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ROUTING AND SINGULAR CONTROL FOR QUEUEING NETWORKS IN HEAVY TRAFFIC

by

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ROUTING AND SINGULAR CONTROL FOR QUEUEING NETWORKS IN HEAVY TRAFFIC

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Abstract

The problem of routing control in an open queueing network under conditions of heavy traffic and finite (scaled) buffers is dealt with. The operating statistics can be state dependent. The sequence of scaled controlled state processes converges to a singularly controlled reflected diffusion (with the associated costs), under broad conditions. Due to the nature of the controls, a 'scaling' method is introduced to get the convergence, since the actual sequence of processes does not necessarily converge in the Skorohod topology. Owing to finite buffers, an extension of the reflection mapping needs to be obtained. The optimal value functions for the physical processes converge to the optimal value function of the limit process, under broad conditions. Approximations to the optimal control for the limit process are obtained, as well as properties of the sequence of physical processes. The optimal or controlled (but not necessarily optimal) limit process can be used to approximate a large variety of functionals of the optimal or controlled (but not necessarily optimal) physical processes.

Key words: routing control, weak convergence, singular control, queues in heavy traffic, reflected controlled diffusions.

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Running head: Routing Control

1. Introduction

We consider the problem of optimal or nearly optimal routing in a queueing system under heavy traffic conditions. The general network model is a "controlled routing" form of the general open network dealt with by Reiman and Harrison [1] or Reiman [2], where each customer eventually leaves the system. We will actually treat two special cases for simplicity in the development. But it should be apparent from these cases that the general open network can be treated in the same way. The treated cases involve all the basic technique that is required for the general case. In [2], one has a finite set of servers, each with an infinite buffer. We bound (and appropriately scale) the buffers here. It is well known [1], [2], that under broad conditions on the service and interarrival times, the vector of queue length processes (with an appropriate amplitude normalization and time scaling) converges weakly to a reflected diffusion, as the traffic intensity goes to unity.

The work in [2] required that the system operating statistics not be state dependent, and used results for the weak convergence of a sequence of sums of mutually independent random variables to a Wiener process, together with a clever method to treat the boundary to get the appropriate limit. The methods are not extendible to the state dependent or to the controlled case, where the required independence no longer holds, and the characterization of the limit processes as well as the proofs of tightness require different methods. The "martingale type" methods for getting limit theorems for wide bandwidth noise driven systems seem to be more appropriate. In reference [4], there is a study of a heavy traffic problem under a control, and the arrival and service processes were allowed to be state dependent. Such state dependence is natural for the controlled problem, since one might want to let the processing depend on what is happening in the system. In addition the methods which are needed to characterize the controls in the limit problem as 'non anticipative', etc., require the use of the same methods that the state dependence requires.

In [4], the processors and the arrival sequences could be shut on or off to control the flows and the costs. The limit problem was an impulsively controlled reflected diffusion of a non classical type, since there was the possibility of multiple 'simultaneous' impulses. It was shown in [4] that any sequence of controlled physical processes with uniformly bounded costs converged to a well defined controlled limit process. Also the sequence of optimally controlled physical processes converged to the optimally controlled limit process, in the sense that the value functions converged. Also a control which was nearly optimal for the limit process could be adapted to become a nearly optimal control for the physical process under heavy traffic, under quite broad conditions. Such results help to justify the use of heavy traffic limit theorems for the purposes of optimal other control. Because of the behavior of the physical process in [4] when the on-off controls were used, the Skorohod topology had to be used with care, because the actual scaled queue length processes did not converge in the Skorohod topology as it is usually used. Also, that reference provided convergent numerical algorithms.

In this paper, we also deal with a controlled heavy traffic problem. In the basic model, the routing of a subset of the external arrivals could be controlled. The aims are similar to those in [4]. The dynamical equations for the scaled queue length process is defined. The sequence of such processes, (as the traffic intensity tends to unity) might not be tight in the Skorohod topology, due to the nature of the routing control. To handle this, we start by working with a rescaling of the time, with which we can get tightness, and a characterization of the weak limits. The rescaling depends on the control. After the limits are obtained, an 'inverse' scaling (dependent on the limit control) yields the process which actually characterizes the limit of the cost functionals. The limit process is a controlled reflected diffusion. But the control is of the 'singular' type in the sense of [8]. The usual reflection mapping which is used to handle the problem of non-negativity of the queue length process must be modified here, due to the presence of the finite buffer. We construct the proper reflection mapping from a sequence of concatenations of the usual one.

The basic problem of interest is defined in Section 2. We work with a system of only two processors for notational simplicity. Also, until Section 7, we do not have feedback. The addition of feedback is straightforward, but it seems to be preferable to present the ideas in as unencumbered a fashion as possible. The extension of the result to the general routing controlled open network is straightforward. Some of the weak convergence arguments and definitions from reference [4] are used, but familiarity with that reference is not necessary. In Section 2, we manipulate the state equations into the 'martingale plus drift' form which will be used in the weak convergence arguments. The reflection mapping result is stated in Section 3 (and proved in Section 8). The required rescaling is defined and the tightness and weak convergence proved in Section 3. We must prove that the limit (singular) controls are non anticipative with respect to the Wiener processes which 'drive' the limit process.

Section 4 is concerned with the convergence of the cost functions. We prove that there is a routing control with a uniformly bounded cost, and show that the liminf of the optimal cost functions for the physical processes is bounded below by the optimal cost for the limit process. In order to show that the limit of the optimal costs for the physical processes is the optimal cost for the limit process, we need to prove various existence and approximation results for the optimal policy for the limit problem. This is done in Section 6, and uses the 'limit form' of the control dependent rescaling introduced in Section 3. An interesting approach to the approximation problem is discussed. The general rescaling and tightness methods are of much wider use for limit and approximation problems where singular controls are involved and where there might not be convergence in the Skorohod topology. The developed approximations are then used to prove the approximate optimality for the physical processes of an appropriate nearly optimal policy for the limit process.

This seems to be the first work which deals with such controlled routing problems. Approximations to singular control problems for wide bandwidth noise driven systems were discussed in [6], but the method used here is rather different and is very natural for the sorts of problems that are being considered. Numerical methods have been developed for the problems of this paper. The proofs of their convergence require methods which are similar to those used here, but since there are many additional details, they will be dealt with in a subsequent paper.

2. Problem Description

Until Section 7, we work with the simple system of Figure 1. This will enable us to develop the main ideas without an excessive notational burden. Also, for notational convenience, we work with a discrete time parameter. The results for the analogous continuous time parameter case are the same. Each of the processors P_0, P_1 and P_2 has its own stream of arrivals from the exterior. P_0 is used only as an (instantaneous) routing node. Its service time is zero. This can readily be changed, and the resulting network would then be a special case of the general network discussed in Section 7. The P_0 routes to either P_1 or P_2, and (until Section 7), the completed services from P_1 and P_2 leave the system. The routing decision is based on the events up to the time of the decision. We suppose that some prior routing is assigned to each new arrival to P_0, but that the routing node can reassign, with an associated profit or loss. Next, we give some simple examples.

Example 1. There are two classes of customers arriving (at random) at P_0. P_i is more efficient for class i, and a prior assignment of class i to P_i is made. But P_0 can reroute to the less efficient processor, depending on the system state. The cost of rerouting might be, for example, a set up cost.

Example 2. The case of Example 1, but with three classes of customers, arriving at random. Class i ($i = 1$ or 2) must be served by P_i. Class 3 can be served by either processor, but one of the P_i is more efficient (cheaper) and a prior assignment to that P_i is made. But P_0 can alter the assignment. For example, let the P_i represent data bases, with some overlap of data files. A subset of the arriving jobs need only the 'overlap' data. But one of the P_i is 'faster' than the other.

Example 3. P_1 is cheaper for all customers arriving at P_0. But, due to the heavy traffic conditions, the mean number of customers routed to each P_i is essentially fixed (modulo some fraction which goes to zero as the traffic intensity goes to unity). Some prior assignment is made but P_0 can reroute at either a cost or a savings if appropriate.

In general, the model can be readily extended to handle rerouting of a customer actually in a queue as well as reneging.

In the modelling of systems under heavy traffic conditions, it has been the usual practice to suppose that the processors 'keep processing' and create departures even if the queues are empty [1]–[4]. Whatever 'fictitious' departures occur due to this convention are compensated for by an added 'reflection term' (our Y below). Thus each P_i ($i = 1, 2$) has associated to it a sequence of service intervals which cover all time. This convention simplifies the analysis. Also, we suppose (as is the usual practice) that if a customer arrives at P_1 or P_2 when the associated queue is empty, then the service time for that customer is just the residual time of the current service time interval for that processor. As in [1]–[4], this convention does not affect the limit processes.

Since we work in discrete time, it is possible that multiple events can occur at the same time at P_1 or P_2. For the sake of precision, we suppose that a departure (real or fictitious) from a processor always occurs 'just before' any arrival to that processor, and that if two arrivals to the same P_i occur at the same time, then the one from P_0 takes precedence. Such a conflict might arise if there is only space for one customer left in some buffer, but there are two arrivals. We ignore these distinctions in the notation, for simplicity. It can be shown that the precedence relations do not affect the limit.

Definitions. We use the notation of [4] whenever possible, although knowledge of that reference is not needed for the reading of this paper. The symbol ϵ indexes the traffic intensity; as $\epsilon \to 0$, the intensity goes to 1. For each $\epsilon > 0$ and $i = 1, 2$, let $\{\Delta_n^{i,\epsilon}, n = 1, 2, \ldots\}$, denote the sequence of service times for P_i and let $\psi_n^{i,\epsilon}$ be the indicator function of the event that a service (real or fictitious) is completed at P_i at time n. For $i = 0, 1, 2$ and each $\epsilon > 0$, let $\{\alpha_n^{i,\epsilon}, n < \infty\}$ denote the sequence of interarrival times to P_i, from the exterior of the system, and let $\xi_n^{i,\epsilon}$, $i = 0, 1, 2$, be the indicator of the event that there is an external arrival to P_i at time n. Write t/ϵ for $[t/\epsilon]$, the largest integer which is no bigger than t/ϵ. Define $X_n^{i,\epsilon} = \sqrt{\epsilon}$ (number of customers waiting for or in service at P_i at time n), and set $X^{i,\epsilon}(t) = X_{t/\epsilon}^{i,\epsilon}$. In general, for a sequence $\{Z_n^\epsilon\}$, define the function $Z^\epsilon(t) = Z_{t/\epsilon}^\epsilon$. The buffer of P_i, $i = 1, 2$, has size $B_i/\sqrt{\epsilon}$, which we assume is always an integer. Let $I_n^{i,\epsilon}$ denote the indicator of the event that an arrival at P_0 at time n has the *prior* assignment to P_i, and let $\rho_n^{ij,\epsilon}$, $j \neq i$, be the indicator of the event that this arrival is *reassigned* to P_j.

We usually use the convention that the superscript ϵ is dropped whenever one of the above terms is used as a summand. Define $(j \neq i)$

$$A_n^{i,\epsilon} = \sqrt{\epsilon} \sum_{m=0}^{n} \xi_m^i, \quad A_n^{0i,\epsilon} = \sqrt{\epsilon} \sum_{m=0}^{n} I_m^i \xi_m^0,$$

$$D_n^{i,\epsilon} = \sqrt{\epsilon} \sum_{m=0}^{n} \psi_m^i,$$

$$J_n^{ij,\epsilon} = \sqrt{\epsilon} \sum_{m=0}^{n} \xi_m^0 I_m^i \rho_m^{ij}$$

$$J_n^{i,\epsilon} = J_n^{ji,\epsilon} - J_n^{ij,\epsilon}$$

$$Y_n^{i,\epsilon} = \sqrt{\epsilon} \sum_{m=0}^{n} \psi_m^i I_{\{X_m^i=0\}}$$

$$U_n^{i,\epsilon} = \sqrt{\epsilon} \sum_{m=0}^{n} \xi_m^i I_{\{X_m^i=B_i\}} + \sqrt{\epsilon} \sum_{m=0}^{n} \xi_m^0 (I_m^i + I_m^j \rho_m^{ji} - I_m^i \rho_m^{ij}) I_{\{X_n^i=B_i\}}$$

The $A_n^{0i,\epsilon}$ is the scaled total number of arrivals (by time n) at P_0 which have been a priori assigned to P_i (they might, of course, be rerouted by P_0).

The $J_n^{ij,\epsilon}$ are the 'rerouting' control terms, the scaled number of customers originally destined for P_i but rerouted to P_j. The $Y_n^{i,\epsilon}$ is the scaled total number of 'fictitious' departures due to our convention of continuing to 'process' even if the queue is empty, and $U_n^{i,\epsilon}$ is the number of customers lost to P_i when its buffer is full.

The *mass balance* equations can be written as (discrete 'real' time and 'interpolated' time, resp.)

$$X_n^{i,\epsilon} = X_0^{i,\epsilon} + A_n^{i,\epsilon} + A_n^{0i,\epsilon} - D_n^{i,\epsilon} + J_n^{i,\epsilon} + Y_n^{i,\epsilon} - U_n^{i,\epsilon}, \tag{2.1}$$

$$X^{i,\epsilon}(t) = X_0^{i,\epsilon} + A^{i,\epsilon}(t) + A^{0i,\epsilon}(t) - D^{i,\epsilon}(t) + J^{i,\epsilon}(t) + Y^{i,\epsilon}(t) - U^{i,\epsilon}(t). \tag{2.2}$$

The cost function. Let $\beta > 0$, $c_i > 0$, $k_i > 0$, and let $k(\cdot)$ be a bounded and continuous function. Define $J^\epsilon = (J^{12,\epsilon}, J^{21,\epsilon})$. We use the cost functional

$$V^\epsilon(x, J^\epsilon) = E_x \int_0^\infty e^{-\beta t} k(X^\epsilon(t))dt + E_x \int_0^\infty e^{-\beta t}[k_1 dJ^{12,\epsilon}(t) + k_2 dJ^{21,\epsilon}(t)$$
$$+ c_1 dU^{1,\epsilon}(t) + c_2 dU^{2,\epsilon}(t)]. \tag{2.3}$$

By Theorem 7 below, there are routing policies $J^{ij,\epsilon}(\cdot)$ for which

$$\sup_\epsilon V^\epsilon(x, J^\epsilon) < \infty. \tag{2.4}$$

Define

$$V^\epsilon(x) = \inf_{J^\epsilon} V^\epsilon(x, J^\epsilon).$$

The $k(\cdot)$ might be non-linear. Such non-linear $k(\cdot)$ occur when we wish to model the costs of reneging or queue switching, or if we wish to limit the possibility of leaving the queue due to a 'long' wait. The second term in (2.3) penalizes the overflows and rerouting. One of the k_i can be negative and we return to this case at the end of Section 6.

Definitions and heavy traffic assumptions. We take many of the definitions from [4] so that the results of that reference can be conveniently used. Define $S_{a,n}^{i,\epsilon} = \sum_{m=1}^{n} \alpha_m^i$. $S_d^{i,\epsilon} = \sum_{m=1}^{n} \Delta_m^i$. Define $\overline{S}_a^{i,\epsilon}(\cdot)$ by $\overline{S}_a^{i,\epsilon}(t) = \max\{\epsilon m : \epsilon S_{a,r_i}^{i,\epsilon} \leq t\}$, and define $\overline{S}_d^{i,\epsilon}(\cdot)$ analogously. These functions are the 'inverses' of the functions $\epsilon S_\alpha^{i,\epsilon}(\cdot)$. Let $E_{a,n}^{i,\epsilon}$ denote the expectation, conditioned on the arrival and departure intervals which started by $S_{a,n}^{i,\epsilon}$ (except for $\alpha_{n+1}^{i,\epsilon}$), and the control (routing) actions taken up to $S_{a,n}^{i,\epsilon}$. Define $E_{d,n}^{i,\epsilon}$ analogously, where $S_{d,n}^{i,\epsilon}$ and $\Delta_{n+1}^{i,\epsilon}$ replace $S_{a,n}^{i,\epsilon}$ and $\alpha_{n+1}^{i,\epsilon}$, resp. Similarly, define the conditional variances $\text{var}_{\alpha,n}^{i,\epsilon}$, $\alpha = a, d$. We use the notation

$$E_{a,n}^{i,\epsilon}\alpha_{n+1}^{i,\epsilon} = \overline{\alpha}_{n+1}^{i,\epsilon}, \quad E_{d,n}^{i,\epsilon}\Delta_{n+1}^{i,\epsilon} = \overline{\Delta}_{n+1}^{i,\epsilon}$$

$$\text{var}_{a,n}^{i,\epsilon}\alpha_{n+1}^{i,\epsilon} = (\sigma_{a,n+1}^{i,\epsilon})^2, \quad \text{var}_{d,n}^{i,\epsilon}\Delta_{n+1}^{i,\epsilon} = (\sigma_{d,n+1}^{i,\epsilon})^2.$$

We will use the following assumptions. A2.1 and A2.4 are the 'usual' heavy traffic assumptions. (A2.4) basically says that (modulo a term which goes to zero as $\epsilon \to 0$) the mean rate of arrivals to P_i equals the 'capacity' of P_i.

A2.1. *There are real $g_{ai} > 0$, $g_{di} > 0$ and bounded and continuous real valued functions $a^i(\cdot)$ and $d^i(\cdot)$ such that*

$$(\overline{\alpha}_{n+1}^{i,\epsilon})^{-1} = g_{ai} + \sqrt{\epsilon}a_{in} + o(\sqrt{\epsilon})$$

$$(\overline{\Delta}_{n+1}^{i,\epsilon})^{-1} = g_{di} + \sqrt{\epsilon}d_{in} + o(\sqrt{\epsilon}),$$

where

$$a_{in} = a^i(X^\epsilon_{S^{i,\epsilon}_{a,n}}), \quad d_{in} = d^i(X^\epsilon_{S^{i,\epsilon}_{d,n}}).$$

Note that $X^\epsilon_{S^{i,\epsilon}_{d,m}}$ is the value of the state at the beginning of the $n+1$st interarrival interval, and so it is the correct argument of the $a^i(\cdot)$ above, and similarly for the $d^i(\cdot)$.

A2.2. $\{|\alpha_n^{i,\epsilon}|^2, |\Delta_n^{i,\epsilon}|^2, i, n, \epsilon > 0\}$ *is uniformly integrable.*

A2.3. *There are $\overline{p}_i$ such that $P\{I_n^{i,\epsilon} = 1 \mid$ all arrival or departure intervals starting by time n and routing actions up to time $n-1\} = \overline{p}_i$.*

This assumption can be weakened in many ways, allowing for batch rerouting and other variations, as well as correlated routings. All that is really needed is that 'loosely speaking', $\overline{p}_i$ be a 'local mean' of the conditional expectations and satisfy (A2.4).

A2.4. *For the $\overline{p}_i$ defined in* (A2.3), $\overline{p}_i g_{a0} + g_{ai} = g_{di}$, $i = 1, 2$.

A2.5. *There are continuous and bounded real valued functions $\sigma_{ai}(\cdot)$, $\sigma_{di}(\cdot)$ such that*

$$\sigma^{i,\epsilon}_{a,n+1} = \sigma_{a,i}(X^\epsilon_{S^{i,\epsilon}_{a,n}}) + \delta_\epsilon$$

$$\sigma^{i,\epsilon}_{d,n+1} = \sigma_{d,i}(X^\epsilon_{S^{i,\epsilon}_{d,n}}) + \delta'_\epsilon$$

where δ_ϵ and $\delta'_\epsilon \xrightarrow{\epsilon} 0$ uniformly in all other variables.

In the sequel, we suppose for simplicity that all $\sigma^2_{\alpha,i}(x) > 0$ for all x, α. The results are true even if this condition is violated.

A more convenient representation for $X^\epsilon(\cdot)$. The 2nd to 4th terms on the right side of (2.2) go to infinity as $\epsilon \to 0$. For purposes of the weak convergence analysis, it is helpful to center these terms so that we can work with martingales and processes of bounded variation. We follow closely the procedure used in [4, Section 3] with a slightly different notation. Access to that paper is not needed. Define the following processes.

$$\tilde{A}_0^{i,\epsilon}(t) = \sqrt{\epsilon} \sum_{m=1}^{t/\epsilon} (1 - \alpha_m^i / \overline{\alpha}_m^i)$$

$$\tilde{A}_0^{0i,\epsilon}(t) = \sqrt{\epsilon} \sum_{m=1}^{t/\epsilon} (I_{S_{a,m}^i}^i - \overline{p}_i \alpha_m^0 / \overline{\alpha}_m^0) \tag{2.5}$$

$$\tilde{D}_0^{i,\epsilon}(t) = \sqrt{\epsilon} \sum_{m=1}^{t/\epsilon} (1 - \Delta_m^i / \overline{\Delta}_m^i).$$

The summand in (2.5) are all centered about their conditional expectations, given the 'past' data. Hence, the sums are martingales. Henceforth, we simply write the indicator function which appears in the second sum as $I_m^{i,\epsilon}$.

As in [4, Section 3], we can write (recall that $A^{i,\epsilon}(t) = \overline{S}_a^{i,\epsilon}(t)/\sqrt{\epsilon}, \ldots$)

$$A^{i,\epsilon}(t) = \tilde{A}_0^{i,\epsilon}(\overline{S}_a^{i,\epsilon}(t)) + \sqrt{\epsilon} \sum_{m=1}^{\frac{1}{\epsilon}\overline{S}_a^{i,\epsilon}(t)} \alpha_m^i / \overline{\alpha}_m^i, \quad i = 1, 2,$$

$$A^{0i,\epsilon}(t) = \tilde{A}_0^{0i,\epsilon}(\overline{S}_a^{0,\epsilon}(t)) + \sqrt{\epsilon}\overline{p}_i \sum_{m=1}^{\frac{1}{\epsilon}\overline{S}_a^{0,\epsilon}(t)} \alpha_m^0 / \overline{\alpha}_m^0, \tag{2.6}$$

$$D^{i,\epsilon}(t) = \tilde{D}_0^{i,\epsilon}(\overline{S}_d^{i,\epsilon}(t)) + \sqrt{\epsilon} \sum_{m=1}^{\frac{1}{\epsilon}\overline{S}_d^{i,\epsilon}(t)} \Delta_m^i / \overline{\Delta}_m^i.$$

The first terms on the right sides of (2.6) are just scaled martingales. The right hand terms in (2.6) 'blow up' as $\epsilon \to 0$. In (2.2), the sum of the first two minus the 3rd term of (2.6) occurs. Subtracting the far right-hand term on the 3rd line of (2.6) from the sum of far right-hand terms of the first two lines of (2.6), and using the heavy traffic assumption (A2.4), the expansion (A2.1), and the fact that

$$\sqrt{\epsilon} \sum_{m=1}^{\overline{S}_{d(t)}^{i,\epsilon}/\epsilon} \Delta_m^i = \frac{t}{\sqrt{\epsilon}} + O(\sqrt{\epsilon}) = \sqrt{\epsilon} \sum_{m=1}^{\overline{S}_a^{i,\epsilon}(t)/\epsilon} \alpha_n^i,$$

yields (as in [4, Section 3]) the expression

$$\epsilon \sum_{m=1}^{\overline{S}_a^{i,\epsilon}(t)/\epsilon} \alpha_m^i a_{im} + \epsilon \sum_{m=1}^{\overline{S}_a^{0,\epsilon}(t)/\epsilon} \overline{p}_i \alpha_m^0 a_{0m} - \epsilon \sum_{m=1}^{\overline{S}_d^{i,\epsilon}(t)/\epsilon} \Delta_m^i d_{im} + \delta^{i,\epsilon}(t), \tag{2.7}$$

where $\delta^{i,\epsilon}(\cdot)$ is such that $\sup_{t \le T} |\delta^{i,\epsilon}(t)| \xrightarrow{\epsilon} 0$, for each $T < \infty$.

For $i = 1, 2$, let $\tilde{A}^{i,\epsilon}(t)$, $\tilde{A}^{0i,\epsilon}(t)$ and $\tilde{D}^{i,\epsilon}(t)$ denote the first terms on the right side of (2.6). Define $b_i(x) = a^i(x) + \overline{p}_i a^0(x) - d^i(x)$ and

$$B^{i,\epsilon}(t) = \int_0^t b_i(X^\epsilon(s))ds.$$

Then, modulo an error (which we absorb into $\delta^{i,\epsilon}(\cdot)$) of order $O(\epsilon)$ due to the approximation of the sum by an integral, (2.7) equals $\delta^{i,\epsilon}(t) + B^{i,\epsilon}(t)$ and

$$X^{i,\epsilon}(t) = X_0^i + [\tilde{A}^{i,\epsilon}(t) + \tilde{A}^{i0,\epsilon}(t) - \tilde{D}^{i,\epsilon}(t)] + B^{i,\epsilon}(t)$$

$$+ \ J^{i,\epsilon}(t) + Y^{i,\epsilon}(t) - U^{i,\epsilon}(t) + \delta^{i,\epsilon}(t). \tag{2.8}$$

3. Weak Convergence

In this section, we deal with the weak convergence of the terms in (2.8), as $\epsilon \to 0$. Let $D^k[0,\infty)$ denote the space of R^k-valued right continuous functions with left-hand limits and $C^k[0,\infty)$ the subspace of continuous functions. For all the weak convergence work, we use $D^k[0,\infty)$ under the Skorohod topology [5, Chapter 3.5]. We will often use the Skorohod representation [5, Theorem 3.1.8] so that we can always assume that if a sequence of processes converges weakly, then the convergence is (w.p.1) also pathwise in the topology of the path space.

There are two main problems. First, little is known about the control terms $J^{i,\epsilon}(\cdot)$. In general, even if bounded, they need not converge in the Skorohod topology. Indeed, their behavior can be quite 'wild'. The pseudopath topology [7] could be used, as it has been in [6] for some approximation and convergence questions arising from systems with wide bandwidth noise disturbances under singular controls. For our purposes, it is more convenient to work directly with the Skorohod topology, but with a rescaled set of processes. (Some comments on the relations between scaling and the pseudopath topology are in [9].) After getting the desired weak convergence, we invert the 'limit' of the rescalings to get the result for (2.8).

The second problem concerns the treatment of the reflection terms $Y^{i,\epsilon}(\cdot)$ and $U^{i,\epsilon}(\cdot)$. Owing to the presence of the upper boundary, the reflection mapping theorem of [1], [2] cannot be used directly. The following extension is proved in Section 8.

Theorem 1. *Let Q be a $k \times k$ probability transition matrix whose spectral* *radius is less than unity. Let* $z(\cdot) \in D^k[0,\infty)$ *and consider the equation:*

$$x(t) = z(t) + (I - Q')y(t) - u(t). \tag{3.1}$$

There is a continuous function (in the topology of uniform convergence on bounded time intervals) $F(\cdot)$ *such that* $(y(\cdot),\ u(\cdot)) = F(z(\cdot))$ *has the following properties:* $F(\cdot)$ *maps* $C^k[0,\infty)$ *into* $C^k[0,\infty)$ *and* $D^k[0,\infty)$ *into* $D^k[0,\infty)$; for $i = 1,2$, $y^i(\cdot)$ *and* $u^i(\cdot)$ *are non-decreasing and increase only when* $x^i(t) = 0$ *and* $x^i(t) = B_i$, *resp. Eqn. (3.1) holds and* $x^i(t) \in [0, B^i]$.

Using the martingale properties of the sums defined in (2.5), it is not hard to prove Theorem 2. In fact, the proof of the first paragraph is given in Lemma 5.2 in [4], and the proof of the second paragraph is in [4, Theorem 5.1].

Theorem 2. *Assume* (A2.1) *to* (A2.5). *Then, for* $\alpha = a$ *or* d, *the processes with values* $\epsilon S^{i,\epsilon}_{\alpha,t/\epsilon}$ *and* $\overline{S}^{i,\epsilon}_{\alpha}(t)$ *converge weakly to the deterministic functions with values* $t/g_{\alpha i}$ *and* $tg_{\alpha i}$, *resp. The processes*

$$\{\tilde{A}^{1,\epsilon}(\cdot), \tilde{A}^{2,\epsilon}(\cdot), (\tilde{A}^{01,\epsilon}(\cdot), \tilde{A}^{02,\epsilon}(\cdot)), \tilde{D}^{1,\epsilon}(\cdot), \tilde{D}^{2,\epsilon}(\cdot), \epsilon > 0\}$$

are tight and the limit of any weakly convergent subsequence of the five sequences (we always pair together $\tilde{A}^{01,\epsilon}(\cdot)$ *and* $\tilde{A}^{02,\epsilon}(\cdot)$*) are orthogonal continuous martingales.*

The quadratic variations of the limit martingales are, resp., the weak limits of

$$\begin{aligned} &\int_0^t \Sigma_{ai}(X^\epsilon(s))ds, \quad i = 1,2,0, \\ &\int_0^t \Sigma_{di}(X^\epsilon(s))ds, \quad i = 1,2, \end{aligned} \tag{3.2}$$

where

$$\Sigma_{ai}(x) = g^3_{ai}\sigma^2_{ai}(x), \quad i = 1,2,$$

$$\Sigma_{a0}(x) = g_{a0}\begin{bmatrix} \bar{p}_1(1-\bar{p}_1) & -\bar{p}_1\bar{p}_2 \\ -\bar{p}_1\bar{p}_2 & \bar{p}_2(1-\bar{p}_2) \end{bmatrix} + g_{a0}^3\sigma_{a0}^2(x)\begin{bmatrix} 1 & \bar{p}_1\bar{p}_2 \\ \bar{p}_1\bar{p}_2 & 1 \end{bmatrix},$$

$$\Sigma_{di}(x) = g_{di}^3\sigma_{di}^2(x), \quad i = 1,2.$$

Since the proofs of an almost identical result is in the cited reference, we omit it and comment only on how (3.2) is calculated in one case.

The quadratic variation of the discrete parameter martingale $\tilde{A}_0^{i,\epsilon}(\cdot)$ is (recalling that the argument of $\sigma_{ai}^2(\cdot)$ is the state at the time of arrival of the mth customer)

$$\epsilon\sum_{m=1}^{t/\epsilon} E_{a,m}^{i,\epsilon}(1-\alpha_m^{i,\epsilon}/\overline{\alpha}_m^{i,\epsilon})^2 = \epsilon\sum_{m=1}^{t/\epsilon} g_{ai}^2\sigma_{ai}^2(X_{S_{a,m}^{i,\epsilon}}^\epsilon)$$
$$+ \text{(small terms)}.$$

Neglecting the small terms (which go to zero, as $\epsilon \to 0$), we can write the quadratic variation of $\tilde{A}^{i,\epsilon}(\cdot)$ as

$$\epsilon\sum_{m=1}^{\overline{S}_a^{i,\epsilon}(t)/\epsilon} \frac{g_{ai}^2\sigma_{ai}^2(X_{S_{a,m}^{i,\epsilon}}^\epsilon)}{\overline{\alpha}_{a,m}^{i,\epsilon}}\alpha_{a,m}^{i,\epsilon}$$

$$+\ \epsilon\sum_{m=1}^{\overline{S}_a^{i,\epsilon}(t)/\epsilon} \frac{g_{ai}^2\sigma_{ai}^2(X_{S_{a,m}^{i,\epsilon}}^\epsilon)}{\overline{\alpha}_{a,m}^{i,\epsilon}}(\overline{\alpha}_{a,m}^{i,\epsilon} - \alpha_{a,m}^{i,\epsilon}). \tag{3.3}$$

The variance of the second term in (3.3) is $O(\epsilon t)$ due to the centering of the summands about the conditional expectations. The first term in (3.3) can be written as (modulo an error of order $O(\sqrt{\epsilon})$)

$$\int_0^t g_{ai}^3\sigma_{ai}^2(X^\epsilon(s))ds. \tag{3.4}$$

Thus, we obtain the first line of (3.2), for $i = 1,2$.

The time rescaling. The weak convergence proofs for the terms in (2.8) is facilitated by means of a rescaling or 'stretching out' of time. Define $T^\epsilon(\cdot)$ by

$$T^\epsilon(n\epsilon) = n\epsilon + \sqrt{\epsilon}\sum_{m=1}^{n}[\rho_m^{12} + \rho_m^{21} - \rho_m^{12}\rho_m^{21}],$$

and for $t \in (n\epsilon, n\epsilon+\epsilon)$, define $T^\epsilon(t)$ to be the piecewise linear interpolation. Let $\hat{T}^\epsilon(\cdot)$ denote the inverse function to $T^\epsilon(\cdot)$. For any function $\phi(\cdot)$ on $[0,\infty)$, define the function $\hat{\phi}^\epsilon(\cdot)$ by $\hat{\phi}^\epsilon(t) = \phi(\hat{T}^\epsilon(t))$. Similarly, define $\hat{A}^{\alpha,\epsilon}(t) = \tilde{A}^{\alpha,\epsilon}(\hat{T}^\epsilon(t))$, etc.

Theorem 3. *Assume* (A2.1) *to* (A2.5). *Then*

$$\{\hat{T}^\epsilon(\cdot), \hat{X}^\epsilon(\cdot), \hat{B}^\epsilon(\cdot), \hat{Y}^{i,\epsilon}(\cdot), \hat{U}^{i,\epsilon}(\cdot), \hat{J}^{12,\epsilon}(\cdot), \hat{J}^{21,\epsilon}(\cdot), \epsilon > 0\} \tag{3.5}$$

is tight and all limits are continuous processes. Also

$$\{\hat{A}^{1,\epsilon}(\cdot), \hat{A}^{2,\epsilon}(\cdot), (\hat{A}^{01,\epsilon}(\cdot), \hat{A}^{02,\epsilon}(\cdot)), \hat{D}^{1,\epsilon}(\cdot), \hat{D}^{2,\epsilon}(\cdot), \epsilon > 0\} \tag{3.6}$$

is tight and the limits of any weakly convergent subsequence of the set of five sequences are orthogonal continuous martingales. Let ϵ index a weakly convergent subsequence of (3.5), (3.6), *and denote the limits by the same letters, but with the ϵ dropped. Then*

$$\hat{X}^i(t) = X^i(0) + \hat{B}^i(t) + [\hat{A}^i(t) + \hat{A}^{0i}(t) - \hat{D}^i(t)] + \hat{Y}^i(t) - \hat{U}^i(t) + \hat{J}^{ji}(t) - \hat{J}^{ij}(t). \tag{3.7}$$

$\hat{Y}^i(\cdot)$ increases only when $\hat{X}^i(t) = 0$ and $\hat{U}^i(\cdot)$ increases only when $\hat{X}^i(t) = B_i$. Also.

$$\hat{B}^i(t) = \int_0^t b_i(\hat{X}(s))d\hat{T}(s). \tag{3.8}$$

The quadratic variations of the martingales are

$$\int_0^t \Sigma_{ai}(\hat{X}(s))d\hat{T}(s), \quad i = 0,1,2,$$

$$\int_0^t \Sigma_{di}(\hat{X}(s))d\hat{T}(s), \quad i = 1,2. \tag{3.9}$$

For the particular chosen weakly convergent subsequence, let $\hat{\mathcal{F}}_t$ denote the minimal σ-algebra which measures $\{\hat{P}(s), s \le t\}$, where

$$\hat{P}(s) = (\hat{X}(s), \hat{J}^{12}(s), \hat{J}^{21}(s), \hat{A}^i(s), \hat{A}^{0i}(s), \hat{D}^i(s), \hat{T}(s),\ i = 1,2).$$

Then the martingales are all $\hat{\mathcal{F}}_t$-martingales.

Proof. The set (3.6) is tight and has the asserted properties by Theorem 2, since (3.6) is just the sequence dealt with in Theorem 2, but with a 'stretched out' time scale. The $\{\hat{T}^\epsilon(\cdot),\ J^{12,\epsilon}(\cdot),\ J^{21,\epsilon}(\cdot),\ \epsilon > 0\}$ are tight since their increments between any $t, t+s$ are bounded by $s + O(\sqrt{\epsilon})$. The set $\{\hat{B}^\epsilon(\cdot),\ \epsilon > 0\}$ is obviously tight.

To treat the $\hat{Y}^{i,\epsilon}(\cdot)$ and $\hat{U}^{i,\epsilon}(\cdot)$, we use the representation of the reflecting terms of Theorem 1. Thus, there is a continuous function (in the sense of Theorem 1) $F_0(\cdot)$ such that

$$(\hat{Y}^\epsilon(\cdot), \hat{U}^\epsilon(\cdot)) = F_0(X_0^\epsilon, \hat{A}^{i,\epsilon}(\cdot), \hat{A}^{0i,\epsilon}(\cdot), \hat{D}^{i,\epsilon}(\cdot), \hat{B}^{i,\epsilon}(\cdot), \hat{J}^{12,\epsilon}(\cdot), \hat{J}^{21,\epsilon}(\cdot),\ i = 1,2).$$

The tightness of $\{\hat{Y}^\epsilon,\ \hat{U}^\epsilon(\cdot),\ \hat{X}^\epsilon(\cdot),\ \epsilon > 0\}$ and the continuity of the weak limits follows from this and the fact that the argument processes of $F_0(\cdot)$ are tight and have continuous weak limits. Also, the properties asserted below (3.7) hold. The representation (3.8) follows from the equality

$$\hat{B}^{i,\epsilon}(t) = \int_0^{\hat{T}^\epsilon(t)} b_i(X^\epsilon(s))ds = \int_0^t b_i(X^\epsilon(\hat{T}^\epsilon(s))d\hat{T}^\epsilon(s), \qquad (3.10)$$

as we will now see. Abusing notation, let ϵ index a weakly convergent subsequence of the sets in (3.5), (3.6), and suppose that the Skorohod representation is used so that we can assume that all weak convergences are convergences w.p.1 and are uniform on each bounded time interval (since the limit processes are continuous w.p.1). Since the $\hat{T}^\epsilon(\cdot)$ satisfy $|\hat{T}^\epsilon(t+s) - \hat{T}^\epsilon(t)| = O(s)$, the uniform convergence (on each $[0,t]$) of $\hat{T}^\epsilon(\cdot)$ to continuous $\hat{T}(\cdot)$ and $\hat{X}^\epsilon(\cdot)$ to continuous $\hat{X}(\cdot)$ and (3.10) yield the assertion. A similar proof yields the analogous assertion for the quadratic variation terms.

The last sentence of the theorem is proved in the same way that (5.4) in [4] is proved, via use of the 'martingale method', and we only do one case. Let $h(\cdot)$ be an arbitrary real valued, bounded and continuous function of its arguments and for arbitrary n, let $t_i \le t \le t+s$, $i \le n$. Define $\hat{P}^\epsilon(t) = (\hat{X}^\epsilon(t), \hat{J}^{12,\epsilon}(t), \hat{J}^{21,\epsilon}(t), \hat{A}^{i,\epsilon}(t), \hat{A}^{0i,\epsilon}(t), \hat{D}^{i,\epsilon}(t), \hat{T}^\epsilon(t), i = 1,2)$. Let ϵ index a weakly convergent subsequence of $\{\hat{P}^\epsilon(\cdot), \epsilon > 0\}$. It can be shown that

$$Eh(\hat{P}^\epsilon(t_i), i \le n)[\hat{A}^{i,\epsilon}(t+s) - \hat{A}^{i,\epsilon}(t)] = 0.$$

This last expression can be shown either by the ideas leading to (5.4) in [4], or by a direct calculation using the definition of the conditional expectation $E^{i,\epsilon}_{a,n}$ and the fact that the summands in $\hat{A}^{i,\epsilon}(\cdot)$ are centered about their conditional expectations, given the 'past'. By the weak convergence and the fact that $\sup_\epsilon E[\hat{A}^{i,\epsilon}(t)]^2 < \infty$ for each $t < \infty$, we have

$$Eh(\hat{P}(t_i), i \le n)[\hat{A}^i(t+s) - \hat{A}^i(t)] = 0.$$

The arbitrariness of $h(\cdot)$, t_i, n, t, $t+s$, implies that

$$E[\hat{A}(t+s) - \hat{A}(t) \mid \hat{P}(u),\ u \le t] = 0,$$

which yields the assertion. Q.E.D.

The inversion of $\hat{T}(\cdot)$. We next deal with the inversion of the time rescaling $\hat{T}(\cdot)$, to get the appropriate 'limits' of the original sets of processes in (2.8). Whether this 'inversion' can be done or not depends on the controls. Clearly if all arrivals at P_0 are rerouted, then for each $t > 0$, $T^\epsilon(t) \to \infty$ as $\epsilon \to 0$ and

$\hat{T}(t) \equiv 0$ and no inversion is possible. However, since the costs associated with this policy go to infinity as $\epsilon \to 0$, such cases can be excluded. It will turn out that for the controls of practical interest, the inversion can be done.

Lemma 4. *Assume* (A2.1)–(A2.5) *and that*

$$\sup_{\epsilon} \sqrt{\epsilon} E \sum_{m=1}^{t/\epsilon} [\rho_m^{12} + \rho_m^{21}] < \infty \tag{3.11}$$

for each $t < \infty$. Then $\hat{T}(t) < \infty$ w.p.1 *for each $t < \infty$ and $\hat{T}(t) \to \infty$* w.p.1, *as $t \to \infty$.*

The proof is easy and is omitted.

For each $t > 0$, define the random variable

$$T(t) = \min\{\tau : \hat{T}(\tau) = t\}.$$

The set $\{T(s), s < \infty\}$ are $\hat{\mathcal{F}}_t$-stopping times, since $\{T(t) \leq \tau\} = \{\hat{T}(\tau) \leq t\} \in \hat{\mathcal{F}}_\tau$ for all τ. Define the σ-algebras $\mathcal{F}_t = \hat{\mathcal{F}}_{T(t)}$. For any process $\hat{\phi}(\cdot)$, define the rescaled process $\phi(\cdot)$ by $\phi(t) = \hat{\phi}(T(t))$, except let $\tilde{A}^\alpha(\cdot)$ and $\tilde{D}^\alpha(\cdot)$ denote $\hat{A}^\alpha(T(\cdot))$ and $\hat{D}^\alpha(T(\cdot))$, resp. Then $\mathcal{F}_t$ is the minimal σ-algebra induced by $\{P(s), s \leq t\} = \{\hat{P}(T(s)), s \leq t\}$.

Theorem 5. *Assume* (A2.1)–(A2.5) *and* (3.11). *Then*

$$X^i(t) = X^i(0) + B^i(t) + [\tilde{A}^i(t) + \tilde{A}^{0i}(t) - \tilde{D}^i(t)] + Y^i(t)$$

$$- U^i(t) + J^{ji}(t) - J^{ij}(t). \tag{3.12}$$

The $Y^i(\cdot)$ and $U^i(\cdot)$ increase only when $X^i(t) = 0$ ($X^i(t) = B_i$, resp.). *The martingales are all $\mathcal{F}_t$-martingales. The quadratic variations are given by* (3.9) *with $\hat{T}(t)$ replaced by t and $\hat{X}(\cdot)$ by $X(\cdot)$.*

The proof is just a consequence of Theorem 3, Lemma 4 and the properties of the $T(t)$. The details are omitted.

Remarks on the representation of the martingales. Since the five processes $\tilde{A}^i(\cdot)$, $\tilde{D}^i(\cdot)$, $i = 1,2$, and $(\tilde{A}^{01}(\cdot), \tilde{A}^{02}(\cdot))$ are mutually orthogonal martingales, we can represent them as stochastic integrals with respect to mutually independent Wiener processes $w_{ai}(\cdot)$, $w_{di}(\cdot)$. If the $\sigma_{\alpha\beta}$ are never zero (which we have assumed for convenience in this paper), then the $w_{\alpha i}(\cdot)$ are all $\mathcal{F}_t$-Wiener processes. Otherwise, we need to augment the probability space and filtration by adding Wiener processes which are independent of all the processes originally defined on the probability space. We can write the martingales in the form

$$\begin{aligned}
\tilde{A}^i(t) &= g_{ai}^{3/2}\int_0^t \sigma_{ai}(X(s))dw_{ai}(s) = \int_0^t \Sigma_{ai}^{1/2}(X(s))dw_{ai}(s), i = 1,2. \\
\tilde{D}^i(t) &= g_{di}^{3/2}\int_0^t \sigma_{di}(X(s))dw_{di}(s) = \int_0^t \Sigma_{di}^{1/2}(X(s))dw_{di}(s). \qquad (3.13) \\
\begin{pmatrix} A^{01}(t) \\ A^{02}(t) \end{pmatrix} &= \int_0^t \Sigma_{a0}^{1/2}(X(s))dw_{a0}(s).
\end{aligned}$$

If the $\{J^{12,\epsilon}(\cdot), J^{21,\epsilon}(\cdot), \epsilon > 0\}$ is tight, then the time change $t \to \hat{T}^\epsilon(t)$ is not needed, and one can work directly with the *original processes* $X^\epsilon(\cdot), \ldots$. We will next give a result for this case which will be useful below. First, we define some new process by a normalization of the summands in the expressions $\tilde{A}_0^{i,\epsilon}(\overline{S}_a^{i,\epsilon}(t))$, $\tilde{A}_0^{0i,\epsilon}(\overline{S}_a^{0,\epsilon}(t))$ and $\tilde{D}_0^{i,\epsilon}(\overline{S}_d^{i,\epsilon}(t))$ appearing in (2.6). These new processes will actually converge weakly to the Wiener processes $w_{\alpha\beta}(\cdot)$. Define

$$W_{ai}^\epsilon(t) = \sqrt{\epsilon}\sum_{m=1}^{\overline{S}_a^{i,\epsilon}(t)/\epsilon}\left[\Sigma_{ai}(X_{S_{a,m}^i}^\epsilon)\right]^{-1/2}(1 - \alpha_m^i/\overline{\alpha}_m^i), \quad i = 1,2,$$

$$W_{di}^\epsilon(t) = \sqrt{\epsilon}\sum_{m=1}^{\overline{S}_d^{i,\epsilon}(t)/\epsilon}\left[\Sigma_{di}(X_{S_{d,m}^i}^\epsilon)\right]^{-1/2}(1 - \Delta_m^i/\overline{\Delta}_m^i), \quad i = 1,2,$$

$$W_{a0}^{\epsilon}(t) = \sqrt{\epsilon} \sum_{m=1}^{\overline{S}_{*}^{0,\epsilon}(t)/\epsilon} \left[\Sigma_{a0}(X_{S_{*,m}^{0}}^{\epsilon}) \right]^{-1/2} \left\{ \begin{array}{l} I_m^{1,\epsilon} - \overline{p}_1 \alpha_m^0 / \overline{\alpha}_m^0 \\ I_m^{2,\epsilon} - \overline{p}_2 \alpha_m^0 / \overline{\alpha}_m^0 \end{array} \right\}.$$

Theorem 6. *Assume* (A2.1) *to* (A2.5) *and suppose that* $\{J^{12,\epsilon}(\cdot),\ J^{21,\epsilon}(\cdot),$ $\epsilon > 0\}$ *is tight. Then (note that we pair the two components of* $W_{a0}^{\epsilon}(\cdot)$*)*

$$\{X^{\epsilon}(\cdot), Y^{\epsilon}(\cdot), U^{\epsilon}(\cdot), \tilde{A}^{i,\epsilon}(\cdot), \tilde{A}^{0i,\epsilon}(\cdot), \tilde{D}^{i,\epsilon}(\cdot), W_{ai}^{\epsilon}(\cdot), W_{di}^{\epsilon}(\cdot),\ \textit{all } i\}$$

is tight. Let ϵ *index a weakly convergent subsequence and denote the limits by the same letters, but without the* ϵ *superscript. Let* $\mathcal{F}_t$ *be the minimal* σ*-algebra which measures the limit process for* $s \le t$. *Then the* $W_{\alpha}(\cdot)$ *are mutually independent standard* $\mathcal{F}_t$*-Wiener processes, and*

$$\begin{aligned} \tilde{A}^{i}(t) &= \int_0^t \Sigma_{ai}^{1/2}(X(s))dW_{ai}(s), \\ \tilde{A}^{0}(t) &= \int_0^t \Sigma_{a0}^{1/2}(X(s))dW_{a0}(s), \\ \tilde{D}^{i}(t) &= \int_0^t \Sigma_{d0}^{1/2}(X(s))dW_{d0}(s). \end{aligned} \tag{3.15}$$

Also (3.12) *holds.*

Proof. It is easy to show the 'Wiener process' result, owing to the centering of the summands and the normalization by the inverse square root of the covariance. The rest is as for Theorem 3, except for the representation (3.15). This can be obtained by using the tightness and a discrete time approximation, and the details are omitted. Q.E.D.

4. Boundedness and Approximation to $V^\epsilon(x)$.

First, we show that there is a control for which the costs are uniformly bounded.

Theorem 7. *Assume* (A2.1) *to* (A2.5), *and let* $V^\epsilon(x,0)$ *denote the cost when* $J^{12,\epsilon}(t) = J^{21,\epsilon}(t) \equiv 0$. *Then*

$$\sup_{\epsilon,x} E_x V^\epsilon(x,0) < \infty.$$

Proof. It is enough to prove that

$$\sup_{\epsilon,n,i} E[U^{i,\epsilon}(n+1) - U^{i,\epsilon}(n)] < \infty.$$

Define $M^{i,\epsilon}(t) = [\tilde{A}^{i,\epsilon}(t) + \tilde{A}^{0i,\epsilon}(t) - \tilde{D}^{i,\epsilon}(t)]$. We let $i = 1$, since the proof is the same for $i = 2$. For an integer n, define the stopping times (omitting the n and ϵ-dependence in the notation)

$$\tau_1 = \min\{t \geq n : X^{1,\epsilon}(t) = B_1\},$$

$$\tau_{2m} = \min\{t > \tau_{2m-1} : X^{1,\epsilon}(t) \leq B_1/2\} \cap (n+1),$$

$$\tau_{2m+1} = \min\{t > \tau_{2m} : X^{1,\epsilon}(t) = B_1\} \cap (n+1).$$

Define $N_n^\epsilon = \min\{m : \tau_{2m} = n+1\}$. Recall that $U^{1,\epsilon}(\cdot)$ can increase only on the intervals $[\tau_{2m-1}, \tau_{2m}]$ and not on (τ_{2m}, τ_{2m+1}). Then

$$\begin{aligned} U^{1,\epsilon}(n+1) - U^{1,\epsilon}(n) = \sum_{m=1}^{N_n^\epsilon+1} [(X^{1,\epsilon}(\tau_{2m}) - X^{1,\epsilon}(\tau_{2m-1})) \\ - (M^{1,\epsilon}(\tau_{2m}) - M^{1,\epsilon}(\tau_{2m-1})) - (B^{1,\epsilon}(\tau_{2m}) - B^{1,\epsilon}(\tau_{2m-1}))]. \end{aligned} \tag{4.1}$$

By (4.1) and the square integrable martingale property of $M^{i,\epsilon}(\cdot)$ and the Lipschitz continuity property of $B^{i,\epsilon}(\cdot)$, there is a constant K_0 such that

$$E|U^{1,\epsilon}(n+1) - U^{1,\epsilon}(n)| \leq K_0 + E(N_n^\epsilon + 1)B_1. \tag{4.2}$$

Thus, to prove the theorem, we only need bound EN_n^ϵ, uniformly in n and ϵ.

Given $\alpha_0 > 0$, there is $\delta_0 > 0$ such that for all bounded stopping times and for small ϵ

$$P\left\{\sup_{\tau+\delta_0 \geq s \geq \tau} \sup[B^{1,\epsilon}(s) + M^{1,\epsilon}(s) - (B^{1,\epsilon}(\tau) + M^{1,\epsilon}(\tau))] \right.$$

$$\left. \geq \frac{B_1}{2} \,|\, \text{data up to } \tau \right\} \leq 1 - \alpha_0. \tag{4.3}$$

This implies that

$$P\{\tau_{2m} - \tau_{2m-1} \geq \delta_0 \,|\, \text{data up to time } \tau_{2m-1}\} \geq \alpha_0. \tag{4.4}$$

Consider the problem of a sequence of 'Bernoulli' trials, where the conditional probability of success, given the past data, is $\geq \alpha_0$ and on each success 'time' advances by δ_0. An upper bound for our EN_n^ϵ is just the mean number of trials that are needed to have $1/\delta_0 = n_1$ (the next largest integer) successes. Since the mean number of required trials is monotonic in the (conditional) probability of success, we get an upper bound by assuming that (4.4) is an equality. Then

$$P\{k \text{ trials needed}\} = \begin{pmatrix} k \\ n_1 \end{pmatrix} (1-\alpha_0)^{k-n_1} \alpha_0^{n_1}$$

which implies that *all* moments of N_n^ϵ are bounded, uniformly in n and ϵ. Q.E.D.

We remark that the proof and the uniform square integrability of the increments in $M^\epsilon(\cdot)$ and $B^\epsilon(\cdot)$ (on unit intervals) implies that

$$\sup_{\epsilon,n} E|U^{i,\epsilon}(n+1) - U^{i,\epsilon}(n)|^2 < \infty. \tag{4.5}$$

The following corollary will be useful later. It is just a consequence of Theorem 7, the structure of the cost and the discounting. Define $V^\epsilon(x) = \inf_{J^\epsilon} V^\epsilon(x, J^\epsilon)$.

Corollary 8. *Assume* (A2.1)–(A2.5). *Given* $\delta > 0$, *there are* $T_0 > 0$ *and a family of* δ*-optimal controls* $J_\delta^\epsilon(\cdot)$ *such that* $J_\delta^\epsilon(\cdot)$ *do not change after time* T_0 (*i.e., after* T_0, *there is no rerouting*).

A very similar proof to that of Theorem 7 yields the following:

Theorem 9. *Assume* (A2.1) *to* (A2.5). *If*

$$\sup_\epsilon E[J^{ij,\epsilon}(t+T) - J^{ij,\epsilon}(T)] < \infty, \quad i \neq j, \ i = 1, 2. \tag{4.6}$$

Then

$$\sup_\epsilon E[U^{i,\epsilon}(t+T) - U^{i,\epsilon}(T)] < \infty.$$

If

$$\{J^{ij,\epsilon}(t+T) - J^{ij,\epsilon}(T), \epsilon > 0\}, i \neq j, \ i = 1, 2, \tag{4.7}$$

is uniformly integrable for each t, *then so is* $\{U^{i,\epsilon}(t+T) - U^{i,\epsilon}(T), \ \epsilon > 0\}$, $i = 1, 2$.

5. The Limit Control Problem

Definition. $J(\cdot) = (J^{12}(\cdot), J^{21}(\cdot))$ is said to be an *admissible control* for the limit controlled reflected diffusion (3.12) if it is non-anticipative with respect to the set of Wiener processes $W(\cdot) = (w_{ai}(\cdot),\ w_{di}(\cdot),\ i = 1,2,\ w_{a0}(\cdot))$ which 'drive' the martingales $(\tilde{A}^i(\cdot),\ \tilde{A}^{0i}(\cdot),\ \tilde{D}^i(\cdot),\ i = 1,2)$ (see the representation (3.13)), and satisfies $J^\alpha(0) = 0$ and $J^\alpha(\cdot)$ is non-decreasing, for $\alpha = 12$ or 21. We often say simply that the pair $(J(\cdot), W(\cdot))$ is admissible. The cost functional for the limit problem is

$$V(x, J, W) = E_x \int_0^\infty e^{-\beta t} k(X(t))dt + E_x \int_0^\infty e^{-\beta t}[k_1 dJ^{12}(t) + k_2 dJ^{21}(t) + c_1 dU^1(t) + c_2 dU^2(t)]. \tag{5.1}$$

The $W(\cdot)$ appears in (5.1) as well as $J(\cdot)$, since the value of the cost function will depend on the *joint distribution* of $(J(\cdot), W(\cdot))$.

Theorem 10. *Assume* (A2.1)–(A2.5) *and that for each* n

$$\sup_{\epsilon,n,i} E[(J^{12,\epsilon}(n+1) - J^{12,\epsilon}(n)) + (J^{21,\epsilon}(n+1) - J^{21,\epsilon}(n))] < \infty. \tag{5.2}$$

Let ϵ index a weakly convergent subsequence of (3.5), (3.6) *with limit denoted by* $(\hat{T}(\cdot), \ldots)$. *Let the retransformed processes defined above and in Theorem* 5 *be denoted by* $(T(\cdot), \ldots)$. *Then*

$$\underline{\lim}_\epsilon V^\epsilon(x, J^\epsilon) \geq V(x, J, W), \tag{5.3}$$

where $W(\cdot) = (w_{ai}(\cdot), w_{di}(\cdot), i = 1,2)$ is the Wiener process which is used to represent the martingales (see (3.13)). *If*

$$\{J^{12,\epsilon}(n+1) - J^{12,\epsilon}(n), J^{21,\epsilon}(n+1) - J^{21,\epsilon}(n), \epsilon > 0, n < \infty\} \tag{5.4}$$

is uniformly integrable, then

$$V^\epsilon(x, J^\epsilon) \to V(x, J, W). \tag{5.5}$$

Proof. The hypothesis (5.2) implies that $\inf_\epsilon E\hat{T}^\epsilon(t) \to \infty$ and $E\hat{T}(t) \to \infty$ as $t \to \infty$. Thus, the 'inverse' transformation $T(\cdot)$ is well defined. It also implies that we need only work on a finite interval (see Corollary 8). For simplicity, we work with only a couple of the terms of the cost functional. We have

$$\int_0^\infty e^{-\beta t} k(X^\epsilon(t))dt = \int_0^\infty e^{-\beta \hat{T}^\epsilon(t)} k(\hat{X}^\epsilon(t))d\hat{T}^\epsilon(t),$$

$$\int_0^\infty e^{-\beta t} dJ^{12,\epsilon}(t) = \int_0^\infty e^{-\beta \hat{T}^\epsilon(t)} d\hat{J}^{12,\epsilon}(t). \tag{5.6}$$

By the weak convergence and the argument of Theorem 3, the right sides of (5.6) converge in distribution to the left sides of

$$\int_0^\infty e^{-\beta \hat{T}(t)} k(\hat{X}(t))d\hat{T}(t) = \int_0^\infty e^{-\beta t} k(X(t))dt,$$

$$\int_0^\infty e^{-\beta \hat{T}(t)} d\hat{J}^{12}(t) = \int_0^\infty e^{-\beta t} dJ^{12}(t). \tag{5.7}$$

The left sides of (5.7) equal the right sides of (5.7) by the rescaling. The theorem follows from the cited convergences (together with those for the other components of the cost) and Fatous' Lemma. Q.E.D.

Theorems 3 and 5 imply that every limit of a weakly convergent subsequence is a legitimate control problem in the sense that the pair $(J(\cdot), W(\cdot))$ which occurs in the representation of the limit is admissible. This fact and Theorem 10 imply the following.

Theorem 11. *Assume* (A2.1)–(A2.5). *Let* $J^{12,\epsilon}(\cdot)$, $J^{21,\epsilon}(\cdot)$ *denote the optimal controls for the physical process. Define*

$$V^\epsilon(x) = \inf_{J^\epsilon} V^\epsilon(x, J^\epsilon), \quad V(x) = \inf_{(J,W)\,\mathrm{adm.}} V(x, J, W).$$

Then

$$\underline{\lim}_\epsilon V^\epsilon(x) \geq V(x). \tag{5.8}$$

Remark. We note that (5.2) can be assumed in Theorem 11. If it doesn't hold for the optimal policy, for each $\delta > 0$ it will hold for the *δ-optimal policy*, owing to the discounting and Corollary 8. We want to prove that (5.8) is an equality. To get the equality, we will need to use the fact that $V^\epsilon(x)$ is actually an *optimal* cost. In order to do this, we need first to study approximations to the control problem for the limit model (3.12), (5.1). We will show that there is an optimal policy for the limit, and that it can be approximated by a policy that we can apply to the $X^\epsilon(\cdot)$ process, and which will be 'recovered' under the weak convergence. Such results will get us the desired equality in (5.8) (Theorem 17), together with a basis for an effective computational approximation. The computational methods and associated proofs will be dealt with in a subsequent paper.

6. Approximations for the Limit Problem, and Convergence of the Costs

In order to prove equality in (5.8), we first establish the existence of an optimal policy for (3.12), (5.1), and then obtain a sequence of approximations to the optimal control. We will use the following assumption.

A6.1. *$k(\cdot)$, $b_i(\cdot)$, $\sigma_{ia}(\cdot)$, $\sigma_{id}(\cdot)$ are continuous.*

Theorem 12. *Assume* (A6.1). *Consider the limit control problem* (3.12), (5.1). *There is an optimal policy $\overline{J}(\cdot)$ in the sense that there is $(\overline{X}(\cdot), \overline{J}(\cdot), \overline{W}(\cdot), \ldots)$ satisfying* (3.12), *where $\overline{W}(\cdot) = \{\overline{w}_{ai}(\cdot), \overline{w}_{di}(\cdot), i = 1, 2\}$ 'drives' the martingales $\tilde{A}^i(\cdot), \ldots$ as in* (3.13) *and the pair $(\overline{J}(\cdot), \overline{W}(\cdot))$ is admissible and*

$$V(x, \overline{J}, \overline{W}) \leq V(x, J, W),$$

for all admissible pairs $(J(\cdot), W(\cdot))$.

Proof. The proof is very similar to those of Theorems 3 and 5, and we make only a few comments. Let $(J^n(\cdot), W^n(\cdot))$ be an admissible pair for (3.12), and write the corresponding form of (3.12) as $(j \neq i)$

$$X^{i,n}(t) = X_0^i + B^{i,n}(t) + [\tilde{A}^{i,n}(t) + \tilde{A}^{0i,n}(t) - \tilde{D}^{i,n}(t)]$$
$$+ [J^{ij,n}(t) - J^{ji,n}(t)] + Y^{i,n}(t) - U^{i,n}(t), \tag{6.1}$$

where $B^{i,n}(t) = \int_0^t b_i(X^{i,n}(s))ds$ and $W^n(\cdot) = \{w_{ai}^n(\cdot), w_{di}^n(\cdot), i = 1, 2\}$ 'drives' the martingales $\tilde{A}^{i,n}(\cdot), \ldots$, as in (3.13). Let $(J^n(\cdot), W^n(\cdot))$ be a minimizing sequence in that $V(x, J^n, W^n) \downarrow V(x)$.

By Theorems 7 and 10, $V(x) < \infty$. Hence

$$\sup_n E[J^{12,n}(t) + J^{21,n}(t)] < \infty,$$

for each $t < \infty$. Define the time change

$$T^n(t) = t + J^{12,n}(t) + J^{21,n}(t),$$

and the inverse $\hat{T}^n(t) = \min\{\tau : T^n(\tau) = t\}$. Analogous to the notation used in Theorem 3, define $\hat{X}^n(\cdot) = X^n(\hat{T}^n(\cdot)), \ldots$. Then

$$\hat{X}^{i,n}(t) = X_0^i + \hat{B}^{i,n}(t) + [\hat{A}^{i,n}(t) + \hat{A}^{i0,n}(t) - \hat{D}^{i,n}(t)]$$

$$+ \hat{J}^{ij,n}(t) - \hat{J}^{ji,n}(t) + \hat{Y}^{i,n}(t) - \hat{U}^{i,n}(t).$$

As in Theorem 3, there is a function $F_0(\cdot)$ which maps $C^k[0,\infty)$ into $C^k[0,\infty)$, for the appropriate integer k and is continuous in the topology of uniform convergence on bounded time intervals and is such that for all n

$$(\hat{Y}^n(\cdot), \hat{U}^n(\cdot)) = F_0(X_0, \hat{A}^{i,n}(\cdot), \hat{A}^{i0,n}(\cdot), \hat{D}^{i,n}(\cdot), \hat{B}^{i,n}(\cdot), \hat{J}^n(\cdot), i = 1,2).$$

The set $\{\hat{T}^n(\cdot), \hat{X}^n(\cdot),\ \hat{J}^n(\cdot),\ \hat{A}^{i,n}(\cdot),\ \hat{A}^{i0,n}(\cdot),\ \hat{D}^{i,n}(\cdot),\ \hat{B}^{i,n}(\cdot),\ i = 1,2,\ n < \infty\}$ is tight. Abusing notation, let n index a weakly convergent subsequence with limit denoted by $(\hat{T}(\cdot),\ \hat{X}(\cdot), \ldots)$. As in Theorem 3 the $(\hat{A}^1(\cdot),\ \hat{A}^2(\cdot),\ (\hat{A}^{10}(\cdot), \hat{A}^{20}(\cdot)),\ \hat{D}^1(\cdot),\ \hat{D}^2(\cdot))$ are orthogonal continuous martingales with quadratic variation defined by (3.9). Define the inverse scaling $T(t) = \min\{\tau : \hat{T}(\tau) = t\}$, and the rescaled processes $X(t) = \hat{X}(T(t)), \ldots$. Then (3.12) holds, and the martingales have the representation (3.13) with respect to some Wiener process $W(\cdot) = (w_{ai}(\cdot),\ w_{di}(\cdot),\ i = 1,2)$ such that the pair $(J(\cdot), W(\cdot))$ is admissible. By an argument which is almost identical to that of Theorem 10, we have

$$\underline{\lim}_n V(x, J^n, W) \geq V(x, J, W). \tag{6.2}$$

We must have equality in (6.2) since $V(x, J^n, W^n) \downarrow V(x)$. Thus, $(J(\cdot),\ W(\cdot))$ is an optimal admissible pair. Q.E.D.

integrable, and so is $\{U^{i,N}(n+1) - U^{i,N}(n),\ n < \infty,\ N < \infty,\ i = 1,2\}$ by Theorem 13. Since $\tau_N \uparrow \infty$, we can suppose that

$$(X_0^N(\cdot), J_0^N(\cdot), W_0^N(\cdot)) \to (X_0(\cdot), J_0(\cdot), W_0(\cdot))$$

pathwise. The theorem follows from this convergence, the cited uniform integrability and an argument similar to that in Theorem 10. Q.E.D.

Definition. A solution $X(\cdot)$ to (3.12), (3.13) is said to be *unique in the weak sense* if the distribution of the admissible pair $(J(\cdot),\ W(\cdot))$ determines that of $(J(\cdot),\ W(\cdot),\ X(\cdot))$.

In order to obtain our approximation results we require that for each $\delta > 0$, there is a δ-optimal control which gives a well defined solution of (3.12), (3.13).

A6.2. *For each $\delta > 0$, there is a δ-optimal control $J(\cdot)$ for which $(J(\cdot), W(\cdot))$ is admissible for some $W(\cdot)$ and the corresponding solution $X(\cdot)$ to* (3.12), (3.13) *is unique in the weak sense.*

In the next theorem, we show that there is a δ-optimal control which is bounded, piecewise constant, and jumps 'in increments'.

Theorem 15. *Assume* (A6.1) *to* (A6.2) *and let $(J(\cdot),\ W(\cdot))$ be a δ-optimal pair satisfying* (A6.2), *with $X(\cdot)$ denoting the corresponding solution process. For $\Delta > 0$ and $\rho > 0$, define the control $J_{\Delta\rho}(\cdot)$ as the piecewise constant control satisfying: $dJ_{\Delta\rho}^{ij}(t) = 0$ on the interval $(n\Delta, n\Delta + \Delta)$ and on $[0, \Delta)$. For $k \geq 0$ and $n \geq 1$, set $dJ_{\Delta\rho}^{ij}(n\Delta) = k\rho$ if $J^{ij}(n\Delta) - J^{ij}(n\Delta - \Delta) \in [k\rho, k\rho + \rho)$. Then*

$$\lim_{\Delta,\rho} V(x, J_{\Delta\rho}, W) = V(x, J, W). \tag{6.4}$$

Proof. By Theorem 13, we can suppose that $J(\cdot)$ is uniformly bounded. By construction, $(J_{\Delta\rho}(\cdot),\ W(\cdot))$ is an admissible pair. A solution to (3.12), The following lemma, whose proof is similar to that of Theorems 7 and 9, will be useful later.

Theorem 13. *Assume* (A6.1), *and let* $(J_n(\cdot),\ W_n(\cdot))$ *be admissible, with* $X_n(\cdot)$, $Y_n(\cdot)$ *and* $U_n(\cdot)$, *be the associated state and reflection process. If*

$$\{J_n^{ij}(t+T) - J_n^{ji}(T), n < \infty\}, j \neq i, \quad j = 1, 2,$$

is uniformly integrable, then so is

$$\{U_n^{i}(t+T) - U_n^{i}(T), n < \infty\}, j \neq i, \quad j = 1, 2.$$

Theorem 14. *Assume* (A6.1) *and, for small* $\delta > 0$ *let* $(J_0(\cdot),\ W_0(\cdot))$ *be a* δ*-optimal admissible pair, with* $X_0(\cdot)$, *being the associated solution to* (3.12). *Define* $\tau_N = \sup\{t : J_0^{12}(t) \leq N,\ J_0^{21}(t) \leq N\}$, *and let* $J^N(\cdot)$ *be the policy which equals* $J_0(\cdot)$ *until* τ_N, *and is constant thereafter. Write the solution to* (3.12) *as*

$$X^{i,N}(t) = X^i(0) + B^{i,N}(t) + [\tilde{A}^{i,N}(t) + \tilde{A}^{0i,N}(t) - \tilde{D}^{i,N}(t)]$$

$$+ Y^{i,N}(t) - U^{i,N}(t) + J^{ji,N}(t) - J^{ij,N}(t).$$

Let $W_0^N(\cdot)$ *be the set of Wiener processes which 'drives' the martingales* $(\tilde{A}^{i,N}(\cdot),\ \ldots)$. *Then, as* $N \to \infty$,

$$V(x, J_0^N, W_0^N) \to V(x, J_0, W_0). \tag{6.3}$$

Proof. We can suppose w.l.o.g. that there is a $T < \infty$ such that $J_0(\cdot)$ is constant after T (by an argument similar to that leading to Corollary 8). Since $J^{i,N}(T) \uparrow J_0^i(T)$, and $EJ_0^i(T) < \infty$, the $\{J_0^N(T),\ N < \infty\}$ is uniformly integrable, and so is $\{U^{i,N}(n+1) - U^{i,N}(n),\ n < \infty,\ N < \infty,\ i = 1, 2\}$ by Theorem 13. Since $\tau_N \uparrow \infty$, we can suppose that

$$(X_0^N(\cdot), J_0^N(\cdot), W_0^N(\cdot)) \to (X_0(\cdot), J_0(\cdot), W_0(\cdot))$$

and reflection processes. The set

$$\{X_{\Delta\rho}(\cdot), J_{\Delta\rho}(\cdot), W(\cdot), U_{\Delta\rho}(\cdot), Y_{\Delta\rho}(\cdot), \Delta > 0, \rho > 0\} \tag{6.5}$$

is tight and the weak limits all satisfy (3.12), (3.13). By the uniqueness (A6.2), the limit of any weakly convergent subsequence of the set (6.5) satisfies (3.12), (3.13). Then (6.4) follows from the weak convergence and the boundedness of $J(\cdot)$ and the consequent uniform integrability of $\{U^i_{\Delta\rho}(n+1) - U^i_{\Delta\rho}(n),\, n < \infty,$ $\Delta > 0,\, \rho > 0,\, i = 1,2\}$. Q.E.D.

For $\Delta > 0$, $\rho > 0$, let $\mathcal{T}_{\Delta\rho}$ denote the set of admissible (with respect to some given Wiener process $W(\cdot)$) controls which are bounded, are constant on each interval $[n\Delta, n\Delta + \Delta)$, jump only at the times $n\Delta$, and $J^{ij}(n\Delta) - J^{ij}(n\Delta^-)$ is an integral multiple of ρ. By Theorem 14 and (A6.2), we know that for each $\delta > 0$ there are $\Delta > 0$, $\rho > 0$ such that there is a δ-optimal control in some $\mathcal{T}_{\Delta\rho}$. We will need to define this control in such a way that it can be used for the physical $X^\epsilon(\cdot)$ process.

Write $k = (k_1, k_2)$, a multi-index, where k_i is either an integer or 0. Fix the Wiener process $W(\cdot)$ and Δ and ρ. For $J(\cdot) \in \mathcal{T}_{\Delta\rho}$, $(J(\cdot), W(\cdot))$ is admissible. For $\gamma > 0$ and integers k and n, define $q_{nk\gamma}(\cdot)$ by

$$\begin{aligned} & q_{nk\gamma}(J(m\Delta), m < n, W(\ell\gamma), \ell\gamma \le n\Delta) \\ = \;& P\{dJ(n\Delta) = k\rho \,|\, J(m\Delta), m < n, W(\ell\gamma), \ell\gamma \le n\Delta\}. \end{aligned} \tag{6.6}$$

By the martingale convergence theorem, as $\gamma \to 0$

$$q_{nk\gamma}(J(m\Delta), m < n, W(\ell\gamma), \ell\gamma \le n\Delta) \to$$

$$P\{dJ(n\Delta) = k\rho \mid J(m\Delta), m < n, W(s), s \le n\Delta\}$$

w.p.1 (Wiener measure) for each k, n and value of $\{J(m\Delta), m < n\}$.

For each $\gamma > 0$, we next choose a control $J_\gamma(\cdot) \in \mathcal{T}_{\Delta\rho}$ recursively by means of the following set of conditional probabilities:

$$P\{dJ_\gamma(n\Delta) = k\rho \mid J_\gamma(m\Delta), m < n, W(s), s \le n\Delta\} =$$

$$= q_{nk\gamma}(J_\gamma(m\Delta), m < n, W(\ell\gamma), \ell\gamma \le n\Delta). \tag{6.7}$$

(6.7) specifies the joint law of admissible $(J_\gamma(\cdot), W(\cdot))$, and there is a solution $X(\cdot)$ on some sample space which is associated with a pair with the same distribution as $(J_\gamma(\cdot), W(\cdot))$.

We will need the following condition.

A6.3. *The uncontrolled system* $(J(t) \equiv 0)$ *has a unique (in the weak sense) solution for each intial condition.*

Theorem 16. *Assume* (A6.1)–(A6.3). *Let* $(J(\cdot), W(\cdot))$ *be admissible with* $J(\cdot) \in \mathcal{T}_{\Delta\rho}$ *for some* $\Delta > 0$, $\rho > 0$. *Define* $J_\gamma(\cdot)$ *as above. Then*

$$V(x, J_\gamma, W) \to V(x, J, W). \tag{6.8}$$

The function $q_{nk\gamma}(J_\gamma(m\Delta), m < n, \cdot)$ *which is used to get* $J_\gamma(\cdot)$ *can be chosen to be continuous for each* n, k, γ, *and values of the set* $\{J_\gamma(m\Delta), m < n\}$.

Proof. The proof of (6.8) follows from the weak convergence $\{J_\gamma(\cdot), W(\cdot), \gamma > 0\} \Rightarrow (J(\cdot), W(\cdot))$, as $\gamma \to 0$ and the uniform boundedness of the controls. By (A6.3), the solution to (3.12), (3.13) is unique in the weak sense for any admissible $J(\cdot)$ in $\mathcal{T}_{\Delta\rho}$. The last sentence of the theorem follows from the fact that for each n, k, γ, and value of $\{J_\gamma(m\Delta), m < \Delta\}$, we can approximate $q_{nk\gamma}(J_\gamma(m\Delta),\ m < n, \cdot)$ by a sequence of continuous distribution functions, which converge to $q_{nk\gamma}(J_\gamma(m\Delta),\ m < n, \cdot)$ w.p.1 (Wiener measure). Q.E.D.

The optimality theorem. We now return to the physical process (2.8), and prove equality in (5.8). Let $(J_\gamma(\cdot), W(\cdot))$ be admissible with $J_\gamma(\cdot)$ chosen by (6.7), where the $q_{nk\gamma}(\cdot)$ have the continuity property asserted in Theorem 15. Recall the definition of $W^\epsilon(\cdot)$ given above Theorem 6.

We would like to define a control $J^\epsilon(\cdot)$ for $X^\epsilon(\cdot)$ such that $\{J^\epsilon(\cdot), W^\epsilon(\cdot)\}$ converges weakly to $(J_\gamma(\cdot), W(\cdot))$ as $\epsilon \to 0$. First, consider the control $\tilde{J}^\epsilon(\cdot)$ defined as follows, where the $q_{nk\gamma}(\cdot)$ are continuous in the w-arguments: $\tilde{J}^\epsilon(\cdot)$ is constant on the intervals $[n\Delta, n\Delta + \Delta)$ and

$$
\begin{aligned}
&P\{d\tilde{J}^\epsilon(n\Delta) = k\rho \mid \tilde{J}^\epsilon(m\Delta), m < n, W^\epsilon(s), s \leq n\Delta\} \\
=\ & P\{d\tilde{J}^\epsilon(n\Delta) = k\rho \mid \tilde{J}^\epsilon(m\Delta), m < n, W^\epsilon(i\gamma), i\gamma \leq n\Delta\} \qquad (6.9) \\
=\ & q_{nk\gamma}(\tilde{J}^\epsilon(m\Delta), m < n, W^\epsilon(i\gamma), i\gamma \leq n\Delta).
\end{aligned}
$$

The control law (6.9) can't quite be realized for $X^\epsilon(\cdot)$, since the controls for $X^\epsilon(\cdot)$ are the result of rerouting decisions and $X^\epsilon(\cdot)$ *cannot* be impulsively controlled. But we can come close enough to realizing the above $\tilde{J}^\epsilon(\cdot)$, as follows.

For notational simplicity, let the $k_i\rho$, $i = 1, 2$, be integral multiples of $\sqrt{\epsilon}$. Let $\Delta_\epsilon \to 0$ as $\epsilon \to 0$ such that $\Delta_\epsilon/\sqrt{\epsilon} \to \infty$. Let Q^n_ϵ denote the event that there are $\geq (B_1 + B_2)/\sqrt{\epsilon}$ arrivals at P_0 on $[n\Delta, n\Delta + \Delta_\epsilon)$. We have $P\{Q^n_\epsilon\} \to 1$ as $\epsilon \to 0$. Define $J^\epsilon(\cdot)$ to be any control with the following properties: $J^\epsilon(\cdot)$ is constant on $[n\Delta + \Delta_\epsilon,\ n\Delta + \Delta)$ and on $[0, \Delta)$; for $n > 0$, the rerouting $[n\Delta, n\Delta + \Delta_\epsilon)$ is such that

$$P\{J^\epsilon(n\Delta + \Delta_\epsilon) - J^\epsilon(n\Delta) = k\rho \,|\, J^\epsilon(m\Delta), m < n, W^\epsilon(s), s \le n\Delta, Q_\epsilon^n\}$$

$$= \; P\{J^\epsilon(n\Delta + \Delta_\epsilon) - J^\epsilon(n\Delta) = k\rho \,|\, J^\epsilon(m\Delta), m < n, W^\epsilon(i\gamma), i\gamma \le n\Delta, Q_\epsilon^n\}$$

$$= \; q_{nk\gamma}(J^\epsilon(m\Delta), m < n, W^\epsilon(i\gamma), i\gamma \le n\Delta). \tag{6.10}$$

The limit of the costs associated with this just constructed $J^\epsilon(\cdot)$ is the same as if the jumps of $J^\epsilon(\cdot)$ are at the time $n\Delta + \Delta_\epsilon, n = 1, 2, \ldots,$ only and not spreadout over $[n\Delta, n\Delta + \Delta_\epsilon)$. This can be easily proved by the "time charge method". Under this "new" $J^\epsilon(\cdot), \{J^\epsilon(\cdot), W^\epsilon(\cdot)\}$ clearly converges weakly to $(J_\gamma(\cdot), W(\cdot))$ as $\epsilon \to 0$. Now, the above discussion and Theorems 6, 10 and 11 yield the following theorem, where we use the $J^\epsilon(\cdot)$ just described. Note that the rescalings are not necessary, due to the fixed form of $J^\epsilon(\cdot)$; we get weak convergence directly in the Skorohod topology.

Theorem 17. *Assume* (A2.1)–(A2.5) *and* (A6.1)–(A6.3). *Then* $\{X^\epsilon(\cdot)$, $Y^\epsilon(\cdot)$, $U^\epsilon(\cdot)$, $W^\epsilon(\cdot)$, $J^\epsilon(\cdot)\}$ *converges weakly to* $(X(\cdot)$, $Y(\cdot)$, $U(\cdot)$, $W(\cdot)$, $J(\cdot))$, *where* $W(\cdot) = (w_{ai}(\cdot), w_{di}(\cdot), i = 0, 1, 2)$ *and the limit satisfies* (3.12), (3.13). *Also*

$$V^\epsilon(x, J^\epsilon) \to V(x, J_\gamma, W), \tag{6.11}$$

$$V^\epsilon(x) \to V(x). \tag{6.12}$$

Remarks. In [4], it was shown that certain forms of nearly optimal controls for the limit process were also nearly optimal for the physical process under heavy traffic. A similar situation holds here, and this partly justifies the use of the heavy traffic limit, but we reserve the comments for a future paper on numerical methods.

Suppose that $k_1 > 0$ but $k_2 < 0$ and $|k_2| < k_1$. Then a very similar analysis can be carried out, with similar results. The costs $V^{\epsilon}(x)$ can be bounded from below since the profit to be made by rerouting from P_1 to P_2 is bounded, due to the limited idle time at P_2.

7. A More General Network Model

The general controlled routing open network version of Figure 1 can be treated, for any number of servers. Because of the notational burden involved in writing all the possible 'rerouting terms', we give the extension only to the model of Figure 2, which differs from Figure 1 only in that feedback is allowed. We continue to use the notation of the previous sections, except for the following additions. Let $I_n^{ij,\epsilon}$ be the indicator of the event that a service completed at P_i $(i \neq 0)$ at real time n is routed to P_j $(j \neq i)$ if $j \neq 0$, and leaves the network if $j = 0$. The input from P_j to P_i is

$$D^{ji,\epsilon}(t) = \sqrt{\epsilon} \sum_{m=1}^{t/\epsilon} \psi_m^j I_m^{ji}, \quad j = 1,2, \; i = 0,1,2.$$

$D^{j0,\epsilon}(t)$ denotes the scaled number of outputs of P_j which leave the system directly. The 'fictitious' outputs from P_j (which are due to our convention of P_j 'processing' even with an empty queue) and which are sent to P_i are

$$Y^{ji,\epsilon}(t) = \sqrt{\epsilon} \sum_{m=1}^{t/\epsilon} \psi_m^j I_m^{ji} I_{\{X_m^{j,\epsilon}=0\}}.$$

The overflow due to a full buffer at P_i is

$$U^{i,\epsilon}(t) = \sqrt{\epsilon} \sum_{m=1}^{t/\epsilon} [\xi_m^i + \xi_m^0 (I_m^i + I_m^j \rho_m^{ji} - I_m^i \rho_n^{ij}) + \psi_m^j I_m^{ji}] I_{\{X_m^{i,\epsilon}=B_i\}}.$$

Then, for $j \neq i$, $j \neq 0$,

$$\begin{aligned} X^{i,\epsilon}(t) = A^{i,\epsilon}(t) + A^{i0,\epsilon}(t) - D^{i0,\epsilon}(t) - D^{ij,\epsilon}(t) + D^{ji,\epsilon}(t) + Y^{i,\epsilon}(t) - Y^{ji,\epsilon}(t) \\ - U^{i,\epsilon}(t) + J^{ji,\epsilon}(t) - J^{ij,\epsilon}(t). \end{aligned} \tag{7.1}$$

We continue to use the cost functional (2.3).

Replace (A2.3) and (A2.4) by

A2.3′. *There are* $\overline{p}_{ij} < 1$ *such that* ($\overline{p}_{0i}$ *replaces the* $\overline{p}_i$ *of* (A2.3))

$$P\{I_n^{ij,\epsilon} = 1 \mid \textit{all arrival and service intervals and routings starting by time } n, \textit{ except for } I_n^{ij,\epsilon}\} = \overline{p}_{ij}$$

A2.4′. $g_{di} = g_{ai} + \overline{p}_{0i} g_{a0} + \overline{p}_{ji} g_{dj},\ j \neq i,\ i, j \neq 0.$

In analogy to the definitions of the centered processes $\tilde{A}_0^{i,\epsilon}(\cdot)$, $\tilde{D}_0^{i,\epsilon}(\cdot)$ given in (2.5), define ($j \neq 0$)

$$\tilde{D}_0^{ji,\epsilon}(t) = \sqrt{\epsilon} \sum_{m=1}^{t/\epsilon} [I^{ji}_{S^i_{d,m}} - \overline{p}_{ji} \Delta^j_m / \overline{\Delta}^j_m].$$

Define the centered reflection process

$$\tilde{Y}^{ji,\epsilon}(t) = \sqrt{\epsilon} \sum_{m=1}^{t/\epsilon} \psi^j_m (I^{ji}_m - \overline{p}_{ji}) I_{\{X^{j,\epsilon}_m = 0\}}.$$

Define $\tilde{D}^{ij}(t) = \tilde{D}_0^{ij,\epsilon}(\overline{S}_d^{i,\epsilon}(t))$. By the same method which was used to get (2.8), we can write (7.1) in the form ($i \neq 0$, $j \neq 0$, $j \neq i$)

$$X^{i,\epsilon}(t) = X^{i,\epsilon}(0) + B^{i,\epsilon}(t) + \tilde{A}^{i,\epsilon}(t) + \tilde{A}^{i0,\epsilon}(t) - \tilde{D}^{i0,\epsilon}(t) - \tilde{D}^{ij,\epsilon}(t)$$

$$+ \tilde{D}^{ji,\epsilon}(t) + Y^{i,\epsilon}(t) - \overline{p}_{ji} Y^{j,\epsilon}(t) - \tilde{Y}^{ji,\epsilon}(t) - U^{i,\epsilon}(t) + [J^{ji,\epsilon}(t) - J^{ij,\epsilon}(t)] + \delta^{i,\epsilon}(t), \tag{7.2}$$

where $\delta^{i,\epsilon}(\cdot)$ is as in (2.7).

Theorem 17. *Assume* (A2.1), (A2.2), (A2.3′), (A2.4′), (A2.5). *Then the five sets of processes (we pair as* (A^{01}, A^{02}), (D^{10}, D^{12}), (D^{20}, D^{21}))

$$\{\tilde{A}^{1,\epsilon}(\cdot), \tilde{A}^{2,\epsilon}(\cdot), (\tilde{A}^{01,\epsilon}(\cdot), \tilde{A}^{02,\epsilon}(\cdot)), (\tilde{D}^{10,\epsilon}(\cdot), \tilde{D}^{12,\epsilon}(\cdot)), (\tilde{D}^{20,\epsilon}(\cdot), \tilde{D}^{21,\epsilon}(\cdot)), \epsilon > 0\} \tag{7.3}$$

are tight. The limits are continuous martingales. All of Theorems 2 *to* 16 *hold, with the obvious changes necessitated by the additional terms in* (7.3). *The limit* *reflected diffusion is*

$$X^i(t) = X^i(0) + \tilde{A}^i(t) + \tilde{A}^{0i}(t) - \tilde{D}^{i0}(t) - \tilde{D}^{ij}(t) + \tilde{D}^{ji}(t) - B^i(t)$$

$$+ J^i(t) - U^i(t) + Y^i(t) - \overline{p}_{ji} Y^i(t), \quad i \neq j. \tag{7.4}$$

Also, for $i \neq j$

$$\text{quad var} \begin{pmatrix} \tilde{D}^{i0}(t) \\ \tilde{D}^{ij}(t) \end{pmatrix} = \int_0^t \Sigma_{dij}(X(s))ds,$$

where

$$\Sigma_{dij}(x) = g_{di} \begin{bmatrix} \overline{p}_{i0}(1-\overline{p}_{i0}) & -\overline{p}_{i0}\overline{p}_{ij} \\ -\overline{p}_{i0}\overline{p}_{ij} & \overline{p}_{ij}(1-\overline{p}_{ij}) \end{bmatrix}$$

$$+ g_{di}^3 \sigma_{di}^2(x) \begin{bmatrix} 1 & \overline{p}_{i0}\overline{p}_{ij} \\ \overline{p}_{i0}\overline{p}_{ij} & 1 \end{bmatrix} \tag{7.5}$$

Proof. All the details are copies of what was done in Theorems 2 to 16, except for the treatment of the $\tilde{Y}^{ij,\epsilon}(\cdot)$ term, and some details in Theorem 7. Define $\hat{Y}^{ij,\epsilon}(t) = \tilde{Y}^{ij,\epsilon}(\hat{T}^\epsilon(t))$. The summands in the $\hat{Y}^{ij,\epsilon}(\cdot)$ are centered about their conditional expectations, given the 'past', and hence $\hat{Y}^{ij,\epsilon}(\cdot)$ is a martingale sum. Its variance is bounded by

$$\epsilon E \sum_{m=0}^{\hat{T}^\epsilon(t)/\epsilon} I_{\{X_m^{i,\epsilon}=0\}} = O(t). \tag{7.6}$$

Since the summands (without the $\sqrt{\epsilon}$ included) in $\hat{Y}^{ij,\epsilon}(\cdot)$ are uniformly square integrable, $\{\hat{Y}^{ij,\epsilon}(\cdot), \epsilon > 0\}$ is tight, and all weak limits are continuous processes. Write the scaled form of (7.2) as

$$\hat{X}^{i,\epsilon}(t) \equiv X_0^{i,\epsilon} + \hat{Z}^{i,\epsilon}(t) + [\hat{Y}^{i,\epsilon} - \overline{p}_{ji}\hat{Y}^{j,\epsilon}(t)] - \hat{U}^{i,\epsilon}(t) \tag{7.7}$$

with the obvious definition of $\hat{Z}^{i,\epsilon}(\cdot)$. $\{\hat{Z}^{i,\epsilon}(\cdot), \epsilon > 0\}$ is tight and all weak limits are continuous processes. Thus, by Theorem 1, $\{\hat{Y}^{i,\epsilon}(\cdot), \hat{U}^{i,\epsilon}(\cdot), \epsilon > 0\}$ is tight and all weak limits are continuous processes. This implies that for each $t < \infty$,

$$\sqrt{\epsilon} \sum_{m=0}^{\hat{T}^{\epsilon}(t)/\epsilon} I_{\{X_m^{1,\epsilon}=0\}}$$

is bounded in probability uniformly in ϵ, for otherwise some subsequence of $\hat{Y}^{i,\epsilon}(t)$ would go to infinity with a positive probability. Hence the left side of (7.6) goes to zero as $\epsilon \to 0$ for each $t < \infty$. Thus $\hat{Y}^{ij,\epsilon}(\cdot) \Rightarrow$ zero process.

Theorem 7 also continues to hold, since in the present case we write (4.1) as

$$[U^{1,\epsilon}(n+1) + Y^{21,\epsilon}(n+1)] - [U^{1,\epsilon}(n) + Y^{21,\epsilon}(n)] = \text{right side of (4.1)}.$$

The left hand side of (4.3) now becomes

$$p\{\sup_{\tau+\delta_0 \ge s \ge \tau} [(B^{1,\epsilon}(s)+M^{1,\epsilon}(s)-Y^{21,\epsilon}(s))-(B^{1,\epsilon}(\tau)+M^{1,\epsilon}(\tau)-Y^{21,\epsilon}(\tau))] \ge \delta_0 \mid \text{data up to } \tau\}.$$

Since $Y^{21,\epsilon}(\cdot)$ is non-decreasing, the expression is still $\le 1-\alpha_0$ for small enough δ_0, and we can continue the proof of Theorem 7. Q.E.D.

8. Proof of Theorem 1

For notational simplicity, we prove the theorem for $k = 2$ and then comment on the general case. The proof in the general case is the same in all essentials. The following result is proved in [1], [2].

Lemma. *Let P be a degenerate Markov transition matrix whose spectral radius is less than unity. Then there is a unique 'non-anticipative' function $\tilde{F}(\cdot)$ with the following properties: $\tilde{F}(\cdot)$ maps $D^k[0,\infty)$ into $D^k[0,\infty)$ and $C^k[0,\infty)$ into $C^k[0,\infty)$, and is continuous in the topology of uniform convergence on bounded intervals. Let $\tilde{z}(\cdot) \in D^k[0,\infty)$ and define $\tilde{y}(\cdot) = (\tilde{y}^i(\cdot),\ i \leq k) = \tilde{F}(\tilde{z}(\cdot))$. Define $\tilde{x}(t) = \tilde{z}(t) + (I - P')\tilde{y}(t)$. The $\tilde{y}^i(\cdot)$ are non-decreasing and $\tilde{y}^i(\cdot)$ can increase only when the $\tilde{x}^i(t) = 0$. Also $\tilde{x}^i(t) \geq 0$, all i, t.*

To prove Theorem 1, we will use the lemma in a 'sequential' way. Refer to Figure 3. We can assume w.l.o.g., that the diagonal entries in Q in (3.1) are zero. There are four different reflection maps which appear in (3.1), depending on which segment of the boundary is involved. On the boundary $(d,a,b) \equiv$ segment 1, our system (3.1) is

$$x(t) = z(t) + \begin{bmatrix} 1 & -q_{21} \\ -q_{12} & 1 \end{bmatrix} y(t), \quad x^i(t) \geq 0. \tag{8.1}$$

For the system (8.1), with the constraint $x^i(t) \geq 0$, the lemma defines a continuous mapping $z(\cdot) \to (y(\cdot), u(\cdot))$, where $u(\cdot) \equiv 0$. Call this mapping $F_1(\cdot)$. On the other segments, the system is

$$x(t) = z(t) + \begin{bmatrix} 0 & -q_{21} \\ 0 & 1 \end{bmatrix} y(t) - \begin{pmatrix} u^1(t) \\ 0 \end{pmatrix}, \text{ segment } 2 = (a,b,c), \tag{8.2}$$

$$x(t) = z(t) - \begin{pmatrix} u^1(t) \\ u^2(t) \end{pmatrix}, \text{ segment } 3 = (b,c,d) \tag{8.3}$$

$$x(t) = z(t) + \begin{bmatrix} 0 & 0 \\ -q_{12} & 0 \end{bmatrix} y(t) - \begin{pmatrix} 0 \\ u^2(t) \end{pmatrix}, \text{ segment } 4 = (c, d, a). \quad (8.4)$$

The reflection maps for (8.2) to (8.4) (with the sides extended to ∞) are trivial, as we will now see, since they are each just concatenations of two one dimensional applications of the lemma. Let $F_2(\cdot)$ denote the map associated with (8.2) which sends $z(\cdot)$ into $(y(\cdot),\ u(\cdot))$. $F_2(\cdot)$ is constructed as follows. First, we have $y^1(\cdot) = u^2(\cdot) = 0$. Then $y^2(\cdot)$ is defined by the lemma for $k = 1$; in particular $y^2(t) = -\min\{0, \inf_{s \le t} z^2(s)\}$. Finally, the $u^1(\cdot)$ is defined by the reflection needed to keep $x^1(t) \le B_1$; i.e., $0 \le B_1 - x^1(t)$ or

$$u^1(t) = -\min\left\{0, \inf_{s \le t}(B_1 - z^1(s) + q_{21}y^2(s))\right\}. \quad (8.5)$$

Similarly, we define the analogous continuous map $F_3(\cdot)$ and $F_4(\cdot)$ associated with (8.3) and (8.4) resp. [The calculation of the $y(\cdot)$ and $u(\cdot)$ always decouples into two separate calculations; first getting $y(\cdot)$ and then getting $u(\cdot)$, even for the general k case as seen below.]

Define $S = [0, B_1] \times [0, B_2]$. Let $z(0) \in S$ w.l.o.g. Define $x(\cdot)$, $y_n(\cdot)$, $u_n(\cdot)$, $z_n(\cdot)$, $\delta y_n(\cdot)$, $\delta u_n(\cdot)$, and τ_n as follows: $\tau_0 = 0$, $z_0(\cdot) = z(\cdot)$, $\delta u_0(\cdot) = \delta y_0(\cdot) = 0$, $\tau_1 = \inf\{t : z(t) \notin S\}$, $u(t) = y(t) = 0$ on $[0, \tau_1]$ and $x(t) = x_0(t) = z(t)$ on $[0, \tau_1]$. In general, for $n \ge 1$, given τ_n, $x_{n-1}(\cdot)$ and $y(\cdot)$ and $u(\cdot)$ on $[0, \tau_n]$, define

$$z_n(t) = z(t) + [I - Q']y(t \cap \tau_n) - u(t \cap \tau_n),$$

$$s_n = \text{a boundary segment } (1, 2, 3 \text{ or } 4) \text{ on which } x_{n-1}(\tau_n) \text{ lies},$$

$$(\delta u_n(\cdot), \delta y_n(\cdot)) = F_{s_n}(z_n(\cdot)), \quad (8.6)$$

$$x_n(t) = z_n(t) + [I - Q']\delta y_n(t) - \delta u_n(t),$$

$$x(t) = x_n(t) \quad \text{for} \quad t \le \tau_{n+1},$$

$$\tau_{n+1} = \inf\{t : x_n(t) \notin S\},$$

$$u(t) = u(\tau_n) + \delta u_n(t), \quad \text{for } t \in [\tau_n, \tau_{n+1}],$$

$$y(t) = y(\tau_n) + \delta y_n(t), \quad \text{for } t \in [\tau_n, \tau_{n+1}].$$

Note that $z_n(t) \in S$ until at least time τ_n and $x_n(t) \in S$ until at least time τ_{n+1}. Hence $(\delta u_n(t), \delta y_n(t)) = 0$ until at least τ_n. Also $x_n(t) = z_n(t)$, $t \le \tau_n$, $z_{n+1}(t) = z_n(t)$ on $[0, \tau_{n+1}]$, $x_{n+1}(t) = x_n(t)$ on $[0, \tau_{n+1}]$. The idea in constructing $x_n(\cdot)$ is that when $x_n(\cdot)$ exits S on a certain boundary segment, we use the reflection map (F_1, F_2, F_3 or F_4, as appropriate) for that segment to get $x_{n+1}(\cdot)$, until $x_{n+1}(\cdot)$ exits S. It must exit on a 'different' segment. $F_{s_n}(\cdot)$ can be any map associated with the boundary segment on which the exit point $x_{n-1}(\tau_n)$ lies. Except for in the corner points, there are two such maps associated with each point on the boundary. Which map is chosen is immaterial. For definiteness, choose $F_1(\cdot)$ on $[e, a, f]$, $F_2(\cdot)$ on $(f, b, g]$, $F_3(\cdot)$ on $(g, c, h]$ and $F_4(\cdot)$ on (h, d, e). We can verify that $\tau_n \to \infty$ and (by induction using the lemma) that the constructed $x(\cdot)$, $y(\cdot)$, $u(\cdot)$ satisfy Theorem 1. To see that the choice of the map which is used at the points f, g, h, e is immaterial, let $x_{n-1}(\tau_n) = f$. Then $F(z_n(\cdot)) = F_2(z_n(\cdot))$ until the infimum of the times that $x_n(\cdot)$ leaves S through $[b, c, d, a]$. An induction argument, based on this observation shows that the choices at the points f, g, h, e, is immaterial.

Remark on the general $k > 2$ case. There is always a decomposition of the construction of $(\delta y_n(\cdot), \delta u_n(\cdot))$ into the two sequential steps: first calculate $\delta y_n(\cdot)$ via the lemma, for a reduced system; then calculate the $\delta u_n^i(\cdot)$ individually via an appropriate analog of (8.5). Just to illustrate this point, consider the case where the space is $S = [0, B]^k$, and focus on the faces of S meeting in the corner $(B, B, 0, 0, \ldots, 0)$. On these faces (excluding the edges which do not touch $(B, B, 0, 0, \ldots, 0)$), (3.1) is

$$x(t) = z(t) + \begin{bmatrix} 0 & 0 & -q_{31} & \cdots & -q_{k1} \\ 0 & 0 & -q_{32} & \cdots & -q_{k2} \\ & & & & \\ 0 & & \tilde{I} - \tilde{Q}' & & \end{bmatrix} y(t) - \begin{pmatrix} u^1(t) \\ u^2(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where $\tilde{I} - \tilde{Q}'$ is a reduced transition matrix. Then first get $(y^2(\cdot), \ldots, y^k(\cdot))$ from the lemma, and then define (for $i = 1, 2$)

$$u^i(t) = -\min\left\{0, \inf_{s \le t}(B - z^i(t) + \sum_{j=3}^{k} q_{ji} y^j(t))\right\}.$$

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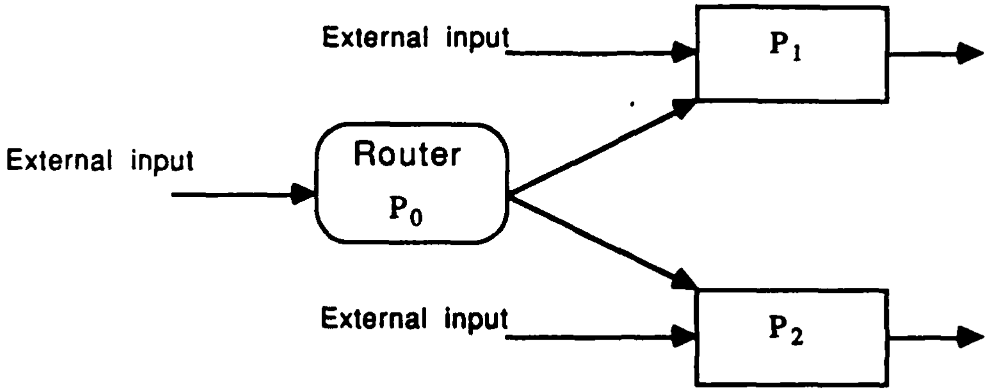

Figure 1.
A Simple Routing Problem

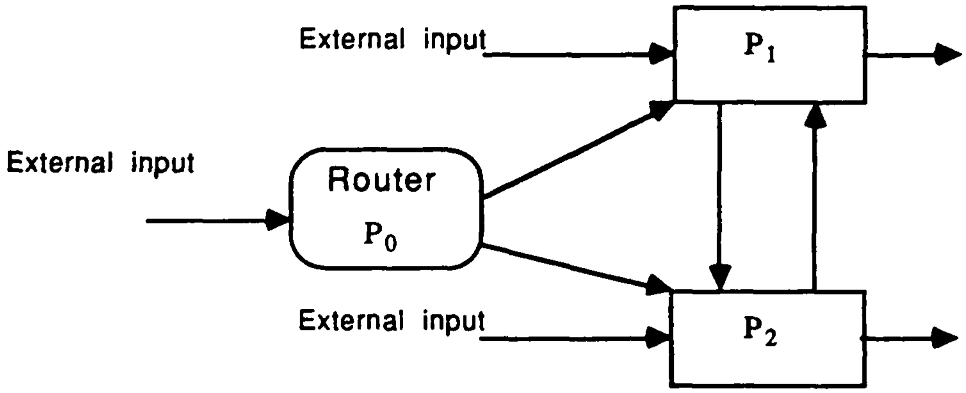

Figure 2
A Routing Problem With Feedback

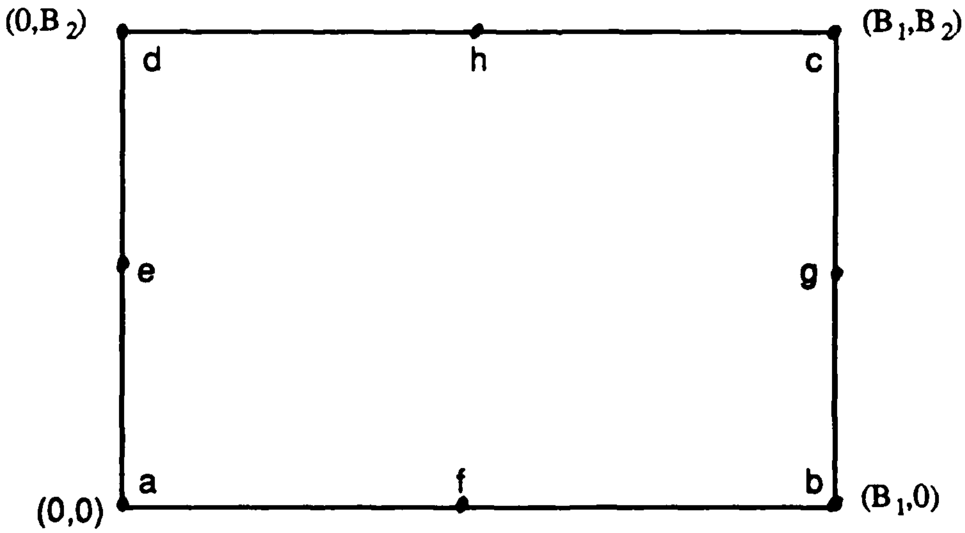

Figure 3
Boundary Sections For The Proof Of Theorem 1