

ROUGHNESS IN SUBTRACTION ALGEBRAS

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ABSTRACT. As a generalization of ideals in subtraction algebras, the notion of rough ideals is discussed.

1. Introduction

B. M. Schein [10] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition “ \circ ” of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction “ \setminus ” (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [11] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [4] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [3], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Y. B. Jun and K. H. Kim [5] introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be a prime/irreducible ideal. In 1982, Pawlak introduced the concept of a rough set (see [8]). This concept is fundamental for the examination of granularity in knowledge. It is a concept which has many applications in data analysis (see [9]). Rough set theory is applied to semigroups and groups (see [6, 7]). In this paper, we apply the rough set theory to subtraction algebras, and we introduce the notion of upper/lower rough subalgebras/ideals which is an extended notion of subalgebras/ideals in a subtraction algebra.

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2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation “ $-$ ” that satisfies the following identities: for any $x, y, z \in X$,

- (S1) $x - (y - x) = x$;
 (S2) $x - (x - y) = y - (y - x)$;
 (S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X : $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [4, 5]):

- (a1) $(x - y) - y = x - y$.
 (a2) $x - 0 = x$ and $0 - x = 0$.
 (a3) $(x - y) - x = 0$.
 (a4) $x - (x - y) \leq y$.
 (a5) $(x - y) - (y - x) = x - y$.
 (a6) $x - (x - (x - y)) = x - y$.
 (a7) $(x - y) - (z - y) \leq x - z$.
 (a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
 (a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
 (a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
 (a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

A nonempty subset S of a subtraction algebra X is called a *subalgebra* of X if $x - y \in S$ whenever $x, y \in S$.

A nonempty subset A of a subtraction algebra X is called an *ideal* of X , denoted by $A \triangleleft X$, if it satisfies

- $0 \in A$
- $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

Note that every ideal of a subtraction algebra X is a subalgebra of X .

LEMMA 2.1. [5] *An ideal A of a subtraction algebra X has the following property:*

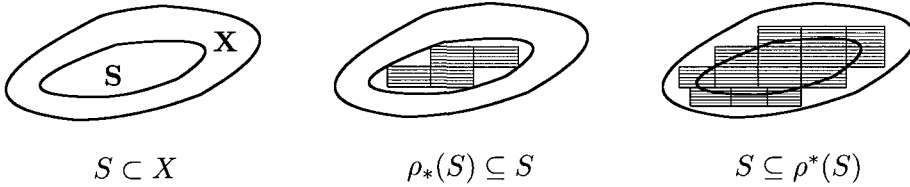
$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

3. Rough sets in subtraction algebras

In what follows let X denote a subtraction algebra unless otherwise specified.

An equivalence relation ρ on X is called a *congruence relation* on X if whenever $(x, y), (u, v) \in \rho$ then $(x - u, y - v) \in \rho$. We denote by $[a]_\rho$ the ρ -congruence class containing the element $a \in X$. Let X/ρ denote the set of all ρ -congruence classes on X , i.e., $X/\rho := \{[a]_\rho \mid a \in X\}$. For any $[x]_\rho, [y]_\rho \in X/\rho$, if we define $[x]_\rho - [y]_\rho = [x - y]_\rho$, then $(X/\rho, -)$ is a subtraction algebra. Let ρ be an equivalence relation on X and let $\mathcal{P}(X)$ denote the power set of X and $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$. For all $x \in X$, let $[x]_\rho$ denote the equivalence class of x with respect to ρ . Define the functions $\rho_*, \rho^* : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ as follows: $\forall S \in \mathcal{P}(X)$,

$$\rho_*(S) = \{x \in X \mid [x]_\rho \subseteq S\} \text{ and } \rho^*(S) = \{x \in X \mid [x]_\rho \cap S \neq \emptyset\}.$$



$\rho_*(S)$ is called the ρ -lower approximation of S while $\rho^*(S)$ is called the ρ -upper approximation of S . For a nonempty subset S of X ,

$$\rho(S) = (\rho_*(S), \rho^*(S))$$

is called a *rough set* with respect to ρ of $\mathcal{P}(X) \times \mathcal{P}(X)$ if $\rho_*(S) \neq \rho^*(S)$. A subset S of X is said to be *definable* if $\rho_*(S) = \rho^*(S)$. The pair (X, ρ) is called an *approximation space*.

The following property is useful for our research (cf. [8]).

PROPOSITION 3.1. *Let ρ and λ be congruence relations on X . Then the following assertions are true.*

- (1) $(\forall F \in \mathcal{P}^*(X)) (\rho_*(F) \subseteq F \subseteq \rho^*(F))$,
- (2) $(\forall F, G \in \mathcal{P}^*(X)) (\rho^*(F \cup G) = \rho^*(F) \cup \rho^*(G))$,
- (3) $(\forall F, G \in \mathcal{P}^*(X)) (\rho_*(F \cap G) = \rho_*(F) \cap \rho_*(G))$,
- (4) $(\forall F, G \in \mathcal{P}^*(X)) (F \subseteq G \Rightarrow \rho_*(F) \subseteq \rho_*(G))$,

- (5) $(\forall F, G \in \mathcal{P}^*(X)) (F \subseteq G \Rightarrow \rho^*(F) \subseteq \rho^*(G)),$
 (6) $(\forall F, G \in \mathcal{P}^*(X)) (\rho_*(F) \cup \rho_*(G) \subseteq \rho_*(F \cup G)),$
 (7) $(\forall F, G \in \mathcal{P}^*(X)) (\rho^*(F \cap G) \subseteq \rho^*(F) \cap \rho^*(G)),$
 (8) $(\forall F \in \mathcal{P}^*(X)) (\rho \subseteq \lambda \Rightarrow \lambda_*(F) \subseteq \rho_*(F), \rho^*(F) \subseteq \lambda^*(F)).$

PROOF. Straightforward. \square

COROLLARY 3.2. *If ρ and λ are congruence relations on X , then*

- (i) $(\forall F \in \mathcal{P}^*(X)) ((\rho \cap \lambda)^*(F) \subseteq \rho^*(F) \cap \lambda^*(F)).$
 (ii) $(\forall F \in \mathcal{P}^*(X)) (\rho_*(F) \cap \lambda_*(F) \subseteq (\rho \cap \lambda)_*(F)).$

PROOF. It follows immediately from Proposition 3.1. \square

For any $F, G \in \mathcal{P}^*(X)$, we define $F - G := \{a - b \mid a \in F, b \in G\}$.

THEOREM 3.3. *If ρ is a congruence relation on X , then*

$$(\forall F, G \in \mathcal{P}^*(X)) (\rho^*(F) - \rho^*(G) \subseteq \rho^*(F - G)).$$

PROOF. Let $c \in \rho^*(F) - \rho^*(G)$. Then there exist $a \in \rho^*(F)$ and $b \in \rho^*(G)$ such that $c = a - b$. It follows that $[a]_\rho \cap F \neq \emptyset$ and $[b]_\rho \cap G \neq \emptyset$ so that $x \in [a]_\rho \cap F$ and $y \in [b]_\rho \cap G$ for some $x, y \in X$. Hence $x - y \in [a]_\rho - [b]_\rho = [a - b]_\rho$ and $x - y \in F - G$, that is, $x - y \in [a - b]_\rho \cap (F - G)$. Thus $c = a - b \in \rho^*(F - G)$, and so $\rho^*(F) - \rho^*(G) \subseteq \rho^*(F - G)$. \square

THEOREM 3.4. *If ρ is a congruence relation on X , then*

$$(\forall F, G \in \mathcal{P}^*(X)) (\rho_*(F - G) \neq \emptyset \Rightarrow \rho_*(F) - \rho_*(G) \subseteq \rho_*(F - G)).$$

PROOF. Let $c \in \rho_*(F) - \rho_*(G)$. Then $c = a - b$ for some $a \in \rho_*(F)$ and $b \in \rho_*(G)$. Thus we get $[a]_\rho \subseteq F$ and $[b]_\rho \subseteq G$. It follows that

$$[a - b]_\rho = [a]_\rho - [b]_\rho \subseteq F - G$$

so that $c = a - b \in \rho_*(F - G)$. Therefore the result is valid. \square

The following example shows the condition that $\rho_*(F - G) \neq \emptyset$ in Theorem 3.4 is necessary.

EXAMPLE 3.5. Let $X = \{0, a, b, c\}$ be a subtraction algebra with the following Cayley table:

-	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Let ρ be a congruence relation on X such that $\{0, a\}$, $\{b\}$, and $\{c\}$ are all ρ -congruences of X . Taking $F = \{b, c\}$ and $G = \{c\}$, we have $F - G = \{0\}$, $\rho_*(F - G) = \emptyset$, $\rho_*(F) = \{b, c\}$, $\rho_*(G) = \{c\}$, and $\rho_*(F) - \rho_*(G) = \{0\}$.

For any congruence relation ρ on X , we note that

- $(\forall F \in \mathcal{P}^*(X)) (\rho_*(F) \subseteq F)$,
- $(\forall F, G \in \mathcal{P}^*(X)) (F \subseteq G \Rightarrow \rho_*(F) \subseteq \rho_*(G))$,
- $(\forall F \in \mathcal{P}^*(X)) (\rho_*(\rho_*(F)) = \rho_*(F))$,

which means that ρ_* is an interior operator on X . This operation induces a topology \mathcal{T} on X such that

$$F \in \mathcal{T} \iff \rho_*(F) = F.$$

LEMMA 3.6. For any congruence relation ρ on X , ρ^* is a closure operator on the topological space (X, \mathcal{T}) .

PROOF. For any $F \in \mathcal{P}^*(X)$ we have

$x \in \rho^*(F) \iff [x]_\rho \cap F \neq \emptyset \iff [x]_\rho \not\subseteq F^c \iff x \notin \rho_*(F^c) \iff x \in (\rho_*(F^c))^c$, that is, $\rho^*(F) = (\rho_*(F^c))^c$, which completes the proof. □

LEMMA 3.7. For any congruence relation ρ on X , we have

- (i) $(\forall F \in \mathcal{P}(X)) (\rho_*(F) = F \iff \rho^*(F^c) = F^c)$,
- (ii) $(\forall F \in \mathcal{P}(X)) (\rho_*(F) = F \iff \rho^*(F) = F)$.

PROOF. Straightforward. □

Based on the above two lemmas we have the following result.

THEOREM 3.8. For any $F \subseteq X$ and a congruence relation ρ on X , the following assertions are equivalent.

- (i) F is definable with respect to ρ .
- (ii) F is open in the topological space (X, \mathcal{T}) .
- (iii) F is closed in the topological space (X, \mathcal{T}) .

According to [7], we say that an open set F of X is said to be *free* in an approximation space (X, ρ) if $x \notin \rho^*(F \setminus \{x\})$ for all $x \in X$. Since $\rho^*(F \setminus \{x\}) = (\rho_*((F \setminus \{x\})^c))^c$, a nonempty subset F of X is free if and only if $x \in \rho_*(F^c \cup \{x\})$, i.e., if and only if $[x]_\rho \subseteq F^c \cup \{x\}$ for every $x \in F$. Thus for a free subset F and any $(x, y) \in \rho \cap (F \times F)$ we have $y \in F$, which together with $y \in [x]_\rho \subseteq F^c \cup \{x\}$ implies that $y = x$. Therefore $\rho \cap (F \times F) = \{(a, a) \mid a \in F\}$. Conversely, let

$$\rho \cap (F \times F) = \{(a, a) \mid a \in F\}$$

and let y be an arbitrary element of $[x]_\rho$. If $y \in F$, then $y = x$, i.e., $y \in \{x\} \subseteq F^c \cup \{x\}$. If $y \notin F$, then $y \in F^c \subseteq F^c \cup \{x\}$. Thus, in each case $[x]_\rho \subseteq F^c \cup \{x\}$, which means that F is free. Consequently, we obtain the following characterization of free subsets.

THEOREM 3.9. $F \subseteq X$ is free if and only if $\rho \cap (F \times F) = \{(a, a) \mid a \in F\}$.

COROLLARY 3.10. If X is free, then any subset of X is free.

4. Roughness of ideals

Let A be an ideal of X . Define a relation \mathcal{R} on X by

$$(\forall x, y \in X) ((x, y) \in \mathcal{R} \Leftrightarrow x - y \in A, y - x \in A).$$

Then \mathcal{R} is an equivalence relation on X related to an ideal A of X . Moreover \mathcal{R} satisfies

$$(\forall x, y, u, v \in X) ((x, y) \in \mathcal{R}, (u, v) \in \mathcal{R} \Rightarrow (x - u, y - v) \in \mathcal{R}).$$

Hence \mathcal{R} is a congruence relation on X . Let A_x denote the equivalence class of x with respect to the equivalence relation \mathcal{R} related to the ideal A of X , and X/A denote the collection of all equivalence classes, that is, $X/A = \{A_x \mid x \in X\}$. Then $A_0 = A$. If $A_x \ominus A_y$ is defined as the class containing $x - y$, that is, $A_x \ominus A_y = A_{x-y}$, then it is easy to verify that $(X/A, -, A_0)$ is a subtraction algebra. Let \mathcal{R} be an equivalence relation on X related to an ideal A of X . For any nonempty subset S of X , the lower and upper approximations of S are denoted by $\underline{\mathcal{R}}(A; S)$ and $\overline{\mathcal{R}}(A; S)$ respectively, that is,

$$\underline{\mathcal{R}}(A; S) = \{x \in X \mid A_x \subseteq S\} \text{ and } \overline{\mathcal{R}}(A; S) = \{x \in X \mid A_x \cap S \neq \emptyset\}.$$

If $A = S$, then $\underline{\mathcal{R}}(A; S)$ and $\overline{\mathcal{R}}(A; S)$ are denoted by $\underline{\mathcal{R}}(A)$ and $\overline{\mathcal{R}}(A)$, respectively.

EXAMPLE 4.1. (1) Let $X = \{0, a, b, c\}$ be a set with the Cayley table as follows:

-	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Then $(X, -, 0)$ is a subtraction algebra. Consider an ideal $A = \{0, a\}$ of X and let \mathcal{R} be an equivalence relation on X related to A . Then $A_0 = A_a = A$, $A_b = \{b\}$, and $A_c = \{c\}$. Hence

- $\underline{\mathcal{R}}(A; \{0, b\}) = \{b\} = \underline{\mathcal{R}}(A; \{b\})$,
- $\underline{\mathcal{R}}(A; \{0, c\}) = \{c\} = \underline{\mathcal{R}}(A; \{c\})$,
- $\underline{\mathcal{R}}(A; \{0, a, b\}) = \{0, a, b\} \triangleleft X$,
- $\underline{\mathcal{R}}(A; \{0, a\}) = \{0, a\} \triangleleft X$,
- $\underline{\mathcal{R}}(A; \{0, c\}) = \{0, a, c\} \triangleleft X$,
- $\underline{\mathcal{R}}(A; \{a\}) = A \triangleleft X$,
- $\underline{\mathcal{R}}(A; \{0\}) = \emptyset = \underline{\mathcal{R}}(A; \{a\})$,
- $\underline{\mathcal{R}}(A; \{0, a, c\}) = \{0, a, c\} \triangleleft X$,
- $\underline{\mathcal{R}}(A; \{0, b, c\}) = \{b, c\}$,
- $\underline{\mathcal{R}}(A; \{0, b\}) = \{0, a, b\} \triangleleft X$,
- $\underline{\mathcal{R}}(A; \{0, a\}) = A \triangleleft X$,
- $\underline{\mathcal{R}}(A; \{b\}) = \{b\}$.

(2) Let $X = \{0, a, b, c, d\}$ be a subtraction algebra with the Cayley table as follows:

$-$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Consider $A = \{0, b, d\} \triangleleft X$ and let \mathcal{R} be an equivalence relation on X related to A . Then the equivalence classes are as follows: $A_0 = A_b = A_d = A$, $A_a = \{a, c, d\}$, and $A_c = \{a, c\}$. Thus

- $\underline{\mathcal{R}}(A; \{0, a\}) = \emptyset$,
- $\underline{\mathcal{R}}(A; \{0, a, d\}) = \emptyset$,
- $\underline{\mathcal{R}}(A; \{0, b, d\}) = A \triangleleft X$,
- $\underline{\mathcal{R}}(A; \{0, b, c, d\}) = A \triangleleft X$,
- $\underline{\mathcal{R}}(A; \{0, b\}) = A \triangleleft X$,
- $\underline{\mathcal{R}}(A; \{0, d\}) = \{0, a, b, d\}$,
- $\underline{\mathcal{R}}(A; \{b\}) = A \triangleleft X$,
- $\underline{\mathcal{R}}(A; \{d\}) = A \triangleleft X$.
- $\underline{\mathcal{R}}(A; \{0, b, c\}) = \emptyset$,
- $\underline{\mathcal{R}}(A; \{0, a, c\}) = \{c\}$,
- $\underline{\mathcal{R}}(A; \{0, a, b, c\}) = \{c\}$,
- $\underline{\mathcal{R}}(A; \{0, a\}) = X$,
- $\underline{\mathcal{R}}(A; \{0, c\}) = X$,
- $\underline{\mathcal{R}}(A; \{0, a, d\}) = X$,
- $\underline{\mathcal{R}}(A; \{c\}) = \{a, c\}$.

In Example 4.1, we know that there exists a non-ideal U of X such that $\underline{\mathcal{R}}(A; U) \triangleleft X$; and there exists a non-ideal V of X such that $\overline{\mathcal{R}}(A; V) \triangleleft X$, where \mathcal{R} is an equivalence relation on X related to $A \triangleleft X$.

PROPOSITION 4.2. *Let \mathcal{R} and \mathcal{Q} be equivalence relations on X related to ideals A and B of X , respectively. If $A \subseteq B$, then $\mathcal{R} \subseteq \mathcal{Q}$.*

PROOF. If $(x, y) \in \mathcal{R}$, then $x - y \in A \subseteq B$ and $y - x \in A \subseteq B$. Hence $(x, y) \in \mathcal{Q}$, and so $\mathcal{R} \subseteq \mathcal{Q}$. □

PROPOSITION 4.3. *Let \mathcal{R} be an equivalence relation on X related to an ideal A of X . Then*

- (1) $(\forall S \in \mathcal{P}(X)) (\mathcal{R}(A; S) \subseteq S \subseteq \overline{\mathcal{R}}(A; S)),$
- (2) $(\forall S, T \in \mathcal{P}(X)) (\overline{\mathcal{R}}(A; S \cup T) = \overline{\mathcal{R}}(A; S) \cup \overline{\mathcal{R}}(A; T)),$
- (3) $(\forall S, T \in \mathcal{P}(X)) (\underline{\mathcal{R}}(A; S \cap T) = \underline{\mathcal{R}}(A; S) \cap \underline{\mathcal{R}}(A; T)),$
- (4) $(\forall S, T \in \mathcal{P}(X)) (S \subseteq T \Rightarrow \underline{\mathcal{R}}(A; S) \subseteq \underline{\mathcal{R}}(A; T), \overline{\mathcal{R}}(A; S) \subseteq \overline{\mathcal{R}}(A; T)),$
- (5) $(\forall S, T \in \mathcal{P}(X)) (\underline{\mathcal{R}}(A; S \cup T) \supseteq \underline{\mathcal{R}}(A; S) \cup \underline{\mathcal{R}}(A; T)),$
- (6) $(\forall S, T \in \mathcal{P}(X)) (\overline{\mathcal{R}}(A; S \cap T) \subseteq \overline{\mathcal{R}}(A; S) \cap \overline{\mathcal{R}}(A; T)),$
- (7) If \mathcal{Q} is an equivalence relation on X related to an ideal B of X and if $A \subseteq B$, then $\overline{\mathcal{R}}(A; S) \subseteq \overline{\mathcal{Q}}(B; S)$ for all $S \in \mathcal{P}(X)$.

PROOF. (1) is straightforward.

(2) For any subsets S and T of X , we have

$$\begin{aligned}
 x \in \overline{\mathcal{R}}(A; S \cup T) &\Leftrightarrow A_x \cap (S \cup T) \neq \emptyset \\
 &\Leftrightarrow (A_x \cap S) \cup (A_x \cap T) \neq \emptyset \\
 &\Leftrightarrow A_x \cap S \neq \emptyset \text{ or } A_x \cap T \neq \emptyset \\
 &\Leftrightarrow x \in \overline{\mathcal{R}}(A; S) \text{ or } x \in \overline{\mathcal{R}}(A; T) \\
 &\Leftrightarrow x \in \overline{\mathcal{R}}(A; S) \cup \overline{\mathcal{R}}(A; T),
 \end{aligned}$$

and hence $\overline{\mathcal{R}}(A; S \cup T) = \overline{\mathcal{R}}(A; S) \cup \overline{\mathcal{R}}(A; T)$.

(3) For any subsets S and T of X we have

$$\begin{aligned}
 x \in \underline{\mathcal{R}}(A; S \cap T) &\Leftrightarrow A_x \subseteq S \cap T \\
 &\Leftrightarrow A_x \subseteq S \text{ and } A_x \subseteq T \\
 &\Leftrightarrow x \in \underline{\mathcal{R}}(A; S) \text{ and } x \in \underline{\mathcal{R}}(A; T) \\
 &\Leftrightarrow x \in \underline{\mathcal{R}}(A; S) \cap \underline{\mathcal{R}}(A; T).
 \end{aligned}$$

Hence $\underline{\mathcal{R}}(A; S \cap T) = \underline{\mathcal{R}}(A; S) \cap \underline{\mathcal{R}}(A; T)$.

(4) Let $S, T \in \mathcal{P}(X)$ be such that $S \subseteq T$. Then $S \cap T = S$ and $S \cup T = T$. It follows from (3) and (2) that

$$\underline{\mathcal{R}}(A; S) = \underline{\mathcal{R}}(A; S \cap T) = \underline{\mathcal{R}}(A; S) \cap \underline{\mathcal{R}}(A; T)$$

and

$$\overline{\mathcal{R}}(A; T) = \overline{\mathcal{R}}(A; S \cup T) = \overline{\mathcal{R}}(A; S) \cup \overline{\mathcal{R}}(A; T),$$

which yield $\underline{\mathcal{R}}(A; S) \subseteq \underline{\mathcal{R}}(A; T)$ and $\overline{\mathcal{R}}(A; S) \subseteq \overline{\mathcal{R}}(A; T)$, respectively.

(5) Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, it follows from (4) that

$$\underline{\mathcal{R}}(A; S) \subseteq \underline{\mathcal{R}}(A; S \cup T) \text{ and } \underline{\mathcal{R}}(A; T) \subseteq \underline{\mathcal{R}}(A; S \cup T).$$

Thus $\underline{\mathcal{R}}(A; S) \cup \underline{\mathcal{R}}(A; T) \subseteq \underline{\mathcal{R}}(A; S \cup T)$.

(6) Since $S \cap T \subseteq S, T$, it follows from (4) that

$$\overline{\mathcal{R}}(A; S \cap T) \subseteq \overline{\mathcal{R}}(A; S) \text{ and } \overline{\mathcal{R}}(A; S \cap T) \subseteq \overline{\mathcal{R}}(A; T)$$

so that $\overline{\mathcal{R}}(A; S \cap T) \subseteq \overline{\mathcal{R}}(A; S) \cap \overline{\mathcal{R}}(A; T)$.

(7) If $x \in \overline{\mathcal{R}}(A; S)$, then $A_x \cap S \neq \emptyset$, and so there exists $a \in S$ such that $a \in A_x$. Hence $(a, x) \in \mathcal{R}$, that is, $a - x \in A$ and $x - a \in A$. Since $A \subseteq B$, it follows that $a - x \in B$ and $x - a \in B$ so that $(a, x) \in \mathcal{Q}$, that is, $a \in B_x$. Therefore $a \in B_x \cap S$, which means $x \in \mathcal{Q}(B; S)$. This completes the proof. \square

PROPOSITION 4.4. *Let \mathcal{R} be an equivalence relation on X related to any ideal A of X . Then $\underline{\mathcal{R}}(A; X) = X = \overline{\mathcal{R}}(A; X)$, that is, X is definable.*

PROOF. It is straightforward. \square

PROPOSITION 4.5. *Let \mathcal{R} be an equivalence relation on X related to the trivial ideal $\{0\}$ of X . Then $\underline{\mathcal{R}}(\{0\}; S) = S = \overline{\mathcal{R}}(\{0\}; S)$ for every nonempty subset S of X , that is, every nonempty subset of X is definable.*

PROOF. Note that $\{0\}_x = \{x\}$ for all $x \in X$, since if $a \in \{0\}_x$ then $(a, x) \in \mathcal{R}$ and hence $a - x = 0$ and $x - a = 0$. It follows that $a = x$. Hence

$$\underline{\mathcal{R}}(\{0\}; S) = \{x \in X \mid \{0\}_x \subseteq S\} = S$$

and

$$\overline{\mathcal{R}}(\{0\}; S) = \{x \in X \mid \{0\}_x \cap S \neq \emptyset\} = S.$$

This completes the proof. \square

REMARK 4.6. Let \mathcal{R} be an equivalence relation on X related to an ideal A of X . If B is an ideal of X such that $A \neq B$, then $\underline{\mathcal{R}}(A; B)$ is not an ideal of X in general. For, consider a subtraction algebra X in Example 4.1(2) and an equivalence relation \mathcal{R} on X related to the ideal $A = \{0, 1, 2\}$. If we take an ideal $B = \{0, 1, 3\}$ of X , then $A \neq B$ and $\underline{\mathcal{R}}(A; B) = \{3\}$ which is not an ideal of X .

DEFINITION 4.7. Let \mathcal{R} be an equivalence relation on X related to an ideal A of X . A nonempty subset S of X is called an *upper* (resp. a *lower*) *rough subalgebra/ideal* of X if the upper (resp. nonempty lower) approximation of S is a subalgebra/ideal of X . If S is both an upper and a lower rough subalgebra/ideal of X , we say that S is a *rough subalgebra/ideal* of X .

THEOREM 4.8. *Let \mathcal{R} be an equivalence relation on X related to an ideal A of X . Then every subalgebra S of X is a rough subalgebra of X .*

PROOF. Let $x, y \in \underline{\mathcal{R}}(A; S)$. Then $A_x \subseteq S$ and $A_y \subseteq S$. Since S is a subalgebra of X , it follows that $A_{x-y} = A_x \ominus A_y \subseteq S$ so that $x - y \in$

$\mathcal{R}(A; S)$. Hence $\mathcal{R}(A; S)$ is a subalgebra of X . Now if $x, y \in \overline{\mathcal{R}}(A; S)$, then $A_x \cap S \neq \emptyset$ and $A_y \cap S \neq \emptyset$, and so there exist $a, b \in S$ such that $a \in A_x$ and $b \in A_y$. It follows that $(a, x) \in \mathcal{R}$ and $(b, y) \in \mathcal{R}$. Since \mathcal{R} is a congruence relation on X , we have $(a - b, x - y) \in \mathcal{R}$. Hence $a - b \in A_{x-y}$. Since S is a subalgebra of X , we get $a - b \in S$, and therefore $a - b \in A_{x-y} \cap S$, that is, $A_{x-y} \cap S \neq \emptyset$. This shows that $x - y \in \overline{\mathcal{R}}(A; S)$, and consequently $\overline{\mathcal{R}}(A; S)$ is a subalgebra of X . This completes the proof. \square

COROLLARY 4.9. *Let \mathcal{R} be an equivalence relation on X related to an ideal A of X . Then $\underline{\mathcal{R}}(A)$ ($\neq \emptyset$) and $\overline{\mathcal{R}}(A)$ are subalgebras of X , that is, A is a rough subalgebra of X .*

PROOF. It is straightforward. \square

THEOREM 4.10. *Let \mathcal{R} be an equivalence relation on X related to an ideal A of X . If U is an ideal of X containing A , then*

- (1) $\underline{\mathcal{R}}(A; U)$ ($\neq \emptyset$) is an ideal of X , that is, U is a lower rough ideal of X .
- (2) $\overline{\mathcal{R}}(A; U)$ is an ideal of X , that is, U is an upper rough ideal of X .

PROOF. Let U be an ideal of X containing A . Let $x \in A_0$. Then $x \in A \subseteq U$, and so $A_0 \subseteq U$. Hence $0 \in \underline{\mathcal{R}}(A; U)$. Let $x, y \in X$ be such that $y \in \underline{\mathcal{R}}(A; U)$ and $x - y \in \underline{\mathcal{R}}(A; U)$. Then $A_y \subseteq U$ and $A_x \ominus A_y = A_{x-y} \subseteq U$. Let $a \in A_x$ and $b \in A_y$. Then $(a, x) \in \mathcal{R}$ and $(b, y) \in \mathcal{R}$, which implies $(a - b, x - y) \in \mathcal{R}$. Hence $a - b \in A_{x-y} \subseteq U$. Since $b \in A_y \subseteq U$ and U is an ideal, it follows that $a \in U$, so that $A_x \subseteq U$. Thus $x \in \underline{\mathcal{R}}(A; U)$. This shows that $\underline{\mathcal{R}}(A; U)$ is an ideal of X , that is, U is a lower rough ideal of X . Now, obviously $0 \in \overline{\mathcal{R}}(A; U)$. Let $x, y \in X$ be such that $y \in \overline{\mathcal{R}}(A; U)$ and $x - y \in \overline{\mathcal{R}}(A; U)$. Then $A_y \cap U \neq \emptyset$ and $A_{x-y} \cap U \neq \emptyset$, and so there exist $a, b \in U$ such that $a \in A_y$ and $b \in A_{x-y}$. Hence $(a, y) \in \mathcal{R}$ and $(b, x - y) \in \mathcal{R}$, which implies $y - a \in A \subseteq U$ and $(x - y) - b \in A \subseteq U$. Since $a, b \in U$ and U is an ideal, we get $y \in U$ and $x - y \in U$; hence $x \in U$. Note that $x \in A_x$, thus $x \in A_x \cap U$, that is, $A_x \cap U \neq \emptyset$. Therefore $x \in \overline{\mathcal{R}}(A; U)$, and consequently U is an upper rough ideal of X . \square

COROLLARY 4.11. *Let \mathcal{R} be an equivalence relation on X related to an ideal A of X . Then $\underline{\mathcal{R}}(A)$ ($\neq \emptyset$) and $\overline{\mathcal{R}}(A)$ are ideals of X , that is, A is a rough ideal of X .*

Theorem 4.10 shows that the notion of an upper (resp. a lower) rough ideal is an extended notion of an ideal in a subtraction algebra.

The following example shows that if A and U are ideals of X such that $A \not\subseteq U$, then $\underline{\mathcal{R}}(A; U)$ may not be an ideal of X .

EXAMPLE 4.12. (1) Let $X = \{0, a, b, c, d\}$ be a subtraction algebra described in Example 4.1(2). Consider two ideals $A = \{0, b\}$ and $U = \{0, d\}$ of X . Then $\underline{\mathcal{R}}(A; U) = \{d\}$ which is not an ideal of X .

(2) Let $X = \{0, a, b, c, d\}$ be a subtraction algebra with the Cayley table as follows:

–	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Consider $A = \{0, a, b\} \triangleleft X$ and let \mathcal{R} be an equivalence relation on X related to A . Then the equivalence classes are as follows: $A_0 = A_a = A_b = A$, $A_c = \{c\}$, and $A_d = \{d\}$. Then $U = \{0, a, c\}$ is an ideal of X which does not contain A , and $\underline{\mathcal{R}}(A; U) = \{c\}$ which is not an ideal of X .

References

- [1] J. C. Abbott, *Sets, Lattices and Boolean Algebras*, Allyn and Bacon, Boston 1969.
- [2] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ., Vol. 25, second edition 1984; third edition, 1967, Providence.
- [3] Y. B. Jun and H. S. Kim, *On ideals in subtraction algebras*, Sci. Math. Jpn. (submitted)
- [4] Y. B. Jun, H. S. Kim, and E. H. Roh, *Ideal theory of subtraction algebras*, Sci. Math. Jpn. Online **e-2004** (2004), 397–402.
- [5] Y. B. Jun and K. H. Kim, *Prime and irreducible ideals in subtraction algebras*, Ital. J. Pure Appl. Math. (submitted)
- [6] N. Kuroki, *Rough ideals in semigroups*, Inform. Sci. **100** (1997), 139–163.
- [7] N. Kuroki and J. N. Mordeson, *Structure of rough sets and rough groups*, J. Fuzzy Math. **5** (1997), no. 1, 183–191.
- [8] Z. Pawlak, *Rough sets*, Int. J. Inform. Comp. Sci. **11** (1982), 341–356.
- [9] ———, *Rough Sets-Theoretical Aspects of Reasoning about Data*, Kluwer Academic, Norwell, MA, 1991.
- [10] B. M. Schein, *Difference Semigroups*, Comm. Algebra **20** (1992), 2153–2169.
- [11] B. Zelinka, *Subtraction Semigroups*, Math. Bohemica **120** (1995), 445–447.

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