# ROW-COLUMN DESIGNS WITH VARYING REPLICATIONS

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## 1. INTRODUCTION

In many situations in agricultural, animal science, forestry and bioassay experimentation, the investigator needs an 'ad hoc' design which eliminates two sources of variation and also wants to study some important comparisons with higher precision than the rest of the comparisons, or wants to assign unequal replications to various treatments. In bioassay the preparation contrast and the combined regression contrast being the most important contrasts need to be estimated with full efficiency. Likewise in plant development programmes one set of treatments (control treatments) is on a different footing from the rest of the treatments and thus two sets of treatments need to be replicated unequally. The RC-designs obtained in this paper are most suitable for these situations.

Let $d(v, b, \mathbf{r}, \mathbf{k})$ denote the block design for v treatments arranged in b blocks of sizes $k_1, k_2, ..., k_b$ such that the ith treatment is replicated $r_i$ ($i$=1,2,...,$v$) times. Let $\mathbf{r} = (r_1, r_2, ..., r_v)'$ and $\mathbf{k} = (k_1, k_2, ..., k_b)'$ denote the replication vector and the block size vector, let $\mathbf{R} = \text{diag}\,(r_1, r_2, ..., r_v)$ and $\mathbf{K} = \text{diag}\,(k_1, k_2, ..., k_b)$ be the diagonal matrices of order $v \times v$ and $b \times b$ respectively and let $\mathbf{N} = (n_{ij})_{v\times b}$ be the $v \times b$ incidence matrix of the design $d$ where $n_{ij}$ is the number of times the ith treatment occurs in the jth block.

Calinski (1971) defines the $\mathbf{M}_o$ matrix for the block designs as

$$\mathbf{M}_o = \mathbf{R}^{-1}\,\mathbf{N}\mathbf{K}^{-1}\,\mathbf{N}' - \mathbf{1}\,\mathbf{r}'/n$$

where $\mathbf{1}$ is the vector of ones and gives the iterative procedure to find the inverse of $\mathbf{\Omega}^{-1}$ matrix defined by Tocher(1952) [$\mathbf{\Omega}^{-1} = \mathbf{R} - \mathbf{N}\mathbf{K}^{-1}\mathbf{N}' + \mathbf{r}\mathbf{r}'/n$] as

$$\mathbf{\Omega} = \mathbf{R}^{-1} + \sum_{h=1}^{\infty} \mathbf{M}_o^h\ \mathbf{R}^{-1}$$

where $\mathbf{\Omega}$ is the pseudo variance covariance matrix of the estimates of the treatment effects. Once the $\mathbf{\Omega}$ matrix is known the best linear estimate of the treatment contrast $\mathbf{p}'\tau$ is given by $\mathbf{\Omega Q}$ and the adjusted sum of the squares attributed to the treatment effects is $\mathbf{Q}'\mathbf{\Omega Q}$ where $\mathbf{Q}$ is the column vector of adjusted treatments totals. Here $\tau$ is the v-component vector of the treatment effects and $\mathbf{p}$ is the v × 1 vector satisfying $\mathbf{p'1}=\mathbf{0}$.

On the lines of Tocher (1952) and Calinski (1971) for the block designs Pearce (1975) gives the expressions for the $\mathbf{M}_o$ and $\mathbf{\Omega}^{-1}$ matrices of the RC-designs having orthogonal rows vs columns classification as

$$\mathbf{M}_o = \mathbf{R}^{-1} [\mathbf{N}_1\mathbf{N}_1'/q + \mathbf{N}_2\mathbf{N}_2'/p] - 2\, \mathbf{1r}'/n \tag{1.1}$$

$$\mathbf{\Omega}^{-1} = \mathbf{R} - \mathbf{N}_1\mathbf{N}_1'/q - \mathbf{N}_2\mathbf{N}_2'/p + 2\, \mathbf{rr}'/n \tag{1.2}$$

where $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{W}$ are the treatments vs rows, treatments vs columns and rows vs columns incidence matrices of orders $v \times p$, $v \times q$ and $p \times q$ respectively of the Row-Column design $d$ ($v$, $p$, $q$, $\mathbf{r}$) obtained by arranging $v$ treatments in a $p \times q$ array. If $\mathbf{W} = \mathbf{J}_{p\times q}$; $\mathbf{J}_{p\times q}$ being the $p \times q$ matrix with each element equal unity, then the RC-design has rows vs columns classification orthogonal and then $n = pq$ is the total number of units.

If $p = q$ and $\mathbf{N}_1 = \mathbf{N}_2$ then the $\mathbf{M}_o$ matrix further simplifies to

$$\mathbf{M}_o = 2\mathbf{R}^{-1} / n\, [p\mathbf{N}_1\, \mathbf{N}_1' - \mathbf{rr}'] \tag{1.3}$$

Singh and Dey (1978) and Pal (1980) have shown that if the $\mathbf{M}_o$ matrix of an RC-design (1.1) has spectral decomposition $\mathbf{M}_o = \sum_{i=1}^{m} \mu_i\, \mathbf{L}_i$ where $\mu_i$'s are the eigenvalues of the $\mathbf{M}_o$ matrix and $\mathbf{L}_i$'s are the mutually orthogonal idempotent matrices of ranks $\rho_i$ ($i$ = 1, 2, ..., $m$). then the RC-design is a partially efficiency balanced Row-Column design (PEB RC-design) with m efficiency classes for which all the $\rho_i$ contrast of the ith class $\left[\sum_{i=1}^{m} \rho_i = v - 1\right]$ are estimated with efficiency $1-\mu_i$ (Puri and Nigam, 1977), Further if $\mu_i$ takes only two values $\mu$ and 0 with multiplicities $\rho$ and $v - \rho - 1$ respectively then the design is called Simple partially efficiency balanced Row-Column design (SPEB RC-design) (Puri and Nigam, 1977). For such designs the $\mathbf{M}_o$ matrix (1.1) takes the form $\mathbf{M}_o = \mu\mathbf{L}$ and the pseudo variance-covariance matrix $\mathbf{\Omega}$ of the estimated treatment effects is given by

$$\mathbf{\Omega} = [\mathbf{I} + (1\text{-}\mu)^{-1}\, \mathbf{M}_o]\, \mathbf{R}^{-1}.$$

Thus the analysis becomes very simple.

## 2. METHOD OF CONSTRUCTION

Let D be an orthogonal RC-design in the form of a $2v \times 2v$ standard cyclic Latin square. Such designs can always be constructed for all values of $v$ taking the first row and the first column in alphabetical order and developing the remaining rows/columns cyclically as shown in the illustration of section 4.

The RC-design D has the following parameters (indicating by $v'$ the number of treatments)

$v' = p = q = 2v$, $r = 2v$, $n = 4v^2$

It can be easily verified that for all values of $v$, the RC-design $D$ so constructed always have diagonal elements as $\mathbf{1}'_2 \otimes (1, 3, 5, ..., 2v - 1)$ in the $2v$ diagonal positions. Where $\mathbf{1}'_2 = (1,1)$. Replace all the diagonal treatments by the treatment '$2v + 1$' and renumber the odd and even treatments as

$1\rightarrow1, 3\rightarrow2, 5\rightarrow3, ..., 2v-3 \rightarrow v-1, 2v-1\rightarrow v$

$2\rightarrow v+1, 4\rightarrow v+2, 6\rightarrow v+3, ..., 2v\text{-}2\rightarrow 2v-1, 2v\rightarrow 2v$

Let $D^*$ be the resultant RC-design. Then we prove the following theorem.

*Theorem.* The design $D^*$ obtained by replacing the diagonal contents of a $2v \times 2v$ Standard cyclic Latin square design by a new treatment '$2v + 1$' is a simple partially efficiency balanced Row-Column design (SPEB-RC-design) with parameters

$v^* = 2v+1$, $p^* = q^* = 2v$, $\mathbf{r}^* = [2(v-1)\mathbf{1}'_v, 2v\mathbf{1}'_{v+1}]'$

$n^* = 4v^2$, $\mu^*=1/v(v-1)$, $e^* = [v(v-1)-1]/v(v-1)$, $\rho^* = v-1$

$$\mathbf{L}^* = diag.\left[\left(\mathbf{I}_v - \frac{1}{v}\mathbf{J}_v\right), \ \mathbf{0}_{v+1}\right]$$

*Proof*: For the RC-design D* the parameters $v^*$, $p^*$, $q^*$, $\mathbf{r}^*$ and $n^*$ are obvious and need no explanation while the treatments vs rows and the treatments vs columns incidence matrices are

$$\mathbf{N}_1^* = \mathbf{N}_2^* = \begin{bmatrix} \mathbf{J}_v - \mathbf{I}_v & \mathbf{J}_v - \mathbf{I}_v \\ \mathbf{J}_{(v+1)xv} & \mathbf{J}_{(v+1)xv} \end{bmatrix} = \mathbf{1}'_2 \otimes \begin{bmatrix} \mathbf{J}_v - \mathbf{I}_v \\ \mathbf{J}_{(v+1)xv} \end{bmatrix}$$

which implies

$$\mathbf{N}_1^*\mathbf{N}_1^{*\prime} = \mathbf{N}_2^*\mathbf{N}_2^{*\prime} = \begin{bmatrix} 2(\mathbf{J}_v - \mathbf{I}_v)(\mathbf{J}_v - \mathbf{I}_v) & 2(\mathbf{J}_v - \mathbf{I}_v)\mathbf{J}_{v(v+1)} \\ 2\mathbf{J}_{(v+1)v}(\mathbf{J}_v - \mathbf{I}_v) & 2v\mathbf{J}_{(v+1)}) \end{bmatrix}$$

and $\mathbf{r}^* = [2(v-1)\mathbf{1}_v', 2v\,\mathbf{1}'_{v+1}]'$

implies

$$\mathbf{r}^*\mathbf{r}^{*\prime} = \begin{bmatrix} 4(v-1)2\ \mathbf{J}_v & 4v(v-1)\ \mathbf{J}_{vx(v+1)} \\ 4v(v-1)\ \mathbf{J}_{(v+1)xv} & 4v2\ \mathbf{J}_{(v+1)} \end{bmatrix}$$

Since for an orthogonal RC-design $D^*$ we have $\mathbf{N}_1^* = \mathbf{N}_2^*$, and $p^* = q^*$ the $Mo^*$ matrix of the RC-design $D^*$ is of the form (1.3).

After simplifying the $\mathbf{M}_o^*$ matrix can be put in the form

$$\mathbf{M}_0^* = \begin{bmatrix} \mathbf{M}_{11}* & \mathbf{M}_{12}* \\ \mathbf{M}_{21}* & \mathbf{M}_{22}* \end{bmatrix}$$

Substituting the values of $\mathbf{R}^*$, $\mathbf{N}_1^*$ $\mathbf{N}_1^{*\prime}$, $p^*$ and $n^*$ in (1.3) and simplifying each component matrix of $\mathbf{M}_o^*$ separately we get

$$\boldsymbol{M}_0^* = [1/v(v-1)] \begin{bmatrix} \mathbf{I}_v - \frac{1}{v}\mathbf{J}_v & \mathbf{0}_{v\times(v+1)} \\ \mathbf{0}_{(v+1)\times v} & \mathbf{0}_{(v+1)\times(v+1)} \end{bmatrix}$$

which is of the form $\mathbf{M}_o^* = \mu^* \mathbf{L}^*$ where $\mu^* = 1/v\ (v-1)$ and $\mathbf{L}^* = diag. \left[\left(\mathbf{I}_v - \frac{1}{v}\mathbf{J}_v\right), \mathbf{0}_{v+1}\right]$. This completes the proof.

## 3. BASIC CONTRASTS

Let $\{\mathbf{S}_i\}$ $(i = 1, 2, ..., v-1)$ be a set of v-1 mutually orthogonal contrasts. Then the $v-1$ contrasts corresponding to the v deleted treatments from the original design $D$

$$\begin{bmatrix} \mathbf{S}_i \\ \mathbf{0}_{(v+1)\times 1} \end{bmatrix} \quad i = 1, 2, ..., v-1$$

are estimated with loss $\mu = 1/v\ (v-1)$ or with efficiency factor $e = 1 - \mu = \{v\ (v-1)\text{-}1\}/v\ (v-1)$, while the contrasts corresponding to the remaining $v$ treatments and the intergroup contrast are estimated with full efficiency.

## 4. ILLUSTRATION

For $v = 4$ consider the following RC-design $D$ as the 8 × 8 standard cyclic Latin Square with parameters

$v' = 2v = p = q = 8$, $r = 8$, $n = 64$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1** | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | **3** | 4 | 5 | 6 | 7 | 8 | 1 |
| 3 | 4 | **5** | 6 | 7 | 8 | 1 | 2 |
| 4 | 5 | 6 | **7** | 8 | 1 | 2 | 3 |
| 5 | 6 | 7 | 8 | **1** | 2 | 3 | 4 |
| 6 | 7 | 8 | 1 | 2 | **3** | 4 | 5 |
| 7 | 8 | 1 | 2 | 3 | 4 | **5** | 6 |
| 8 | 1 | 2 | 3 | 4 | 5 | 6 | **7** |

It may be noted that diagonal elements of above RC-design D are $\mathbf{1}'_2 \otimes (\mathbf{1}, \mathbf{3}, \mathbf{5}, \mathbf{7})$.

By replacing all the diagonal elements (**1**, **3**, **5**, **7**, **1**, **3**, **5**, **7**) by the treatment number '**9**' and renumbering the treatments as follows:

$1 \to 1 \quad 3 \to 2 \quad 5 \to 3 \quad 7 \to 4$
$2 \to 5 \quad 4 \to 6 \quad 6 \to 7 \quad 8 \to 8$ and $9 \to 9$

we get the following resultant RC-design $D^*$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **9** | 5 | 2 | 6 | 3 | 7 | 4 | 8 |
| 5 | **9** | 6 | 3 | 7 | 4 | 8 | 1 |
| 2 | 6 | **9** | 7 | 4 | 8 | 1 | 5 |
| 6 | 3 | 7 | **9** | 8 | 1 | 5 | 2 |
| 3 | 7 | 4 | 8 | **9** | 5 | 2 | 6 |
| 7 | 4 | 8 | 1 | 5 | **9** | 6 | 3 |
| 4 | 8 | 1 | 5 | 2 | 6 | **9** | 7 |
| 8 | 1 | 5 | 2 | 6 | 3 | 7 | **9** |

with parameters

$$v^* = 9, p^* = q^* = 8, \mathbf{r}^* = [6\mathbf{1}'_4, 8\mathbf{1}'_5]', n^* = 64$$

$$\mu^* = 1/12, e^* = 11/12, \quad \mathbf{L}^* = diag\left[\mathbf{I}_4 - \frac{1}{4}\ \mathbf{J}_4,\ \mathbf{0}_{5\times5}\right]$$

Some of the useful SPEB-RC designs with varying replications that can be constructed from the above series for different values of v are presented in Table 1.

TABLE 1

| $v_1$ | $v_2$ | $r_1^*$ | $r_2^*$ | $p^*$ | $q^*$ | $e^*$ |
|---|---|---|---|---|---|---|
| 2 | 3 | 2 | 4 | 4 | 4 | 0.500 |
| 3 | 4 | 4 | 6 | 6 | 6 | 0.833 |
| 4 | 5 | 6 | 8 | 8 | 8 | 0.916 |
| 5 | 6 | 8 | 10 | 10 | 10 | 0.950 |
| 6 | 7 | 10 | 12 | 12 | 12 | 0.966 |
| 7 | 8 | 12 | 14 | 14 | 14 | 0.976 |
| 8 | 9 | 14 | 16 | 16 | 16 | 0.982 |
| 9 | 10 | 16 | 18 | 18 | 18 | 0.986 |
| 10 | 11 | 18 | 20 | 20 | 20 | 0.988 |
| 11 | 12 | 20 | 22 | 22 | 22 | 0.990 |

## 6. APPLICATIONS IN AGRICULTURAL SCIENCE

Freeman (1975) has discussed the application of such RC-designs in field experiments with one set of treatments as chemicals to control plant diseases and the other set being standard chemicals, while the rows and columns are either two directions in the field or blocks and residual effects of chemicals of previous trial laid in a randomized block design. Federer and Raghavarao (1975) and Federer *et al.* (1975) have discussed the applications of RC-designs in plant development programmes. Hanusz (1995) has applied the RC-designs in the study of relative potency of two preparations eliminating two sources of variation like ages and weights of the subjects as rows and columns. The RC-designs obtained by Mehta and Puri (1990) by deleting a column in Youden Squares, the General efficiency balanced RC- designs by Gupta and Gandhi Prasad (1990), the Latin square type Row- Column designs by Niwas and Mehta (1997), the Augmented Row- Column designs by Mehta and Gupta (1998) and the Supplemented Row- Column designs with varying replications by Kumari *et al.* (2000) also come in the category of RC- designs in two sets of treatments with different replications.

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RIASSUNTO

*Disegni Riga-Colonna con differenti replicazioni*

Sostituendo i trattamenti diagonali di un quadrato latino ciclico standard $2v$ x 2v con un nuovo trattamento '2v+1' è stata costruita una classe di disegni Riga-Colonna in due insiemi di trattamenti aventi differenti replicazioni. I disegni così ottenuti sono stati identificati come "Simple partially efficiency balanced Row-Column designs" (SPEB RC-designs) (Singh and Dey, 1978; Pal, 1980).

SUMMARY

*Row-Column designs with varying replications*

A class of Row-Column designs in two sets of treatments having different replications has been constructed by replacing the diagonal treatments of a $2v$ x 2v standard cyclic Latin Square by a new treatment '2v+1'. The designs so obtained have been identified as Simple partially efficiency balanced Row-Column designs (SPEB RC-designs) (Singh and Dey, 1978; Pal, 1980).