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The RS-IMEX splitting for the isentropic Euler equations

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Abstract. *Approximating solutions to singularly perturbed differential equations necessitates the use of stable integrators. One famous approach is to split the equation into stiff and non-stiff parts. Treating stiff parts implicitly, non-stiff ones explicitly leads to so-called IMEX schemes. These schemes are supposed to exhibit very good accuracy and uniform stability, however, not every (seemingly reasonable) splitting induces a stable algorithm. In this paper, we present a new IMEX-splitting - based on a reference solution (RS) - applied to the isentropic Euler equations.*

Keywords: singular perturbation problems; low Mach number flow; IMEX; flux splitting; von Neumann stability

1 INTRODUCTION

In this work, we consider the isentropic Euler equations at low Mach number ε, given by

$$\rho_t + \nabla \cdot (\rho u) = 0 \tag{1}$$

$$(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \frac{\nabla p(\rho)}{\varepsilon^2} = 0. \tag{2}$$

Pressure p is set as $p(\rho) = \rho^\gamma$, where γ is a constant larger than one. Consequently, (1)-(2) is a conservation law, and it can therefore be written as

$$w_t + \nabla \cdot f(w) = 0 \tag{3}$$

for $w = (\rho, \rho u)^T$. The equations are equipped with suitable initial and boundary conditions that allow for a limit as $\varepsilon \to 0$, so-called *well-prepared* conditions. To simplify the presentation, we only use periodic boundary conditions in this work. The eigenvalues of the Jacobian of f in normal direction $\vec{n}$ are given by

$$\lambda_0 = u \cdot \vec{n}, \qquad \lambda_\pm = u \cdot \vec{n} \pm \frac{1}{\varepsilon}\sqrt{\frac{\gamma p}{\rho}}.$$

It is well-known that for explicit finite volume-type schemes, the time-step restriction depends on the Courant-Friedrichs-Levy (CFL) condition in a way such that $\Delta t \lesssim \varepsilon \Delta x$ which, for $\varepsilon \ll 1$, is an undesirable feature. Fully implicit schemes offer a (partial) remedy, as they usually tend to remove the severe CFL condition. However, these schemes are, at least for low approximation orders, overly diffusive. Therefore, the use of an implicit / explicit (IMEX) mixture has been developed for this type of problem, see, e.g., [1, 2, 3, 4] and the references therein. All these schemes rely on a splitting of the convective function f into a stiff part $\widetilde{f}$ (with eigenvalues of $\mathcal{O}(\varepsilon^{-1})$) and a non-stiff part $\widehat{f}$. The former one is treated implicitly, while the latter one is treated explicitly. However, not every splitting of f yields a stable IMEX scheme, see [5, 6]. Based on our work in [8], we therefore present a new flux splitting for (1)-(2) which is based on what we call the *reference solution*, the (formal) solution for $\varepsilon \to 0$. Expanding all quantities ρ, u and p in (1) in terms of ε (i.e., $\rho = \rho_{(0)} + \varepsilon\rho_{(1)} + \ldots$), one can - given suitable initial

conditions [7] - show that the limit equation is given by

$$\rho_{(0)} \equiv \text{const}, \qquad \nabla \cdot u_{(0)} = 0 \tag{4}$$

$$(u_{(0)})_t + \nabla \cdot (u_{(0)} \otimes u_{(0)}) + \frac{\nabla p_{(2)}}{\rho_{(0)}} = 0. \tag{5}$$

More formally, this has been shown in [7]. Equations (4)-(5) constitutes the incompressible Euler equations. As before, we abbreviate $w_{(0)} = (\rho_{(0)}, \rho_{(0)} u_{(0)})$.

2 SPLITTING

Based on an earlier idea being momentarily under development [9], we consider a splitting of the convective flux f into stiff and non-stiff terms based on the reference solution. For singularly perturbed ODEs, we have already applied this idea successfully in [8].

Definition 1. *RS-IMEX splitting: The* reference-solution *(RS) IMEX splitting for f in* (3) *is given by*

$$\widetilde{f}(w) = f(w_{(0)}) + f'(w_{(0)})(w - w_{(0)}), \qquad \widehat{f}(w) = f(w) - \widetilde{f}(w). \tag{6}$$

Let us note that $\widetilde{f}(w)$ is a linear function in w. In a numerical scheme, this will be beneficial in terms of computing time. It follows from (4) that $\rho_{(0)}$ is a constant, which implies the following simple corollary:

Corollary 1. *For $\widehat{f}$ and $\widetilde{f}$ as defined in Definition 1, there holds*

$$\nabla \cdot \widetilde{f}(w) = \nabla \cdot f'(w_{(0)}) w. \tag{7}$$

Furthermore, the eigenvalues of $\widehat{f}'(w) \cdot \vec{n}$ are given by

$$\widehat{\lambda}_0 = 0, \qquad \widehat{\lambda}_1 = u \cdot \vec{n} - u_{(0)} \cdot \vec{n}, \qquad \widehat{\lambda}_2 = 2 \left(u \cdot \vec{n} - u_{(0)} \cdot \vec{n} \right),$$

in particular, they are independent of ε.

With this being said, the semi-discrete first order IMEX scheme can be written as

$$w_{\Delta t}^{n+1} = w_{\Delta t}^{n} - \Delta t \, \nabla \cdot \left(\widetilde{f}(w_{\Delta t}^{n+1}) + \widehat{f}(w_{\Delta t}^{n}) \right). \tag{8}$$

An important consistency requirement of such a scheme is *asymptoticity preservation*, see [1] for a thorough introduction. Roughly speaking, this means that as $\varepsilon \to 0$, the numerical scheme (8) is a consistent discretization of the $\varepsilon \to 0$ continuous equations. Indeed, the scheme investigated has this property:

Lemma 1. *The semi-discrete IMEX scheme* (8) *is* asymptotic preserving *(AP), i.e., plugging*

$$w_{\Delta t} = w_{\Delta t,(0)} + \varepsilon w_{\Delta t,(1)} + \dots$$

into (8)*, the lowest order terms constitute a consistent discretization of equations* (4)-(5)*.*

Proof. We assume that $w_{\Delta t}^0$ (the initial datum) is such that both $\rho_{\Delta t,(0)}^0$ and $\rho_{\Delta t,(1)}^0$ are constants and that $u_{\Delta t,(0)}^0$ is solenoidal. This makes sense because it is only in this setting that the continuous equations have a limit as $\varepsilon \to 0$, see [7]. In literature, such a choice of initial data is called *well-prepared*. We conclude inductively, showing that w^{n+1} is well-prepared given that w^n is. With these preliminaries, there holds $\nabla \cdot \widehat{f}(w^n) = \mathcal{O}(1)$, i.e., the only terms of order ε^{-2} and ε^{-1} can be found in $\nabla \cdot \widetilde{f}(w^{n+1})$. Leaving out the tedious details, one obtains that those terms are

$$\gamma {\rho_{(0)}}^{\gamma-1} \nabla \left(\rho_{\Delta t,(0)}^{n+1} \right) = 0, \qquad \gamma {\rho_{(0)}}^{\gamma-1} \nabla \left(\rho_{\Delta t,(1)}^{n+1} \right) = 0,$$

respectively, which yields the fact that both $\rho_{\Delta t,(0)}^{n+1}$ and $\rho_{\Delta t,(1)}^{n+1}$ are constant in space. Due to conservation of mass and periodic boundary conditions, it must be the same constant as on time slab n. Then again, considering the continuity equation, one can deduce that $\nabla \cdot {u_{\Delta t,(0)}}^{n+1} = 0$. This means that at time slab $n+1$, the data is again well-prepared. Now, realizing that $\widehat{f}(w^n) + \widetilde{f}(w^{n+1}) = f(w^n) + \mathcal{O}(\|w^{n+1} - w^n\| + \|w_{(0)}(t^{n+1}) - w_{(0)}(t^n)\|)$, and further realizing that taking the divergence removes the $\mathcal{O}(\varepsilon^{-2})$ and $\mathcal{O}(\varepsilon^{-1})$ dependencies, the proof that the first-order expansion (8) is a consistent approximation of (4)-(5) follows. □

3 FULL DISCRETIZATION

So far, we have only been concerned with the temporal discretization of (3). To simplify matters, the spatial discretization is chosen as a standard first-order finite volume scheme with a numerical flux with constant numerical viscosity both for the explicit and for the implicit part of the equation. More precisely, we discretize (1)-(2) with flux splitting as in Definition 1 by

$$w_j^{n+1} = w_j^n - \frac{1}{|\Omega_j|} \sum_{e_{jk} \in \partial\Omega_j} |e_{jk}| \left(\widehat{f}_{\mathrm{num}}(w_k^n, w_j^n, \vec{n}_{jk}) + \widetilde{f}_{\mathrm{num}}(w_k^{n+1}, w_j^{n+1}, \vec{n}_{jk}) \right). \tag{9}$$

As is standard in finite volumes, w_j^n is an approximation of the mean-value of $w(x, t^n)$ on cell Ω_j. e_{jk} is an edge connecting Ω_j to Ω_k; n_{jk} is the associated normal facing into Ω_k. The numerical flux $\widehat{f}_{\mathrm{num}}$ is set to

$$\widehat{f}_{\mathrm{num}}(w_k^n, w_j^n, \vec{n}_{jk}) := \frac{1}{2} \left(\widehat{f}(w_k^n) + \widehat{f}(w_j^n) \right) \vec{n}_{jk} - \frac{\widehat{\alpha}}{2}(w_k^n - w_j^n),$$

similarly for $\widetilde{f}_{\mathrm{num}}$, ensuring conservativity. In our numerical results, we set the viscosities $\widehat{\alpha}$ and $\widetilde{\alpha}$ to positive constants. Note that $\widetilde{\alpha}$ could be equally well set to zero. This work only constitutes a proof of concept as to what extend the new splitting is able to cope with singular perturbations, which explains the simple choices of both discretization and parameters. Extensions to more advanced methods, such as the discontinuous Galerkin method, are currently underways.

The reference solution (4)-(5) is also approximated using a finite volume scheme. It is natural to treat the convection term $\nabla \cdot (u_{(0)} \otimes u_{(0)})$ explicitly, and the remaining terms implicitly. The procedure is completely analogous to (9), with the obvious exception that $\nabla \cdot u_{(0)} = 0$ does not have a time-derivative, and so (9) has to be modified accordingly.

4 NUMERICAL RESULTS

In order to assess the stability properties of the proposed splitting, we consider a test case with periodic boundaries on the unit square $(0, 1)^2$ taken from [4], originally proposed in [3]. The adiabatic coefficient is set to $\gamma = 2$, and initial data are given by

$$\begin{aligned} \rho(x, y, t = 0) &= 1 + \varepsilon^2 \sin(2\pi(x + y))^2 \\ u(x, y, t = 0) = v(x, y, t = 0) &= \sin(2\pi(x - y)). \end{aligned}$$

Our numerical viscosities are set to $\widehat{\alpha} = 2$ and $\widetilde{\alpha} = 1$. The choice for the explicit part is motivated by a rough approximation of the eigenvalues of the explicit convective flux. As already stated, $\widetilde{\alpha}$ could be equally well set to zero.

Obviously, the initial data are well-prepared, i.e., they allow for a limit as $\varepsilon \to 0$. As outlined before, an explicit scheme is not able to cope with this problem given a time step restriction *independent* of ε. With the scheme presented here, we do not observe any ε restrictions on the CFL number, and can use a uniform CFL number for all ranges of ε. See Figure 1 for an image of the approximated u for $\varepsilon = 10^{-3}$ (left) and $\varepsilon = 10^{-6}$ (right). As one can see, there is hardly any difference between those plots, which is clear from the fact that the solution should converge as $\varepsilon \to 0$.

5 CONCLUSION AND OUTLOOK

We have presented a way of splitting a hyperbolic equation into stiff and non-stiff contributions. The splitting relies on the reference solution $w_{(0)}$. We have shown that this splitting is AP, and we have shown preliminary numerical results.

Current work is twofold: First, mathematical analysis is performed to prove that the scheme is indeed stable independent of ε for $\varepsilon \to 0$. Second, we extend this concept to higher-order discretization schemes in both time and space. In particular in that scenario, a better splitting can certainly yield improvements over existing splittings.

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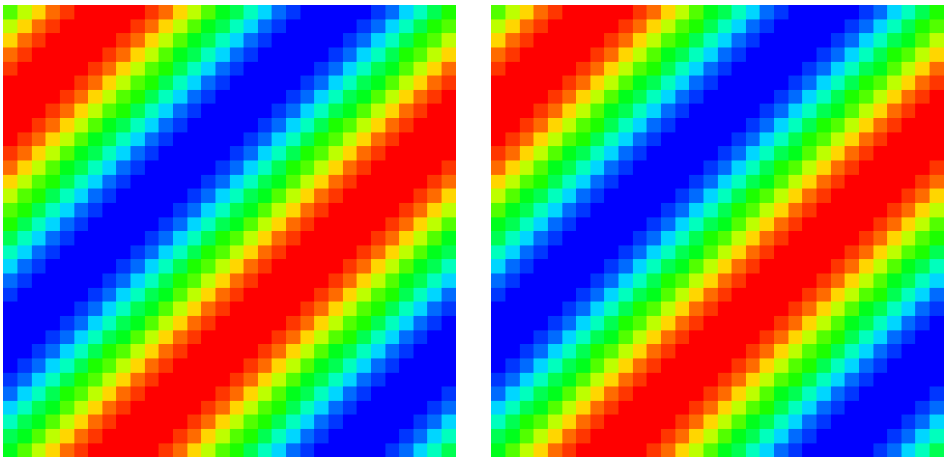

Figure 1: u for $\varepsilon = 10^{-3}$ (left) and $\varepsilon = 10^{-6}$ (right). Note that because the solution converges with $\varepsilon \to 0$, the plots are practically identical. Time step restriction amounts to $\Delta t \leq 0.3\Delta x$.

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