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Rounding Errors in Algebraic Processes

J. H. Wilkinson

by brief résumés of the definitions, formulas, and theorems. The hints occupy 58 pages, and the complete solutions occupy **141** pages.

The subject matter is the differential and integral calculus of real functions of one real variable, with a small amount of material on convergence of sequences and infinite series. This problem book and the text which it is intended to accompany can be highly recommended for reference use by teachers of calculus. These books should be readily accessible in college libraries for the benefit of interested undergraduates.

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Rounding Errors in Algebraic Processes. By J. H. Wilkinson. Prentice-Hall, Englewood Cliffs, N. J., 1964. vi+161 pp., $6.00.

This book, as does Dr. Wilkinson's work in recent years, stands almost alone in the analysis of round off errors arising in digital computations dealing with polynomials and matrices. It should be required reading for all numerical analysts and a particular source book of insight and illustration for those who would carry on in research.

Dr. Wilkinson develops, primarily, the so-called "backward analysis" of round off errors in which rigorous (determinate) error bounds are derived. The backward analysis technique, as the name suggests, attempts to deduce bounds on perturbations of input data which, with perfect operations, could have produced the rounded numerical results. Then, the round off errors can be bounded by results of a sensitivity analysis. In his words, "Suppose we have shown that a practical (i.e. digital) eigenvalue technique gives, for the eigenvalues of (matrix) A, the exact eigenvalues of some $A+E$ and that we can give bounds for the elements of E. Then to complete the analysis we must give bounds for the effect of the perturbation matrix E on the eigenvalues of A." (The method of "forward analysis," in contrast, attempts to bound accumulated round off errors in terms of bounds on the local round off errors that arise at each arithmetic step. This method has been used in studying numerical solutions for differential equations most effectively, e.g., in Peter Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, Wiley, 1962).

The book contains three chapters, entitled 1) The Fundamental Arithmetic Operations, 2) Computations Involving Polynomials, 3) Matrix Computations. There are literally scores of separate analyses for different problems and subproblems faced by a numerical analyst in getting practical insights and results for computation. Dr. Wilkinson exhibits an unusual capability for "asking the right questions" in a most difficult topic.

Chapter 1 contains preliminary concepts and definitions; error analyses for fixed, floating, and block floating point computation, dealing with repeated additions, repeated multiplication and inner products. It concludes with a discussion of the concept of an ill-conditioned computing problem and its relation to round off errors. In brief, ill-conditioned problems have solutions which are highly sensitive to initial data. If a subproblem is ill-conditioned, round off errors introduced into it (through previous computation) will be greatly amplified in the solution of the subproblem (which may then become initial data for further computation).

Chapter 2 contains error analyses for the evaluation of truncated power series and polynomials, a discussion of the condition of zeros of polynomials and a series of error analyses on zero finding procedures including the methods of Bisection, Newton, Polynomial Deflation, Root Squaring, and Bairstow. All of these methods involve recursive procedures which, with perfect operations, are independent, in their regions of convergence, of initial values, and are, therefore, self correcting. Notwithstanding these properties, however, the accuracy of such a recursive method is not necessarily comparable to the accuracy of an iteration because of round off and ill conditioning problems.

Chapter 3 is especially noteworthy and divided into two main parts, one dealing with solving linear systems and inverting matrices, the other with eigenvalue computations. The first part includes a discussion of matrix norms; error analyses of matrix multiplication and vector orthogonalization. Then the condition of a linear system is discussed and the method of Gaussian elimination and variants are analyzed for solving linear systems and inverting matrices in a sequence of subanalyses. These subanalyses consider the reduction of linear systems to triangular form, solving triangular linear systems (with both single and double precision inner product operations), a discussion of left and right numerical inverses (they are surprisingly interchangeable), an analysis of Cholesky's method for inverting positive definite matrices and its generalization, and finally the use of an approximate inverse in an iterative procedure.

The remainder of Chapter 3 considers eigenvalue computations. The condition of eigenvalues is discussed, and then methods of estimating the accuracy of a proposed eigenvalue and eigenvector are developed. An error analysis for the calculation of eigenvectors of a tri-diagonal matrix with known eigenvalues is given (this handles the general case with known reductions). Finally, an analysis of the determination of eigenvalues of a lower Hessenberg matrix by the method of Hyman is given.

Dr. Wilkinson is a very clear writer and has supplied a profusion of numerical examples worked out in decimal arithmetic. The work is accessible, with some diligence, to almost any level of background expected in a practicing numerical analyst. And, as indicated above, the rewards will merit the diligence required. Much of the material of this book has been taken from notes and papers prepared separately and the organization and some repetition shows in this. This is not all deficit, however, because the various analyses can be studied almost independently once a few basic ideas are grasped from the first chapter.

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Interpolation and Approximation. By Philip J. Davis. Blaisdell, New York, 1963. x+393 pp. $12.50.

The author is well known for his work in interpolation and approximation