N_{<r>}(x) \cup N_{<l>}(x)$.
(8) $N_{<i>}(x) = N_{<r>}(x) \cap N_{<l>}(x)$.

3 Rough approximation model via graphs using *j*-neighborhood systems

In this section, we extent Definition 1 of Yao in terms of j-neighborhood systems which are defined in Definition 4. We first give an example to illustrate j-neighborhood systems from a simple graph.

Example 1 Let G be a simple graph as shown in Fig. 1. j-neighborhood systems are defined as follows:
Take $j = \{r\}$ and $j \in \{l, <r>, <l>, u, i, <u>, <i>\}$, we have

(i) If $j = \{r, l, u, i\}$, then $N_j(a) = \{b, e\}$, $N_j(b) = \{a, c, d\}$, $N_j(c) = \{b, d\}$, $N_j(d) = \{b, c, e\}$, $N_j(e) = \{a, d\}$.
(ii) If $j = \{<r>, <l>, <u>, <i>\}$, then $N_j(a) = \{a, d\}$, $N_j(b) = \{b\}$, $N_j(c) = \{c\}$, $N_j(d) = \{d\}$, $N_j(e) = \{b, e\}$.

Definition 5 Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. Define a first type of lower approximations and upper approximations of H which are denoted by $\underline{N_j}(V(H))$ and $\overline{N_j}(V(H))$, respectively.

$\underline{N_j}(V(H)) = \{x \in V(G) : N_j(x) \subseteq V(H)\}$, and
$\overline{N_j}(V(H)) = V(H) \bigcup \{x \in V(G) : N_j(x) \cap V(H) \neq \phi\}$.

Remark 1 If $j = r$ in Definition 5, then we have approximations in Definition 1.

Definition 6 Let G= ($V(G)$, $E(G)$) be a graph and H be a subgraph of G. Define the j-boundary, j-positive, j-negative regions and j-accuracy measure of H in terms of j-adhesion neighborhood which will be denoted by BND_{P_j}, POS_{P_j}, NEG_{P_j} and α_{Nj}, respectively.

(1) $BND_{N_j}(V(H)) = \overline{N_j}(V(H)) - \underline{N_j}(V(H))$.
(2) $POS_{N_j}(V(H)) = \underline{N_j}(V(H))$.
(3) $NEG_{N_j}(V(H)) = V(G) - \overline{N_j}(V(H))$.
(4) $\alpha_{Nj}(V(H)) = \frac{|\underline{N_j}(V(H))|}{|\overline{\overline{N_j}}(V(H))|}$, $|\overline{N_j}(V(H))| \neq 0$.

Example 2 (Contine from Example 1). Take $j = r$, If $V(H) = \{b, d, e\}$, we have

(i) $\overline{N_r}(V(H)) = V(G)$ and $\underline{N_r}(V(H)) = \{a, c\}$.
(ii) $BND_{N_r}(V(H)) = \{b, d, e\}$, $POS_{N_r}(V(H)) = \{a, c\}$, $NEG_{N_r}(V(H)) = \{b, d, e\}$ and $\alpha_{Nj}(V(H)) = \frac{2}{5}$. For $j = l, <r>, <l>, u, i, <u>, <i>$, the results are also by the same manner.

Theorem 1 *Let $G = (V(G), E(G))$ be a graph and H and K be subgraphs of G. Then, the following properties are held.*

(1) $\underline{N_j}(V(G)) = V(G)$.
(2) If $V(H) \subseteq V(K)$, then $\underline{N_j}(V(H)) \subseteq \underline{N_j}(V(K))$.
(3) $\underline{N_j}(V(H) \cap V(K)) = \underline{N_j}(V(H)) \cap \underline{N_j}(V(K))$.
(4) $\underline{N_j}(V(H)) \cup \underline{N_j}(V(K)) \subseteq \underline{N_j}(V(H) \cup V(K))$.
(5) $\underline{N_j}(V(H)) = (\overline{N_j}(V(H))^c)^c$.
(6) $\overline{N_j}(\phi) = \phi$.
(7) If $V(H) \subseteq V(K)$, then $\overline{N_j}(V(H)) \subseteq \overline{N_j}(V(K))$.
(8) $\overline{N_j}(V(H) \cup V(K)) = \overline{P_j}(V(H)) \cup \overline{P_j}(V(K))$.
(9) $\overline{N_j}(V(H) \cap V(K)) \subseteq \overline{N_j}(V(H)) \cap \overline{N_j}(V(K))$.
(10) $\overline{N_j}(V(H)) = (\underline{N_j}(V(H))^c)^c$.

Proof It is sufficient to prove (1), (2), (3), (4), and (5) and the other proofs are obvious.

(1) Follows from Definition 5.
(2) If $V(H) \subseteq V(K)$, then we have $\underline{N_j}(V(H)) = \{v \in V(G) : N_j(v) \subseteq V(H)\} \subseteq \{v \in V(G) : N_j(v) \subseteq V(K)\} = \underline{N_j}(V(K))$.
(3) $\underline{N_j}(V(H \cap K)) = \{v \in V(G) : P_j(v) \subseteq V(H \cap K)\}$. Since $V(H \cap K) \subseteq V(H)$ and $V(H \cap K) \subseteq V(K)$, then $N_j(v) \subseteq V(H)$ and $N_j(v) \subseteq V(K)$. Thus, we have $\underline{N_j}(V(H \cap K)) \subseteq \underline{N_j}(V(H))$ and $\underline{N_j}(V(H \cap K)) \subseteq \underline{N_j}(V(K))$. Therefore, $\underline{N_j}(V(H)) \cap \underline{N_j}(V(K)) = \{v \in V(G) : N_j(v) \subseteq V(H)\}$

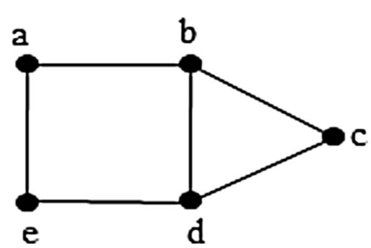

Fig. 1 A simple graph G

$$\cap \{v \in V(G) : N_j(v) \subseteq V(K)\}$$
$$= \{v \in V(G) : N_j(v) \subseteq (V(H) \cap V(K))\}$$
$$= \{v \in V(G) : N_j(v) \subseteq (V(H \cap K))\}$$
$$= \underline{N_j}(V(H \cap K)) = \underline{N_j}(V(H)) \quad \cap \underline{N_j}(V(K)).$$

(4) The proof is similar to (3).

(5) If $v \in \underline{N_j}(V(H))$ for every $v \in V(H)$, there exists $N_j(v) \subseteq V(H)$. Then, for every $v \in V(G) - [V(G) - V(H)]$, there exists $N_j(v)$ such that $N_j(v) \cap [V(G) - V(H)] = \phi$. So, $v \notin \overline{N_j}[V(G) - V(H)]$, $v \in V(G) - [\overline{N_j}(V(G) - V(H))]$. Therefore, $\underline{N_j}(V(H)) = V(G) - [\overline{N_j}(V(G) - V(H))] = (\overline{N_j}(V(H))^c)^c$.

□

Example 3 (Continue for Example 1). Take $j = l$ (and also $j \in \{r, <r>, <l>, u, i, <u>, <i>\}$ are similar).

(1) If $V(H) = \{c, d\}$, then $\overline{N_l}(V(H)) = \{b, c, d, e\}$ and $\underline{N_l}(V(H)) = \phi$.

(2) If $V(H) = \{a\}$ and $V(K) = \{a, d\}$, $V(H) \subseteq V(K)$, then $\overline{N_l}(V(H)) = \{b, e\}$, $\overline{N_l}(V(K)) = \{b, c, e\}$.

(3) If $V(H) = \{a, b\}$ and $V(K) = \{a, b, d\}$, $V(H) \subseteq V(K)$, then $\underline{N_l}(V(H)) = \phi$, $\underline{N_l}(V(K)) = \{c, e\}$.

(4) If $V(H) = \{b\}$ and $V(K) = \{a, d\}$, then $\overline{N_l}(V(H \cap K)) = \phi$. Hence, $\overline{N_l}(V(H)) = \{b, c, e\}$ and $\overline{N_l}(V(K)) = \{a, b, c, d, e\}$. So, $\overline{P_l}(V(H)) \cap \overline{P_l}(V(K)) = \{b, c, e\}$. Also, $\underline{P_l}(V(H \cup K)) = \{c, e\}$. Thus, $\underline{P_l}(V(H)) = \phi$ and $\underline{P_l}(V(K)) = \{e\}$. Therefore, $\underline{P_l}(V(H)) \cup \underline{P_l}(V(K)) = \{e\}$.

Remark 2 According to Nicoletti et al. (2001); Zafar and Akram (2018), we can construct new types of rough sets, say, j-rough graph. So, we can also establish new j-approximation graphs which will be denoted by $(V(G), N_j)$, $\forall$ $j \in \{r, l, <r>, <l>, u, i, <u>, <i>\}$. All properties of Pawlak rough approximation can also be satisfied by the same manner.

4 Generalized rough approximations via graphs using *j*-adhesion neighborhoods

In this section, j-adhesion neighborhoods on graphs are introduced. Also, new types of j-lower (respectively, j-upper) approximations will be presented and studied.

Definition 7 Let $G = (V(G), E(G))$ be a graph. For each $x \in V(G)$, j-adhesion neighborhoods are defined $\forall$ $j \in \{r, l, <r>, <l>, u, i, <u>, <i>\}$ as follows:

(1) $P_r(x) = \{y \in V(G) : xR = yR\}$.
(2) $P_l(x) = \{y \in V(G) : Rx = Ry\}$.
(3) $P_{<r>}(x) = \{y \in V(G) : \bigcap_{x \in yR} yR = \bigcap_{y \in xR} xR\}$.
(4) $P_{<l>}(x) = \{y \in V(G) : \bigcap_{x \in Ry} Ry = \bigcap_{y \in Rx} Rx\}$.
(5) $P_u(x) = P_r(x) \cup P_l(x)$.
(6) $P_i(x) = P_r(x) \cap P_l(x)$.
(7) $P_{<u>}(x) = P_{<r>}(x) \cup P_{<l>}(x)$.
(8) $P_{<i>}(x) = P_{<r>}(x) \cap P_{<l>}(x)$.

To illustrative Definition 7, we introduce Examples 4 and 5.

Example 4 (Continue for Example 1) Take $j = \{r, l, <r>, <l>, u, i, <u>, <i>\}$, we have $P_j(a) = \{a\}, P_j(b) = \{b\}, P_j(c) = \{c\}, P_r(d) = \{d\}, P_r(e) = \{e\}$.

Example 5 Let G be a directed graph as shown in Fig. 2. Then, the j-adhesion neighborhoods are

(i) If $j \in \{r\}$, then $P_j(a) = \{a, d\}, P_j(b) = \{b\}, P_j(c) = \{c\}$, $P_j(d) = \{a, d\}$.
(ii) If $j \in \{l, <r>, <i>\}$, then $P_j(a) = \{a\}, P_j(b) = \{b, c\}, P_j(c) = \{b, c\}$, $P_j(d) = \{d\}$.
(iii) If $j \in \{i\}$, then $P_j(a) = \{a\}, P_j(b) = \{b\}, P_j(c) = \{c\}$, $P_j(d) = \{d\}$.
(iv) If $j \in \{u, <l>, <u>\}$, then $P_j(a) = \{a, d\}$, $P_j(b) = \{b, c\}, P_j(c) = \{b, c\}$, $P_j(d) = \{a, d\}$.

Definition 8 Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. The second type of lower approximations and upper approximations of H which will be denoted by $\underline{P_j}(V(H))$ and $\overline{P_j}(V(H))$, respectively, is defined by

$$\underline{P_j}(V(H)) = \{x \in V(G) : P_j(x) \subseteq V(H)\}, \text{ and}$$
$$\overline{P_j}(V(H)) = V(H) \bigcup \{x \in V(G) : P_j(x) \cap V(H) \neq \phi\}.$$

Definition 9 Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. The j-boundary, j-positive, j-negative regions and j-accuracy measure of H in terms of j-adhesion

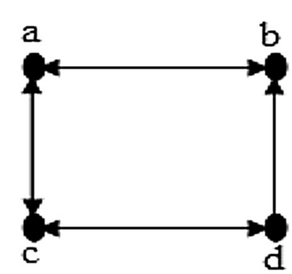

Fig. 2 A directed graph G

neighborhood which will be denoted by BND_{P_j}, POS_{P_j}, NEG_{P_j} and α_{Pj}, respectively, are defined by

(i) $BND_{P_j}(V(H)) = \overline{P_j}(V(H)) - \underline{P_j}(V(H))$.
(ii) $POS_{P_j}(V(H)) = \underline{P_j}(V(H))$.
(iii) $NEG_{P_j}(V(H)) = V(G) - \overline{P_j}(V(H))$.
(iv) $\alpha_{Pj}(V(H)) = \frac{|\underline{P_j}(V(H))|}{|\overline{P_j}(V(H))|}$, $|\overline{P_j}(V(H))| \neq 0$.

Example 6 (Continue from Example 5). Take $j = r$. If $V(H) = \{a, c\}$, then we have

(i) $\overline{P_r}(V(H)) = \{a, c, d\}$ and $\underline{P_r}(V(H)) = \{c\}$.
(ii) BND_{P_r} $(V(H)) = \{a, d\}$, POS_{P_r} $(V(H)) = \{c\}$, NEG_{P_r} $(V(H)) = \{b\}$ and α_{Pj} $(V(H)) = \frac{1}{3}$.
For $j = l, <r>, <l>, u, i, <u>, <i>$, we have the results by similarity.

Theorem 2 *Let $G = (V(G), E(G))$ be a graph and H and K be subgraphs of G. Then, the following properties are held.*

(1) $\underline{P_j}(\phi) = \phi$.
(2) $\underline{P_j}(V(G)) = V(G)$.
(3) $\underline{P_j}(V(H)) \subseteq V(H)$.
(4) If $V(H) \subseteq V(K)$, then $\underline{P_j}(V(H)) \subseteq \underline{P_j}(V(K))$.
(5) $\underline{P_j}(\underline{P_j}(V(H)))=\underline{P_j}(V(H))$.
(6) $\underline{P_j}(V(H) \cap V(K))=\underline{P_j}(V(H)) \cap \underline{P_j}(V(K))$.
(7) $\underline{P_j}(V(H)) \cup \underline{P_j}(V(K)) \subseteq \underline{P_j}(V(H) \cup V(K))$.
(8) $\underline{P_j}(V(H)) = (\overline{P_j}(V(H))^c)^c$.
(9) $\overline{P_j}(\phi) = \phi$.
(10) $\overline{P_j}(V(G)) = V(G)$.
(11) $V(H) \subseteq \overline{P_j}(V(H))$.
(12) If $V(H) \subseteq V(K)$, then $\overline{P_j}(V(H)) \subseteq \overline{P_j}(V(K))$.
(13) $\overline{P_j}(\overline{P_j}(V(H))) = \overline{P_j}(V(H))$.
(14) $\overline{P_j}(V(H) \cup V(K)) = \overline{P_j}(V(H)) \cup \overline{P_j}(V(K))$.
(15) $\overline{P_j}(V(H) \cap V(K)) \subseteq \overline{P_j}(V(H)) \cap \overline{P_j}(V(K))$.
(16) $\overline{P_j}(V(H)) = (\underline{P_j}(V(H))^c)^c$.

Proof It is sufficient to prove properties (1), (2), (3), (4), (5), (6), (7), and (8) and other proofs are obvious.

(1) $\underline{P_j}(\phi) = \{v \in V(G) : P_j(v) \subseteq \phi\} = \phi$.
(2) Follows from property (1) and Definition 8.
(3) Follows from Definition 8.
(4) If $V(H) \subseteq V(K)$, then we have $\underline{P_j}$ $(V(H)) = \{v \in V(G) : P_j(v) \subseteq V(H)\} \subseteq \{v \in V(G) : P_j(v) \subseteq V(K)\} = \underline{P_j}(V(K))$.
(5) Follows from property (3) and Definition 8.
(6) $\underline{P_j}(V(H \cap K)) = \{v \in V(G) : P_j(v) \subseteq V(H \cap K)\}$. Since $V(H \cap K) \subseteq V(H)$ and $V(H \cap K) \subseteq V(K)$, then $P_j(v) \subseteq V(H)$ and $P_j(v) \subseteq V(K)$. Thus, by property (4), we have $\underline{P_j}(V(H \cap K)) \subseteq \underline{P_j}$ (V(H)) and $\underline{P_j}(V(H \cap K)) \subseteq \underline{P_j}(V(K))$. Therefore, $\underline{P_j}(V(H)) \cap \underline{P_j}(V(K)) = \{v \in V(G) : P_j(v) \subseteq V(H)\} \cap \{v \in V(G) : P_j(v) \subseteq V(K)\} = \{v \in V(G) : P_j(v) \subseteq (V(H) \cap V(K))\} = \{v \in V(G) : P_j(v) \subseteq (V(H \cap K))\} = \underline{P_j}$ $(V(H \cap K)) = \underline{P_j}(V(H)) \cap \underline{P_j}(V(K))$.
(7) The proof is similar to property (6).
(8) If $v \in \underline{P_j}$ (V(H)) for every $v \in V(H)$, there exists $P_j(v) \subseteq V(H)$. Then, for every $v \in V(G) - [V(G) - V(H)]$, there exists $P_j(v)$ such that $P_j(v) \cap [V(G) - V(H)] = \phi$. So, $v \notin \overline{P_j}$ $[V(G) - V(H)]$, $v \in V(G) - [\overline{P_j}(V(G) - V(H))]$. Therefore, $\underline{P_j}(V(H)) = V(G) - [\overline{P_j}(V(G) - V(H))] = (\overline{P_j}(V(H))^c)^c$.

□

Example 7 (Continuing from Example 5) Take $j = l$. Then

(i) If $V(H) = \{c, d\}$, then $\overline{P_l}$ $(V(H)) = \{b, c, d\}$ and $\underline{P_l}$ $(V(H)) = \{d\}$.
(ii) If $V(H) = \{a\}$ and $V(K) = \{a, b\}$, $V(H) \subseteq V(K)$, then $\overline{P_l}$ $(V(H)) = \{a\}$, $\overline{P_l}(V(K)) = \{a, b, c\}$.
(iii) If $V(H) = \{a, b\}$ and $V(K) = \{a, b, d\}$, $V(H) \subseteq V(K)$, then $\underline{P_l}$ $(V(H)) = \{a\}$, $\underline{P_l}$ $(V(K)) = \{a, d\}$.
(iv) If $V(H) = \{b\}$ and $V(K) = \{a, c\}$, then $\overline{P_l}$ $(V(H \cap K)) = \phi$. Hence, $\overline{P_l}(V(H)) = \{b, c\}$ and $\overline{P_l}$ $(V(K)) = \{a, b, c\}$. So, $\overline{P_l}$ $(V(H)) \cap \overline{P_l}$ $(V(K)) = \{b, c\}$. Also, $\underline{P_l}$ $(V(H \cup K)) = \{a, b, c\}$. Thus, $\underline{P_l}$ $(V(H)) = \phi$ and $\underline{P_l}(V(K)) = \{a\}$. Therefore, $\underline{P_l}(V(H)) \cup \underline{P_l}$ $(V(K)) = \{a\}$.
For $j = l, <r>, <l>, u, i, <u>, <i>$, we have the results by similarity .

Theorem 3 *Let $G = (V(G), E(G))$ be a graph and $H, K \subseteq G$. Then, the following properties are held.*

(1) $BND_{P_j}(V(H)) = \overline{P_j}$ (V(H)) $\cap \overline{P_j}$ $(V(G) - V(H))$.
(2) BND_{P_j} $(V(H)) = BND_{P_j}$ $(V(G) - V(H))$.
(3) $\overline{P_j}$ $(V(H)) = V(H) \cup BND_{P_j}$ (V(H)).
(4) $\underline{P_j}$ $(V(H)) = V(H) - BND_{P_j}$ (V(H)).
(5) BND_{P_j} $(V(H)) \cap \underline{P_j}(V(H)) = \phi$.
(6) BND_{P_j} $(V(H) \cup V(K)) \subseteq BND_{P_j}$ (V(H)) $\cup$ BND_{P_j} (V(K)).
(7) BND_{P_j} $(V(H) \cap V(K)) \subseteq BND_{P_j}(V(H)) \cup BND_{P_j}$ (V(K)).
(8) BND_{P_j} $(\overline{P_j}$ $(V(H))) \subseteq BND_{P_j}(V(H)$.

(9) $BND_{P_j}(\underline{P_j}(V(H))) \subseteq BND_{P_j}(V(H)$.

(10) $BND_{P_j}(BND_{P_j}(V(H))) \subseteq BND_{P_j}(V(H))$.

Proof

(1) $BND_{P_j}(V(H)) = \overline{P_j}(V(H)) - \underline{P_j}(V(H)) = \overline{P_j}(V(H)) \cap (\underline{P_j}(V(H)))^c = \overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(H))$.

(2) $BND_{P_j}(V(H)) = \overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(H)) = \overline{P_j}(V(G) - (V(G) - V(H))) \cap \overline{P_j}(V(G) - V(H)) = BND_{P_j}(V(G) - V(H))$.

(3) $V(H) \cup BND_{P_j}(V(H)) = V(H) \cup (\overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(H))) = [V(H) \cup \overline{P_j}(V(H))] \cap [V(H) \cup \overline{P_j}(V(G) - V(H))] = \overline{P_j}(V(H)) \cap [V(H) \cup (\underline{P_j}(V(H)))^c] = \overline{P_j}(V(H)) \cap V(G) = \overline{P_j}(V(H))$.

(4) $V(H) - BND_{P_j}(V(H)) = V(H) - [\overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(H))] = V(H) \cap [\overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(H))]^c = V(H) \cap [\overline{P_j}(V(H))]^c \cup [\overline{P_j}(V(G) - V(H))]^c = [V(H) \cap \underline{P_j}(V(G) - V(H))] \cup [V(H) \cap \underline{P_j}(V(H))] = \phi \cup \underline{P_j}(V(H)) = \underline{P_j}(V(H))$.

(5) Follows from Definitions 8 and 9.

(6) $BND_{P_j}(V(H) \cup V(K)) = \overline{P_j}(V(H) \cup V(K)) \cap \overline{P_j}(V(G) - (V(H) \cup V(K))) \subseteq [\overline{P_j}(V(H)) \cup \overline{P_j}(V(K))] \cap [\overline{P_j}(V(G) - V(H)) \cap \overline{P_j}(V(G) - V(K))] = [\overline{P_j}(V(H)) \cup \overline{P_j}(V(K)) \cap \overline{P_j}(V(G) - V(H))] \cap [\overline{P_j}(V(G) - V(K))] = [(\overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(H))) \cup (\overline{P_j}(V(K)) \cap \overline{P_j}(V(G) - V(K)))] \cap \overline{P_j}(V(G) - V(H)) = [(\overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(H))) \cap \overline{P_j}(V(G) - V(K))] \cup [(\overline{P_j}(V(K)) \cap \overline{P_j}(V(G) - V(K))) \cap \overline{P_j}(V(G) - V(K))] = [BND_{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(K))] \cup [BND_{P_j}(V(K)) \cap \overline{P_j}(V(G) - V(K))] \subseteq BND_{P_j}(V(H)) \cup BND_{P_j}(V(K))$. Therefore, $BND_{P_j}(V(H) \cup V(K)) \subseteq BND_{P_j}(V(H)) \cup BND_{P_j}(V(K))$.

(7) $BND_{P_j}(V(H) \cap V(K)) = \overline{P_j}(V(H) \cap V(K)) \cap \overline{P_j}(V(G) - (V(H) \cap V(K))) \subseteq [\overline{P_j}(V(H)) \cap \overline{P_j}(V(K))] \cap [\overline{P_j}(V(G) - V(H)) \cup \overline{P_j}(V(G) - V(K))] = [\overline{P_j}(V(H)) \cap \overline{P_j}(V(K)) \cap \overline{P_j}(V(G) - V(H))] \cup [\overline{P_j}(V(H)) \cap \overline{P_j}(V(K)) \cap \overline{P_j}(V(G) - V(K))] = [BND_{P_j}(V(H)) \cap \overline{P_j}(V(K))] \cup [BND_{P_j}(V(K)) \cap \overline{P_j}(V(H))] \subseteq BND_{P_j}(V(H)) \cup BND_{P_j}(V(K))$.

(8) $BND_{P_j}(\overline{P_j}(V(H))) = \overline{P_j}(\overline{P_j}(V(H))) \cap \overline{P_j}(V(G) - \overline{P_j}(V(H))) = \overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - \overline{P_j}V(H)) \subseteq \overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(H)) = BND_{P_j}(V(H))$. Since $V(H) \subseteq \overline{P_j}(V(H))$, then $(\overline{P_j}(V(H)))^c \subseteq (V(H))^c$ and hence $\overline{P_j}(V(G) - \overline{P_j}(V(H))) \subseteq \overline{P_j}(V(G) - V(H))$. Thus, $BND_{P_j}(\overline{P_j}(V(H))) \subseteq BND_{P_j}(V(H)$.

(9) $BND_{P_j}(\underline{P_j}(V(H))) = \overline{P_j}(\underline{P_j}(V(H))) \cap \overline{P_j}(V(G) - \underline{P_j}(V(H))) \subseteq \overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - \underline{P_j}(V(H))) \subseteq \overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(H)) = BND_{P_j}(V(H))$. Since $\underline{P_j}(V(H)) \subseteq V(H)$, then $\overline{P_j}(\underline{P_j}(V(H))) \subseteq \overline{P_j}(V(H))$. So, $BND_{P_j}(\underline{P_j}(V(H))) \subseteq BND_{P_j}(V(H)$.

(10) $BND_{P_j}(BND_{P_j}(V(H))) = BND_{P_j}(\overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - \underline{P_j}(V(H)))) = \overline{P_j}[\overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - \underline{P_j}(V(H)))] \cap \overline{P_j}[V(G) - (\overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - \underline{P_j}(V(H))))] \subseteq \overline{P_j}[V(H) \cap \overline{P_j}(V(G) - V(H))] \cap \overline{P_j}[\overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - \overline{P_j}(V(H))) \cup \overline{P_j}((V(G)) \cap \overline{P_j}(V(G) - V(H)))] = \overline{P_j}[\overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(H))] \cap [\overline{P_j}(\underline{P_j}(V(G) - V(H) \cup \overline{P_j}(\underline{P_j}(V(H)))))] = \overline{P_j}(V(H)) \cap \overline{P_j}(V(G) - V(H)) = BND_{P_j}(V(H))$.

□

Remark 3 According to Nicoletti and Zafar results in (Nicoletti et al. 2001; Zafar and Akram 2018), new types of rough sets ,say, j-rough graphs and new j-approximation graphs, say, $(V(G), P_j)$, $\forall \; j \in \{r, l, <r>, <l>, u, i, <u>, <i>\}$ can be constructed. Also, all Pawlak's rough approximation properties can be studied by the same manner.

In algorithm 1, we establish the system of j-adhesion neighborhoods from any graphs in terms of adjacent vertices and their adjacent matrix. Moreover, two vertices are neighbors if they are adjacent.

Algorithm1. An algorithm for the j-adhesion neighborhoods

```
Input: A graph G = (V, E).
Output: j-adhesion neighborhoods of V(G).
for (i ∈ |V(G)|)
    Enter V(G).
endfor
for (i ∈ |V(G)|)
    for (j ∈ |V(G)|)
        Enter Adjacency Matrix of V(G).
    endfor
endfor
for (i ∈ |V(G)|)
    for (j ∈ |V(G)|).
        if(A[i, j]! = 0)
        Class[i, j]=x_j.
        if (Neighbor == r)
            P_r(x_i)= {v ∈ V(G) : x_j = v_j}.
            if(v_j == x_j)
                P_r(x_i)=v_i.
            endif
        endif
        else if (Neighbor == l)
            P_l(x_j)= {v ∈ V(G) : x_i = v_i}.
            if(v_i == x_i)
                P_l(x_j)=v_j.
            endif
        else if (Neighbor == u)
            P_u(x_i)= {P_r(x_i) ∪ P_l(x_j)}.
        else if (Neighbor == i)
            P_i(x_i)= {P_r(x_i) ∩ P_l(x_j)}.
        else if (Neighbor == R)
            P_R(x_i)= {v_i ∈ V(G) : ∩P_r(v_i)}.
        else if (Neighbor == L)
            P_L(x_i)= {v ∈ V(G) : ∩P_l(v)}.
        else if (Neighbor == U)
            P_U(x_i)= {P_R(x_i) ∪ P_L(x_i)}.
        else if (Neighbor == I)
            P_I(x_i)={P_R(x_i) ∩ P_L(x_i)}.
        endif
    endfor
endfor
```

5 Reformulation for Pawlak's properties via graphs

In this section, we construct some of Pawlak's concepts in terms of graphs.

Definition 10 Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. Then, we have

(i) H_{Pj}-definable (H_{Pj}-exact) if $\overline{P_j}\ (V(H)) = \underline{P_j}\ (V(H))$.

(ii) H_{Pj}-rough if $\overline{P_j}\ (V(H)) \neq \underline{P_j}\ (V(H))$, $\forall\ j \in \{r, l, <r>, <l>, u, i, <u>, <i>\}$.

Example 8 (Continue from Example 5) Take $j = l$. If $V(H) = \{a\}$, then H_{Pl}-exact. While, $V(K) = \{a, b\}$ is K_{Pl}-rough. The results for $j \in \{r, <r>, <l>, u, i, <u>, <i>\}$ are similar.

Proposition 1 *Let $G = (V(G), E(G))$ be a graph. Then for all $j \in \{r, <r>, <l>, u, i, <u>, <i>\}$, we have*

(i) Every exact graph is H_{Pj}-exact.

(ii) Every rough graph is H_{Pj}-rough.

Proof Obviously, by Definition 10. □

Definition 11 Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. Then, H is called

(i) Roughly H_{Pj}-definable, if $\underline{P_j}\ (V(H)) \neq \phi$ and $\overline{P_j}\ (V(H)) \neq V(G)$,

(ii) Internally H_{Pj}-undefinable, if $\underline{P_j}\ (V(H)) = \phi$ and $\overline{P_j}\ (V(H)) \neq V(G)$,

(iii) Externally H_{Pj}-undefinable, if $\underline{P_j}\ (V(H)) \neq \phi$ and $\overline{P_j}(V(H)) = V(G)$,

(iv) Totally H_{Pj}-undefinable, if $\underline{P_j}\ (V(H)) = \phi$ and $\overline{P_j}(V(H)) = V(G)$.

Definition 12 Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. A membership function $\underline{\in}_j$ is called a j-strong if $x \in \underline{P_j}\ (V(H))$. It is called a j-weak membership if $x \in \overline{P_j}\ (V(H))$, $\forall\ j \in \{r, l, <r>, <l>, u, i, <u>, <i>\}$.

Lemma 1 *Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. Then, we have*

(i) If $x \underline{\in}_j V(H)$, then $x \in V(H)$.

(ii) If $x \in V(H)$, then $x \overline{\in}_j V(H)$.

Proof Obviously, by Definition 12. □

The inverse implication of Lemma 1 may not be true, in general.

Example 9 (Continue from Example 5) Take $j = r$. Suppose that $V(H) = \{b, c, d\}$. Then, $\underline{P_r}(V(H)) = \{b, c\}$ and $\overline{P_r}\ (V(H)) = V(G)$. It is clear that $d \in V(H)$, while, $\underline{\notin}_r$ $V(H)$ and $a \overline{\in}_r\ V(H)$ and $a \notin V(H)$. We have results for $j \in \{l, <r>, <l>, u, i, <u>, <i>\}$ by the same manner.

Proposition 2 *Let $G = (V(G), E(G))$ be a graph and H be a subgraph of G. Then, the following implications are held:*

(1) $x \underline{\in}_u V(H) \Rightarrow x \underline{\in}_r V(H) \Rightarrow x \underline{\in}_i\ V(H)$.

(2) $x \underline{\in}_u V(H) \Rightarrow x \underline{\in}_l V(H) \Rightarrow x \underline{\in}_i V(H)$.

(3) $x \underline{\in}_{<u>}\ V(H) \Rightarrow x \underline{\in}_{<r>}\ V(H) \Rightarrow x \underline{\in}_{<i>}\ V(H)$.

(4) $x \underline{\in}_{<u>}\ V(H) \Rightarrow x \underline{\in}_{<l>}\ V(H) \Rightarrow x \underline{\in}_{<i>}\ V(H)$.

(5) $x\overline{\in}_i V(H) \Rightarrow x\overline{\in}_r V(H) \Rightarrow x\overline{\in}_u V(H)$.
(6) $x\overline{\in}_i V(H) \Rightarrow x\overline{\in}_l V(H) \Rightarrow x\overline{\in}_u V(H)$.
(7) $x\overline{\in}_{<i>} V(H) \Rightarrow x\overline{\in}_{<r>} V(H) \Rightarrow x\overline{\in}_{<u>} V(H)$.
(8) $x\overline{\in}_{<i>} V(H) \Rightarrow x\overline{\in}_{<l>} V(H) \Rightarrow x\overline{\in}_{<u>} V(H)$.

The converse may not be true, in general.

Example 10 (Continue from Example 5). Suppose that $V(H) = \{a, b\}$. Then $\underline{P}_u(V(H)) = \phi$, $\underline{P}_r(V(H)) = \{b\}$, $\underline{P}_l(V(H)) = \{a\}$ and $\underline{P}_i(V(H)) = \{a, b\}$. Accordingly, $b \in \underline{\in}_r(V(H))$ and $a \in \underline{\in}_l$ $(V(H))$. But, $b \in \underline{\notin}_u(V(H))$ and $a \in \underline{\notin}_u$ $(V(H))$. Also, $a \in \underline{\in}_i$ $(V(H))$ and $b \in \underline{\in}_i$ $(V(H))$. While, $a \in \underline{\notin}_r$ $(V(H))$ and $b \in \underline{\notin}_l$ $(V(H))$.

According to Definitions 8 and 9, we give an algorithm 2 to determine $\underline{P_j}(V(H))$, $\overline{P_j}(V(H))$, $BND_{P_j}(V(H))$, $POS_{P_j}(V(H))$ and $NEG_{P_j}(V(H))$.

Algorithm2. Generate approximations on graphs

Input: j-adhesion neighborhoods.
Output: Approximations on a graph G.
1: Compute the Neighborhoods of vertices $V(G)$, denote $P_j(x)$ $\forall j \in \{r, l, R, L, u, i, U, I\}$.
2: Input $V(H) \subseteq V(G)$.
3: **for** $(j \in |V(G)|)$
4: $\overline{P_j}(V(H)) = V(H) \bigcup \{x \in V(G) : P_j(x) \cap V(H) \neq \phi\}$.
5: **endfor**
6: **for** $(j \in |V(G)|)$
7: $\underline{P_j}(V(H)) = \{x \in V(G) : P_j(x) \subseteq V(H)\}$.
8: **endfor**
9: **for** $(j \in |V(G)|)$
10: $BND_{P_j}(V(H)) = Step(4) - Step(7)$.
11: $\alpha_{Pj}(V(H)) = \frac{|Step(7)|}{|Step(4)|}$.
12: **endfor**

6 A comparison between approach of Nada and our study

The comparison in Table 1 between Nada's method (Nada et al. 2018) and the proposed method aims to increase the accuracy measure and reduce the boundary region by increasing lower approximations and decreasing the upper approximations. So, Example 11 will be studied at $j = r$.

Example 11 (Continue for Example 5) Lower approximations and upper approximations in Definitions 5 and 8 are given. Also, the j-boundary and j-accuracy are evaluated the comparison between them are discussed in Table 1.

In Example 12, we apply Definitions 5, 6, 8 and 9 in Walczak's example in Chemistry. We take five amino acids as a sample in Table 2. From Table 3 and Fig. 4, we show that the vertices of subgraphs $V(H_1) = \{v_1, v_4\}$ and $V(H_2) = \{v_2, v_5\}$ are necessary to determine the high energy of unfolding.

Table 2 Quantitative attributes of five amino acids

X	a_1	a_2	a_3	a_4	a_5
v_1	0.23	254.2	2.126	0.02	82.2
v_2	0.48	303.6	2.994	1.24	112.3
v_3	0.61	287.9	2.994	1.08	103.7
v_4	0.45	282.9	2.933	0.11	99.1
v_5	0.11	335.0	3.458	0.19	127.5

Table 1 Comparison between the boundary region and accuracy measure for Nada approachs and our study

$V(H)$	Nada's method (Nada et al. 2018) (as in Definition 1) (i.e., Definition 5 when $j = j_r$)				The current method in Definition 8			
	$\underline{N_r}(V(H))$	$\overline{N_r}(V(H))$	$BND_{N_r}(V(H))$	$\alpha_{N_r}(V(H))$	$\underline{P_r}(V(H))$	$\overline{P_r}(V(H))$	$BND_{P_r}(V(H))$	$\alpha_{P_r}(V(H))$
$\{a\}$	ϕ	$\{a, b, c\}$	$\{a, b, c\}$	0	ϕ	$\{a, d\}$	$\{a, d\}$	0
$\{b\}$	ϕ	$\{a, b, d\}$	$\{a, b, d\}$	0	$\{b\}$	$\{b\}$	ϕ	1
$\{c\}$	ϕ	$\{a, c, d\}$	$\{a, c, d\}$	0	$\{c\}$	$\{c\}$	ϕ	1
$\{d\}$	ϕ	$\{c, d\}$	$\{c, d\}$	0	ϕ	$\{a, d\}$	$\{a, d\}$	0
$\{a, b\}$	$\{b\}$	$V(G)$	$V(G)$	$\frac{1}{4}$	$\{b\}$	$\{a, b, d\}$	$\{a, d\}$	$\frac{1}{3}$
$\{a, c\}$	ϕ	$V(G)$	$V(G)$	0	$\{c\}$	$\{a, c, d\}$	$\{a, d\}$	$\frac{1}{3}$
$\{b, d\}$	ϕ	$V(G)$	$V(G)$	0	$\{b\}$	$\{a, b, d\}$	$\{a, d\}$	$\frac{1}{3}$
$\{c, d\}$	ϕ	$\{a, c, d\}$	$\{a, c, d\}$	0	$\{c\}$	$\{a, c, d\}$	$\{a, d\}$	$\frac{1}{3}$
$\{a, b, c\}$	$\{a, b\}$	$V(G)$	$\{c, d\}$	$\frac{1}{2}$	$\{b, c\}$	$V(G)$	$\{a, d\}$	$\frac{1}{2}$
$\{a, c, d\}$	$\{c\}$	$V(G)$	$\{a, b, d\}$	$\frac{1}{4}$	$\{a, c, d\}$	$\{a, c, d\}$	ϕ	1
$\{b, c, d\}$	$\{d\}$	$V(G)$	$\{a, b, c\}$	$\frac{1}{4}$	$\{b, c\}$	$V(G)$	$\{a, d\}$	$\frac{1}{2}$
$V(G)$	$V(G)$	$V(G)$	0	1	$V(G)$	$V(G)$	0	1

Table 3 Comparison between boundary region and accuracy measure for Nada et al's method and our study

$V(H)$	Nada's method (Nada et al. 2018) (as in Definition 1) (i.e., Definition 5 when $j = j_r$)				Our proposed method (as in Definition 8)			
	$\underline{N_r}(V(H))$	$\overline{N_r}(V(H))$	$BND_{N_r}(V(H))$	$\alpha_{N_r}(V(H))$	$\underline{P_r}(V(H))$	$\overline{P_r}(V(H))$	$BND_{P_r}(V(H))$	$\alpha_{P_r}(V(H))$
$\{v_1\}$	ϕ	$\{v_1\}$	$\{v_1\}$	0	$\{v_1\}$	$\{v_1\}$	ϕ	1
$\{v_2\}$	ϕ	$\{v_2\}$	$\{v_2\}$	0	$\{v_2\}$	$\{v_2\}$	ϕ	1
$\{v_3\}$	ϕ	$\{v_3\}$	$\{v_3\}$	0	$\{v_3\}$	$\{v_3\}$	ϕ	1
$\{v_4\}$	$\{v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3\}$	$\frac{1}{3}$	$\{v_4\}$	$\{v_4\}$	ϕ	1
$\{v_5\}$	$\{v_5\}$	$\{v_2, v_3, v_5\}$	$\{v_2, v_3\}$	$\frac{1}{3}$	$\{v_5\}$	$\{v_5\}$	ϕ	1
$\{v_1, v_4\}$	$\{v_1, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_3\}$	$\frac{2}{3}$	$\{v_1, v_4\}$	$\{v_1, v_4\}$	ϕ	1
$\{v_2, v_5\}$	$\{v_2, v_5\}$	$\{v_2, v_3, v_5\}$	$\{v_3\}$	$\frac{2}{3}$	$\{v_2, v_5\}$	$\{v_2, v_5\}$	ϕ	1
$\{v_3, v_4\}$	$\{v_4\}$	$\{v_1, v_2, v_3, v_4\}$	$\{v_1, v_2, v_3\}$	$\frac{1}{4}$	$\{v_3, v_4\}$	$\{v_3, v_4\}$	ϕ	1
$\{v_3, v_5\}$	$\{v_5\}$	$\{v_2, v_3, v_5\}$	$\{v_2, v_3\}$	$\frac{1}{3}$	$\{v_3, v_5\}$	$\{v_3, v_5\}$	ϕ	1
$V(G)$	$V(G)$	$V(G)$	ϕ	1	$V(G)$	$V(G)$	ϕ	1

Example 12 (A chemical example) Let $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ be five amino acids (AAs) which described in terms of five attributes: $a_1 = PIE$, $a_2 = SAC =$ surface area, $a_3 = MR =$ molecular refractivity, $a_4 = LAM =$ the side chain polarity and $a_5 = Vol =$ molecular volume (Walzak et al. 1999) as shown in Table 2. We illustrate five graphs G_k, where $k = 1, 2, \cdots, 5$ on $V(H) \subseteq V(G)$ in Fig. 3 such that $R_k = \{(x_i, x_j) \in X \times X : x_i(a_k) - x_j(a_k) < \frac{\sigma_k}{2},\ i, j, k = 1, 2, \cdots, 5\}$, where σ_k represents the standard of the quantitative attributes a_k, $k = 1, 2, 3, 4, 5$. From intersection of five graphs, we have a new graph G in Fig. 4 which uses to construct j-neighborhood systems and j-adhesion neighborhoods for adjacent vertices.

Take $j = r$. Then, we have

(i) $N_r(x_1) = \{x_1, x_4\}$, $N_r(x_2) = \{x_2, x_5\}$, $N_r(x_3) = \{x_3, x_4, x_5\}$, $N_r(x_4) = \{x_4\}$, $N_r(x_5) = \{x_5\}$.
(ii) $P_r(x_1) = \{x_1\}$, $P_r(x_2) = \{x_2\}$, $P_r(x_3) = \{x_3\}$, $P_r(x_4) = \{x_4\}$, $P_r(x_5) = \{x_5\}$.

Similarily, we obtain the results of $j = \{l, <r>, <l>, u, i, <u>, <i>\}$.

Table 3 gives a comparison between the r-lower approximations, r-upper approximations, r-boundary regions and r-accuracy measures for Nada method at $j = r$ and our proposed method in Definition 8. We prove that

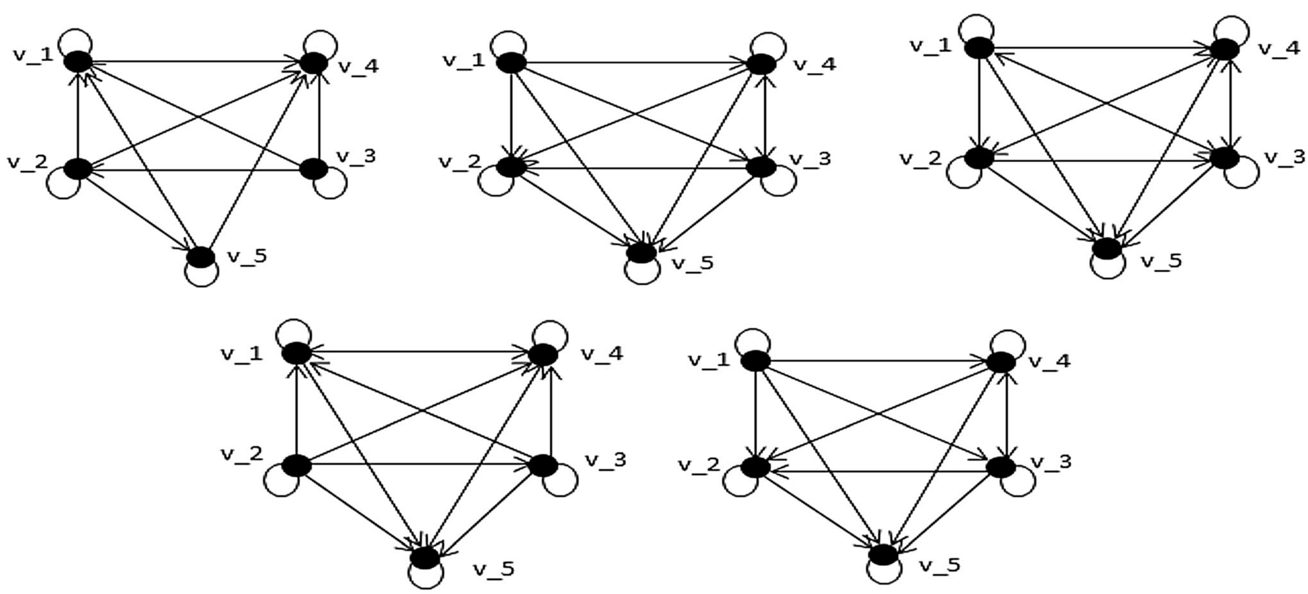

Fig. 3 A graph G_k for R_k, where $k = 1, 2, 3, 4, 5$

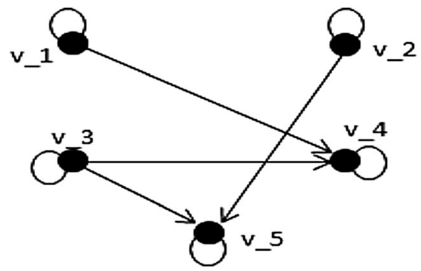

Fig. 4 Intersection of graphs G_k to generate a graph G

our study method has more accurate than the previous approaches.

7 Conclusion and future work

j-Adhesion neighborhoods on general graphs are important tools to approximate graphs as finite structures. Different eight types of j-adhesion neighborhoods are introduced and discussed for each $j \in \{r, l, i, u, <r>, <l>, <i>, <u>\}$. By these neighborhoods, j-lower approximations and j-upper approximations will be constructed. Moreover, the relationships among j-approximations are superposed. Furthermore, we show that boundary regions are decreased through increasing the j-lower approximations and decreasing the j-upper approximations. So, the j-accuracy is more accurate than the other type defined in (Nada et al. 2018). The results in this article are very significant in decision-making, especially, to classify the family of coronavirus (Lai et al. 2020; Kampf et al. 2020) which is a topological space and type of Stone–$\breve{C}$ech compactification.

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