

Rough Stochastic PDEs

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Abstract

In this article, we show how the theory of rough paths can be used to provide a notion of solution to a class of nonlinear stochastic PDEs of Burgers type that exhibit too high spatial roughness for classical analytical methods to apply. In fact, the class of SPDEs that we consider is genuinely ill-posed in the sense that different approximations to the nonlinearity may converge to different limits. Using rough paths theory, a pathwise notion of solution to these SPDEs is formulated, and we show that this yields a well-posed problem, which is stable under a large class of perturbations, including the approximation of the rough driving noise by a mollified version and the addition of hyperviscosity.

We also show that under certain structural assumptions on the coefficients, the SPDEs under consideration generate a reversible Markov semigroup with respect to a diffusion measure that can be given explicitly.

1 Introduction

This article is devoted to the study of the following class of Burgers-like SPDEs:

$$du = \partial_x^2 u dt + f(u) dt + g(u) \partial_x u dt + \sigma dW(t). \quad (1.1)$$

Here, $\sigma \in \mathbf{R}_+$, the spatial variable x takes values in $[0, 2\pi]$, the linear operator ∂_x^2 is endowed with periodic boundary conditions, u takes values in \mathbf{R}^n , and $f: \mathbf{R}^n \rightarrow \mathbf{R}^n$, $g: \mathbf{R}^n \rightarrow \mathbf{R}^{n \times n}$ are C^∞ functions. We assume that the driving noise W gives rise to space-time white noise; in other words that W is a standard cylindrical Wiener process on $L^2([0, 2\pi], \mathbf{R}^n)$ [DPZ92]. One motivation for studying such equations arises from the theory of path sampling: for f and g of some specific form, (1.1) does *formally* arise as a gradient system with the law of a diffusion process as invariant measure, see [HSV07] and Section 4 below.

The problem with (1.1) that we address in this article is that of making sense of the nonlinearity $g(u) \partial_x u$ in equations of this type. To appreciate the difficulty of the problem, we note that the solution ψ to the linearised equation

$$d\psi = \partial_x^2 \psi dt + \sigma dW(t), \quad (1.2)$$

is not differentiable in x for fixed t . Actually, these solutions have a spatial regularity akin to the temporal regularity of Brownian motion: they are almost surely α -Hölder

continuous for every $\alpha < \frac{1}{2}$, but not more in the sense that they are almost surely *not* $\frac{1}{2}$ -Hölder continuous [Wal86].

This usually doesn't cause any serious problem: the standard procedure in this case is to consider weak (in the PDE sense) solutions of the form

$$d\langle \varphi, u \rangle = \langle \partial_x^2 \varphi, u \rangle dt + \langle \varphi, f(u) \rangle dt + \langle \varphi, g(u) \partial_x u \rangle dt + \sigma \langle \varphi, dW(t) \rangle ,$$

for sufficiently regular test functions φ and to make sense of the term $\langle \varphi, g(u) \partial_x u \rangle$ by performing one integration by parts. However, this is only possible if there exists a function $G: \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $g = DG$. Assuming the existence of such a function G would impose non-trivial structural conditions on g as soon as $n > 1$, which is not something that we wish to do. Now if it were the case that, for fixed $t > 0$, u was α -Hölder continuous for some exponent α strictly *greater* than $\frac{1}{2}$, then we could rewrite the nonlinearity in the suggestive form

$$\langle \varphi, g(u) \partial_x u \rangle = \int_0^{2\pi} \varphi(x) g(u(x)) du(x) , \quad (1.3)$$

for smooth periodic test functions φ , and interpret this integral as a simple Riemann-Stieltjes integral. By Young's theory of integration [You36], this expression would indeed be well-defined in this case. Unfortunately, as already mentioned, we expect our solutions to fall just slightly short of this kind of regularity, so that there is a priori no obvious way in which to make sense of (1.3). From this perspective, the problem at hand is very strongly reminiscent of the problem of making sense of solutions to ordinary stochastic differential equations. Actually, similarly to the case of SDEs, different numerical approximations to (1.1) converge to different solutions, which differ by a correction term similar to the classical Itô-Stratonovich correction term, see [HV11] for a numerical exploration of this phenomenon.

Motivated by this observation, let us try to apply the standard theory of stochastic integration to this problem. For this, we need to first specify what type of stochastic integral we wish to consider. Since we would like to recover the usual concept of weak / mild solutions for the Burgers equations in the case where g is a total derivative, it is natural to look for a kind of "Stratonovich integral" interpretation of (1.3). Since we expect u to behave like ψ at "small scales", it is arguably sufficient to make sense of the expression

$$\langle \varphi, g(u) \partial_x \psi \rangle = \int_0^{2\pi} \varphi(x) g(u(x)) \circ d\psi(x) . \quad (1.4)$$

This seems promising since it can easily be seen that for fixed t , the law of ψ differs from that of a Brownian bridge only by the addition of some random C^∞ function. The problems with this approach seem twofold:

1. There is no "arrow of time". In particular, the process $g(u(x))$ is not adapted with respect to the filtration generated by ψ .
2. For any fixed $t > 0$, both $\psi(\cdot, t)$ and $u(\cdot, t)$ have a very complicated dependence on the driving space-time white noise for times $s < t$. This would make it a highly non-trivial task to develop a Malliavin calculus of u with respect to ψ and to study the dependence of this calculus on the time parameter t .

All of these problems can be solved in an elegant way if the integral in (1.4) can be interpreted in a pathwise sense. The theory of rough paths developed by Lyons [Lyo98]

provides just such an interpretation! The twist here is that we will use the theory of rough paths in order to make sense of a driving noise that leads to solutions that are rough *in space* rather than in time.

One may wonder at this stage whether the notion of solution to (1.1) given by rough paths theory is in any way natural. This question will be answered by the affirmative in two different ways. First, it is natural to consider a smoothed version of (1.1) where the noise W is hit by a mollifier with lengthscale $\varepsilon > 0$ and to study the limit $\varepsilon \rightarrow 0$. We will see in Section 3.3 that the stability properties of our solution, together with known approximation results for Gaussian rough paths, imply that the sequence of classical solutions obtained in this way does indeed converge as $\varepsilon \rightarrow 0$ to the solution constructed in this article. Secondly, we will come back to the original motivation for the study of (1.1), which is to provide an SPDE with invariant measure given by a certain diffusion process. We will show in Section 4 that it is indeed the case that if we consider (1.1) with a particular structure for the nonlinearities f and g derived formally in [HSV07], then the process constructed in this article is reversible with respect to the expected invariant measure.

It is of course not the first time that the theory of rough paths has been applied to stochastic PDEs. To our knowledge, three groups of authors have considered such problems in quite different contexts. Friz and coauthors showed in [CF09, CFO09] that rough paths theory can be used to provide meaning (and solutions) to a class of nonlinear stochastic PDEs via the method of stochastic characteristics. This is essentially a variant of the type of problems that have been considered by Souganidis and Lions [LS98], and the emphasis in these problems is the treatment of temporally rough driving signals. Concurrently, Gubinelli and Tindel developed a theory of stochastic PDEs driven by rough paths which allows to treat semilinear problems of “Da Prato & Zabczyk type”, see [GT10]. There, the emphasis is not just on treating temporally rough driving noise, but also on understanding the interplay between temporal and spatial regularity. This theory is based on the ideas developed in [Gub04], combined with the insights obtained in the more regular case in [GLT06], but it relies on classical Sobolev calculus to treat the spatial roughness of the solutions. Finally, a more recent result was obtained by Teichmann [Tei10], where Szökefalvi-Nagy’s dilation theorem for contraction semigroups is used to provide a simple and elegant way of constructing solutions to a class of semilinear SPDEs when the corresponding linear problem generates a semigroup of contractions on a Hilbert space. We also refer to the works [Gub06, BGN10] for examples of *deterministic* PDEs that can be tackled using rough paths theory.

The main novelty of the present work is the ability to give meaning to a class of stochastic PDEs such that the *deterministic part* of the equation does not have any classical meaning. While this has been achieved in a number of equations using renormalisation techniques [JLM85, BG97, Cha00, DPD03], to the best of our knowledge, this is the first time that rough paths theory is used in such an endeavour. The advantage of rough paths theory in that context is that it allows to treat nonlinearities that do not exhibit a “polynomial” structure, as is required by renormalisation techniques. It is also the first time that rough path theory is used to provide meaning to an equation which is classically ill-posed due to a lack of *spatial* regularity, rather than a lack of temporal regularity.

1.1 A motivation: path sampling

Besides its interest from a purely mathematical perspective, one of the original motivations of the present article is to be able to treat the type of stochastic PDEs arising in the context of path sampling, as in [HSV07]. The aim of this work is to construct an

algorithm allowing to sample from the distribution of an elliptic diffusion, conditioned on having not only a fixed initial condition x_- , but also a prescribed terminal condition x_+ at time 1, say.

The idea is to find a stochastic PDE associated to the problem (with the “time” variable of the original diffusion now being interpreted as a “space” variable for the stochastic PDE) with the property that its invariant measure is precisely given by the conditional law of diffusion that one wishes to sample. It is well-known that if the original diffusion is of gradient type, namely

$$dX = -\nabla V(X) dt + dB, \quad X(t) \in \mathbf{R}^n,$$

with an n -dimensional driving noise B , then the corresponding SPDE is given by

$$du = \partial_x^2 u dt + \Phi(u) dt + \sqrt{2} dW(t), \quad \Phi(u) = \frac{1}{2}(|\nabla V(u)|^2 - \Delta V(u))$$

where W is space-time white noise as before, endowed with the boundary conditions $u(0, t) = x_-$ and $u(1, t) = x_+$. The situation gets more complicated when the original diffusion is not of gradient type. In this case, a formal calculation led to the conjecture in [HSV07] that, if the underlying diffusion is given by

$$dX = f(X) dt + dB, \quad X(t) \in \mathbf{R}^n, \quad (1.5)$$

then the corresponding stochastic PDE is given by

$$du = \partial_x^2 u dt + \Phi(u) dt + (Df(u) - (Df(u))^T) \partial_x u dt + \sqrt{2} dW(t), \quad (1.6)$$

with

$$\Phi(u) = \frac{1}{2}(|f(u)|^2 + \operatorname{div} f(u)). \quad (1.7)$$

This equation is precisely of the form (1.1), so that this article yields a way of interpreting it. We will see in Section 4 below that this allows us to verify the conjecture from [HSV07], namely we show in Theorem 4.1 that (1.6) does indeed admit the law of (1.5) as its invariant measure.

Remark 1.1 The last statement is slightly incorrect. Indeed, we will only consider (1.6) with *periodic* boundary conditions instead of the Dirichlet boundary conditions arising from the application to path sampling. We expect this to be a mere technical detail; the technique developed in the present article should carry over to the case of Dirichlet boundary conditions without major changes.

Actually, the periodicity of the solutions were mainly used in Section 5. It should however be possible to simplify those arguments with the help of the results obtained in [FR11, CLL11] after the redaction of the present work was completed.

Remark 1.2 One may wonder how the invariant measure μ of (1.6) with periodic boundary conditions relates to (1.5). The invariant measure for the equation with Dirichlet boundary conditions has an explicit density with respect to the Brownian bridge measure pinned at 0. The invariant measure for the equation with periodic boundary conditions has the *same* density, but with respect to the Brownian bridge measure with endpoint distributed with respect to Lebesgue measure.

As a consequence, we see that μ can be constructed in the following way. For an arbitrary probability measure ϱ on \mathbf{R}^n , denote by μ_ϱ the law of the solutions to (1.5) with initial condition $X_0 \sim \varrho$ and conditioned to satisfy $X_1 = X_0$. Then μ is given by μ_ϱ , where ϱ is the only measure on \mathbf{R}^n such that μ_ϱ is invariant under rotations of the circle.

Another conjecture was given in [HSV07] regarding the SPDE associated to the problem of data assimilation. This corresponds to the underlying SDE being given by

$$\begin{aligned} dX &= f(X) dt + dB, & X(t) &\in \mathbf{R}^n, \\ dY &= g(X, Y) dt + d\tilde{B}, & Y(t) &\in \mathbf{R}^m, \end{aligned} \quad (1.8)$$

where B and \tilde{B} are independent Brownian motions. The problem now is to sample from the law of X , conditional on a realisation of Y over a fixed time interval, say $[0, 1]$. In this case, one expects the corresponding SPDE to be given by

$$\begin{aligned} du &= \partial_x^2 u dt + \Phi(u) dt + (Df(u) - (Df(u))^T) \partial_x u dt \\ &\quad + 2(D_u g(u, Y))^T (\partial_x Y + g(u, Y)) dt + \sqrt{2} dW(t), \end{aligned} \quad (1.9)$$

with $\Phi(u)$ as in (1.7), and where Y denotes the instance of the observation. (Recall that the spatial variable for the stochastic PDE corresponds to the temporal variable of the original problem, this is why Y is a function of x in (1.9).) Here, one faces the additional problem of making sense of the term proportional to $\partial_x Y$.

While this does not quite fit the framework developed in this article, it can be treated with minor modifications. Indeed, all we need to do to make sense of (1.9) is to incorporate \tilde{B} into the reference rough path $(\Psi, \mathbf{\Psi})$ that is used in Section 3 to define our notion of solution. This is not a problem since W and \tilde{B} are independent.

1.2 Notations

We denote by C^α the space of all α -Hölder continuous functions on $[0, 2\pi]$ and by $\|\cdot\|_\alpha$ the corresponding seminorm, namely

$$\|u\|_\alpha \stackrel{\text{def}}{=} \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

We will also make a slight abuse of notation by writing $\|\cdot\|_\infty$ for the supremum norm and we set $\|u\|_{C^\alpha} = \|u\|_\alpha + \|u\|_\infty$ which, on bounded intervals, is also equivalent to $\|u\|_\alpha + |u(0)|$. For integer values of n , we set $\|u\|_{C^n} = \sum_{k=0}^n \|D^k u\|_\infty$, where $D^k u$ denotes the k th derivative of u .

Our main fixed point argument will take place in the space of space-time functions with norm

$$\|v\|_{1,T} \stackrel{\text{def}}{=} \sup_{t \leq T} \|v_t\|_{C^1}.$$

Note also that throughout this article, we will work with periodic boundary conditions. We will therefore make the usual identification of the interval $[0, 2\pi]$ with the circle S^1 without further comment.

1.3 Article overview

The remainder of this article is organised as follows. First, in Section 2, we give a short overview of those elements of rough path theory that are being used in this work. While this is of course by no means a general introduction to the theory (we refer for this to the monographs [LQ02, FV10b] and the lecture notes [LCL07]), it is intended to be sufficiently self-contained so that even a reader without prior knowledge of rough paths theory should be able to follow the subsequent arguments. Section 3 provides the definition of a solution to (1.1), as well as the proof that this equation is locally well-posed (globally if f and g are sufficiently bounded) and that its solutions are

stable with respect to perturbations in the initial condition and the driving noise. In Section 4, we then show that under the structural assumptions derived in [HSV07], one can explicitly exhibit an invariant measure for (1.1), and the corresponding Markov process is reversible. Finally, Section 5 contains a uniform exponential integrability result which is essential in the proofs of Section 4.

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2 Elements of rough path theory

We will mostly make use of the notations introduced by Gubinelli in [Gub04] since the estimates given in that work seem to be the ones that are most suitable for the present undertaking. This is because Gubinelli essentially builds a theory of integration for quite general integrands against a given rough path, whereas Lyons mostly considers integrands that are the composition of a smooth (local!) function with the rough path. This restriction could in principle be overcome by a slight reformulation of the problem (just as it can be overcome when one wishes to use the theory to solve SDEs), but this appears to be more cumbersome in our setting.

We denote by $\mathcal{C}_2([0, T], \mathbf{R}^n)$ the space of continuous functions from $[0, T]^2$ into \mathbf{R}^n that vanish on the diagonal. Very often, we will omit the time interval $[0, T]$ and the target space \mathbf{R}^n in our notations for the sake of simplicity. We also define a difference operator $\delta: \mathcal{C} \rightarrow \mathcal{C}_2$ by

$$\delta X_{s,t} = X_t - X_s .$$

A rough path on an interval $[0, T]$ then consists of two parts: a continuous function $X: [0, T] \rightarrow \mathbf{R}^n$, as well as a continuous ‘‘area process’’ $\mathbf{X}: [0, T]^2 \rightarrow \mathbf{R}^{n \times n}$ such that $\mathbf{X}_{t,t} = 0$ for every t and such that the algebraic relations

$$\mathbf{X}_{s,t}^{ij} - \mathbf{X}_{u,t}^{ij} - \mathbf{X}_{s,u}^{ij} = \delta X_{s,u}^i \delta X_{u,t}^j , \quad (2.1)$$

hold for every triple of times (u, s, t) and every pair of indices (i, j) . One should think of \mathbf{X} as ‘‘postulating’’ the value of the quantity

$$\int_s^t \delta X_{s,r}^i dX_r^j \stackrel{\text{def}}{=} \mathbf{X}_{s,t}^{ij} , \quad (2.2)$$

where we take the right hand side as a *definition* for the left hand side. (And not the other way around!) Note that the algebraic relations (2.1) are by themselves not sufficient to determine \mathbf{X} as a function of X . Indeed, for any matrix-valued function F , the substitution $\mathbf{X}_{s,t}^{ij} \mapsto \mathbf{X}_{s,t}^{ij} + F_t^{ij} - F_s^{ij}$ leaves the left hand side of (2.1) invariant. The aim of imposing (2.1) is to ensure that (2.2) does indeed behave like an integral when considering it over two adjacent intervals.

For $\alpha \in (\frac{1}{3}, \frac{1}{2})$, we will denote by \mathcal{D}^α the space of those rough paths (X, \mathbf{X}) such that $X \in \mathcal{C}^\alpha$ and

$$\|\mathbf{X}\|_{2\alpha} := \sup_{s \neq t \in [0, T]} \frac{|\mathbf{X}_{s,t}|}{|x - y|^{2\alpha}} < \infty. \quad (2.3)$$

At this stage, it is important to note that while it is a closed subset of a vector space, the space \mathcal{D}^α is *not* itself a vector space because of the nonlinear constraint (2.1). One rather unpleasant consequence of this fact is that the natural norm on \mathcal{D}^α given by $\|X\|_{\mathcal{C}^\alpha} + \|\mathbf{X}\|_{2\alpha}$ does not reflect its geometry, since the natural dilatation on \mathcal{D}^α is given by $(X, \mathbf{X}) \mapsto (\lambda X, \lambda^2 \mathbf{X})$. Note also that the quantities defined in (2.3) are merely seminorms since they vanish for constants.

2.1 Controlled rough paths

Another important notion taken from [Gub04] is that of a path Y *controlled* by a rough path X . Given a rough path $X \in \mathcal{D}^\alpha([0, T], \mathbf{R}^d)$, we say that a pair of functions $(Y, Y') \in \mathcal{C}_X^\alpha([0, T], \mathbf{R}^m)$ is a rough path controlled by X if $Y \in \mathcal{C}^\alpha([0, T], \mathbf{R}^m)$, $Y' \in \mathcal{C}^\alpha([0, T], \mathbf{R}^{m \times d})$, and the “remainder term” R given by

$$R_{s,t} = \delta Y_{s,t} - Y'_s \delta X_{s,t}, \quad (2.4)$$

satisfies $\|R\|_{2\alpha} < \infty$. Here, $R_{s,t} \in \mathbf{R}^m$ and the second term is a matrix-vector multiplication. We endow the space \mathcal{C}_X^α with the norm

$$\|(Y, Y')\|_{X, \alpha} = \|Y\|_{\mathcal{C}^\alpha} + \|Y'\|_{\mathcal{C}^\alpha} + \|R\|_{2\alpha}. \quad (2.5)$$

Note that since we assumed that X is α -Hölder continuous, it immediately follows from these definitions that the same is true for Y with

$$\|Y\|_\alpha \leq C \|R\|_{2\alpha} + \|Y'\|_\infty \|X\|_\alpha.$$

The term $\|Y\|_{\mathcal{C}^\alpha}$ in (2.5) is therefore used only to control the supremum of Y .

Remark 2.1 We will sometimes make an abuse of notation and simply write $\|Y\|_{X, \alpha}$ instead of the more correct expression $\|(Y, Y')\|_{X, \alpha}$. Since Y' will always be constructed from X and regular functions by using the rules laid out in the next subsections, this will hopefully not cause any confusion.

Note that in general, there could be many “derivative processes” Y' associated to a given path Y . However, in dimension $d = 1$, if for some given $s \in (0, T)$ there exists a sequence of times $t_n \rightarrow s$ such that $|X_{t_n} - X_s|/|t_n - s|^{2\alpha} \rightarrow \infty$, then Y'_s is uniquely determined from Y by (2.4) and the condition that $\|R\|_{2\alpha} < \infty$. A similar condition can also be written down for higher dimensions. In most cases of interest, such as when it is given by the sample path of a (fractional) Brownian motion, the function X will have such a property at a dense set of points, thus determining Y' uniquely as a function of Y .

In the sequel, we will sometimes omit to explicitly mention the derivative process Y' . We hope that this will not cause any ambiguity since all the controlled paths that we are going to consider will be constructed using the following list of operations.

2.1.1 Canonical lift of X

It is easy to see that the process X itself can be interpreted as a process “controlled by X ”. Indeed, we can identify X with the element $(X, I) \in \mathcal{C}_X^\alpha$, where I is the process which is equal to the identity matrix for all times.

2.1.2 Lifting of regular functions.

There is a canonical embedding $\iota: \mathcal{C}^{2\alpha} \hookrightarrow \mathcal{C}_X^\alpha$ given by $\iota Y = (Y, 0)$, since in this case $R = \delta Y$ does indeed satisfy $\|R\|_{2\alpha} < \infty$ (recall that we are only interested in the case $\alpha < \frac{1}{2}$). If one actually has $Y \in \mathcal{C}^1$, then one can define the integral of Y against X by setting

$$Z_t = \int_0^t Y_s \otimes dX_s = \dot{Y}_t \otimes X_t - \dot{Y}_0 \otimes X_0 - \int_0^t \dot{Y}_s \otimes X_s ds, \quad (2.6)$$

where \dot{Y} denotes the time derivative of Y . One can check quite easily that this integral has the property that $(Z, Y \otimes I)$, where I is the identity matrix, is itself a controlled rough path belonging to \mathcal{C}_X^α .

2.1.3 Composition with regular functions.

Let $\varphi: \mathbf{R}^m \times [0, T] \rightarrow \mathbf{R}^n$ be a function which is uniformly \mathcal{C}^2 in its first argument (i.e. φ is bounded and both $D_y \varphi$ and $D_y^2 \varphi$ are bounded, where D_y denotes the derivative with respect to the first argument) and uniformly $\mathcal{C}^{2\alpha}$ in its second argument. Let furthermore $(Y, Y') \in \mathcal{C}_X^\alpha([0, T], \mathbf{R}^m)$, then one can define a controlled path $(\varphi(Y), \varphi(Y')) \in \mathcal{C}_X^\alpha([0, T], \mathbf{R}^n)$ by

$$\varphi(Y)_t = \varphi(Y_t, t), \quad \varphi(Y)'_t = D_y \varphi(Y_t, t) Y'_t. \quad (2.7)$$

(Here, the path $\varphi'(Y)Y'$ is to be interpreted as the pointwise an $n \times m$ and an $m \times d$ matrix-valued path.) It is straightforward to check that the corresponding remainder term does indeed satisfy the required bound. It is also straightforward to check that this definition is consistent in the sense that $(\varphi \circ \psi)(Y, Y') = \varphi(\psi(Y, Y'))$. Furthermore, we have the bound:

Lemma 2.2 *Let φ be as above, let $(Y, Y') \in \mathcal{C}_X^\alpha([0, T], \mathbf{R}^m)$, and let $(\varphi(Y), \varphi(Y')) \in \mathcal{C}_X^\alpha([0, T], \mathbf{R}^n)$ be given by (2.7). Then, there exists a constant C such that one has the bound*

$$\|\varphi(Y)\|_{X, \alpha} \leq C(\|D_y^2 \varphi\|_\infty + \|\varphi\|_\infty + \|\varphi\|_{2\alpha; t})(1 + \|Y\|_{X, \alpha})^2,$$

where we denote by $\|\varphi\|_{2\alpha; t}$ the supremum over y of the 2α -Hölder norm of $\varphi(y, \cdot)$.

Proof. We start by showing that there exists a constant C such that

$$\|D_y \varphi\|_{\alpha; t}^2 \leq C \|D^2 \varphi\|_\infty \|\varphi\|_{2\alpha; t}. \quad (2.8)$$

We consider the case $m = n = 1$, the general case follows in a similar way. There are two times $s \neq t$ and a point x such that

$$\varphi'(x, s) - \varphi'(x, t) = \|\varphi'\|_{\alpha; t} |t - s|^\alpha \stackrel{\text{def}}{=} \varepsilon.$$

Therefore, for y such that $|y - x| \leq \varepsilon / (4\|\varphi''\|_\infty)$, we have

$$\varphi'(y, s) - \varphi'(y, t) \geq \frac{\varepsilon}{2}.$$

Integrating this inequality from x to y , we obtain

$$\varphi(y, s) - \varphi(x, s) - \varphi(y, t) + \varphi(x, t) \geq \frac{\varepsilon^2}{8\|\varphi''\|_\infty},$$

so that

$$\frac{\|\varphi'\|_{\alpha;t}^2 |t-s|^{2\alpha}}{8\|\varphi''\|_\infty} = \frac{\varepsilon^2}{8\|\varphi''\|_\infty} \leq 2\|\varphi\|_{2\alpha;t} |t-s|^{2\alpha},$$

which is precisely the claim (2.8).

It follows from (2.7) and elementary properties of the Hölder norms that

$$\begin{aligned} \|\varphi(Y)\|_\alpha &\leq C(\|\varphi\|_\infty + \|D_y\varphi\|_\infty)\|Y\|_{\mathcal{C}^\alpha} + \|\varphi\|_{\alpha;t}\|Y\|_\infty, \\ \|\varphi(Y)'\|_\alpha &\leq C(\|D\varphi\|_\infty + \|D_y^2\varphi\|_\infty)\|Y\|_{\mathcal{C}^\alpha}\|Y'\|_{\mathcal{C}^\alpha} + \|D_y\varphi\|_{\alpha;t}\|Y'\|_\infty \\ &\leq C(\|\varphi\|_\infty + \|D_y^2\varphi\|_\infty)\|Y\|_{X,\alpha}^2 + C\sqrt{\|D^2\varphi\|_\infty}\|\varphi\|_{2\alpha;t}\|Y'\|_\infty. \end{aligned}$$

Concerning the remainder, we have the bound

$$|\varphi(Y_t, t) - \varphi(Y_s, s) - D_y\varphi(Y_s, s)\delta Y_{s,t}| \leq \|\varphi\|_{2\alpha;t} |t-s|^{2\alpha} + \frac{1}{2}\|D_y^2\varphi\|_\infty |\delta Y_{s,t}|^2.$$

Since on the other hand $|\delta Y_{s,t} - Y'_s \delta X_{s,t}| = R_{s,t}$ by definition, we then have for the remainder term R^φ of the controlled rough path $(\varphi(Y), \varphi(Y)')$ the bound

$$|R_{s,t}^\varphi| \leq \|D_y\varphi\|_\infty |R_{s,t}| + \|\varphi\|_{2\alpha;t} |t-s|^{2\alpha} + \frac{1}{2}\|D_y^2\varphi\|_\infty \|Y\|_\alpha^2 |t-s|^{2\alpha}.$$

The claim now follows from the assumptions on φ . \square

In particular, this shows that if $f \in \mathcal{C}^{2\alpha}$ and $(Y, Y') \in \mathcal{C}_X^\alpha$, then both $f \cdot Y$ and $f + Y$ are well-defined elements of \mathcal{C}_X^α . In that case, one can slightly improve over the general bound given in Lemma 2.2, namely one has

$$\begin{aligned} \|(fY, fY')\|_{X,\alpha} &\leq 2\|f\|_{\mathcal{C}^{2\alpha}} \|(Y, Y')\|_{X,\alpha}, \\ \|(f + Y, Y')\|_{X,\alpha} &\leq 2\|f\|_{\mathcal{C}^{2\alpha}} + \|(Y, Y')\|_{X,\alpha}. \end{aligned} \quad (2.9)$$

It also shows immediately that \mathcal{C}_X^α is an algebra for every reference rough path X .

2.2 Integration of controlled rough paths.

The aim of this section is to give a meaning to the expression $\int Y_t \otimes dX_t$, when $X \in \mathcal{D}^\alpha$ and $(Y, Y') \in \mathcal{C}_X^\alpha$. A natural approach would be to try to define it as a limit of Riemann sums, that is

$$\int_0^1 Y_t \otimes dX_t = \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[s,t] \in \mathcal{P}} Y_s \otimes \delta X_{s,t}, \quad (2.10)$$

where \mathcal{P} denotes a partition of $[0, 1]$ (interpreted as a finite collection of intervals) and $|\mathcal{P}|$ denotes the length of the largest element of \mathcal{P} . Unfortunately, this does not converge in general. The next best approximation to the integral is given by making use of the approximation $Y_t \approx Y_s + Y'_s \delta X_{s,t}$ suggested by (2.4) and combining this with (2.2). This suggests that instead of (2.10), one should rather define the integral as

$$\int_0^1 Y_t \otimes dX_t = \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[s,t] \in \mathcal{P}} (Y_s \otimes \delta X_{s,t} + Y'_s \mathbf{X}_{s,t}). \quad (2.11)$$

With these notations at hand, we quote the following result, which is a slight reformulation of [Gub04, Prop 1]:

Theorem 2.3 *Let $(X, \mathbf{X}) \in \mathcal{D}^\alpha$ for some $\alpha > \frac{1}{3}$ and fix $T > 0$. Then, the map*

$$(Y, Y') \mapsto \left(\int_0^\cdot Y_t \otimes dX_t, Y \otimes I \right), \quad (2.12)$$

with the integral defined as in (2.11) is continuous from \mathcal{C}_X^α to \mathcal{C}_X^α and one has the bound

$$\left\| \int_0^\cdot \delta Y_{0,t} \otimes dX_t \right\|_\alpha \leq C(\|X\|_\alpha \|R\|_{2\alpha} + \|\mathbf{X}\|_{2\alpha} \|Y'\|_{\mathcal{C}^\alpha}), \quad (2.13)$$

for some constant $C > 0$ depending only on the final time. If furthermore $(\bar{X}, \bar{\mathbf{X}}) \in \mathcal{D}^\alpha$ and $(\bar{Y}, \bar{Y}') \in \mathcal{C}_{\bar{X}}^\alpha$, then there exists a constant C such that the bound

$$\begin{aligned} \left\| \int_0^\cdot \delta Y_{0,t} \otimes dX_t - \int_0^\cdot \delta \bar{Y}_{0,t} \otimes d\bar{X}_t \right\|_\alpha &\leq C\|X - \bar{X}\|_\alpha (\|R\|_{2\alpha} + \|\bar{R}\|_{2\alpha}) \\ &\quad + C\|\mathbf{X} - \bar{\mathbf{X}}\|_{2\alpha} (\|Y'\|_{\mathcal{C}^\alpha} + \|\bar{Y}'\|_{\mathcal{C}^\alpha}) \\ &\quad + C\|R - \bar{R}\|_{2\alpha} (\|X\|_\alpha + \|\bar{X}\|_\alpha) \\ &\quad + C\|Y' - \bar{Y}'\|_{\mathcal{C}^\alpha} (\|\mathbf{X}\|_{2\alpha} + \|\bar{\mathbf{X}}\|_{2\alpha}) \end{aligned} \quad (2.14)$$

holds.

Remark 2.4 The bound (2.13) does behave in a very natural way under dilatations. Indeed, the integral is invariant under the transformation

$$(Y, X, \mathbf{X}) \mapsto (\lambda^{-1}Y, \lambda X, \lambda^2 \mathbf{X}). \quad (2.15)$$

The same is true for right hand side of (2.13), since under this dilatation, we also have $(Y', R) \mapsto (\lambda^{-2}Y', \lambda^{-1}R)$.

2.3 Integration against a scaled function

While the bound (2.13) is well-behaved under (2.15), it is very badly behaved if the integrand is multiplied by a smooth function that is rescaled in its argument. However, when acting onto the nonlinearity of our equation with the heat semigroup, this is precisely the type of expression that we encounter, and sharp bounds are essential in order to obtain the well-posedness of our problem. In this subsection, we give such a bound. Let (X, \mathbf{X}) be a rough path belonging to \mathcal{D}^α , let $(Y, Y') \in \mathcal{C}_X^\alpha$ be a rough path controlled by X , and let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a \mathcal{C}^1 function such that

$$\|f\|_{1,1} := \sum_{n \in \mathbf{Z}} \sup_{0 \leq t \leq 1} (|f(n+t)| + |f'(n+t)|) < \infty. \quad (2.16)$$

We then have the following bound, which is crucial in what follows.

Proposition 2.5 *Let $\alpha > \frac{1}{3}$. With the same notations as above, there exists a universal constant C_α such that the bound*

$$\left| \int_0^1 f(\lambda t) Y(t) dX(t) \right| \leq C_\alpha \lambda^{-\alpha} \|Y\|_{X,\alpha} \|f\|_{1,1} (\|X\|_\alpha + \|\mathbf{X}\|_{2\alpha}), \quad (2.17)$$

holds for all $\lambda \geq 1$.

Remark 2.6 Recall that if $Y \in \mathcal{C}_X^\alpha$, then $t \mapsto f(\lambda t)Y_t$ also belongs to the same space. Therefore, the integral appearing in (2.17) is well-defined in the sense of [Gub04].

Proof. We assume without loss of generality that λ is an integer and we write

$$\int_0^1 f(\lambda t)Y(t) dX(t) = \sum_{k=0}^{\lambda-1} \int_0^1 f(t+k)Y_{\lambda,k}(t) dX_{\lambda,k}(t),$$

where we have set

$$X_{\lambda,k}(t) = X((t+k)/\lambda), \quad \mathbf{X}_{\lambda,k}(s,t) = \mathbf{X}((s+k)/\lambda, (t+k)/\lambda).$$

Similarly, the path $Y_{\lambda,k}$ is considered to be controlled by $X_{\lambda,k}$ with $Y'_{\lambda,k}(t) = Y'((t+k)/\lambda)$. With these notations, it is straightforward to check from the definitions that one has the bounds $\|X_{\lambda,k}\|_\alpha \leq \lambda^{-\alpha}\|X\|_\alpha$ and $\|\mathbf{X}_{\lambda,k}\|_{2\alpha} \leq \lambda^{-2\alpha}\|\mathbf{X}\|_{2\alpha}$, and that furthermore

$$\|Y_{\lambda,k}\|_{X_{\lambda,k},\alpha} \leq \|Y\|_{X,\alpha}.$$

Finally, setting $f_k(t) = f(t+k)$ with $t \in [0, 1]$, we can view f_k as a path $F_k \in \mathcal{C}_X^\alpha$ just as in Section 2.1.2. Setting

$$\alpha_k = \|F_k\|_{\mathcal{C}^{2\alpha}},$$

it follows from (2.16) that there exists a constant C such that $\sum_{k \geq 0} \alpha_k \leq C\|f\|_{1,1} < \infty$. Combining these bounds, it follows from Theorem 2.3 and (2.9) that

$$\begin{aligned} \left| \int_0^1 f(t+k)Y_{\lambda,k}(t) dX_{\lambda,k}(t) \right| &\leq C\alpha_k \|Y_{\lambda,k}\|_{X_{\lambda,k},\alpha} (\|X_{\lambda,k}\|_\alpha + \|\mathbf{X}_{\lambda,k}\|_{2\alpha}) \\ &\leq C\lambda^{-\alpha}\alpha_k \|Y\|_{X,\alpha} (\|X\|_\alpha + \|\mathbf{X}\|_{2\alpha}), \end{aligned}$$

so that the requested bound follows at once by summing over k . \square

2.4 Gaussian rough paths

Finally, let us conclude this introductory section by a short discussion of rough paths constructed from Gaussian processes. In the context of this article, Gaussian rough paths are relevant because our solution will turn out to be sufficiently “close” to the solution to the linearised equation (the one with $f = g = 0$), which is indeed a Gaussian process. In the wider context of rough paths theory, Gaussian processes provide one of the rare examples where, under some regularity assumptions that are considerably weaker than those required for Young’s theory of integration to be applicable, there exists a canonical choice for \mathbf{X} .

Note at this point that this is where probability theory enters the game. So far, the theory of rough paths appeared like a purely deterministic theory. However, an abstract theory is only as useful as the examples to which it applies, and besides some rare examples like [Gub06], the theory of rough paths really shines when applied to problems that have Gaussian processes as basic building blocks.

It turns out that the “correct” way of measuring the regularity of the covariance function for a Gaussian process that ensures the existence of an associated rough path is that of two-dimensional ϱ -variation. Following [FV10a], for $K: [0, 1]^2 \rightarrow \mathbf{R}$ the covariance of a Gaussian process X , we define its ϱ -variation on the rectangle $[s, \bar{s}] \times [t, \bar{t}] \subseteq [0, 1]^2$ by

$$|K|_\varrho([s, \bar{s}] \times [t, \bar{t}]) = \left(\sup \sum_{i,j} |\mathbf{E} \delta X_{s_i, s_{i+1}} \delta X_{t_j, t_{j+1}}|^\varrho \right)^{\frac{1}{\varrho}}, \quad (2.18)$$

where the supremum is taken over all finite partitions $s = s_1 \leq \dots \leq s_m = \bar{s}$ and $t = t_1 \leq \dots \leq t_n = \bar{t}$.

Remark 2.7 The notion of finite two-dimensional 1-variation does *not* coincide with the classical concept of being of bounded variation (BV). Actually a function is BV if its gradient is a vector-valued measure, whereas K is of finite 2-dimensional 1-variation if $\partial_x \partial_y K$ is a measure.

The following lemma is a slightly modified version of the existence and continuity results from [FV10a, Thm 35] and [FV10b, Cor. 15.31].

Lemma 2.8 *Assume that $X = (X_t^1, \dots, X_t^n)$, $t \in [0, 1]$ is a centred Gaussian process with mutually independent components and denote by K_i the covariance function of the i -th component.*

Assume that for every $0 \leq s \leq t \leq 1$

$$|K_i^X|_{\varrho}([s, t]^2) \leq M|t - s|^{\frac{1}{\varrho}}. \quad (2.19)$$

Then for every $\alpha < \frac{1}{2\varrho}$, the process X can be lifted canonically to an α rough path (X, \mathbf{X}) and for every $p \geq 1$ there exists a constant C_p such that

$$\mathbf{E}(\|X\|_{\alpha}^{2p} + \|\mathbf{X}\|_{2\alpha}^p) < C_p M^p. \quad (2.20)$$

Remark 2.9 Given a Gaussian process X_t , the area process $\mathbf{X}_{s,t}$ that is canonically associated to it is given by

$$\mathbf{X}_{s,t} = \lim_{\varepsilon \rightarrow 0} \int_s^t \delta X_{s,r}^{\varepsilon} \dot{X}_r^{\varepsilon} dr,$$

where X^{ε} is any suitable sequence of smooth approximations to X . It follows from the results in [FV10a, Theorem 35] that if the covariance of X has finite two-dimensional p -variation for $p < 2$, then this limit exists and is independent of the choice of approximation for a large class of approximations (mollification, piecewise linear approximation, Karhunen-Loève expansion).

Note also that the condition (2.19) implies that the classical Kolmogorov criterion

$$\mathbf{E}|X_t - X_s|^2 \leq C|t - s|^{\frac{1}{\varrho}}, \quad (2.21)$$

for α -Hölder continuous sample paths is satisfied. However, (2.19) is a stronger condition in general.

3 Definition of solutions and well-posedness

In this section, we show that it is possible to give a meaning to (1.1) by using rough path theory. For this, we first denote by ψ the *stationary* solution to the linearised SPDE

$$d\psi = (\partial_x^2 - 1)\psi + \sigma dW. \quad (3.1)$$

We then have the following result, which is a straightforward consequence of the results in [CQ02, FV10a]:

Lemma 3.1 *For every $\alpha < \frac{1}{2}$, the stochastic process ψ given by (3.1) can be lifted canonically to a process $\Psi: \mathbf{R}_+ \rightarrow \mathcal{D}^\alpha$ which has almost surely Hölder continuous sample paths.*

Proof. The process ψ is a centred Gaussian process with covariance function given, for fixed t , by

$$\mathbf{E}\psi(x, t)\psi(y, t) = K(x - y),$$

where K is given by

$$K(x) = \frac{\sigma^2}{2\pi} \sum_{k \in \mathbf{Z}} \frac{\cos kx}{1 + k^2} = \frac{\sigma^2}{2} \frac{\cosh(|x| - \pi)}{\sinh \pi},$$

for $x \in [-\pi, \pi]$. It is then extended periodically for the remaining values of x . In particular, K is C^∞ away from the origin, with a jump discontinuity in its first derivative at the origin.

By Lemma 2.8, we conclude that for every fixed $t > 0$, the Gaussian process $\psi(\cdot, t)$ can be lifted canonically to a rough path Ψ_t . (See Remark 2.9 below on the meaning of “canonical” in this context.) We stress once again that the spatial variable x plays the rôle of “time” here, while t remains fixed!

Furthermore, there exists a constant $C > 0$ such that, given any two times $s, t > 0$, we have

$$\mathbf{E}|\psi(x, t) - \psi(x, s)|^2 \leq C|t - s|^{1/2}.$$

This can be seen from the bound

$$\mathbf{E}|\psi(x, t) - \psi(x, s)|^2 = \frac{\sigma^2}{2\pi} \sum_{k \in \mathbf{Z}} \frac{1 - e^{-(1+k^2)|t-s|}}{1 + k^2} \leq C \sum_{k \geq 1} |t - s| \wedge k^{-2},$$

and bounding this sum by an integral. By Lemma 2.8, this shows that there exists an exponent $\theta > 0$ and constants C_q such that

$$\mathbf{E}(\|\Psi_s - \Psi_t\|_\alpha^{2q} + \|\Psi_s - \Psi_t\|_{2\alpha}^q) \leq C_q |t - s|^{\theta q},$$

for every $q \geq 1$ and every $s, t \in [0, 1]$, say. The claim then follows at once from Kolmogorov’s continuity criterion, as for example in [RY91, p. 26]. \square

We are now finally in a position to formulate what we mean exactly by a solution u to (1.1):

Definition 3.2 A continuous stochastic process u is a solution to (1.1) if the process $v_t = u_t - \psi_t$ belongs to $\mathcal{C}([0, T], \mathcal{C}) \cap \mathbf{L}^1([0, T], \mathcal{C}^1)$ and is such that the identity

$$\begin{aligned} \langle v_t, \varphi \rangle &= \langle u_0 - \psi_0, \varphi \rangle + \int_0^t \langle (\partial_x^2 - 1)\varphi, v_s \rangle ds + \int_0^t \langle \varphi, g(u_s) \partial_x v_s \rangle ds \\ &\quad + \int_0^t \int_0^{2\pi} \varphi(x) g(v_s(x) + \Psi_s(x)) d\Psi_s(x) ds + \int_0^t \langle \varphi, \hat{f}(u_s) \rangle ds, \end{aligned} \quad (3.2)$$

holds almost surely for every smooth periodic test function $\varphi: [0, 2\pi] \rightarrow \mathbf{R}$. Here, we have set $\hat{f}(u) = f(u) + u$.

Remark 3.3 Since we assume that $v_s \in \mathcal{C}^1$, the path $x \mapsto \varphi(x)g(v_s(x) + \Psi_s(x))$ is controlled by Ψ_s , so that the inner integral on the second line is well-defined as a rough integral in the sense of [Gub04]. Furthermore, it yields a measurable function of s , being the pointwise limit of measurable functions. Its value is bounded by a constant depending on $\|\Psi_s\|_\alpha$ and $\|\varphi\|_{\mathcal{C}^1}$, and depending linearly on $\|v_s\|_{\mathcal{C}^1}$ (by (2.13) and Lemma 2.2), so that the outer integral always makes sense as well.

Remark 3.4 At first sight, one could think that this notion of solution is dependent on the arbitrary choice of the constant “1” in (3.1), which in turn accounts for the presence of the function \hat{f} in (3.2). This constant is present for the sole purpose of actually having a stationary solution to (3.1). It is however a straightforward exercise to check that the notion of a weak solution (with the obvious modifications in (3.2)) is independent of this choice.

3.1 Mild solutions

It is clear that the notion of a “weak solution” given in Definition 3.2 has a “mild solution” counterpart. We denote by S_t the heat semigroup generated by ∂_x^2 endowed with periodic boundary conditions and we define the heat kernel p_t as being the periodic function such that

$$(S_t u)(x) = \int_0^{2\pi} p_t(x-y)u(y) dy ,$$

holds for every continuous function u .

With these notations in place, we say that v is a “mild solution” to (1.1) if it satisfies the same conditions as in Definition 3.2, but with (3.2) replaced by the requirement that the identity

$$\begin{aligned} v_t(x) &= (S_t(u_0 - \psi_0))(x) + \int_0^t (S_{t-s}(g(u_s)\partial_x v_s + \hat{f}(u_s)))(x) ds \\ &\quad + \int_0^t \int_0^{2\pi} p_{t-s}(x-y)g(u(y,s)) d\Psi_s(y) ds , \end{aligned} \quad (3.3)$$

holds almost surely for every $x \in [0, 2\pi]$ and $t \in (0, T]$.

Before we show that mild solutions exist, we show that (as expected) the concepts of mild and weak solutions do agree.

Proposition 3.5 *Every mild solution is a weak solution and vice-versa.*

Proof. For fixed t , the rough integral provides a way of interpreting

$$F_s(v) = g(v_s + \Psi_s)\partial_x \Psi_s$$

as an element of the space \mathcal{S}' of Schwartz distributions. Furthermore, it follows from Lemma 2.2 that the map $v \mapsto g(v + \Psi_s)\partial_x \Psi_s$ is continuous as a map from \mathcal{C}^1 to \mathcal{S}' . The claim then follows by standard techniques. \square

From now on, we will only use the concept of a mild solution.

3.2 Existence and uniqueness

Our main result in this section is the following well-posedness result:

Theorem 3.6 *Let $\beta \in (\frac{1}{3}, \frac{1}{2})$ and let $u_0 \in \mathcal{C}^\beta$. Then, for almost every realisation of the driving process Ψ , there exists $T > 0$ such that equation (1.1) has a unique mild solution taking values in $\mathcal{C}([0, T], \mathcal{C}^\beta)$. If furthermore g is bounded and all derivatives of f and g are bounded, then this solution is global (i.e. one can choose T arbitrary, independently of Ψ).*

Remark 3.7 Once $\Psi: \mathbf{R}_+ \rightarrow \mathcal{D}^\alpha$ is fixed, our construction is completely deterministic. The ill-posedness of the equation (1.1) is then a consequence of the fact that the area process Ψ_t is not uniquely determined by Ψ_t . Care needs to be taken since different numerical approximations to (1.1) may converge to solutions corresponding to different choices of the area process. However, the canonical choice given by Lemma 3.1 is natural, as we will see in Section 4.

Before we turn to the proof of this result, we show that:

Lemma 3.8 *Let $\alpha \in (\frac{1}{3}, \frac{1}{2})$, let $\Psi \in \mathcal{D}^\alpha([0, 2\pi], \mathbf{R}^d)$ and let $(Y, Y') \in \mathcal{C}_\Psi^\alpha$. Then, there exists a constant C independent of Ψ and Y such that*

$$\left| \int_0^{2\pi} \partial_x p_t(x-y) Y_y d\Psi(y) \right| \leq C t^{\frac{\alpha}{2}-1} \|Y\|_{\Psi, \alpha} (\|\Psi\|_\alpha + \|\Psi\|_{2\alpha}),$$

uniformly for $x \in [0, 2\pi]$.

Proof. From the identity

$$\partial_x p_t(x) = - \sum_{n \in \mathbf{Z}} \frac{x}{\sqrt{2\pi t}^{\frac{3}{2}}} \exp\left(-\frac{(x-2\pi n)^2}{2t}\right),$$

it is a simple exercise to check that for $t \in (0, 1]$ there exist functions f_t such that $\sup_{t \in (0, 1]} \|f_t\|_{1,1} \leq \infty$ and such that, for $x \in [-\pi, \pi]$, one has the identity

$$\partial_x p_t(x) = \frac{1}{t} f_t\left(\frac{x}{\sqrt{t}}\right).$$

The result then follows at once from Proposition 2.5. \square

With this bound in hand, we can now turn to the

Proof of Theorem 3.6. We perform a classical Picard iteration scheme for the fixed point equation (3.3). Fix $\alpha \in (\frac{1}{3}, \beta)$, and let the process $\Psi: [0, 1] \rightarrow \mathcal{D}^\alpha([0, 2\pi], \mathbf{R}^d)$ be given as in Lemma 3.1.

We then consider a Picard iteration in the space of continuous functions from $[0, T]$ to $\mathcal{C}^1([0, 2\pi], \mathbf{R}^d)$ (with $T \leq 1$ to be determined), endowed with the norm

$$\|v\|_{1,T} = \sup_{t \leq T} \|v_t\|_{\mathcal{C}^1}.$$

Denote this space by \mathcal{C}_T^1 for the sake of conciseness. We also fix an initial condition $u_0 \in \mathcal{C}^\beta$ and we use the shorthand notation $U_t = S_t(u_0 - \psi_0)$.

It turns out to be advantageous to subtract the contribution of the initial condition so that we set $v_t = u_t - \Psi_t - U_t$. With this definition, we have $v_0 = 0$ and we solve for the obvious modification of (3.3). Note also that even though we consider Ψ as a process with values in \mathcal{D}^α , one actually has $\psi_0 \in \mathcal{C}^\beta$ almost surely, and we will make use of the additional leeway that this provides. Given Ψ , u_0 and T , we then consider the map

$$\mathcal{M}_{T,\Psi}: \mathcal{C}_T^1 \rightarrow \mathcal{C}_T^1,$$

given by

$$\begin{aligned} (\mathcal{M}_{T,\Psi}v)(x, t) &= \int_0^t (S_{t-s}(g(u_s)(\partial_x v_s + \partial_x U_s) + \hat{f}(u_s)))(x) ds \\ &\quad + \int_0^t \int_0^{2\pi} p_{t-s}(x-y)g(v(y, s) + \Psi_s(y) + U_s(y)) d\Psi_s(y) ds \\ &\stackrel{\text{def}}{=} (\mathcal{M}_{T,\Psi}^{(1)}v)(x, t) + (\mathcal{M}_{T,\Psi}^{(2)}v)(x, t), \end{aligned} \quad (3.4)$$

where we use the shorthand notation $u_s = v_s + \psi_s + U_s$. We now fix a realisation of Ψ and we set $K > 1$ such that

$$\|u_0\|_{\mathcal{C}^\beta} \leq K, \quad \|\psi_0\|_{\mathcal{C}^\beta} \leq K.$$

We also consider v, \bar{v} such that

$$\|v\|_{1,T} < K, \quad \|\bar{v}\|_{1,T} < K,$$

and we set $\|\Psi\| = \sup_{t \leq 1} (\|\Psi_t\|_{\mathcal{C}^\alpha} + \|\Psi_t\|_{2\alpha})$.

We then have a constant c such that $\|u_s\|_\infty \leq cK$ for $s \leq T$ and such that $\|U_s\|_{\mathcal{C}^1} \leq cKs^{\frac{\beta-1}{2}}$. Since furthermore S_t is bounded by $Ct^{-\frac{1}{2}}$ as a linear operator from L^∞ into \mathcal{C}^1 , this immediately implies that $\mathcal{M}_{T,\Psi}^{(1)}v$ belongs to \mathcal{C}_T^1 and that

$$\|\mathcal{M}_{T,\Psi}^{(1)}v\|_{1,T} \leq C_K(1 + \|\Psi\|)T^{\frac{\beta}{2}}, \quad (3.5)$$

for some constant C_K . Note that if g and Df are bounded, then we can take C_K proportional to K .

Regarding the modulus of continuity of the map $\mathcal{M}_{T,\Psi}^{(1)}$, we have the identity

$$\begin{aligned} (\mathcal{M}_{T,\Psi}^{(1)}v - \mathcal{M}_{T,\Psi}^{(1)}\bar{v})_t &= \int_0^t S_{t-s}(g(u_s)(\partial_x v_s - \partial_x \bar{v}_s) + \hat{f}(u_s) - \hat{f}(\bar{u}_s)) ds \\ &\quad + \int_0^t S_{t-s}(g(u_s) - g(\bar{u}_s))(\partial_x \bar{v}_s + \partial_x U_s) ds. \end{aligned}$$

Since $\|g(u_s) - g(\bar{u}_s)\|_\infty \leq C_K\|v_s - \bar{v}_s\|_\infty \leq C_K\|v_s - \bar{v}_s\|_{\mathcal{C}^1}$, and similarly for f , we obtain as before the bound

$$\|\mathcal{M}_{T,\Psi}^{(1)}v - \mathcal{M}_{T,\Psi}^{(1)}\bar{v}\|_{1,T} \leq C_K(1 + \|\Psi\|)\|v - \bar{v}\|_{1,T}T^{\frac{\beta}{2}},$$

where $C_K \propto K$ if g, Dg , and Df are bounded. Let us now turn to the second term. Here, the integrand of the inner integral should be interpreted as a rough path controlled by Ψ_s , which is built from Ψ_s, v_s , and U_s by making use of Lemma 2.2.

More explicitly, the integrand (without the prefactor $p_{t-s}(x-y)$) is the controlled rough path $(Y_s, Y'_s) \in \mathcal{C}_{\Psi_s}^\alpha$ given by

$$Y_s(x) = g(v(x, s) + \psi(x, s) + U_s(x)), \quad (3.6)$$

$$Y'_s(x) = Dg(v(x, s) + \psi(x, s) + U_s(x)) .$$

(We stress again that we view s here simply as an index, with the “temporal” variable of our controlled rough path being given by x .) It then follows from Lemma 2.2 that there exists a constant \hat{C}_K such that

$$\|Y_s\|_{\Psi_s, \alpha} \leq \hat{C}_K(\|v_s\|_{2\alpha} + \|U_s\|_{2\alpha})(1 + \|\Psi\|)^2 .$$

Since $\|U_s\|_{2\alpha} \leq Cs^{-\frac{2\alpha-\beta}{2}}$ by standard properties of the heat semigroup, we have the bound

$$\|Y_s\|_{\Psi_s, \alpha} \leq C_K(1 + \|\Psi\|)^2(1 + s^{-\frac{2\alpha-\beta}{2}}) ,$$

Again, it is straightforward to check that if the first two derivatives of g are bounded, then we can take \hat{C}_K independent of K , and therefore we have C_K proportional to K .

It now immediately follows from Lemma 3.8 there exists a constant C_K such that

$$\left| \partial_x \int_0^{2\pi} p_{t-s}(x-y) Y_s(y) d\Psi_s(y) \right| \leq C_K(1 + \|\Psi\|)^3(1 + s^{-\frac{2\alpha-\beta}{2}})(t-s)^{\frac{\alpha}{2}-1} ,$$

so that one has the bound

$$\|\mathcal{M}_{T, \Psi}^{(2)} v\|_{1, T} \leq C_K(1 + \|\Psi\|)^3 T^{\frac{\beta-\alpha}{2}} , \quad (3.7)$$

where C_K is proportional to K if Dg and D^2g are bounded.

In order to obtain control over the modulus of continuity of $\mathcal{M}_{T, \Psi}^{(2)}$, we denote by (\bar{Y}_s, \bar{Y}'_s) the controlled rough path associated to \bar{v}_s , so that

$$\bar{Y}_s(x) - Y_s(x) = \int_0^1 Dg(\psi_s(x) + U_s(x) + v_s(x) + \lambda \delta v_s(x))(\bar{v}_s(x) - v_s(x)) d\lambda . \quad (3.8)$$

Applying Lemma 2.2 to the integrand of this expression, we obtain as before the bound

$$\|\bar{Y}_s - Y_s\|_{\Psi_s, \alpha} \leq \hat{C}_K(1 + \|\Psi\|)^2(1 + s^{-\frac{2\alpha-\beta}{2}})\|\bar{v}_s - v_s\|_{C^{2\alpha}} ,$$

so that there exists a constant C_K such that

$$\|\mathcal{M}_{T, \Psi}^{(2)} v - \mathcal{M}_{T, \Psi}^{(2)} \bar{v}\|_{1, T} \leq C_K(1 + \|\Psi\|)T^{\frac{\beta-\alpha}{2}}\|v - \bar{v}\|_{1, T} .$$

Note that even if the derivatives of g are all bounded, this constant this time actually grows quadratically in K , but this turns out not to be a problem. Combining these bounds and using the fact that $\beta > \alpha$ by assumption, it follows immediately that for T sufficiently small, $\mathcal{M}_{T, \Psi}$ maps the ball of radius K in \mathcal{C}_T^1 into itself and satisfies $\|\mathcal{M}_{T, \Psi} v - \mathcal{M}_{T, \Psi} \bar{v}\|_{1, T} \leq \frac{1}{2}\|v - \bar{v}\|_{1, T}$, so that it admits a unique fixed point in this space. Iterating this argument in the usual way, we construct a local solution up to some blow-up time τ with $\lim_{t \rightarrow \tau} \|u_t\|_{C^\beta} = +\infty$.

It remains to show that the solution constructed in this way is global if $g, Dg, D^2g,$ and Df are bounded. This only uses the fact that in this case, as a consequence of (3.5) and (3.7), there exists $T_\star > 0$ depending on $\|\Psi\|$ but *independent of K* such that

$$\|\mathcal{M}_{T, \Psi} v\|_{1, T} \leq \frac{K}{2} , \quad (3.9)$$

for every $T \leq T_\star$, provided that $\|u_0\|_{C^\beta} \leq K$, $\|\psi_0\|_{C^\beta} \leq K$, and $\|v\|_{1, T} \leq K$. Let now $\hat{T} = \inf\{t \geq 0 : \|v_t\|_1 \geq K\}$. If $\hat{T} \leq T_\star$, then the assumptions for (3.9) to hold

are satisfied by construction so that, since v is a fixed point of $\mathcal{M}_{T,\Psi}$, we conclude that $\|v_{\hat{T}}\|_1 \leq K/2$, a contradiction. Therefore, we must have $\hat{T} > T_*$, from which we conclude that $\tau > T_*$. Since T_* is independent of the initial condition u_0 , this argument can be iterated up to arbitrarily long times, thus yielding the existence and uniqueness of global solutions. \square

Remark 3.9 Inspection of the proof reveals that we actually only need $g \in \mathcal{C}^3$ and $f \in \mathcal{C}^1$ for the existence and uniqueness of local solutions.

3.3 Stability of the solution

As an almost immediate corollary of the results obtained in the previous section, we obtain the stability of the solutions under perturbations of the driving noise and of the initial condition. We have the following result:

Corollary 3.10 *Let f and g be smooth and let $u_0, \bar{u}_0 \in \mathcal{C}^\beta$ and $\Psi, \bar{\Psi} \in \mathcal{C}(\mathbf{R}_+, \mathcal{D}^\alpha) \cap \mathcal{C}(\mathbf{R}_+, \mathcal{C}^\beta)$ with $\frac{1}{3} < \alpha < \beta < \frac{1}{2}$. Denote the corresponding local solutions by u, \bar{u} and the blow-up times by $\tau, \bar{\tau}$. Then, for every such u_0 and Ψ , every $T < \tau$, and every $\varepsilon > 0$, there exists $\delta > 0$ such that $\bar{\tau} \geq T$ and*

$$\sup_{t \leq T} \|u_t - \bar{u}_t\|_{\mathcal{C}^\beta} \leq \varepsilon,$$

for all \bar{u}_0 and $\bar{\Psi}$ such that

$$\Delta_{u,\Psi} \stackrel{\text{def}}{=} \|u_0 - \bar{u}_0\|_{\mathcal{C}^\beta} + \sup_{t \leq T} \|\Psi_t - \bar{\Psi}_t\|_\beta + \sup_{t \leq T} \|\Psi_t - \bar{\Psi}_t\|_{2\alpha} \leq \delta,$$

where Ψ_t denotes the area process of Ψ_t as before.

Proof. Denote by $\mathcal{M}_{u_0,\Psi}$ the same map as in (3.4), but where we change notation in order to suppress the dependency on T (which is not relevant here), and show instead the dependency on the initial condition u_0 . The claim then follows from standard arguments if we can show that, for every $K > 0$, there exists a constant C_K such that the bound

$$\|\mathcal{M}_{u_0,\Psi} v - \mathcal{M}_{\bar{u}_0,\bar{\Psi}} v\|_{1,T} \leq C_K \Delta_{u,\Psi},$$

holds provided that $\Delta_{u,\Psi} \leq 1$, and that

$$\|u_0\|_{\mathcal{C}^\beta} + \|\Psi\| + \|v\|_{1,T} \leq K.$$

This in turn follows immediately from considerations similar to those in the proof of Theorem 3.6. \square

In particular, it follows from this that the notion of a solution given by Definition 3.2 coincides with those solutions that are obtained by mollifying the noise in (1.1) and passing to the limit. We can formulate this more precisely as:

Corollary 3.11 *Let $\varphi: \mathbf{R} \rightarrow \mathbf{R}$ be a smooth compactly supported function such that $\int_{\mathbf{R}} \varphi(x) dx = 1$ and set $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)/\varepsilon$. Define the operator $Q_\varepsilon: \mathbf{L}^2(S^1) \rightarrow \mathbf{L}^2(S^1)$ by $(Q_\varepsilon u)(x) = \int_{S^1} \varphi_\varepsilon(x-y) u(y) dy$ and consider the solution u_ε to*

$$du_\varepsilon = \partial_x^2 u_\varepsilon dt + f(u_\varepsilon) dt + g(u_\varepsilon) \partial_x u_\varepsilon dt + \sigma Q_\varepsilon dW(t),$$

with blow-up time τ_ε . Then, there exists a sequence $\hat{\tau}_\varepsilon$ of stopping times converging almost surely to τ such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \leq \hat{\tau}_\varepsilon} \|u_\varepsilon(t) - u(t)\|_{C^\beta} = 0,$$

where u is the solution to (1.1) as given by Theorem 3.6.

Proof. We first note that in the particular case where Ψ_t is a smooth function of x for every t and Ψ_t is given by (2.2) (this time reading it from right to left as a definition for Ψ), then the rough integral against Ψ_t coincides with the usual Riemann integral, so that the notion of a solution given by Definition 3.2 coincides with the usual notions of solution as given in [DPZ92, Hai09] for example.

The claim then follows from Corollary 3.10, noting that $\Psi^\varepsilon \rightarrow \Psi$ uniformly in $\mathcal{C}([0, T], \mathcal{D}^\alpha) \cap \mathcal{C}([0, T], \mathcal{C}^\beta)$ as a consequence of [FV10a, Theorem 37]. \square

4 Invariant measures

In this section, we show that in some cases, the invariant measure for equations of the type studied above can be exhibited explicitly, due to the fact that the equation has a type of gradient structure. Indeed, let ν be the Gaussian probability measure on $\mathcal{C}([0, 2\pi], \mathbf{R}^n)$ with covariance operator given by

$$K_\nu = (I - \partial_x^2)^{-1}, \quad (4.1)$$

where I is the identity matrix and ∂_x^2 acts independently on every component. Since, when restricted to any strict subinterval of $[0, 2\pi]$, this covariance differs from that of an Ornstein-Uhlenbeck process only by a smooth function, we see that on such subintervals it is equivalent to Wiener measure (provided that we start the Wiener process with a Gaussian initial condition). In particular, the expression

$$\int_0^{2\pi} G(w_x) \circ dw_x,$$

is well-defined as a Stratonovich integral for every smooth function $G: \mathbf{R}^n \rightarrow \mathbf{R}^n$ and ν -almost every w . If the function G has sublinear growth and a bounded derivative, then this quantity actually has exponential moments (see Section 5 below), so that we can define a probability measure μ as a change of measure from ν by

$$\frac{d\mu}{d\nu}(w) = Z^{-1} \exp\left(\int_0^{2\pi} G(w_x) \circ dw_x + \int_0^{2\pi} F(w_x) dx\right) \stackrel{\text{def}}{=} Z^{-1} \exp(\Xi(w)), \quad (4.2)$$

where Z is a normalisation constant that ensures that μ is a probability measure. Here, F could be any measurable function with subquadratic growth to ensure that this expression is integrable with respect to ν . Were it not for the periodic boundary conditions, the particular choice $F = -\frac{1}{2}(\text{div } G + |G|^2)$ would yield for μ the law of a, not necessarily reversible, diffusion with drift G .

If G happens to be a gradient, so that the underlying diffusion is reversible, the stochastic integral is reduced to a boundary term, and these equations were treated in [HSV07]. The main contribution of the present article is to be able to treat the non-gradient case.

Let us now suspend our disbelief for a moment and pretend just for the sake of the argument that Ξ is differentiable as a function from $L^2([0, 2\pi], \mathbf{R}^n)$ into \mathbf{R} (which of course it is not!). Formally, we can then compute the L^2 -derivative of Ξ , which yields

$$(D\Xi(w))_i(x) = \partial_i G_j(w(x)) \frac{dw_j(x)}{dx} - \partial_j G_i(w(x)) \frac{dw_j(x)}{dx} + \partial_i F(w(x)) .$$

Since on the other hand, the reference measure ν is invariant for the damped stochastic heat equation

$$du = \partial_x^2 u dt - u dt + \sqrt{2} dW(t) ,$$

this suggests that the measure μ given by (4.2) is invariant for the equation

$$du = \partial_x^2 u dt + g(u) \partial_x u dt + f(u) dt + \sqrt{2} dW(t) , \quad (4.3)$$

where f and g are given by

$$f_i(u) = \partial_i F(u) - u , \quad g_{ij}(u) = \partial_i G_j(u) - \partial_j G_i(u) , \quad (4.4)$$

and where W is an L^2 -cylindrical Wiener process. (Which is just another way of stating that the driving noise is space-time white noise, see [DPZ92].) The aim of this section is to give a rigorous justification of this fact, which we encompass in the following theorem:

Theorem 4.1 *Let F and G be C^∞ functions with bounded derivatives of all orders such that G is bounded and define f and g by (4.4). Then, the mild solution to (4.3) generates a Markov process that is reversible with respect to the measure μ defined in (4.2).*

We postpone the proof of this result to the end of the section and we first lay out the technique and prove a number of intermediate results. Our technique will be to first consider the smoother problem with reference measure ν_ε having covariance operator K_ν^ε given by

$$K_\nu^\varepsilon = (I - \partial_x^2 + \varepsilon^2 \partial_x^4)^{-1} . \quad (4.5)$$

One can check that the measure ν_ε charges paths that are C^1 . Furthermore, the map Ξ is continuous from C^1 to \mathbf{R} (with the ‘‘stochastic integral’’ now being a simple Riemann integral), so that we can define a sequence of probability measures μ_ε by

$$\frac{d\mu_\varepsilon}{d\nu_\varepsilon}(w) = Z_\varepsilon^{-1} \exp(\Xi(w)) , \quad (4.6)$$

where Z_ε is a suitable normalisation constant. One then has the following result:

Proposition 4.2 *Let F and G have bounded derivatives of all orders and let f and g be defined as in (4.4). Then, for every fixed value $\varepsilon > 0$, the stochastic PDE*

$$du_\varepsilon = -\varepsilon^2 \partial_x^4 u_\varepsilon dt + \partial_x^2 u_\varepsilon dt + g(u_\varepsilon) \partial_x u_\varepsilon dt + f(u_\varepsilon) dt + \sqrt{2} dW(t) , \quad (4.7)$$

has unique global solutions in C^1 . Furthermore, it admits the measure μ_ε as an invariant measure and the corresponding Markov process is reversible.

Proof. The local well-posedness of solutions is standard and follows for example from [Hai09]. The global well-posedness and the invariance of μ_ε then follow as in [HSV11, Prop. 26], see also [Zab88]. \square

Furthermore, one has the following convergence result:

Proposition 4.3 *One has $\mu_\varepsilon \rightarrow \mu$ weakly in the space \mathcal{C}^α for every $\alpha < \frac{1}{2}$.*

Proof. Denote by $\tilde{\nu}_\varepsilon$ the lift of ν_ε to a measure on \mathcal{D}^α . Since ν_ε charges \mathcal{C}^1 functions, this lift is performed by simply associating to each element its ‘‘area process’’ given by a standard Riemann integral. On the other hand, we can lift ν to a measure $\tilde{\nu}$ on \mathcal{D}^α in a canonical way as in Lemma 3.1. (Note that this yields the same measure as if we were to construct the area process by Stratonovich integration.) It then follows from [FV10a, Theorem 35] that $\tilde{\nu}_\varepsilon \rightarrow \tilde{\nu}$ weakly in \mathcal{D}^α as $\varepsilon \rightarrow 0$.

Since Ξ is continuous as a map from \mathcal{D}^α to \mathbf{R} , the claim then follows from the uniform exponential integrability of Ξ with respect to $\tilde{\nu}_\varepsilon$. Unfortunately, this uniform exponential integrability turns out to be a highly non-trivial fact, the proof of which is postponed to Theorem 5.1 below. \square

Our final ingredient is the convergence of solutions to (4.7) to those of (4.3). Note that this is not a completely straightforward application of the approximation result given in Corollary 3.10, since we change the linear part of the equation, rather than the noise. However, the statement is quite similar:

Proposition 4.4 *Let f and g be as in the statement of Theorem 4.1, let $u_0 \in \mathcal{C}^\beta$ for some $\beta \in (\frac{1}{3}, \frac{1}{2})$ and let $\{u_0^\varepsilon\}$ be a sequence such that $u_0^\varepsilon \rightarrow u_0$ in \mathcal{C}^β . Then, for every $T > 0$ and every $\alpha < \beta$, we have*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{t \leq T} \|u_t - u_t^\varepsilon\|_{\mathcal{C}^\alpha} = 0, \quad (4.8)$$

where u_t^ε is the solution to (4.7) and u_t is the solution to (4.3), driven by the same realisation of W .

Proof. It follows from the assumptions on f and g that both (4.3) and (4.7) have unique global solutions by Theorem 3.6. Denote by S_t^ε the semigroup generated by the operator $\partial_x^2 - 1 - \varepsilon^2 \partial_x^4$, and denote by ψ^ε the solution to the corresponding linear equation, namely

$$\psi^\varepsilon(t) = \int_{-\infty}^t S_{t-s}^\varepsilon dW(s).$$

We can lift this canonically to a rough-path valued process Ψ_t^ε as before. It is straightforward to show that the two-dimensional p -variation (for any $p > 1$) of the covariance of ψ_t^ε is bounded, uniformly in ε , and that $\mathbf{E}|\psi^\varepsilon(t, x) - \psi(t, x)|^2 \leq C\varepsilon$ for some C , uniformly in x and t (see for example [Hai11, Prop 3.10]). It then follows as before that $\Psi^\varepsilon \rightarrow \Psi$ uniformly over bounded intervals in the rough path topology.

We set as before $v_t = u_t - \psi_t - S_t(u_0 - \psi_0)$ and $v_t^\varepsilon = u_t^\varepsilon - \psi_t^\varepsilon - S_t^\varepsilon(u_0^\varepsilon - \psi_0^\varepsilon)$. It follows from Corollary A.2, [Hai11, Prop 3.10], and standard interpolation inequalities that there exists some constant C and an exponent κ such that

$$\mathbf{E} \sup_{t \leq T} \|\psi_t - \psi_t^\varepsilon + S_t^\varepsilon(u_0^\varepsilon - \psi_0^\varepsilon) - S_t(u_0 - \psi_0)\|_{\mathcal{C}^\alpha} \leq C(\varepsilon^\kappa(1 + \|u_0\|_{\mathcal{C}^\beta}) + \|u_0^\varepsilon - u_0\|_{\mathcal{C}^\beta}),$$

so that it suffices to show (4.8) with u_t and u_t^ε replaced by v_t and v_t^ε respectively.

This then follows like in the proof of Corollary 3.10 once we can show that

$$\|\mathcal{M}_{u_0, \Psi} v - \mathcal{M}_{u_0^\varepsilon, \Psi^\varepsilon} v\|_{1, T} \rightarrow 0, \quad (4.9)$$

uniformly over bounded sets, where \mathcal{M}^ε is the map defined like \mathcal{M} , but with S_t replaced by S_t^ε . Since we already have a bound on $\|\mathcal{M}_{u_0, \Psi} v - \mathcal{M}_{u_0^\varepsilon, \Psi^\varepsilon} v\|_{1, T}$ from the proof of Corollary 3.10, it suffices to bound $\|\mathcal{M}_{u_0, \Psi} v - \mathcal{M}_{u_0^\varepsilon, \Psi}^\varepsilon v\|_{1, T}$. We break this into two terms as in (3.4). For the first term, we note that for $t \leq T$, we have from (A.1) and (A.3) that

$$\begin{aligned} \|S_t^\varepsilon u - S_t u\|_{C^1} &= \|S_{t/2}(\hat{S}_{\varepsilon^2 t} - 1)S_{t/2} u\|_{C^1} \leq Ct^{-1/2} \|(\hat{S}_{\varepsilon^2 t} - 1)S_{t/2} u\|_\infty \\ &\leq C\varepsilon^{\frac{\alpha}{2}} t^{-1/2} \|S_{t/2} u\|_{C^\alpha} \leq C\varepsilon^{\frac{\alpha}{2}} t^{-(1+\alpha)/2} \|u\|_\infty. \end{aligned}$$

Since this singularity is integrable, the requested bound follows.

In order to bound the term involving the rough integral, we need to perform a preliminary computation. Recall that we can write $(S_t^\varepsilon u)(x) = \int p_t^\varepsilon(x-y) u(y) dy$, where p_t^ε is given by

$$p_t^\varepsilon(x) = \sum_{k \in \mathbf{Z}} \exp(-t - k^2 t - \varepsilon^2 k^4 t) f_k(x),$$

where $f_k(x) = (2\pi)^{-1} e^{ikx}$. Let now $\delta p_t^\varepsilon(x) = p_t(x) - p_t^\varepsilon(x)$. With this notation, it then follows from the bound $|f_k(x) - f_k(y)| \leq 2(1 \wedge k|x-y|)$ that

$$\begin{aligned} |\delta p_t^\varepsilon(x) - \delta p_t^\varepsilon(y)| &\leq 2 \sum_{k \in \mathbf{Z}} e^{-k^2 t} (1 \wedge \varepsilon^2 k^4 t) (1 \wedge k|x-y|) \\ &\leq 2 \sum_{k \in \mathbf{Z}} e^{-k^2 t} (1 \wedge \varepsilon^2 k^4 t \wedge k|x-y|). \end{aligned} \quad (4.10)$$

Note now that, by bounding the sum by an integral, one obtains the bound

$$\sum_{k \in \mathbf{Z}} e^{-k^2 t} k^n \leq Ct^{-(n+1)/2},$$

valid for every $n > 0$. Combining this bound with (4.10), we obtain

$$|\delta p_t^\varepsilon(x) - \delta p_t^\varepsilon(y)| \leq \frac{C}{\sqrt{t}} (1 \wedge |x-y| t^{-1/2} \wedge \varepsilon^2 t^{-1})$$

so that, for any pair of exponents $\alpha, \kappa \geq 0$ such that $2\alpha + \kappa \leq 1$, we have

$$|\delta p_t^\varepsilon(x) - \delta p_t^\varepsilon(y)| \leq C|x-y|^{2\alpha} \varepsilon^{2\kappa} t^{-\frac{1}{2}-\alpha-\kappa}. \quad (4.11)$$

It then follows from (2.9) and (4.11) that for $w_s = v_s + U_s \in \mathcal{C}^{2\alpha}$, we have

$$\int_0^{2\pi} \delta p_{t-s}^\varepsilon(x-y) g(w_s + \Psi_s(y)) d\Psi_s(y) \leq C\varepsilon^{2\kappa} (t-s)^{-\frac{1}{2}-\alpha-\kappa} (1 + \|w_s\|_{\mathcal{C}^{2\alpha}}),$$

where C depends on the size of Ψ . Since $\alpha < \frac{1}{2}$ and the 2α -Hölder norm of $w_s = v_s + U_s$ behaves like $s^{-\alpha+\frac{\beta}{2}}$, the right hand side of the above expression is integrable for every fixed $T > 0$, provided that κ is made sufficiently small. (However the value of the integral diverges in general as $T \rightarrow 0$!) Combining all these bounds, we conclude that (4.9) holds, which then implies the result. \square

It is now rather straightforward to combine all of these ingredients in order to prove Theorem 4.1:

Proof of Theorem 4.1. It follows from Proposition 4.3 and Skorokhod's representation theorem [Bil99] that one can construct a sequence of random variables u_0^ε with law μ_ε and a random variable u_0 with law μ such that $u_0^\varepsilon \rightarrow u_0$ in \mathcal{C}^β almost surely.

Denoting by u_t^ε the solution to (4.7) with initial condition u_0^ε and similarly for u_t , it then follows from Proposition 4.4 that $u_t^\varepsilon \rightarrow u_t$ almost surely for every $t \geq 0$. Since, by Proposition 4.2, the law of u_t^ε is given by μ_ε for every $t > 0$, we conclude from Proposition 4.3 that the law of u_t is given by μ for all t . The reversibility of u_t follows in the same way from the reversibility of the u_t^ε by considering the joint distributions at any two times. \square

5 Weak convergence of approximating measures

The aim of this section is to prove the following uniform exponential integrability result, which is essential for the convergence result of the previous section:

Theorem 5.1 *Let $G: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a \mathcal{C}^3 function which is bounded, with bounded first and second derivatives. Then,*

$$\sup_{\varepsilon < 1} \mathbf{E}_\varepsilon \exp\left(\int_0^{2\pi} G(w) \dot{w} dx\right) < \infty,$$

where \mathbf{E}_ε is a shorthand notation for the expectation with respect to the Gaussian measure ν_ε with covariance given by (4.5).

Proof. Since ν_ε charges the set of \mathcal{C}^1 functions for every $\varepsilon > 0$, the quantity under the expectation is defined for ν_ε -almost every w . Furthermore, the argument of the exponential is bounded by $C\|\dot{w}\|_{L^\infty}$ for some $C > 0$, so that it follows from Fernique's theorem that

$$\sup_{\varepsilon \in [\varepsilon_0, 1]} \mathbf{E}_\varepsilon \exp\left(\int_0^{2\pi} G(w) \dot{w} dx\right) < \infty,$$

for every $\varepsilon_0 > 0$, implying that it suffices to obtain a uniform bound for small values of ε .

Note now that it follows from Cauchy-Schwarz and the translation invariance of ν_ε that

$$\mathbf{E}_\varepsilon \exp\left(\int_0^{2\pi} G(w) \dot{w} dx\right) \leq \mathbf{E}_\varepsilon \exp\left(2 \int_0^\pi G(w) \dot{w} dx\right) < \infty,$$

so that it suffices to bound exponential moments of the integral up to time π .

Our proof then proceeds by constructing a sequence of measures $\bar{\nu}_\varepsilon$ that are equivalent to ν_ε and such that there exists $\alpha > 1$ and $\varepsilon_0 > 0$ with

$$\sup_{\varepsilon < \varepsilon_0} \int \left(\frac{d\nu_\varepsilon}{d\bar{\nu}_\varepsilon}\right)^\alpha d\bar{\nu}_\varepsilon < \infty. \quad (5.1)$$

These measures will furthermore have the property that

$$\sup_{\varepsilon < \varepsilon_0} \int \exp\left(\int_0^\pi G(w) \dot{w} dx\right) \bar{\nu}_\varepsilon(dw) < \infty, \quad (5.2)$$

for every function G as in the statement of the result. The construction of $\bar{\nu}_\varepsilon$ together with the uniform bound (5.1) is the content of Lemma 5.9 and Corollary 5.5 below.

The uniform exponential integrability with respect to $\bar{\nu}_\varepsilon$, namely (5.2), is the content of Proposition 5.10.

Setting $\mathcal{E}(w) = \exp(\int_0^\pi G(w) \dot{w} dx)$, we now use Hölder's inequality to write

$$\begin{aligned} \int \mathcal{E}(w) \nu_\varepsilon(dw) &= \int \mathcal{E}(w) \left(\frac{d\nu_\varepsilon}{d\bar{\nu}_\varepsilon} \right)(w) \bar{\nu}_\varepsilon(dw) \\ &\leq \left(\int \left(\frac{d\nu_\varepsilon}{d\bar{\nu}_\varepsilon} \right)^\alpha d\bar{\nu}_\varepsilon \right)^{1/\alpha} \left(\int \mathcal{E}^p(w) \bar{\nu}_\varepsilon(dw) \right)^{1/p}, \end{aligned}$$

where p is the exponent conjugate to α . The claim then follows from (5.1) and (5.2) by noting that \mathcal{E}^p is again of the same form as \mathcal{E} . The remainder of this section is devoted to the construction of $\bar{\nu}_\varepsilon$ and to the proof that (5.1) and (5.2) do indeed hold. \square

Remark 5.2 Retracing the steps of the proof, it is not difficult to check that Theorem 5.1 still holds if, in addition to having bounded first and second derivatives, G is only assumed to have sublinear growth, namely $|G(u)| \leq C(1 + |u|)^\alpha$ for some $\alpha < 1$.

5.1 Construction of $\bar{\nu}_\varepsilon$

Essentially, we will construct $\bar{\nu}_\varepsilon$ as the law of the integral of an Ornstein-Uhlenbeck process with timescale ε , which is reflected in a suitable way around $t = \pi$. Once we know that two Gaussian measures are mutually equivalent, it is easy to show that a bound of the type (5.1) holds for some $\alpha > 1$. The difficult part is to show that it holds uniformly in ε with the same value α . Our main tool in this endeavour is the following standard result from Gaussian measure theory:

Proposition 5.3 *Let $\bar{\nu}$ be a centred Gaussian measure on a separable Banach space \mathcal{B} with covariance operator $\bar{C}: \mathcal{B}^* \rightarrow \mathcal{B}$ and let ν be a centred Gaussian measure on \mathcal{B} with covariance operator C . We assume that both measures have full support, so that the range of both C and \bar{C} is dense in \mathcal{B} .*

Let \mathcal{H} be a separable Hilbert space and let $\mathcal{A}: \mathcal{B} \rightarrow \mathcal{H}$ be an unbounded operator such that $C^{-1} = \mathcal{A}^ \mathcal{A}$. Then, $\bar{\nu}$ and ν are equivalent if and only if the operator $\Lambda = 1 - \mathcal{A} \bar{C} \mathcal{A}^*$ is Hilbert-Schmidt as an operator from \mathcal{H} to \mathcal{H} and has no eigenvalue 1.*

In that case, denoting by $\{\lambda_n\}_{n \geq 0}$ the eigenvalues of Λ , one has the identity

$$\int_{\mathcal{B}} \left(\frac{d\nu}{d\bar{\nu}} \right)^\alpha d\bar{\nu} = \prod_{n \geq 0} \frac{(1 - \lambda_n)^{\frac{\alpha}{2}}}{\sqrt{1 - \alpha \lambda_n}}, \quad (5.3)$$

for all values $\alpha > 1$ such that $\lambda_n \leq 1/\alpha$ for all n .

Proof. The first statement is the content of the Feldman-Hájek theorem. The identity (5.3) is straightforward to check in the case $\mathcal{B} = \mathbf{R}^n$ by using the fact that the left hand side is invariant under changes of coordinates, so that we can reduce ourselves to the case $C = 1$. The general case then follows by approximation. \square

Remark 5.4 A canonical choice of \mathcal{A} and \mathcal{H} is to take for \mathcal{H} the Cameron-Martin space \mathcal{H}_ν of ν and for \mathcal{A} the restriction operator (with domain $\mathcal{H}_\nu \in \mathcal{B}$). This operator can however be multiplied from the left by an arbitrary unitary operator without changing the statement, a fact that we will use in the sequel.

An important remark is that the right hand side of (5.3) is continuous in the Hilbert-Schmidt topology on the set of operators Λ for which the expression makes sense. As a consequence, we have the following:

Corollary 5.5 *Let ν_ε and $\bar{\nu}_\varepsilon$ be a sequence of centred Gaussian measures with covariances C_ε and \bar{C}_ε such that $C_\varepsilon^{-1} = \mathcal{A}_\varepsilon \mathcal{A}_\varepsilon^*$ and such that the range of \mathcal{A}_ε is some common Hilbert space \mathcal{H} . Let $\Lambda_\varepsilon = 1 - \mathcal{A}_\varepsilon \bar{C}_\varepsilon \mathcal{A}_\varepsilon^*$ as before and assume that $\Lambda_\varepsilon \rightarrow \Lambda$ in the Hilbert-Schmidt operator topology, where Λ is a Hilbert-Schmidt operator with $\lambda_n < 1$ for all n .*

Then, there exist $\alpha > 1$ and $\varepsilon_0 > 0$ such that $\sup_{\varepsilon < \varepsilon_0} \int_{\mathcal{B}} \left(\frac{d\nu_\varepsilon}{d\bar{\nu}_\varepsilon}\right)^\alpha d\bar{\nu}_\varepsilon < \infty$.

Proof. Set $1/\alpha = (1 + \sup_n \lambda_n)/2 < 1$, so that the right hand side of (5.3) is finite. The claim then follows from the continuity of that expression in the Hilbert-Schmidt operator topology. \square

A final remark is that as an immediate consequence of Proposition 5.3, one has the following result, where we identify \mathcal{H} with its dual in the usual way in order to consider the covariance operators as self-adjoint Hilbert-Schmidt operators on \mathcal{H} :

Corollary 5.6 *Let ν and $\bar{\nu}$ be two centred Gaussian measures with covariance operators C and \bar{C} on a common Hilbert space \mathcal{H} . If C and \bar{C} are simultaneously diagonalisable with respective eigenvalues $a_n > 0$ and $\bar{a}_n > 0$, then $\nu \sim \bar{\nu}$ provided that $\sum_{n \geq 1} (1 - \frac{a_n}{\bar{a}_n}) < \infty$.* \square

Before we proceed with the construction of the sequence of measures $\bar{\nu}_\varepsilon$, we construct their limit $\bar{\nu}$, which is equivalent to the measure ν given by (4.1).

Lemma 5.7 *Let ν be the Gaussian measure on $L^2(S^1, \mathbf{R})$ with covariance given by $(1 - \partial_t^2)^{-1}$, and let $\bar{\nu}$ be the Gaussian measure with covariance*

$$\bar{C}(x, y) = \begin{cases} 1 + (x \wedge y) & \text{if } x \leq \pi \text{ and } y \leq \pi, \\ 1 + 2\pi - (x \vee y) & \text{if } x \geq \pi \text{ and } y \geq \pi, \\ 1 + 2(x \wedge y) - \frac{xy}{\pi} & \text{otherwise.} \end{cases} \quad (5.4)$$

Then, we have $\bar{\nu} \sim \nu$.

Proof. In principle, the easiest way to check that the operator Λ satisfies the assumptions of Proposition 5.3 is to show that it is Hilbert-Schmidt and has norm strictly less than 1. Unfortunately, it turns out that in our case, this operator does have negative eigenvalues close to -1 , so we use the trick of considering instead a rank one perturbation of \bar{C} , which does give rise to a measure which is obviously equivalent to that given by \bar{C} . By tuning the parameter of that perturbation, we will see that Λ can be made arbitrarily close to an operator which is explicitly diagonalisable.

Let η_κ be the Gaussian measure on $L^2(S^1, \mathbf{R})$ with covariance given by $(\kappa^2 - \partial_t^2)^{-1}$, and let $\bar{\eta}_\kappa$ be the Gaussian measure with covariance

$$\bar{C}_\kappa(x, y) = \begin{cases} \kappa^{-2} + (x \wedge y) & \text{if } x \leq \pi \text{ and } y \leq \pi, \\ \kappa^{-2} + 2\pi - (x \vee y) & \text{if } x \geq \pi \text{ and } y \geq \pi, \\ \kappa^{-2} + 2(x \wedge y) - \frac{xy}{\pi} & \text{otherwise.} \end{cases}$$

Then, we will show that $\eta_\kappa \sim \bar{\eta}_\kappa$ for κ sufficiently large. Since one obviously has $\bar{\eta}_\kappa \sim \eta_{\kappa'}$ for every $\kappa, \kappa' > 0$ (the two covariance operators differ by a rank 1 perturbation) and since it follows easily from Corollary 5.6 that $\eta_\kappa \sim \eta_{\kappa'}$, the claim then follows.

Setting $\mathcal{B} = \mathcal{H} = L^2(S^1, \mathbf{R})$ and $\mathcal{A} = \kappa - \partial_t$, we see that \mathcal{A} has the required property since $\mathcal{A}^* \mathcal{A} = \kappa^2 - \partial_t^2$. Furthermore, the corresponding operator $\Lambda_\kappa = 1 - \mathcal{A} \bar{C}_\kappa \mathcal{A}^*$ is an integral operator with kernel given in the sense of distributions by

$$\Lambda_\kappa(x, y) = \delta(y - x) - (\kappa - \partial_t)(\kappa - \partial_s) \bar{C}_\kappa(x, y).$$

An explicit calculation shows that

$$\Lambda_\kappa(x, y) = \Lambda_0(x, t) + \mathcal{O}(\kappa),$$

where $\mathcal{O}(\kappa)$ means that the remainder term is uniformly bounded by some constant times κ . The kernel Λ_0 is given by

$$\Lambda_0(x, y) = \begin{cases} -\pi^{-1} & \text{if } (x, y) \in [0, \pi]^2 \cup [\pi, 2\pi]^2, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that the operator Λ_0 (we identify an operator with the corresponding integral kernel) is negative definite, since it can be diagonalised by considering linear combinations of step functions. The claim then follows at once, since the Hilbert-Schmidt norm of the remainder term is of order κ . \square

Remark 5.8 The covariance \bar{C} is realised by the following construction. Take a standard Wiener process \hat{W} , an independent Brownian bridge B , and an independent normal random variable ξ . Then, the process X defined by

$$X_x = \xi + \hat{W}_x, \quad x \in [0, \pi], \quad X_x = \xi + (2\pi - x)\hat{W}_\pi + B_{2\pi-x}, \quad x \in [\pi, 2\pi],$$

does have \bar{C} as its covariance operator.

We now make the following construction for $\varepsilon > 0$. Let Z^ε be a stationary Ornstein-Uhlenbeck process with characteristic time ε and variance $\frac{1}{2\varepsilon}$, so that its covariance is given by

$$\mathbf{E} Z_x^\varepsilon Z_y^\varepsilon = \frac{1}{2\varepsilon} \exp\left(-\frac{|x-y|}{\varepsilon}\right).$$

We then define X^ε as the integral of Z^ε , so that $X_x^\varepsilon = \int_0^x Z_y^\varepsilon dy$. The covariance of X^ε is then given by

$$\mathbf{E} X_x^\varepsilon X_y^\varepsilon = (x \wedge y) - \frac{\varepsilon}{2} (1 + e^{-|x-y|/\varepsilon} - e^{-x/\varepsilon} - e^{-y/\varepsilon}) \stackrel{\text{def}}{=} K^\varepsilon(x, y). \quad (5.5)$$

Denote now by A^ε the vector consisting of the ‘‘boundary data’’ $(\sqrt{2\varepsilon} Z_0^\varepsilon, \sqrt{2\varepsilon} Z_\pi^\varepsilon, X_\pi^\varepsilon)$, so that the covariance of A^ε is given by

$$Q_\varepsilon = \begin{pmatrix} 1 & \delta & \sqrt{\frac{\varepsilon}{2}}(1-\delta) \\ \delta & 1 & \sqrt{\frac{\varepsilon}{2}}(1-\delta) \\ \sqrt{\frac{\varepsilon}{2}}(1-\delta) & \sqrt{\frac{\varepsilon}{2}}(1-\delta) & \pi - \varepsilon(1-\delta) \end{pmatrix}, \quad \delta = e^{-\pi/\varepsilon}.$$

Here, we have normalised Z^ε in such a way that it is of order one. We also note that the covariance of X_t^ε with A^ε is given by

$$v_x^\varepsilon := \mathbf{E} X_x^\varepsilon A^\varepsilon = \left(\sqrt{\frac{\varepsilon}{2}}(1 - e^{-x/\varepsilon}), \sqrt{\frac{\varepsilon}{2}}(e^{-(\pi-x)/\varepsilon} - \delta), K^\varepsilon(x, \pi) \right), \quad (5.6)$$

so that there exists a process Y^ε independent of A^ε such that

$$X_x^\varepsilon = Y_x^\varepsilon + \langle Q_\varepsilon^{-1} v_x^\varepsilon, A^\varepsilon \rangle. \quad (5.7)$$

The covariance of the process Y^ε is then given by

$$\mathbf{E}Y_x^\varepsilon Y_y^\varepsilon = K^\varepsilon(x, y) - \langle Q_\varepsilon^{-1} v_y^\varepsilon, v_x^\varepsilon \rangle.$$

Let now \tilde{Y}^ε be a process independent of Y^ε and A^ε , but identical in law to Y^ε , and define \tilde{X}^ε to be the stochastic process given by

$$\tilde{X}_x^\varepsilon = \tilde{Y}_x^\varepsilon + \langle Q_\varepsilon^{-1} v_x^\varepsilon, JA^\varepsilon \rangle, \quad (5.8)$$

where J is the matrix given by $J = \text{diag}(-1, -1, 1)$. Note that \tilde{X}^ε is *not* identical in law to X^ε because, while we force the identity $\tilde{X}^\varepsilon(\pi) = X^\varepsilon(\pi)$, we force the derivatives at the boundary points to satisfy the relations

$$\partial_x \tilde{X}^\varepsilon(x) = -\partial_x X^\varepsilon(x), \quad x \in \{0, \pi\}. \quad (5.9)$$

Actually, it follows from (5.7) and (5.8) that the covariance of \tilde{X} can be written as

$$\mathbf{E}\tilde{X}_x^\varepsilon \tilde{X}_y^\varepsilon = \mathbf{E}X_x^\varepsilon X_y^\varepsilon + \langle Q_\varepsilon^{-1} v_y^\varepsilon, (JQ_\varepsilon J - Q_\varepsilon)Q_\varepsilon^{-1} v_x^\varepsilon \rangle. \quad (5.10)$$

With all of these notations at hand, we let ξ be a $\mathcal{N}(0, 1)$ distributed random variable independent of X^ε and \tilde{X}^ε , and we define a process W^ε by

$$W_x^\varepsilon = \begin{cases} \xi + X_x^\varepsilon & \text{if } x \leq \pi, \\ \xi + \tilde{X}_{2\pi-x}^\varepsilon & \text{otherwise.} \end{cases}$$

We denote by \bar{C}_ε the covariance of W^ε and we let $\bar{\nu}_\varepsilon$ be the law of W^ε . Our aim now is to show that the sequence $(\bar{\nu}_\varepsilon, \nu_\varepsilon)$, where ν_ε has covariance operator $(1 - \partial_x^2 + \varepsilon^2 \partial_x^4)^{-1}$, satisfies the assumptions of Corollary 5.5. To this end, we note that we can write

$$1 - \partial_x^2 + \varepsilon^2 \partial_x^4 = \mathcal{A}_\varepsilon \mathcal{A}_\varepsilon^*, \quad \mathcal{A}_\varepsilon = 1 + \sqrt{1 - 2\varepsilon} \partial_x - \varepsilon \partial_x^2,$$

so that the integral operator $\Lambda^\varepsilon = 1 - \mathcal{A}_\varepsilon \bar{C}_\varepsilon \mathcal{A}_\varepsilon^*$ associated to the pair $(\nu_\varepsilon, \bar{\nu}_\varepsilon)$ is given by the integral kernel

$$\Lambda^\varepsilon(x, y) = \delta(y - x) - \mathcal{A}_{\varepsilon; y} \mathcal{A}_{\varepsilon; x} \bar{C}_\varepsilon(x, y),$$

where we denote by $\mathcal{A}_{\varepsilon; x}$ the differential operator \mathcal{A}_ε acting on the x variable and similarly for y . Recall that on the other hand, the operator Λ associated in the same way to $(\nu, \bar{\nu})$ is given by

$$\Lambda(x, y) = \delta(y - x) - \mathcal{A}_{0; y} \mathcal{A}_{0; x} \bar{C}(x, y),$$

with \bar{C} defined as in (5.4). With these notations at hand, we then have

Lemma 5.9 *We have $\|\Lambda - \Lambda^\varepsilon\| = \mathcal{O}(\sqrt{\varepsilon})$, where $\|\cdot\|$ denotes the L^2 -norm on $[0, 2\pi]^2$, which is identical to the Hilbert-Schmidt norm when identifying kernels with the corresponding integral operators.*

Proof. Note first that \bar{C}_ε , viewed as a 2π -periodic function in both arguments, is smooth everywhere, except at $x = y$ and at $x, y \in \{0, \pi\}$. We first analyse the singularities and then proceed to the proof of the bound on $\Lambda - \Lambda^\varepsilon$.

We argue that the singularities at $\{0, \pi\}$ are harmless. Indeed, it follows from the gluing condition (5.9) that \bar{C}_ε is globally \mathcal{C}^1 , with possible jump-discontinuities in its second derivatives. Considering the singularity $x = \pi$ for example, it then follows that $\mathcal{A}_{\varepsilon; y} \bar{C}_\varepsilon$ is still \mathcal{C}^1 in the vicinity of this discontinuity line, so that $\mathcal{A}_{\varepsilon; x} \mathcal{A}_{\varepsilon; y} \bar{C}_\varepsilon$ has at most a jump discontinuity.

Let us now turn to the singularity at $x = y$. It follows immediately from (5.5) that one has

$$\bar{C}_\varepsilon(x, y) = \frac{|y - x|^3}{12\varepsilon^2} + R_\varepsilon^1(x, y),$$

where R_ε^1 is \mathcal{C}^4 in a vicinity of this singularity. It follows immediately that

$$\mathcal{A}_{\varepsilon; y} \bar{C}_\varepsilon(x, y) = -\frac{|y - x|}{2\varepsilon} + R_\varepsilon^2(x, y),$$

where R_ε^2 is \mathcal{C}^1 with a jump discontinuity in its second derivative in a vicinity of the singularity. It follows from these considerations that we do indeed have $\mathcal{A}_{\varepsilon; x} \mathcal{A}_{\varepsilon; y} \bar{C}_\varepsilon = \delta(y - x)$, up to a smooth function with jump discontinuities along the singularity lines, so that it suffices to bound $\Lambda - \Lambda^\varepsilon$ away from the singularities.

In order to do so, we first introduce the vector-valued function Ψ given by

$$\Psi_x = (1, t, \sqrt{\varepsilon}e^{-x/\varepsilon}, \sqrt{\varepsilon}e^{-(\pi-x)/\varepsilon}).$$

With this notation, it follows from (5.5), (5.10), and (5.6) that there exist matrices \mathcal{K}^ε and $\mathcal{K}_{**}^\varepsilon$ such that, for $s, t \in [0, \pi]$,

$$\begin{aligned} \mathbf{E}X_y X_x &= (x \wedge y) - \frac{\varepsilon}{2}e^{-|x-y|/\varepsilon} + \sqrt{\varepsilon}\langle \Psi_y, \mathcal{K}^\varepsilon \Psi_x \rangle, \\ \mathbf{E}\tilde{X}_y \tilde{X}_x &= (x \wedge y) - \frac{\varepsilon}{2}e^{-|x-y|/\varepsilon} + \sqrt{\varepsilon}\langle \Psi_y, \mathcal{K}_{**}^\varepsilon \Psi_x \rangle, \end{aligned}$$

where \mathcal{K}^ε and $\mathcal{K}_{**}^\varepsilon$ both have all of their entries bounded by some constant independent of ε . For X , this is obvious by inspection, and for \tilde{X} it follows from the fact that $JQ_\varepsilon J = Q_\varepsilon + \mathcal{O}(\sqrt{\varepsilon})$.

In order to obtain a similar expression for $\mathbf{E}X_y \tilde{X}_x$, we note from (5.6) that there exists a vector V^ε with all entries bounded by a constant independent of ε such that

$$v_x^\varepsilon = \left(-\sqrt{\frac{\varepsilon}{2}}e^{-x/\varepsilon}, \sqrt{\frac{\varepsilon}{2}}e^{-(\pi-x)/\varepsilon}, t \right) + \sqrt{\varepsilon}\langle V^\varepsilon, \Psi_x \rangle.$$

Since furthermore $JQ_\varepsilon^{-1} = \text{diag}(-1, -1, \pi^{-1}) + \mathcal{O}(\sqrt{\varepsilon})$, we conclude that there exist a matrix $\mathcal{K}_*^\varepsilon$ with uniformly bounded entries such that

$$\mathbf{E}X_y \tilde{X}_x = \langle v_y^\varepsilon, JQ_\varepsilon^{-1} v_x^\varepsilon \rangle = \frac{xy}{\pi} - \frac{\varepsilon}{2}(e^{-\frac{y+x}{\varepsilon}} + e^{-\frac{2\pi-y-x}{\varepsilon}}) + \sqrt{\varepsilon}\langle \Psi_y, \mathcal{K}_*^\varepsilon \Psi_x \rangle.$$

Combining these expressions, we conclude that there exists a matrix $\bar{\mathcal{K}}^\varepsilon$ uniformly bounded in ε such that

$$\bar{C}_\varepsilon(x, y) = \bar{C}(x, y) - \frac{\varepsilon}{2}(e^{-|x-y|/\varepsilon} + e^{-|y-x+2\pi|/\varepsilon} + e^{-|y-x-2\pi|/\varepsilon}) + \sqrt{\varepsilon}\langle \bar{\Psi}_y, \bar{\mathcal{K}}^\varepsilon \bar{\Psi}_x \rangle,$$

where $\bar{\Psi}_x$ consists of Ψ_x restricted to $[0, \pi]$, as well as its translate to $[\pi, 2\pi]$. Writing $\mathcal{A}_\varepsilon = \mathcal{A}_0 + \delta\mathcal{A}_\varepsilon$, we conclude that

$$\begin{aligned} \Lambda - \Lambda_\varepsilon &= (\mathcal{A}_{0;y}\delta\mathcal{A}_{\varepsilon;x} + \mathcal{A}_{0;y}\delta\mathcal{A}_{\varepsilon;x} + \delta\mathcal{A}_{\varepsilon;y}\delta\mathcal{A}_{\varepsilon;x})\bar{C}(x, y) \\ &\quad - \frac{\varepsilon}{2}\mathcal{A}_{\varepsilon;x}\mathcal{A}_{\varepsilon;y}(e^{-|x-y|/\varepsilon} + e^{-|y-x+2\pi|/\varepsilon} + e^{-|y-x-2\pi|/\varepsilon}) \\ &\quad + \sqrt{\varepsilon}\langle(\mathcal{A}_\varepsilon\bar{\Psi})_y, \bar{\mathcal{K}}^\varepsilon(\mathcal{A}_\varepsilon\bar{\Psi})_x\rangle. \end{aligned} \quad (5.11)$$

Since \bar{C} is smooth outside of the singular set and since $\delta\mathcal{A}_\varepsilon$ is a differential operator with coefficients of order ε , the first term is bounded by $\mathcal{O}(\varepsilon)$ uniformly in x, y . In order to bound the third term, we can check by inspection that the L^2 -norm of $\mathcal{A}_\varepsilon\Psi$ is bounded, uniformly in ε (this is how the scaling of the various terms was chosen in the first place), so that this term has L^2 -norm bounded by $\mathcal{O}(\sqrt{\varepsilon})$. It therefore remains to bound the terms appearing on the second line in (5.11). Since these terms are just translates of each other, it is sufficient to bound the first one and, by symmetry, it suffices to consider the region $s \leq t$, so that the term in question is given by a constant multiple of

$$\begin{aligned} \varepsilon\mathcal{A}_{\varepsilon;x}e^{x/\varepsilon}\mathcal{A}_{\varepsilon;y}e^{-y/\varepsilon} &= \varepsilon\left(1 + \frac{\sqrt{1-2\varepsilon}}{\varepsilon} - \frac{1}{\varepsilon}\right)\left(1 - \frac{\sqrt{1-2\varepsilon}}{\varepsilon} - \frac{1}{\varepsilon}\right)e^{(x-y)/\varepsilon} \\ &= \varepsilon e^{(x-y)/\varepsilon}. \end{aligned}$$

This term has an L^2 -norm of order $\mathcal{O}(\varepsilon^{3/2})$, so that the claim follows. \square

5.2 Exponential moment bounds for μ_ε

Let Z^ε be an n -dimensional Ornstein-Uhlenbeck process with time-scale ε , that is

$$Z_y^\varepsilon = \int_{-\infty}^y f_\varepsilon(y-x)dw(x), \quad f_\varepsilon(y) = \varepsilon^{-1}e^{-y/\varepsilon},$$

where w is a standard n -dimensional Wiener process. Note that f_ε is an approximation to the Dirac δ -function, so that for small values of ε , Z^ε is an approximation to white noise. We will also use the shorthand notation $F_\varepsilon(x) = 1 - \exp(-x/\varepsilon)$ in the sequel, so that $F_\varepsilon(x) = \int_0^x f_\varepsilon(y)dy$. We then define

$$w_x^\varepsilon = \int_0^x Z_y^\varepsilon dy,$$

which can also be rewritten as

$$w_x^\varepsilon = \int_{-\infty}^0 (F_\varepsilon(x-y) - F_\varepsilon(-y))dw(y) + \int_0^x F_\varepsilon(x-y)dw(y).$$

Note that for small values of ε , this is a good approximation to Brownian motion. In particular, the variance of any component of w_x^ε is given by

$$\mathbf{E}|w_x^{\varepsilon,i}|^2 = x - \varepsilon F_\varepsilon(x). \quad (5.12)$$

The aim of this section is to show that if $G: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a sufficiently nice function, then the quantity

$$H_\varepsilon := \int_0^\pi \langle G(w_x^\varepsilon), Z_x^\varepsilon \rangle dx, \quad (5.13)$$

is uniformly exponentially integrable as $\varepsilon \rightarrow 0$. Indeed, we have the following result:

Proposition 5.10 *Let $G: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a bounded C^2 function with bounded first and second derivatives. Then, there exists $\varepsilon_0 > 0$ such that*

$$\sup_{\varepsilon < \varepsilon_0} \mathbf{E} \exp \int_0^\pi \langle G(w_x^\varepsilon), Z_x^\varepsilon \rangle dx < \infty .$$

Proof. Let H_ε be as in (5.13). It follows from the Clark-Ocone formula [Nua95] that we can represent H_ε as a stochastic integral:

$$H_\varepsilon = \mathbf{E}H_\varepsilon + \int_{-\infty}^\pi \mathbf{E}(\mathcal{D}_x^j H_\varepsilon | \mathcal{F}_x) dw^j(x) , \quad (5.14)$$

(here and in the sequel, summation over repeated indices is implicit) where \mathcal{D}_x^j denotes the Malliavin derivative at point x with respect to w^j and \mathcal{F}_x is the filtration generated by the increments of w . Here, the convention we use in the definition of \mathcal{D}_x is such that if f is any smooth deterministic function, then

$$\mathcal{D}_x^j \int_0^\pi \langle f(y) dw(y) \rangle = f^j(x) , \quad x \in [0, \pi] .$$

Since the second term in (5.14) is a martingale, the result then follows if we can show that $\mathbf{E}H_\varepsilon$ is uniformly bounded as $\varepsilon \rightarrow 0$ and that the quantity

$$J_\varepsilon^j \stackrel{\text{def}}{=} \int_{-\infty}^\pi |\mathbf{E}(\mathcal{D}_y^j H_\varepsilon | \mathcal{F}_y)|^2 dy \quad (5.15)$$

is uniformly exponentially integrable for every G satisfying the assumptions of the theorem.

The expectation of H_ε is relatively straightforward to bound. Indeed, we have the identity

$$\mathbf{E}(Z_x^\varepsilon | w_x^\varepsilon) = w_x^\varepsilon \frac{\mathbf{E}Z_x^\varepsilon w_x^\varepsilon}{\mathbf{E}|w_x^\varepsilon|^2} = \frac{w_x^\varepsilon}{2} \frac{F_\varepsilon(x)}{t - \varepsilon F_\varepsilon(x)} ,$$

which implies that

$$|\mathbf{E}H_\varepsilon| \leq \frac{1}{2} \int_0^\pi \frac{F_\varepsilon(x)}{x - \varepsilon F_\varepsilon(x)} \mathbf{E}(\langle G(w_x^\varepsilon), w_x^\varepsilon \rangle) dx \leq C \int_0^\pi \frac{F_\varepsilon(x)}{\sqrt{x - \varepsilon F_\varepsilon(x)}} dx ,$$

where we have used the boundedness of G together with (5.12) to obtain the second bound. Since $F_\varepsilon(x) \approx \frac{x}{\varepsilon} - \frac{x^2}{2\varepsilon^2}$ for $x \ll \varepsilon$ and $F_\varepsilon(x) \approx 1$ for $x \gg \varepsilon$, one can check that the integrand is uniformly bounded by C/\sqrt{t} for some $C > 0$, so that $|\mathbf{E}H_\varepsilon|$ is indeed bounded by a constant independently of the value of $\varepsilon \leq 1$.

Let us now turn to the bounds on $\mathcal{D}_x H_\varepsilon$. It follows immediately from the definition of the Malliavin derivative that one has the identities

$$\mathcal{D}_y^i w_x^{\varepsilon, j} = \delta_{ij} (F_\varepsilon(x - y) \mathbf{1}_{y \leq x} - F_\varepsilon(-y) \mathbf{1}_{y \leq 0}) , \quad \mathcal{D}_y^i Z_x^{\varepsilon, j} = \delta_{ij} f_\varepsilon(x - y) \mathbf{1}_{y \leq x} .$$

We treat the case $y < 0$ separately from the case $y \geq 0$. For $y \geq 0$, we have

$$\begin{aligned} \mathbf{E}(\mathcal{D}_y^j H_\varepsilon | \mathcal{F}_y) &= \int_y^\pi \mathbf{E}(\partial_j G_i(w_x^\varepsilon) F_\varepsilon(x - y) Z_\varepsilon^i(x) + G_j(w_x^\varepsilon) f_\varepsilon(x - y) | \mathcal{F}_y) dx \\ &\stackrel{\text{def}}{=} J_1^j(y) + J_2^j(y) . \end{aligned}$$

The term J_2^j is easy to bound since G is bounded and the variation of f_ε is also bounded, uniformly in ε , so that

$$|J_2(y)| \leq \|G\|_\infty \int_y^\pi f_\varepsilon(x-y) dx \leq \|G\|_\infty, \quad (5.16)$$

holds almost surely. The first term is much more tricky to bound. We write

$$J_1^j(y) = \mathbf{E} \left(\int_y^{2\pi} \partial_j G_i(w_x^\varepsilon) F_\varepsilon(x-y) \mathbf{E}(Z_x^{\varepsilon,i} | \mathcal{F}_y \vee w_x^\varepsilon) dx \middle| \mathcal{F}_y \right), \quad (5.17)$$

and we note that we have the identities

$$\begin{aligned} Z_x^\varepsilon &= \partial_x a_y(x) + \int_y^x f_\varepsilon(x-z) dw(z), \\ w_x^\varepsilon &= a_s(x) + \int_y^x F_\varepsilon(x-z) dw(z), \end{aligned}$$

where we introduced the shorthand notation

$$a_y(x) = w_y^\varepsilon + \varepsilon F_\varepsilon(x-y) Z_y^\varepsilon.$$

This implies that

$$\begin{aligned} \mathbf{E}(Z_x^\varepsilon | \mathcal{F}_y \vee w_x^\varepsilon) &= \partial_x a_y(x) + M_\varepsilon(x-y)(w_x^\varepsilon - a_y(x)), \\ M_\varepsilon(x) &= \frac{F_\varepsilon^2(x)}{2x - \varepsilon F_\varepsilon(x)(2 + F_\varepsilon(x))}. \end{aligned} \quad (5.18)$$

An important fact is that the function M_ε has the property that there exists a constant C such that the bound

$$|M_\varepsilon(x)| \leq \frac{C}{x}, \quad (5.19)$$

holds for every $x \geq 0$, uniformly in $\varepsilon > 0$. Note that the scaling properties of F_ε imply that $M_\varepsilon(x) = \varepsilon^{-1} M_1(x/\varepsilon)$, so that this needs to be shown only for M_1 . On the one hand, (5.19) clearly holds for $x \rightarrow \infty$. On the other hand, one can check that $\lim_{x \rightarrow 0} x M_\varepsilon(x) = \frac{3}{2}$. The claim then follows by noting that the numerator in the definition of M_ε is bounded and that the denominator is an increasing function of x .

We now introduce the further shorthand notation

$$T_\varepsilon(x) = \int_0^x F_\varepsilon^2(y) dy,$$

and we denote by \mathcal{P}_T the heat semigroup on \mathbf{R}^n . With these shorthands, we can combine (5.18) and (5.17) to obtain the following explicit expression for J_1 :

$$\begin{aligned} J_1^j(y) &= \int_y^\pi \partial_x a_y^i(x) (\mathcal{P}_{T_\varepsilon(x-y)} \partial_j G_i)(a_y(x)) F_\varepsilon(x-y) dx \\ &\quad + \frac{1}{2} \int_y^\pi M_\varepsilon(x-y) \sqrt{T_\varepsilon(x-y)} (\mathcal{P}_{T_\varepsilon(x-y)} \partial_{ij}^2 G_i)(a_y(x)) F_\varepsilon(x-y) dx \\ &\stackrel{\text{def}}{=} J_{1,1}^j(y) + J_{1,2}^j(y). \end{aligned}$$

Since $\partial_j G_i$ is bounded by assumption, it follows immediately from the definition of $J_{1,1}^j(y)$ that there exists a constant C such that the bound

$$|J_{1,1}^j(y)| \leq \varepsilon |Z_y^\varepsilon| \|DG\|_\infty, \quad (5.20)$$

holds almost surely. In order to bound $J_{1,2}^j$, we note that $M_\varepsilon(x-y)\sqrt{T_\varepsilon(x-y)} \leq \frac{3}{2}|x-y|^{-1/2}$ so that the bound

$$|J_{1,2}^j(y)| \leq C \int_y^\pi \frac{|(\mathcal{P}_{T_\varepsilon(x-y)} \partial_{ij}^2 G_i)(a_y(x))|}{\sqrt{x-y}} dx \leq C \sqrt{\pi-y} \|D^2G\|_\infty, \quad (5.21)$$

holds almost surely.

We now turn to the case $y < 0$. For this case, we have the identity

$$\begin{aligned} \mathbf{E}(\mathcal{D}_s^j H_\varepsilon | \mathcal{F}_s) &= \int_0^\pi \mathbf{E}(\partial_j G_i(w_x^\varepsilon)(F_\varepsilon(x-y) \\ &\quad - F_\varepsilon(-y))Z_\varepsilon^i(x) + G_j(w_x^\varepsilon)f_\varepsilon(x-y) | \mathcal{F}_y) dx \\ &\stackrel{\text{def}}{=} J_3^j(y) + J_4^j(y). \end{aligned}$$

The term J_4^j is easy to bound as before, yielding

$$|J_4^j(y)| \leq e^{y/\varepsilon} \|G\|_\infty, \quad (5.22)$$

so that

$$\int_{-\infty}^0 |J_4^j(y)|^2 \leq \frac{\varepsilon}{2} \|G\|_\infty^2.$$

In order to bound J_3^j , we note that we have the identities

$$Z_x^\varepsilon = e^{-(x-y)/\varepsilon} Z_y^\varepsilon + \int_s^t f_\varepsilon(t-z) dw(z),$$

as well as $F_\varepsilon(x-y) - F_\varepsilon(-y) = e^{y/\varepsilon} F_\varepsilon(x)$. It follows that

$$\begin{aligned} |J_3^j| &\leq e^{2y/\varepsilon} \|\partial_j G\|_\infty \left(\varepsilon |Z_y^\varepsilon| + \int_0^\pi \sqrt{\mathbf{E}\left(\left(\int_y^x f_\varepsilon(x-z) dw(z)\right)^2 \middle| \mathcal{F}_y\right)} dx \right) \\ &\leq e^{2y/\varepsilon} \|\partial_j G\|_\infty (\varepsilon |Z_y^\varepsilon| + \varepsilon^{-1/2}) \stackrel{\text{def}}{=} e^{2y/\varepsilon} \tilde{J}_3^j(y). \end{aligned}$$

In order to conclude, note that by (5.15) it remains to show that $K \int_0^\pi |J_k^j(y)|^2 dy$ is uniformly exponentially integrable for every $K > 0$ and for $k \in \{1, 2\}$, and similarly for $K \int_{-\infty}^0 |J_k^j(y)|^2 dy$ with $k \in \{3, 4\}$. These bounds are trivial for $k \in \{2, 4\}$ by (5.16) and (5.22). To bound the term involving J_1^j , we deduce from Jensen's inequality that

$$\mathbf{E} \exp\left(K \int_0^\pi |J_1^j(y)|^2 dy\right) \leq \frac{1}{\pi} \int_0^\pi \mathbf{E} \exp(\pi K |J_1^j(y)|^2) dy.$$

Since Z_y^ε is Gaussian with variance ε^{-1} , we deduce that for any $K > 0$ we can choose ε_0 small enough so that the bound $\mathbf{E} \exp(\varepsilon^2 K |Z_y^\varepsilon|^2) < 2$ holds for every $\varepsilon < \varepsilon_0$. The requested bound for J_1 thus follows from (5.20) and (5.21).

It remains to bound the term involving J_3^j . We similarly obtain from Jensen's inequality that

$$\mathbf{E} \exp\left(K \int_{-\infty}^0 e^{4y/\varepsilon} |\tilde{J}_3^j(y)|^2 dy\right) \leq \frac{4}{\varepsilon} \int_{-\infty}^0 e^{4y/\varepsilon} \mathbf{E} \exp\left(\frac{K\varepsilon}{4} |\tilde{J}_3^j(y)|^2\right) dy .$$

The requested bound then follows as before, using the explicit form of \tilde{J}_3 . \square

Appendix A Semigroup bounds

In this appendix, we collect some elementary results on the way that the semigroup S_t^ε introduced in Section 4 approximates the damped heat semigroup S_t . To investigate this, we consider the semigroup \hat{S}_t generated by $-\partial_x^4$ (always with periodic boundary conditions), and we use the fact that one has the identity

$$S_t^\varepsilon = \hat{S}_{\varepsilon^2 t} S_t . \quad (\text{A.1})$$

The semigroup \hat{S} can be described explicitly with the help of the kernel φ defined by

$$\varphi(x) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{-k^4 - ikx} dk .$$

With this definition, one has the identity

$$(\hat{S}_t u)(x) = t^{-1/4} \int_{\mathbf{R}} \varphi(yt^{-1/4}) u(x+y) dy , \quad (\text{A.2})$$

where we identify u with its periodic continuation. As a consequence of this representation, one has:

Proposition A.1 *One has, $\sup_{t \leq 1} \|\hat{S}_t\|_{\mathcal{C}^\alpha \rightarrow \mathcal{C}^\alpha} < \infty$.*

Proof. Since convolution with a periodic function ψ is an operation on \mathcal{C}^α with norm bounded by the L^1 -norm of ψ , the claim now follows from (A.2), using the fact that φ is in L^1 and that the scaling by $t^{-1/4}$ does not change its L^1 norm. \square

It follows that one has the following approximation result:

Corollary A.2 *For every $\beta \in (0, 1)$, every $T > 0$, and every $\alpha \in (0, \beta)$, there exists a constant $C > 0$ such that*

$$\|S_t^\varepsilon u - S_t u\|_{\mathcal{C}^\alpha} \leq C \varepsilon^{\frac{\beta-\alpha}{2}} ,$$

uniformly for all $t \in [0, T]$ and all $u \in \mathcal{C}^\beta$.

Proof. We note that it follows from (A.2) that

$$|(\hat{S}_\varepsilon u - u)(x)| \leq \varepsilon^{\beta/4} \|u\|_\beta \int_{\mathbf{R}} \varepsilon^{-1/4} |\varphi(x\varepsilon^{-1/4})| |x\varepsilon^{-1/4}|^\beta dx \leq C \varepsilon^{\beta/4} \|u\|_\beta . \quad (\text{A.3})$$

On the other hand, we know from Proposition A.1 that the bound

$$\|\hat{S}_\varepsilon u - u\|_{\mathcal{C}^\beta} \leq C \|u\|_{\mathcal{C}^\beta} ,$$

is valid for $\varepsilon < 1$, say. Combining these bounds, we conclude that

$$\|\hat{S}_\varepsilon u - u\|_{\mathcal{C}^\alpha} \leq C \|\hat{S}_\varepsilon u - u\|_{\mathcal{C}^\beta}^{\alpha/\beta} \|\hat{S}_\varepsilon u - u\|_\infty^{(\beta-\alpha)/\beta} \leq C \varepsilon^{\frac{\beta-\alpha}{4}} \|u\|_{\mathcal{C}^\beta} .$$

The claim now follows from (A.1). \square

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