

RTN lectures on the non AdS/non CFT Correspondence

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We review some aspects of the AdS/CFT correspondence for non conformal, QCD-like theories. We discuss the general features of holographic duals of confining theories and we then focus on two specific subjects: 1) the inclusion of flavors in the probe approximation, and 2) the controllable case of $\mathcal{N} = 1$ supersymmetric theories, which is discussed in details. A brief discussion of the relation between the supergravity solutions and string compactifications is also included.

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1. Introduction

The AdS/CFT correspondence [1, 2, 3] is the first explicit realization of the old idea that the strongly coupled dynamics of a gauge theory has a description in terms of an effective theory of strings. The correspondence also naturally implements the 't Hooft large N expansion, thus providing a verification of many ideas about gauge theories at large N . Despite its name, the correspondence can be extended to non conformal gauge theories where we are naturally led to study QCD-like theories. Flavors have been recently added to the correspondence, but the road to truly realistic theories is still long. Classical supergravity solutions give a quite accurate description of theories that are not pure YM theories, but contain infinite additional fields. It is a general expectation that classical supergravity alone cannot describe realistic gauge theories, which contain higher spin glueballs. The dual of pure QCD is therefore expected to be a strongly coupled string model. However, the supergravity duals provide many exactly solvable models exhibiting confinement and other phenomena typical of the pure gauge theory. Thus, even if not quantitatively relevant for QCD, they provide a good laboratory for studying the mechanism of confinement and the qualitative properties of QCD.

Considering the huge literature on the subject (for reviews see [4] and for more recent developments regarding non conformal theories [5, 6, 7]), here we will only discuss some recent topics. In particular, we will focus on two main subjects. The first one (addressed in Section 3) is the introduction of flavors in the AdS/CFT correspondence and the possibility, at least in a probe approximation, of studying the mesonic spectrum. The addition of flavors is certainly a first step towards more realistic theories. The probe approximation is a sort of quenched approximation where

the effect of the glue on the dynamical quarks is included but the backreaction of the quarks is not. The introduction of really dynamical quarks in the fundamental representation in the AdS/CFT correspondence is still an important open problem. The second subject we will address (in Section 4) is the discussion of supergravity duals for $\mathcal{N} = 1$ gauge theories. In this context, we will explore in details the vacuum properties of the two known regular solutions (Klebanov-Strassler (KS) and Maldacena-Nunez (MN)) [8, 9]. We have now an improved understanding of the IR dynamics of the KS solution. Surprisingly enough, the KS and MN solutions are continuously connected by a supergravity flow. We will discuss to which extent these solutions can be considered as convenient toy models for $\mathcal{N} = 1$ pure SYM. The last part of these lectures is a brief introduction to the relation of AdS-like supergravity solutions with (warped) string compactifications. The KS solution, in particular, has been used for extra-dimensions phenomenology and cosmology. We refer to other lectures in this school for a detailed discussion of string compactifications with fluxes and related problems.

The material in these lectures should be understandable by readers knowing the very basic facts about the AdS/CFT correspondence. We will try to be self-contained as possible but we will be certainly not exhaustive. The reader will be referred to the appropriate literature when needed. These lectures can be considered in a sense as a companion of [7].

2. Generalities about non conformal backgrounds

The reader is suppose to know the basic facts about the conformal case; a thoughtful introduction to the *AdS/CFT* correspondence [1, 2, 3] can be found in [4]. In this Section, we will briefly discuss the general philosophy of the gauge/gravity duality for the non-conformal case. We will mostly proceed by examples, in order to emphasize the basic facts we will use in the following. More details can be found in [4] and (for more recent developments about non-conformal backgrounds) in [5, 6, 7].

The basic principle is: systems of D-branes in Type II string theory (or systems of M-branes in M theory) admit a complementary description in terms of gauge theories on their world-volume on one side and curved supergravity backgrounds on the other side. To apply the AdS/CFT correspondence, the gauge theory of interest must be engineered on a stack of branes. The gravity dual is then obtained as the near horizon geometry of the stack. More precisely, the near horizon limit is obtained by sending $\alpha' \rightarrow 0$, rescaling the dimensional parameters on the branes in a zooming process and keeping the dimensionless parameters fixed.

Example 2.1: Consider N parallel D3-branes filling the four-dimensional space-time in a background of the form $\mathbf{R}^{(1,3)} \times M_6$ in type IIB, where M_6 is a non-compact Calabi-Yau. At low energies, the system is described by the gauge fields on the branes coupled to the massless fields of Type IIB supergravity in the bulk. The low energy Lagrangian for the coupled brane/bulk system schematically reads

$$-\frac{1}{8\pi g_s} \int d^4x \sqrt{g} \text{Tr}(F^2) + \frac{1}{(2\pi)^7 \alpha'^4 g_s^2} \int d^{10}x \sqrt{g} R + \dots \quad (2.1)$$

The gauge group and the matter content of the world-volume theory depend on the manifold M_6 and are not always simple to determine. Roughly, the number of branes N will determine the average

number of colors of the various group factors and g_s will determine the average gauge coupling $g_{YM}^2 = 4\pi g_s$. The string frame metric generated by a stack of D3-branes is

$$\begin{aligned} ds^2 &= h(r)^{-1/2} dx_\mu dx^\mu + h(r)^{1/2} (ds_6^2), \\ h(r) &= 1 + \frac{4\pi g_s N \alpha'^2}{r^4}. \end{aligned} \quad (2.2)$$

In this description the decoupling limit can be realized by sending $\alpha' \rightarrow 0$ while keeping N and g_s fixed. The $\alpha' \rightarrow 0$ limit will decouple closed string modes and excited open string modes. However, one must be careful in doing this limit. The low energy dynamics of the gauge theory (moduli space of vacua, Higgs phenomenon,...) is determined by open strings connecting different branes. For example, we can Higgs the gauge group by separating the branes. The masses of the W-bosons are of order $\Delta r/\alpha'$, where Δr is the distance between the branes. We have to rescale distances in order to keep these masses finite. We then send $\alpha' \rightarrow 0$ keeping g_s and $r/\alpha' \equiv u$ fixed. In the limit we have just described, we can discard the 1 in the expression (2.2) for h ,

$$h(r) = 1 + \frac{4\pi g_s N}{\alpha'^2 u^4} \rightarrow \frac{4\pi g_s N}{\alpha'^2 u^4} \quad (2.3)$$

This familiar trick eliminates the asymptotically flat region at infinity.

Example 2.2 (CFTs): a) for D3-branes in flat space, $ds_6^2 = dr^2 + r^2 \Omega_5^2$ and we obtain the familiar metric

$$ds^2 = \left\{ R^2 \frac{du^2}{u^2} + \frac{\alpha'^2 u^2}{R^2} dx_\mu dx^\mu + R^2 d\Omega_5^2 \right\}. \quad (2.4)$$

The metric is the direct product of two spaces of constant curvature, $AdS_5 \times S^5$, with the same radius $R^2 = \sqrt{g_{YM}^2 N \alpha'}$. The matching of parameters on the two sides of the correspondence reads:

$$\begin{aligned} 4\pi g_s &= g_{YM}^2 = \frac{x}{N} \\ \frac{R^2}{\alpha'} &= \sqrt{g_{YM}^2 N} = \sqrt{x} \end{aligned} \quad (2.5)$$

where we defined the t'Hooft coupling $x = g_{YM}^2 N$. The string theory is weakly coupled when N is large and $x \gg 1$. The latter condition means that the large- N gauge theory is strongly coupled. As well known, the double perturbative expansion of string theory, in powers of g_s (strings loop) and α' (higher derivatives terms) is associated respectively with the $1/N$ expansion and the $1/x$ expansion at fixed N .

b) The amount of supersymmetry can be reduced by placing D-branes in curved geometries. Since the AdS/CFT correspondence focuses on the near brane region and every smooth manifold is locally flat, we will find new models only when the branes are placed at a singular point of the transverse space [10, 11, 12, 13]. An interesting class of theories makes use of conifold singularities. We place branes at the singularity of Ricci-flat manifolds C_6 whose metric has the conical form

$$ds_{C_6}^2 = dr^2 + r^2 ds_{M_5}^2. \quad (2.6)$$

One can prove that C_6 is Ricci-Flat if M_5 is a five-dimensional Einstein manifold [11, 12]. The AdS/CFT correspondence is then formulated with the background $AdS_5 \times M_5$, which is the near

horizon geometry of the previous metric. Although some methods are known for particular cases, no general way to derive the world-volume theory from the geometry of M_5 is known.

The gauge theory is maximally supersymmetric $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ in case a) and a less supersymmetric gauge theory depending on M_5 in case b). The presence of an AdS_5 factor implies that both theories are conformal. A word of caution is needed for the gauge group. A stack of N D3 branes leads to a $U(N)$ gauge theory; however, as well known [4], all abelian factors are not described by the AdS/CFT correspondence.

Example 2.3 (The Black Brane): The extremal D3-branes can be replaced by black three-branes [14]. In this case we can consider theories defined on a space-time with topology $\mathbf{R}^3 \times S^1$ and break conformal invariance and supersymmetry by compactification. The near horizon geometry of an Euclidean black three-brane is given by,

$$ds^2 = R^2 \left[\left(u^2 \sum_{i=1}^3 dx_i^2 + u^2 \left(1 - \frac{u_0^4}{u^4} \right) d\tau^2 \right) + \frac{du^2}{u^2 \left(1 - \frac{u_0^4}{u^4} \right)} + d\Omega_5 \right] \quad (2.7)$$

The geometry has an horizon at $u = u_0$. To avoid a conical singularity at $u = u_0$, τ should be considered an angular variable with radius $R_0 = \frac{1}{2u_0}$. Since the world-volume topology is $\mathbf{R}^3 \times S^1$, the natural candidate for the dual gauge theory is $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ on $\mathbf{R}^3 \times S^1$. The metric admits a spin structure where fermions change sign along S^1 . Supersymmetry is broken, the fermions get masses through these boundary conditions and the scalars get masses through loops of fermions. For $R_0 \rightarrow 0$ all fermions and scalars decouple and we are left with pure YM in three-dimensions. This way, one obtains a non-supersymmetric and non-conformal theory in three-dimensions that can be studied using a weakly coupled supergravity dual. We are obviously more interested in four-dimensional theories. There is a refined version of this construction that gives pure YM in four-dimensions [14]. In this case, one has to start with the (2,0) superconformal theory in six dimensions, realized on the world-volume of a stack of M5 branes; the corresponding M theory dual is $AdS_7 \times S^4$. The four dimensional theory is obtained by compactifying two dimensions. One can alternatively start with the reduction of the system to type IIA, a stack of D4 branes compactified on a circle. The IIA background is a black hole geometry corresponding to non-extremal D4 branes

$$ds^2 \sim \left(\frac{u}{R} \right)^{3/2} \left(dx_\mu^2 + \left(1 - \frac{u_0^3}{u^3} \right) d\tau^2 \right) + \left(\frac{R}{u} \right)^{3/2} \frac{du^2}{1 - \frac{u_0^3}{u^3}} + R^{3/2} \sqrt{u} \Omega_4. \quad (2.8)$$

with a non constant dilaton $e^\phi = g_s (u/R)^{3/4}$. In this solution τ has period $4\pi^2 R^{3/2} / (3u_0^{1/2})$.

A crucial ingredient in all the models obtained by the AdS/CFT correspondence is the identification of the radial coordinate in the supergravity solution with an energy scale in the dual field theory. Let us first consider a conformal field theory and its AdS dual. The identification between radius and energy follows from the form of the AdS metric (we put $R = 1$ when no confusion is possible)

$$ds^2 = u^2 dx_\mu^2 + \frac{du^2}{u^2}. \quad (2.9)$$

A dilatation $x_\mu \rightarrow \lambda x_\mu$ in the boundary CFT corresponds in AdS to the $SO(4,2)$ isometry

$$x_\mu \rightarrow \lambda x_\mu, \quad u \rightarrow \frac{u}{\lambda}. \quad (2.10)$$

We see that we can roughly identify u with an energy scale μ . The boundary region of AdS ($u \gg 1$) is associated with the UV regime in the CFT , while the horizon region ($u \ll 1$) is associated with the IR. This is more than a formal identification: holographic calculations of Green functions or Wilson loops associated with a specific reference scale μ are dominated by bulk contributions from the region $u = \mu$. Examples and further references can be found in [4]. Obviously, a change of scale in a CFT has little physical meaning. In a non conformal theory, however, the quantum field theory couplings run with the scale. This suggests that we can interpret the running couplings in terms of a specific radial dependence of the fields in the supergravity solution. This interpretation works very well at the qualitative level. As in the AdS case, the region with large (small) radius will be associated with the UV (IR) dynamics of the gauge theory. However, the quantitative identification of the radius with the scale can be difficult to find. For non-conformal theories the precise form of the relation depends on the physical process we use to determine it [15]. The radius/energy relation can be found for instance by considering the warp factor multiplying the flat four-dimensional part of the metric $ds^2 = e^{2A(u)} dx_\mu dx^\mu + \dots$, since $e^{2A(u)}$ is a redshift factor ¹ connecting the energies of observers at different points in the bulk: $e^{-A(u')} E' = e^{-A(u)} E$. Alternatively, we can compute a Wilson loop in supergravity [4]: the energy of a string stretched between the boundary and a fixed IR reference radius represents in the gauge theory the self-energy of a quark. Finally, one can also extract the radius/energy relation by analyzing the equation of motion of a supergravity mode with fixed four-dimensional momentum. While for conformal theories all the different methods give the same result, this is no longer true for gravity duals of non-conformal theories. Also in the relatively well understood case of the Klebanov-Strassler solution (see Section 4), the different prescriptions give different results [16, 17].

Our main interest in these lectures are non-conformal theories that reduce at low energy to pure YM theories. Pure glue theories with $\mathcal{N} = 1$ or $\mathcal{N} = 0$ supersymmetry confine, have a mass gap and a discrete spectrum of massive glueballs. We will now describe how these features are realized in the gravity dual, using as a toy model the black three-brane of Example 2.3:

- **Confinement:** this issue is usually investigated using a Wilson loop [4]. The criterion for confinement is the following: the warp factor e^{2A} multiplying the four-dimensional part of the metric

$$ds^2 = e^{2A(u)} dx_\mu^2 + \dots \quad (2.11)$$

must be bounded above zero. The theory has then stable finite tension strings which can be identified with the type IIB fundamental string. They will live in the region of the solution where the warp factor has its minimum value e^{2A_0} and their tension will be given by $\frac{e^{2A_0}}{2\pi\alpha'}$. In the black brane example, the warp factor reaches its minimum $e^{2A_0} = \sqrt{4\pi g_s N} u_0^2$ at the horizon u_0 . Heavy external quark sources can be introduced at the boundary of AdS. The potential energy between two external sources at distance L on the boundary is determined by the energy of the string connecting them. The string will minimize its energy by deeply penetrating in the interior of the geometry and by reaching u_0 where the metric components $\sqrt{g_{xx}g_{tt}}$ have a minimum. For large L , the minimal energy configuration consists of three straight segments: two long strings at fixed x_μ connecting the boundary to the point u_0 , and a

¹Both functions $h(u)$ and $A(u)$ are used in the literature. The relation between them is $h(u) = e^{-4A(u)}$.

string at fixed u_0 stretching for a distance L along the four-dimensional spacetime directions (see Figure 1). The infinite energy of the long string from $u = \infty$ to $u = u_0$ is interpreted as the bare mass of the external source. All the relevant contribution to the potential energy between two external sources is then due to a string localized at $u = u_0$ and stretched in the x direction. The total energy

$$E(L) = m_q + m_{\bar{q}} + \frac{e^{2A_0}}{2\pi\alpha'} L \tag{2.12}$$

then gives the linear increasing potential characteristic of confinement. This situation should

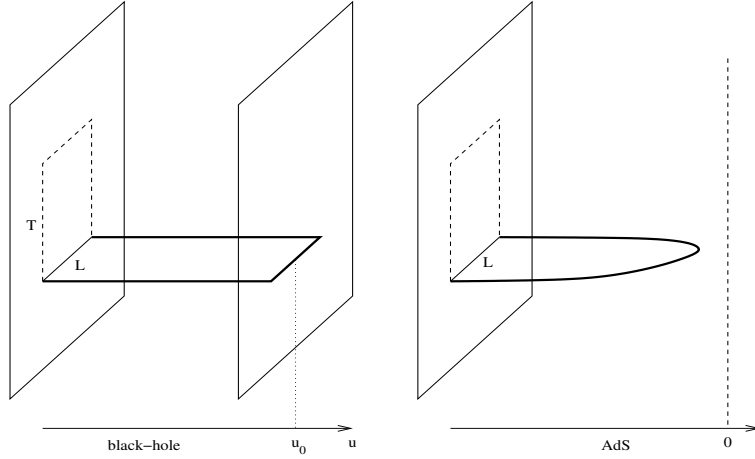


Figure 1: The quark-antiquark potential in the black-hole and in the conformal case.

be contrasted with the *AdS* case, where $e^{2A(u)}$ vanishes at $u = 0$. In this case, an explicit computation of the the minimal area configuration gives an energy $E(L) = m_q + m_{\bar{q}} + \frac{1}{L}$ [4], appropriate for a conformal theory.

- **Glueball Spectrum:** Masses of bound states can be extracted from correlation functions of gauge invariant operators,

$$\langle O(x)O(y) \rangle \sim \sum a_i e^{-M_i|x-y|}, \quad |x-y| \gg 1 \tag{2.13}$$

We can also extract M_i from the equation of motion for the fields ϕ dual to O by looking at AdS-normalizable solution of the form [4]

$$\phi(x_\mu, u) = \phi(u)e^{ikx}, \quad k^2 = -M^2 \tag{2.14}$$

where k is the four-dimensional momentum. For example, for a minimally coupled scalar field

$$\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) - \sqrt{g}m^2\phi = 0 \tag{2.15}$$

which propagates in the AdS-black hole, the spectrum is determined by the Schroedinger type equation

$$\partial_u(u(u^4 - u_0^4)\partial_u\phi) + M^2u\phi - m^2u^3\phi = 0 \tag{2.16}$$

This equation should be solved with the following boundary conditions: regularity at the horizon and normalizability at large u ². With these boundary conditions, the previous equation determines a discrete spectrum of strictly positive masses $M_n^2 > 0$. This result is interpreted as the existence of confinement and a mass gap. Once again this result should be contrasted with the AdS case: in the AdS metric the equation for a minimally coupled scalar field determines a continuum spectrum of masses $M^2 > 0$ not bounded above zero. Notice that the typical glueball mass in the black hole background is given by $\sim \frac{1}{R_0^2}$ while the string tension is $\sqrt{4\pi g_s N} u_0^2 \sim \frac{\sqrt{g_s N}}{R_0^2}$. The two scales are different, with the glueball masses much smaller than the string tension. This is a typical discrepancy which is present in many gauge/gravity dualities.

- **The Decoupling Problem:** In all the gauge/gravity dualities we will be eventually able to perform predictive calculations only in the limit where the supergravity approximation is valid. It is difficult to study the complete dynamics of “realistic” theories such as pure Yang-Mills in this context. For example, in the black three-brane case, the compactification on a circle of radius R_0 of N=4 SYM with coupling g_{YM} breaks conformal invariance and leads at low energy to a non conformal 3d YM theory with coupling constant $\frac{1}{g_{3d}^2 N} = \frac{2\pi R_0}{g_{YM}^2 N}$. The limit where the low energy theory decouples from the CFT leaving a finite three dimensional coupling is $R_0 \rightarrow 0$ with $x = Ng_{YM}^2 \rightarrow 0$. However, we can trust supergravity in the opposite limit $x \gg 1$. Thus the description of the low energy pure YM theory requires the knowledge of the full string theory. Similar arguments apply to all the non-conformal models constructed so far. In many other examples, we work directly in four dimensions and we obtain pure YM by adding a mass deformation M to a CFT that possesses a holographic dual. In this case, the mass parameter induces a dimensional scale $\Lambda \sim Me^{-1/Ng_{YM}^2}$. Once again the decoupling limit is $M \rightarrow \infty$, $x = Ng_{YM}^2 \rightarrow 0$, with Λ fixed, which is the opposite of the supergravity limit. The expectation that the spectrum of bound states in any realistic model should contain higher spin glue-balls suggests that more than supergravity is required to describe the pure YM theory. In the previous example, it would be sufficient to re-sum all world-sheet α' corrections in the string background to correctly describe pure YM in the large N limit. World-sheet corrections are, in principle, more tractable than loop corrections. In flat space, for example, all the α' corrections are computable. In the AdS case, the analogous computation is made difficult by the presence of RR-fields. We may take various attitudes towards the supergravity duals. In the previous example, we may consider the supergravity solution as a description of pure YM with a finite cut-off $\Lambda \sim M$. The situation is similar, in spirit, to a lattice computation at strong coupling. In general, in all the models discussed so far, the supergravity solution describes a YM theory with many non-decoupled massive modes. These theories can be considered as cousins of pure YM, and they have often the same qualitative behavior. At present we have many examples of theories that are, in a cer-

²Normalizability here means the following. At large u the theory is asymptotically AdS and a minimally coupled field of mass m behaves as $\phi(u) \sim Au^{\Delta-4} + Bu^{-\Delta}$, where $m^2 = \Delta(\Delta-4)$ is the familiar relation between mass and conformal dimension valid in the conformal case [4]. The contribution proportional to A is non-normalizable in AdS while the term proportional to B is normalizable ($\int \sqrt{g} |\phi|^2 < \infty$ for every field satisfying the unitarity bound $\Delta > 1$). When studying the spectrum, we shall only consider normalizable wave-functions.

tain sense, generalization of pure glue theories. They are interesting as exactly solvable toy models. Moreover, it is interesting to investigate which properties of pure YM, that are not consequences of symmetries, are realized in these generalized models.

3. Adding flavors

Real QCD contains matter in the fundamental representation. Confinement can be thought as a property of the pure gluonic theory and studied, in the large N limit, as in the previous Section. However the introduction of matter fields leads to new features: the flux tubes become unstable and flavor symmetries can be spontaneously broken. It is therefore important to include flavors in the AdS/CFT correspondence. Fundamental fields can be added by introducing other branes beside D3. The problem of finding smooth solutions containing different type of branes is still an open one. A great simplification can be achieved in the limit where the number of flavors is much less than the number of colors [19]. In this case, other branes can be introduced as probes. The probe approximation is a sort of quenched approximation where the effect of the glue on the dynamical quarks is included but the backreaction of the quarks is not. At large t'Hooft coupling, this quenched QCD behaves qualitatively as real QCD, exhibiting a spectrum of mesons and baryons and chiral symmetry breaking. Mesons correspond to fluctuations of the probe branes and their spectrum is computable along the lines discussed in the previous Section. Obviously, in the strongly coupled regime, the mesonic spectrum may give surprises when compared with the expectations at weak coupling.

Let us start by discussing the standard brane set-up for introducing fundamental matter fields.

Example 3.1: Consider a system of parallel D3 and D7 branes, with N_c D3 along the directions (0123) and N_f D7 along (01234567). On the world-volume of the D3 branes, we obtain an $\mathcal{N} = 2$ gauge theory with gauge group $U(N_c)$, N_f hypermultiplet matter fields transforming in the fundamental representation and one transforming in the adjoint representation (Figure 2). The R symme-

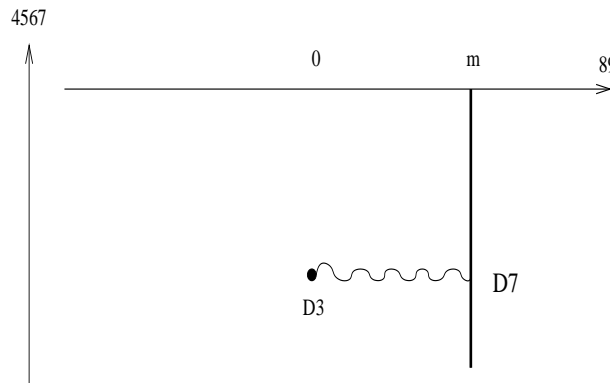


Figure 2: The $U(N_c)$ gauge fields and the adjoint hypermultiplet are obtained from the D3-D3 open strings, while the fundamental hypermultiplets from the D3-D7 open strings.

try of the theory is $SU(2)_R \times U(1)_R$, where $SU(2)_R$ is a subgroup of the $SO(4)$ that rotates (4567) and $U(1)_R$ rotates the plane (89). The relative distance between D7 and D3 branes determines the mass of the hypermultiplet.

There exist supergravity solutions associated to D3 and D7 branes [18]. Consistent solutions also requires O7 planes since the D7 branes backreaction generates a logarithmic deformation in the (89) components of the metric. With this type of construction we can obtain USp gauge groups. The situation simplifies in the probe approximation. For $N_f \ll N_c$, the back-reaction of the D7 can be neglected. The near horizon geometry associated to such gauge theories is still $AdS_5 \times S^5$ with D7 introduced as probes [19]. This is a sort of quenched approximation: the glue effect on the quarks are included (the D7 feels a modified geometry) but no quark effect on the glue is included (no backreaction).

A D7 brane in $AdS_5 \times S^5$ at $x_8 = x_9 = 0$ will fill AdS_5 and wrap an S^3 inside S^5 . The situation is more complicated if $x_8 = x_9 \neq 0$ (i.e. $m \neq 0$) and the D7 brane will acquire a non trivial shape. To investigate the D7 embedding, we choose metric coordinates for space transverse to the D3 as follows

$$ds_6^2 = dr^2 + r^2(d\psi^2 + \cos\psi^2 d\theta^2 + \sin\psi^2 \Omega_3) = d\rho^2 + \rho^2 \Omega_3 + dx_8^2 + dx_9^2 \quad (3.1)$$

where $r^2 = x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2$ while (ρ, Ω_3) give spherical coordinates for the four directions (4567).

Consider a D7 brane located at $x_8 + ix_9 = m$.

- For $m = 0$, the D7 brane wraps $S^3 \subset S^5$. The induced metric is $AdS_5 \times S^3$. The presence of an AdS_5 factor suggests that the theory is still conformal. In the absence of D7 branes the theory is $\mathcal{N} = 4$ SYM. With the addition of $N_f \neq 0$ massless flavors the beta function is no more vanishing. However, the corrections to the beta function for the t'Hooft coupling depend on N_f/N_c , which is negligible in the probe approximation.
- For $m \neq 0$, conformal invariance is broken even in the probe approximation. The induced metric on a D7 brane is

$$ds^2 = (\rho^2 + |m|^2) dx_\mu^2 + \frac{d\rho^2}{\rho^2 + |m|^2} + \frac{\rho^2}{\rho^2 + |m|^2} \Omega_3 \quad (3.2)$$

where we have used $r^2 = \rho^2 + m^2$. The induced metric is $AdS_5 \times S^3$ only asymptotically ($\rho \gg 1$), that is in the UV where the mass can be effectively neglected. The embedding of the D7 brane in the background geometry is highly non trivial. The S^3 radius indeed depends on ρ and vanishes for $\rho = 0$ (this means $r = |m|$). Therefore, the D7 brane does not fill the entire AdS_5 (see figure 3) [19]. Notice also that the D7 wraps a contractible cycle; this avoids problems with charge conservation and the necessity of including other D7 branes or O7 planes.

We can better study the shape of the D7 brane by looking at the Born-Infeld action. By choosing the obvious embedding $x^I = (x_\mu, \rho, \Omega_3, x_8(\rho), x_9(\rho))$ and defining $\Phi = x_8 + ix_9$ the BI action reads

$$T_7 \int \sqrt{G_{IND}} \sim \int d^8 x \sqrt{g} \sqrt{1 + g_{\Phi\bar{\Phi}} g^{ab} (\partial_a \Phi) (\partial_b \bar{\Phi})} = \int d^8 x \rho^3 \sqrt{1 + |\partial_\rho \Phi|^2} \quad (3.3)$$

The equations of motion for $x_8(\rho)$ (we put for simplicity $x_9 = 0$)

$$\frac{d}{d\rho} \left(\frac{\rho^3}{\sqrt{1 + (\partial_\rho x_8)^2}} \frac{dx_8}{d\rho} \right) = 0 \quad (3.4)$$

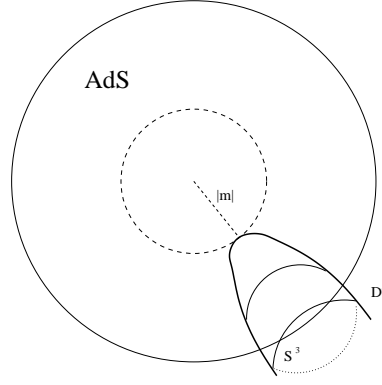


Figure 3: The topologically trivial cycle wrapped by the D7 brane.

has asymptotic solution for $\rho \gg 1$

$$x_8(\rho) = |m| + \frac{c}{\rho^2} \quad (3.5)$$

The two terms in (3.5) have a simple interpretation that is obscured by the fact that the field x_8 is not canonically normalized. We then switch to the angular variable ψ related to x_8 by $x_8 = r \cos \psi$. The expansion of ψ for $r \gg 1$ is

$$\psi(r) = \frac{\pi}{2} + \left(\frac{|m|}{r} + \dots \right) + \frac{c}{r^3} \quad (3.6)$$

The fluctuation $\tilde{\psi} = \psi - \pi/2$ is governed by the linearized BI action

$$\int r^3 \sin \psi^3 \sqrt{1 + r^2 (\partial_r \psi)^2} \sim \int \sqrt{g} \left(1 + \frac{(\partial \tilde{\psi})^2}{2} - \frac{3}{2} \tilde{\psi}^2 \right) \quad (3.7)$$

We see that $\tilde{\psi}$ propagates in the background geometry AdS_5 as a canonically normalized scalar field with mass squared -3 (in units of the AdS radius). The AdS/CFT correspondence predicts the following relation between the mass m of a scalar field in the bulk and the conformal dimension Δ of the dual operator [4]

$$m^2 = \Delta(\Delta - 4) \quad (3.8)$$

$\tilde{\psi}$ thus corresponds to an operator of dimension three that we can identify with the quark bilinear operator $q\bar{q}$. According to the standard AdS/CFT philosophy [4], the two terms in (3.6) can be interpreted as follows³

- m/r is the non-normalizable solution of the equation of motion ($r^{\Delta-4}$). Its presence indicates that the Lagrangian has been deformed with the mass term $m q\bar{q}$.
- c/r^3 is the normalizable mode ($r^{-\Delta}$). Its presence indicates that the gauge theory is in a vacuum where the operator $q\bar{q}$ has a non-zero VEV. Notice that this option is forbidden in supersymmetric theories where the quark bilinear must vanish.

³Due to the non linearities of the BI action, there are subtleties in correctly identifying the VEV in equation (3.6) [19]. Notice, for example, that a supersymmetric zero-VEV configuration with $x_8 = |m|$ is mapped to $\psi = \arccos \frac{|m|}{r} = \frac{\pi}{2} - \frac{|m|}{r} - \frac{|m|^3}{r^3} + \dots$

The inclusion of D7 branes introduces in the $AdS_5 \times S^5$ background a set of new fields whose fluctuations are dual to mesonic operators. These fields come from D7-D7 open strings and are localized on the eight-dimensional world-volume of the D7 branes. The mesonic spectrum can be studied using the Born-Infeld action for D7 branes in the type IIB background.

Example 3.2. The mesonic spectrum in the $\mathcal{N} = 2$ theory: The mesons in this theory are obtained by reducing the D7 worldvolume fields on S^3 . The mesons will fit in multiplets of the $\mathcal{N} = 2$ supersymmetry and, without loss of generality, we can restrict our attention to the bosonic modes. We get KK towers of scalar fields both from the reduction of $\Phi = x_8 + ix_9$ and of the gauge fields with components along S^3 . We also obtain a tower of KK vector modes from the reduction of the D7 gauge fields. All these KK modes can be classified according to the S^3 quantum numbers (l, l') . From Φ , for example, one obtains a tower of scalar KK modes with quantum numbers (l, l) dual to operators with dimension $\Delta = 3 + l$. The simplest case with $l = 0$ is the quark bilinear described above. The cases $l \neq 0$ correspond to operators of the form $q\phi^l\bar{q}$, where $\phi_i, i = 1, 2, 3, 4$ are the scalar components of the adjoint hypermultiplet. Consider, for example, the equation of motion for the field Φ : it follows from the linearization of the action (3.3)

$$\frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}\frac{g^{ab}}{r^2}\partial_b\Phi\right) = 0 \quad (3.9)$$

For a KK mode with four-dimensional momentum k and corresponding to a spherical harmonic (l, l) on S^3

$$\Phi = \phi(\rho)e^{ikx}Y_l \quad (3.10)$$

we obtain

$$\frac{1}{\rho^3}\partial_\rho(\rho^3\partial_\rho\phi) + \left(\frac{R^4M^2}{(\rho^2 + |m|^2)^2} - \frac{l(l+2)}{\rho^2}\right)\phi = 0 \quad (3.11)$$

where $M^2 = -k^2$ is the four dimensional mass. We also reintroduced the AdS radius R . As discussed in Section 2, the mesons correspond to solutions of this equation that are regular and normalizable at infinity. The previous equation can be explicitly solved [20], giving a spectrum of scalar mesons with masses

$$M_n^2 = \frac{2|m|}{R^2}\sqrt{(n+l+1)(n+l+2)} \quad (3.12)$$

One can repeat the analysis for the other KK towers obtaining scalar and vector mesons with comparable masses [20]. Notice that the lightest meson has a mass $M \sim m/\sqrt{g_s N}$, much smaller than the mass m of the quarks since the supergravity approximation requires $g_s N \gg 1$. The meson mass will be given by $M = 2m - E_{binding}$. Since we are inserting quarks in an $\mathcal{N} = 4$ gauge theory, their interaction is mainly due to Coulomb forces. The problem is similar to that of a hydrogenic atom. At weak coupling $E_{binding} \sim (g_s N)^2$ and M is of order m . At strong coupling we see that the binding energy cancels most of the rest energy of the quarks [20].

Example 3.3. QCD₄: Our principal interest are non-conformal theories. We can then consider the background of non extremal D4 branes discussed in Section 2 and add D6 branes [21]. The discussion is completely parallel to the D3-D7 case. We consider N_c D4 branes along (01234) and N_f D6 along (0123567). Since the number of mutual Dirichlet-Neumann directions is four, the system is

$N = 2$ supersymmetric: $U(N_c)$ SYM with N_f flavors plus an adjoint hypermultiplet. A compactification in the direction 4 gives masses to all the fermions and scalars of the gauge multiplet and to the scalars of the flavor multiplets. The N_f fundamental quarks are however protected by the $U(1)_A$ symmetry that rotates the (89) plane and remain massless. The result is QCD_4 with N_f quarks. In the probe approximation for the D6 we are considering QCD in the quenched approximation. The D4 background is as in equation (2.8)

$$ds^2 \sim \left(\frac{u}{R}\right)^{3/2} \left(dx_\mu^2 + \left(1 - \frac{u_0^3}{u^3}\right) d\tau^2\right) + \left(\frac{R}{u}\right)^{3/2} \frac{du^2}{1 - \frac{u_0^3}{u^3}} + R^{3/2} \sqrt{u} \Omega_4. \quad (3.13)$$

The simple case $N_f = 1$ has been studied in [21]. In Figure 4, we plotted $x_8 + ix_9$ as a function of

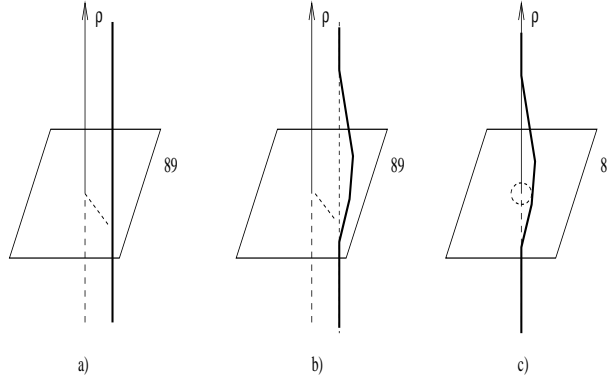


Figure 4: The three cases: the supersymmetric one, the non-supersymmetric one with $m \neq 0$ and the non-supersymmetric one with $m = 0$.

ρ . Case a) in the Figure corresponds to the supersymmetric solution with $u_0 = 0$. In this case there is no quark condensate and $c = 0$. This is consistent with the fact that supersymmetry forbids a quark condensate. Case b) and c) correspond to the QCD_4 case. We see from case b) that for non zero mass there is a non vanishing condensate $c = c(m)$; the axial symmetry $U(1)_A$ -rotation in the plane (89)- is explicitly broken by the mass term. In case c), although $m = 0$, $x_8 + ix_9$ has a non trivial profile. This means that there exists a quark condensate at zero mass. This is the standard expectation for QCD: the chiral symmetry is spontaneously broken. Moreover, the spectrum for the mesons can be computed by analyzing the equation of motion for the fluctuation $x_8 + ix_9$. This can be done using the Born-Infeld action in the background (3.13). In [21] a massless meson was found in the spectrum for $m = 0$: this is the massless pion, or better, the η' particle.

We can flavor other supersymmetric and non-supersymmetric backgrounds [22]. In all these cases, one can study chiral symmetry breaking and the spectrum of mesons. In the supersymmetric case one must check that the probe wraps a supersymmetric cycle. As usual, all the qualitative properties of QCD are well described by these dual solutions. We always need to remember that we are studying phenomena in strongly coupled gauge theories where the spectrum may exhibit an unexpected behavior, as we saw, for instance, in Example 3.2.

4. N=1 theories

Possibly regular 10 dimensional supersymmetric type IIB solutions dual to $\mathcal{N} = 1$ SYM the-

ories are

- **KS:** The Klebanov-Strassler solution [8]. It is obtained using fractional branes on a conifold. The dual gauge theory has $SU(N + M) \times SU(N)$ gauge group and bi-fundamental matter fields.
- **MN:** The Maldacena-Nunez solution [9]. It is obtained from wrapped branes. The dual theory is a compactification to four dimensions of the six-dimensional Little String Theory; at low energy the theory is effectively four dimensional with gauge group $SU(M)$.
- **PS:** The Polchinski-Strassler solution [23]. It includes brane sources – three-branes polarized into a five-brane by a dielectric effect [24]. The complete solution is not known, although it is believed to exist. The dual theory is the so-called $\mathcal{N} = 1^*$ theory, $\mathcal{N} = 4$ SYM with a mass for the three adjoint chiral fields.

We will focus on the KS and MN cases, which are explicitly known and regular. They also have a classical chiral $U(1)_R$ symmetry as pure $\mathcal{N} = 1$ SYM. There is indeed a limit where the dual gauge theories reduce to pure $\mathcal{N} = 1$ SYM in the IR. As discussed in the previous Section, this must necessarily be a stringy limit. In the regime of parameters where we can control the solution, these theories appear as deformations of pure $\mathcal{N} = 1$ SYM, which nevertheless share various properties of the same. We briefly recall here the main properties of pure $SU(N)$ $\mathcal{N} = 1$ SYM. There is a classical $U(1)_R$ symmetry, rotating the gaugino, which is broken to a discrete \mathbb{Z}_{2N} subgroup by instantons. This theory has N vacua associated with the spontaneous breaking of the \mathbb{Z}_{2N} symmetry to \mathbb{Z}_2 by gaugino condensation,

$$\langle \lambda\lambda \rangle \sim N\Lambda^3 e^{2\pi i n/N}, \quad (4.1)$$

where Λ is the physical, RG invariant mass scale, and may be written in terms of the bare coupling τ at some UV scale as $\Lambda = \Lambda_{UV} e^{2\pi i \tau/3N}$. The integer $n = 0, \dots, N - 1$ in (4.1) labels the different vacua. In presence of a spontaneous breaking of the \mathbb{Z}_{2N} symmetry, we expect the existence of domain walls (classical field configurations of codimension one) separating different vacua [25, 26]. The domain walls in $\mathcal{N} = 1$ gauge theories are BPS saturated and their tension is determined by a central charge of the supersymmetry algebra [25, 26], in terms of holomorphic data. The tension of a domain wall connecting the vacua i and j is determined by

$$T_{DW} \sim N |(\lambda\lambda)_i - (\lambda\lambda)_j| \sim N^2 \Lambda^3 \sin \frac{(i-j)\pi}{N}. \quad (4.2)$$

In the large N limit the tension is then linear in N . By analogy with D-branes, it was conjectured that the QCD strings can end on $\mathcal{N} = 1$ domain walls [26]. This typically happens also in all stringy-inspired generalization of pure $\mathcal{N} = 1$ SYM, where domain walls are realized in terms of branes and the QCD strings in terms of fundamental or D-strings. It is widely believed that pure $\mathcal{N} = 1$ SYM confines and has a mass gap. The characteristic scale of the theory Λ is set by the tension of the color flux tubes, or briefly QCD strings. They are not BPS objects and the value of their tensions cannot be fixed in terms of central charges or symmetries. Strings connecting external sources in different representations of the gauge group are, in general, different physical objects.

They are classified by the center of the gauge group. In a confining $\mathcal{N} = 1$ $SU(N)$ SYM theory, we can define $N - 1$ different types of QCD strings, since there are exactly $N - 1$ representations of the gauge group that are not screened by gluons. A k -string, $k = 1, \dots, N - 1$, connects external sources in the k -fold antisymmetric representation of $SU(N)$. It is then interesting to ask what is the ratio of the tensions for k -strings.

As we will see in the following, in the KS and MN theories there is chiral symmetry breaking and colored flux tubes, identified with the fundamental strings. Domain walls can be identified as wrapped five-branes and the QCD strings can end on them. The k -string tension can be explicitly computed [29]. In many stringy-inspired models one can indeed derive the sine formula for the ratio of k -strings

$$\frac{T_k}{T_{k'}} = \frac{\sin k\pi/N}{\sin k'\pi/N}. \quad (4.3)$$

This formula, or mild modifications of it, is valid in a variety of toy models exhibiting confinement, from softly broken $\mathcal{N} = 2$ SYM [27] to MQCD [28]. It is also realized in the MN solution (and, with a small correction, in the KS model) [29]. It is certainly not an universal formula. There are many quantum field theory counterexamples showing that it can have corrections [30]. It would be quite interesting to understand if this formula is valid in pure YM theories. Unfortunately, since the QCD strings are not BPS, there is no known method of performing a rigorous computation in $\mathcal{N} = 1$ SYM. Interestingly, the sine formula has been supported by recent lattice computations for pure non supersymmetric YM [31]. As a difference with pure $\mathcal{N} = 1$ SYM where the same scale Λ determines the scale of chiral symmetry breaking and the string tensions ($T_{DW} \sim N\lambda^3$, $T_s \sim \Lambda^2$), in the KS and MN theories T_s and T_{DW} are distinct.

The KS theory is not really in the same phase of pure $\mathcal{N} = 1$ SYM. The KS vacuum belongs indeed to a one-dimensional moduli space that is obtained by varying the VEV of some baryonic operator [32, 33]. It is in a phase with confinement, chiral symmetry breaking but no mass gap. There is indeed a family of regular type IIB solutions deforming KS [33, 34]. As we will see in the following, with a specific choice of boundary conditions, the same family interpolates between KS and MN [34].

4.1 Physical branes at conical singularity: the conformal case

Let us start with a stack of D3-branes at the conical singularity of a Calabi-Yau C_6 . As discussed in Section 2, this will give rise to a class of superconformal gauge theories on the D3 branes. A Calabi-Yau threefold reduces the amount of supersymmetry from 32 to 8 supercharges. The presence of D3-branes further breaks to 4 supercharges. This set-up thus generically describes $\mathcal{N} = 1$ gauge theories. In the near horizon geometry, the conformal supersymmetries enhance the number of supercharges to 8. If C_6 is a cone over the five-dimensional manifold M_5 , the metric has the form (2.6) and the near horizon geometry is $AdS_5 \times M_5$.

C_6 is a Calabi-Yau (a Ricci flat Kahler manifold) if and only if M_5 is a Sasaki-Einstein manifold [13]. The Einstein condition means $R_{\mu\nu}^{(5)} = \Lambda g_{\mu\nu}^{(5)}$ and is necessary for C_6 to be Ricci-flat [11, 12]. The Sasaki condition is more complicated; it is a set of conditions that are equivalent to the fact that C_6 is Kahler. Until recently, the only known regular Sasaki-Einstein manifolds in five dimensions were homogeneous spaces: $S^5 = SO(6)/SO(5)$ with 32 supercharges, corresponding to

$\mathcal{N} = 4$ SYM, and $T^{1,1} = (SU(2) \times SU(2))/U(1)$ with 8 supercharges. Infinite new Sasaki-Einstein manifolds $Y^{p,q}$ and $L^{p,q,r}$ with topology $S^2 \times S^3$ were recently constructed [35, 36].

The interesting features of all these metrics is that the gauge theory living on the D3-brane world-volume can be explicitly determined. Let us start with the well known case of $T^{1,1}$ [12]. The manifold C_6 relevant for this example can be written as a singular quadric in \mathbb{C}^4 , $\sum_{a=1}^4 w_a^2 = 0$, or equivalently

$$\det W = 0, \quad (W \equiv \sigma^a w^a, \sigma = (\sigma^i, i\mathbf{1})), \quad (4.4)$$

σ^i being Pauli matrices. This equation is invariant under $SO(4) \times U(1)_R \sim SU(2) \times SU(2) \times U(1)_R$. The constraint (4.4) can be solved in terms of complex doublets A_i, B_j ($W_{ij} \sim A_i B_j$) satisfying

$$|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2, \quad A_i \sim e^{i\alpha} A_i, B_i \sim e^{-i\alpha} B_i. \quad (4.5)$$

C_6 is a cone over $T^{1,1}$. The base of the cone is obtained by intersecting C_6 with the sphere $\sum_{a=1}^4 |w_a|^2 = 1$, or, equivalently, by restricting $\sum |A_i|^2 = \sum |B_i|^2 = 1$ in eq. (4.5). In this way we obtain an equation for $(S^3 \times S^3)/U(1) = (SU(2) \times SU(2))/U(1) = T^{1,1}$. An Einstein metric on $T^{1,1}$ is

$$ds_{T^{1,1}}^2 = \frac{1}{9} (d\psi + \sum_{i=1}^2 \cos \theta_i d\phi_i)^2 + \frac{1}{6} \sum_{i=1}^2 (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2). \quad (4.6)$$

The angular variable ψ ranges from 0 to 4π , while (θ_1, ϕ_1) and (θ_2, ϕ_2) parameterize two S^2 's in the standard way. The expression above shows that $T^{1,1}$ is an S^1 bundle over $S^2 \times S^2$. The metric is invariant under $SU(2) \times SU(2) \times U(1)_R$, where the $SU(2)$ factors act on the two S^2 and $U(1)_R$ shifts the angle ψ . By forgetting an $SU(2)$, $T^{1,1}$ can be also written as an S^3 bundle over S^2 . It can be proved that such bundle is topologically trivial (see for instance [12]), so that $T^{1,1}$ is isomorphic to $S^3 \times S^2$. In particular, $T^{1,1}$ has non-trivial two and three cycles where we could wrap D-branes.

It is difficult, in general, to determine the world-volume theory of branes sitting at singularities different from the orbifold ones. A powerful hint in this direction is provided by the observation that the space transverse to the branes should describe the moduli space of the gauge theory. In our case, equations (4.5) can be viewed as the D-terms of an $\mathcal{N} = 1$ abelian gauge theory [12] $U(1) \times U(1)$ with two sets of chiral multiplets A_i and B_i with charges $(\mathbf{1}, -\mathbf{1})$ and $(-\mathbf{1}, \mathbf{1})$, respectively. Here the diagonal $U(1)$ factor is decoupled while the other linearly independent combination of the $U(1)$'s acts as in eq. (4.5). We identify this theory with that living on the world-volume of a brane placed at the conifold singularity. The moduli space of vacua of such abelian $\mathcal{N} = 1$ theory is in fact identical to C_6 . When we consider a stack of N parallel D3-branes at the singularity, we have to extrapolate this result to the non-abelian case. We then consider a $U(N) \times U(N)$ theory with two sets of chiral fields A_i, B_i transforming in the representations (N, \bar{N}) and (\bar{N}, N) . We must also add to the theory the superpotential

$$W = h \varepsilon_{ij} \varepsilon_{pq} \text{Tr}(A_i B_p A_j B_q). \quad (4.7)$$

Such superpotential respects all the symmetries of the model and is crucial for avoiding a proliferation of geometrically-redundant non-abelian modes [12]: with W one can check that the classical moduli space of vacua corresponds to N copies of (4.5), corresponding to N branes free to move on C_6 . The global symmetry of the CFT is $SU(2) \times SU(2) \times U(1)_R$, which corresponds to the

isometry of $T^{1,1}$. When considering the AdS dual, the $U(1)$ factors become invisible and the gauge group is $SU(N) \times SU(N)$.

There are various strong checks that the identification is correct. First of all, the theory has to be conformal. Using the results of [12] it can be rigorously proved that this non-abelian gauge theory flows at low energies to an interacting conformal field theory. Indeed, even though the theory depends on various parameters, the couplings $g_{YM,i}$ and h , the conditions for conformal invariance [37] impose a single relation among them [12]. For both groups, the vanishing of the exact NSVZ beta functions [38]⁴ gives the relation

$$\mu \frac{d}{d\mu} \frac{8\pi^2}{g_{YM,i}^2} \sim 3T(G) - \sum T(R_a)(3 - 2\Delta_a) = N(2(\Delta_A + \Delta_B) - 3) = 0, \quad (4.8)$$

where $\Delta_{A,B}(g_{YM,i}, h)$ are the dimensions of the fields A_i and B_j . These dimensions do not depend on the indices i, j due to the $SU(2) \times SU(2)$ invariance. When (4.8) is satisfied, the last condition, which requires that the superpotential has scaling dimension three [37], is automatically satisfied. We are thus left with a manifold of fixed points, defined implicitly by the requirement that the dimension of the gauge invariant operator $Tr(AB)$ is $3/2$. This manifold has complex dimension two: the two exactly marginal parameters of the CFT are identified in the gravity dual with the dilaton and the value $\int_{S^2} B_{(2)}$ of the B-field on the non-trivial two cycle. These two are free parameters in the type IIB solution since they only enter the equations of motion under derivatives.

There are many other impressive checks of the identification:

- *The spectrum:* the complete KK spectrum of Type IIB compactified on $T^{1,1}$ has been computed [39], finding a complete agreement with *CFT* expectations.
- *The central charge:* it is well known that the conformal anomaly in four dimensions is determined by two central charges a and c which multiply the two independent invariants in the trace of the stress-energy tensor

$$T_\mu^\mu = -\frac{a}{16\pi^2}(R_{ijkp}^2 - 4R_{IJ}^2 + R^2) + \frac{c}{16\pi^2}(R_{ijkp}^2 - 2R_{IJ}^2 + \frac{1}{3}R^2) \quad (4.9)$$

c can be also extracted from the two point function $\langle T(x)T(0) \rangle \sim c/|x|^8$. For conformal theories with an AdS dual, the central charges satisfy $c = a$ [40]. The explicit value of $c = a$ is proportional to the inverse volume of M_5 [41]⁵ and, therefore, can be determined through

⁴In $\mathcal{N} = 1$ gauge theories, if we use a holomorphic scheme, the beta function is completely determined at 1-loop. From this result one can then deduce the following beta function for the IPI coupling $\mu \frac{d}{d\mu} \frac{8\pi^2}{g_{YM}^2} = f(g_{YM})(3T(G) - \sum T(R_a)(3 - 2\Delta_a))$ where $T(G)$ is the second Casimir of the group G , and $T(R_a)$ are the Casimirs for the representations R_a of the matter fields; we use the conventions that $T(\text{adjoint}) = N_c$ and $T(\text{fundamental}) = 1/2$ for an $SU(N_c)$ group. Here, $f(g_{YM})$ is a positive scheme dependent function of the coupling. The knowledge of $f(g_{YM})$ is not necessary when imposing the scheme independent condition $\beta(g_{YM}) = 0$.

⁵This scaling is better seen by writing the solution in the form $ds^2 = L^2 ds_{AdS}^2 + L^2 ds_{M_5}^2$ and $F_5 \sim N \text{vol}_{M_5} / \text{Vol}(M_5)$ (so that $\int_{M_5} F_5 = N$). The relation between L and $\text{Vol}(M_5)$ then follows from the Einstein equation $R \sim F_{ijkl} F^{ijkl}$ and reads $L^4 \sim N / \text{Vol}(M_5)$. The effective five dimensional action for g_{AdS} with all factors of L and $\text{Vol}(M_5)$ rescaled is then proportional to $L^8 \text{Vol}(M_5) \sim 1 / \text{Vol}(M_5)$, and this factor will appear multiplicatively in all correlation functions computed from the AdS/CFT prescription.

the relation $c/c_{N=4} = \text{Vol}(S^5)/\text{Vol}(M_5)$. The value of $c = a$ is explicitly computable with quantum field theory methods from the R-charge R using the formula [42, 43]

$$a(R) = \frac{3}{32}(3\text{Tr}R^3 - \text{Tr}R) \quad (4.10)$$

where the trace is taken over all the fermionic fields ⁶. Formula (4.10) predicts the ratio $c/c_{N=4} = 27/16$ which remarkably agrees with the computation using the volumes since $\text{Vol}(S^5) = \pi^3$ and $\text{Vol}(T^{1,1}) = 16\pi^3/27$ [41].

- *The baryons:* $U(1)$ factors are not described by the AdS/CFT duality. In our case one of them is decoupled while the other reduces to a global baryonic symmetry. The corresponding gauge field in $AdS_5 \times T^{1,1}$ is obtained by reducing the RR four-form on the three sphere S^3 . This vector multiplet is known in the supergravity literature as the Betti multiplet. Baryons in this theory are then constructed as D3-branes wrapped on S^3 . The two baryonic operators A^N and B^N can be identified as D3-branes wrapped over the two supersymmetric cycles at $(\theta_1 = \phi_1 = 0)$ and $(\theta_2 = \phi_2 = 0)$. The conformal dimension of these chiral operators is large, being equal to $3N/4$. This formula can be explicitly tested: Δ can be computed in purely geometrical terms from the volume of the three cycle Σ using the formula [44]

$$\Delta = \frac{\pi N \text{Vol}(\Sigma)}{2 \text{Vol}(T^{1,1})} \quad (4.11)$$

This formula is a consequence of the fact that the mass of a wrapped D3-brane can be computed from the volume of the three cycle. Moreover, for operators with large dimension, the familiar relation between mass and dimension becomes linear $m = \sqrt{\Delta(\Delta - 4)}R^2 \sim \Delta R$. This piece of information and some attention to normalizations leads to formula (4.11). By using the explicit metric for $T^{1,1}$ given above, it is easy to check that formula (4.11) reproduces the correct result $\Delta = 3N/4$.

It is remarkable that a similar analysis can be carried out for the manifolds $Y^{p,q}$ and $L^{p,q,r}$. We refer for a detailed treatment to [46, 47, 48, 49, 50, 51]. In Figure 1 we pictured in quiver language the gauge theory corresponding to the simplest case $Y^{2,1}$. The check of baryonic dimensions and central charges (computable both from field theory - using the recently developed technique of a-maximization - and from supergravity from volumes of cycles) gives a remarkable agreement. We

⁶To use this formula, one has to remember that, for chiral fields, the conformal dimension is related to the R-charge by $\Delta = 3R/2$. Normalizations are chosen such as the $\mathcal{N} = 1$ supercharge Q has charge -1 and the gaugino (with dimension $3/2$) has charge $+1$. For a chiral field $\Phi = \phi + \theta\psi + \dots$ with dimension Δ the associated fermion ψ has charge $2\Delta/3 - 1$. The difficulty in using formula (4.10) is to find the correct superconformal R-charge, since we can always add any combination of existing anomaly free $U(1)$ global symmetries. Actually, the a-maximization procedure discovered in [43] implies that the R-charges of an $\mathcal{N} = 1$ superconformal fixed point are the ones that locally maximize the functional (4.10). The maximization is done with respect to the most general assignment of R-charges r_i to the chiral fields that i) respects the global symmetries, ii) gives R-charge 2 to each term in the superpotential and iii) implies the vanishing of the numerator of the NSVZ beta function for each group (this last condition, written as $\beta \sim T(G) + \sum T(R_a)(r_i - 1)$, is equivalent to the vanishing of the t'Hooft anomalies for the $U(1)_R$ charge). In the $T^{1,1}$ case, we have a global $U(1)$ baryonic symmetry that may mix with R; the most general assignment is $r_A = x$ and $r_B = 1 - x$ and $a(R)$ has a local maximum exactly for $r_A = r_B = 1/2$. This result is obviously consequence of the Z_2 symmetry between A and B . More generally, it is easy to prove that baryonic symmetries (or symmetries J that satisfy $\text{Tr}J = 0$) cannot mix with the superconformal R-charge.

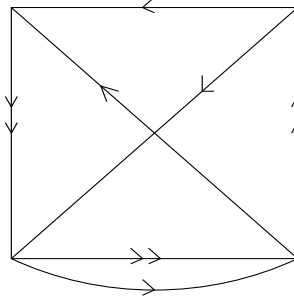


Figure 5: The quiver for $Y^{2,1}$. Each node is a gauge group $U(N)$ and each link, counted with arrow multiplicity, a chiral bi-fundamental. For each closed loop of oriented arrows (triangle or rectangle) there is a term in the superpotential containing the product of the fields along the loop.

should also notice that some remarkable progress has been made recently in the identification of the correspondence between toric singularities and dual gauge theories [52] using the *brane tiling*, an ingenious generalization of the brane boxes construction [53, 54].

4.2 Wrapped and fractional branes

Most of the recently proposed duals of non-conformal theories are based on wrapped and fractional branes. The philosophy may be exemplified in the four dimensional case as follows. Consider a geometry with a non-trivial two-cycle S^2 on which we wrap a D5-brane. The world-volume of the brane is thus of the form $\mathbb{R}^4 \times S^2$, and at energies lower than the inverse radius of S^2 the theory living on the world-volume is effectively four dimensional. String theory has many moduli, some geometrical in nature and some related to the bundles of antisymmetric forms which are always present in string theory. For simplicity, we focus on two specific moduli associated with S^2 : the volume of S^2 and the integral of the B -field over the cycle. Only the first modulus has a geometrical meaning. These moduli appear in the Born-Infeld action for the D-brane

$$\begin{aligned}
 & -\frac{1}{(2\pi)^5 \alpha'^3} \int dx^6 e^{-\Phi} \sqrt{G + 2\pi\alpha' F + B} = \\
 & = -\frac{1}{(2\pi)^5 \alpha'^3} \int dx^4 \left[e^{-\Phi} \int_{S^2} d\Omega_2 \sqrt{(G+B)_{S^2}} \right] \sqrt{(G + 2\pi\alpha' F + B)_{\mathbb{R}^4}}. \quad (4.12)
 \end{aligned}$$

We see, by expanding the last square root, that the four dimensional gauge theory has an effective coupling which reads

$$\frac{1}{g^2} \sim e^{-\Phi} \int_{S^2} d\Omega_2 \sqrt{(G+B)_{S^2}}. \quad (4.13)$$

Whenever the quantity on the r.h.s. of this equation runs, also the coupling does, and the resulting theory is non-conformal. We can then have two basic different models:

- **Wrapped branes** [55, 56]: configurations of D5-branes wrapped in a supersymmetric fashion on a non-vanishing two-cycle $Vol(S^2) \neq 0$. There is no need to introduce a B -field.
- **Fractional branes** [57]: configurations of D5-branes wrapped on collapsed cycles. If $2\pi b = \frac{1}{2\pi\alpha'} \int_{S^2} B \neq 0$, the corresponding four-dimensional theory has still a non-vanishing well-defined coupling constant. Manifolds with collapsed cycles are singular, and fractional branes must live at the singularity.

The amount of supersymmetry preserved in these kinds of model depends on how the S^2 is embedded in the background geometry.

In particular, Calabi-Yau cones over bases with non-trivial two cycles have collapsing spheres. Thus both $T^{1,1}$ and $Y^{p,q}$ admit fractional branes [58, 59]. Let us, for simplicity, consider the case of $T^{1,1}$ which has a single collapsed two cycle. A D5-brane or anti-D5-brane wrapped over an S^2 in M_5 will minimize its volume sitting at the tip of the cone $r = 0$. Since the sphere is collapsed to a point, these five-branes have a complementary interpretation as three-branes. There are two different RR four-forms after the reduction on C_6 : the original type IIB $C_{(4)}$ and $C_{(4)}^T$, the reduction of the ten dimensional RR six-form $C_{(6)}$ along S^2 . Consequently, we expect to classify three branes using two RR charges and we expect the existence of two different types of branes, that we name fractional. The two types of fractional branes can be represented as a D5-brane wrapped on the collapsed two-cycle, or an anti-D5-brane with one unit of flux for the gauge field living on it: $\int_{S^2} F = -2\pi$ [60, 57, 61]. From the Born-Infeld and Wess-Zumino action for D5 or anti-D5 branes

$$-\frac{1}{(2\pi)^5 \alpha'^3} \left[\int dx^6 e^{-\Phi} \sqrt{G + 2\pi\alpha' F + B} \right. \\ \left. \pm \int (C_{(6)} + C_{(4)} \wedge (2\pi\alpha' F + B) + \frac{1}{2} C_{(2)} \wedge (2\pi\alpha' F + B)^2 + \frac{1}{6} C_{(0)} \wedge (2\pi\alpha' F + B)^3) \right]. \quad (4.14)$$

we see that the charges under $(C_{(4)}^T, C_{(4)})$ are $(1/2, b)$ and $(-1/2, 1-b)$, respectively. Since the tensions are proportional to $|b|$ and $|1-b|$, for $b \in [0, 1)$, these values satisfy the BPS condition. A physical D3-brane is made as a bound state of two different types of fractional branes.

Each D5 or anti-D5 brane gives rise in four dimensions to a gauge field. From this point of view the $U(N) \times U(N)$ gauge theory living on N physical D3-branes at the conifold singularity can be reinterpreted as follows: each physical D3-brane is actually divided in two fractional branes and each type of three brane is responsible for one of the two gauge groups. Extrapolating from this, we conclude that, in the presence of n_1 and n_2 fractional branes of the two types we realize a $U(n_1) \times U(n_2)$ gauge theory with two pairs of bi-fundamental fields A_i, B_i interacting with the superpotential (4.7) [58]. Fractional branes can only live at the singularity, but two fractional branes of different type can join to form a physical brane that can move freely on the conifold: this motion corresponds to the Higgs branch of the gauge theory. The complexified gauge couplings of the two groups, $\tau_i = \frac{\theta_i}{2\pi} + i \frac{4\pi}{g_{YM}^2}$, are determined (for $b \in [0, 1)$) in terms of the space-time fields by equation (4.14)

$$\tau_1 = (b\tau + c), \quad \tau_2 = (1-b)\tau - c, \quad (4.15)$$

where $\tau = C_{(0)} + ie^{-\Phi}$ is the complex dilaton of Type IIB. For $n_1 = n_2$ the theory is conformal and the two coupling constants correspond to two exactly marginal parameters in $AdS_5 \times T^{1,1}$: the dilaton and the value of the B -field on S^2 . For $n_1 = N + M$ and $n_2 = M$ the theory is no longer conformal. One of the two gauge factor is not asymptotically free in the UV. We will discuss the UV completion of this theory soon.

4.3 The gravity duals

We now investigate the KS solution corresponding to a set of N physical D3-branes and M fractional branes at a conifold singularity. The corresponding gauge group is $SU(N+M) \times SU(N)$.

As usual, all the $U(1)$ factors are not described by the AdS/CFT correspondence. As we will see the KS solution is also continuously connected to MN.

The type IIB solution corresponding to N physical branes is $AdS_5 \times T^{1,1}$. The addition of M fractional branes induces the presence of a RR $F_{(3)}$ flux. Since we have a D5-brane wrapping the S^2 in a space-time with topology $\mathbf{R}^{1,3} \times \mathbf{R} \times S^2 \times S^3$, where the \mathbf{R} direction is interpreted as the radial one, we expect a flux supported on S^3 :

$$\int_{S^3} F_{(3)}^{RR} = M \quad (4.16)$$

Since the type IIB equations of motion require

$$d * (e^{-\phi} H_{(3)}) \sim F_{(3)} \wedge F_{(5)} \quad (4.17)$$

$B_{(2)}$ cannot vanish: this will induce a radial dependence of the couplings τ_1 and τ_2 in (4.15) which is interpreted in the quantum field theory as the running of the couplings with the scale. We may expect to find a metric of warped form

$$ds_{10}^2 = h^{-1/2}(t) dx_4^2 + h^{1/2}(t) ds_6^2. \quad (4.18)$$

KS found a regular solution that is compatible with all these requirements [8].

A supersymmetric solution with this minimal set of fields and internal metric given by the conifold one, was found in [59], but it has a naked singularity in the IR. In [8], a regular solution was found by considering a deformed conifold instead of the original singular one. We will see that the deformation of the conifold corresponds to the chiral symmetry breaking in the dual field theory. In terms of complex geometry, the deformation of the singular conifold $\sum w_a^2 = 0$ is described by the equation in \mathbb{C}^4

$$\sum w_a^2 = \epsilon^2. \quad (4.19)$$

The deformation consists in blowing-up an S^3 at the apex of the conifold, so to obtain a smooth manifold. The deformed conifold metric can be written as

$$ds_6^2 = \frac{\epsilon^{4/3}}{2} K(t) \left[\frac{(dt^2 + (\tilde{\epsilon}_3)^2)}{3K^3(t)} + \frac{\cosh t}{2} (e_1^2 + e_2^2 + \epsilon_1^2 + \epsilon_2^2) + \frac{1}{2} (e_1 \epsilon_1 + e_2 \epsilon_2) \right], \quad (4.20)$$

where $K(t) = (2^{1/3} \sinh t)^{-1} (\sinh(2t) - 2t)^{1/3}$. In this formula $e_1 = d\theta_1$ and $e_2 = -\sin\theta_1 d\phi_1$. Similarly $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ are the left-invariant forms on S^3 with Euler angle coordinates ψ, θ_2, ϕ_2

$$\begin{aligned} \epsilon_1 &\equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2, \\ \epsilon_2 &\equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2, \\ \epsilon_3 &\equiv d\psi + \cos\theta_2 d\phi_2, \\ d\epsilon_i &= -\frac{1}{2} \epsilon_{ijk} \epsilon_j \wedge \epsilon_k. \end{aligned} \quad (4.21)$$

We also defined, for convenience, $\tilde{\epsilon}_3 = \epsilon_3 + \cos\theta_1 d\phi_1$.

The KS solution consists of a metric of the form (4.18), with ds_6 as in (4.20), warp factor given by

$$h(t) = (g_s M \alpha')^2 2^{2/3} \epsilon^{-8/3} \int_t^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh 2x - 2x)^{1/3}, \quad (4.22)$$

and antisymmetric fields

$$\begin{aligned}
B_{(2)} &= \frac{g_s M \alpha'}{4} [(f(t) + k(t))(e_2 \wedge e_1 + \varepsilon_2 \wedge \varepsilon_1) + (k(t) - f(t))(e_2 \wedge \varepsilon_1 - e_1 \wedge \varepsilon_2)], \\
F_{(3)} &= \frac{M \alpha'}{4} [\tilde{\varepsilon}_3 \wedge (e_2 \wedge e_1 + \varepsilon_2 \wedge \varepsilon_1) + (1 - 2F)\tilde{\varepsilon}_3 \wedge (e_2 \wedge \varepsilon_1 - e_1 \wedge \varepsilon_2) + F' dt \wedge (e_1^2 + e_2^2 + \varepsilon_1^2 + \varepsilon_2^2)], \\
F_{(5)} &= \mathcal{F}_{(5)} + \star \mathcal{F}_{(5)}, \\
\mathcal{F}_{(5)} &= B_{(2)} \wedge F_{(3)} = \frac{g_s M^2 (\alpha')^2}{16} [f(1 - F) + kF] e_1 \wedge e_2 \wedge \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3.
\end{aligned} \tag{4.23}$$

The functions of t appearing in the previous formulae read

$$\begin{aligned}
F(t) &= \frac{\sinh t - t}{2 \sinh t}, \\
f(t) &= \frac{t \coth t - 1}{2 \sinh t} (\cosh t - 1), \\
k(t) &= \frac{t \coth t - 1}{2 \sinh t} (\cosh t + 1).
\end{aligned} \tag{4.24}$$

The complex dilaton of Type IIB is constant and this allows for a small string coupling everywhere.

KS belongs to a special class of backgrounds where the internal metric is conformally Calabi-Yau. For such metrics, the condition for supersymmetry requires a constant dilaton and a self-dual flux $\star G = iG$, where $G_{(3)} = H_{(3)} + ie^{-\phi} F_{(3)}$ [63]. It is easy to check that the condition $\star G = iG$ is satisfied by KS.

4.4 Properties of the KS solution

For large values of t (which correspond to the UV limit of the dual gauge theory) it is convenient to introduce the radial coordinate $r \sim e^{t/3}$. The metric reduces to

$$ds_{10}^2 \rightarrow h^{-1/2}(r) dx_4^2 + h^{1/2}(r) (dr^2 + r^2 ds_{T^{1,1}}^2), \tag{4.25}$$

with $h(r) = \frac{81(\alpha' g_s M)^2}{8r^4} \log(r/r_s)$. It can be viewed, in some sense, as a logarithmic deformation of $AdS_5 \times T^{1,1}$. This was the solution first found in [59]. If we would allow r to range in $[0, \infty)$ it would be singular for $r = r_s$. In this limit, the RR and NSNS forms reduce to

$$F_{(3)} \rightarrow \frac{M \alpha'}{4} \tilde{\varepsilon}_3 \wedge (e_2 \wedge e_1 + \varepsilon_2 \wedge \varepsilon_1), \quad B_{(2)} \rightarrow \frac{3g_s M \alpha'}{4} \log(r/r_s) (e_2 \wedge e_1 + \varepsilon_2 \wedge \varepsilon_1). \tag{4.26}$$

It is believed that the $SU(N+M) \times SU(N)$ theory exhibits a series of Seiberg dualities until it eventually reduces in the deep IR to pure $SU(M)$. At each step of the cascade, the group is $SU(N+M-kM) \times SU(N-kM)$. The strongly coupled factor $SU(N+M-kM)$ undergoes a Seiberg duality to $SU(N-M-kM)$, while the other factor remains inert⁷. As a result, k is increased by one unit.

⁷We refer to [66] for a detailed discussion of Seiberg duality. We simply mention that this duality occurs for a $SU(N)$ theory with $N_f > N + 1$ flavors of quark chiral superfields A_i, \bar{A}_i , $i = 1, \dots, N_f$, in the $\mathbf{N}, \bar{\mathbf{N}}$ representations. In this case the theory is dual to another $\mathcal{N} = 1$ SYM with $SU(N_f - N)$ gauge group, N_f flavors C_i, \bar{C}_i , and an extra gauge singlet chiral superfield N^{ij} interacting by the superpotential $W = CNC$.

In the KS solution, this can be seen from the UV limit of the RR five-form field strength which can be rewritten as

$$\mathcal{F}_{(5)} \sim N_{eff}(r) \text{vol}(T^{1,1}), \quad N_{eff}(r) = N + \frac{3}{2\pi} g_s M^2 \log(r/r_0). \quad (4.27)$$

We introduced a convenient reference scale r_0 defined so that the effective D3-charge $N_{eff}(r_0) = N$. The logarithmic decreasing of N_{eff} with the radius was interpreted in [59] as a decreasing in the rank of the dual gauge theory group as the theory flows to the IR. At the UV scale $r = r_0$, $N_{eff} = N$ and the dual field theory has $SU(N+M) \times SU(N)$ as gauge group. At $r_k = r_0 \exp(-2\pi k/3g_s M)$, with k integer, the dual gauge group is $SU(N-kM+M) \times SU(N-kM)$. If $N = kM$, we thus find that after k cascade steps the gauge group flows to $SU(M)$, with a subtlety that we will discuss later on. The UV completion of the theory is somewhat peculiar. The inverse cascade never stops. In a sense, the UV limit is a $SU(\infty) \times SU(\infty)$ gauge theory.

The metric in eq. (4.25) can be used to study the UV properties of the $SU(N+M) \times SU(N)$ gauge theory when $M \ll N$. Indeed, the curvature, which is determined by Ng_s at the reference scale r_0 , decreases for larger values of r . Moreover, if Mg_s is sufficiently small the cascade steps are well separated. In these conditions, the singular metric (4.25), which is a logarithmic deformation of $AdS_5 \times T^{1,1}$, will give a convenient description of the almost conformal theory $SU(N+M) \times SU(N)$. As shown in formula (4.15), the gauge couplings are related to some of the supergravity moduli

$$\frac{1}{g_1^2} + \frac{1}{g_2^2} = \frac{1}{4\pi g_s}; \quad \frac{1}{g_1^2} - \frac{1}{g_2^2} = \frac{1}{4\pi^2 g_s} \left(\frac{1}{2\pi\alpha'} \int_{S^2} B_{(2)} - \pi \right). \quad (4.28)$$

We can consider the D5 branes as wrapped on the S^2 with $\theta_1 = \theta_2, \phi_1 = -\phi_2$. In the large r limit, we thus find that the sum of the gauge couplings is constant while (see (4.26)) the difference runs as

$$\frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2} = 3M \log(r/r_s) = 3M \log(\mu/\Lambda). \quad (4.29)$$

The last equality in the above equation requires a specific choice of how to relate the radial coordinate to the energy scale of the field theory. We use the same relation as for the conformal $AdS_5 \times T^{1,1}$ solution, $r/r_s = \mu/\Lambda$, Λ being the IR scale⁸. One can show that eq. (4.29) reproduces, up to orders M/N , the UV gauge theory result obtained from the exact NSVZ beta function [38] for $\mathcal{N} = 1$ gauge theories. Using formula (4.8) we can indeed write

$$\begin{aligned} \frac{4\pi^2}{g_1^2} &= \frac{1}{2} (3(N+M) - N(6 - 2\Delta_A - 2\Delta_B)) \ln(\mu/\Lambda), \\ \frac{4\pi^2}{g_2^2} &= \frac{1}{2} (3N - (N+M)(6 - 2\Delta_A - 2\Delta_B)) \ln(\mu/\Lambda). \end{aligned} \quad (4.30)$$

At leading order in M/N , $\Delta_A + \Delta_B = 3/2$, which is the result for the conformal case. The difference of the two equations in (4.30) then reproduces the supergravity result.

For small t the metric approximates to

$$ds_{10}^2 \rightarrow \frac{\varepsilon^{4/3}}{bg_s M \alpha'} dx_\mu dx^\mu + c(g_s M \alpha') \left[\frac{1}{2} (dt^2 + t^2 \Omega_2) + d\Omega_3 \right], \quad (4.31)$$

⁸The different methods for computing the radius/energy relation give different results [16]. However, if we are only interested in the leading logarithmic UV behavior all methods agree.

where b and c are numerical constants and we introduce the parameter ε related to the value of h for $t = 0$. The angular part splits in a non-vanishing S^3 and a shrinking S^2 . The curvature is controlled by the value of $g_s M$, and it is small when this parameter is large. The antisymmetric fields $B_{(2)}, F_{(5)}$ go to zero in the limit, while F_3 is supported only by the non-vanishing S^3 . The fact that $F_{(5)} = 0$ implies that all the physical branes disappear in the IR. This suggests to interpret the solution as describing the $N = kM$ case, where the final point of the cascade is an $SU(M)$ theory. The case where $N = kM + p, p = 1, \dots, M - 1$ reduces to an IR theory $SU(p) \times SU(M + p)$ and it is believed to be described by a solution with explicit brane sources.

The theory confines and presents the standard pattern of chiral symmetry breaking. The natural candidate for a QCD string is a fundamental string. As discussed in Section 2, confinement is expected because the space-time components of the metric at $t = 0$ are non-vanishing. The value of a Wilson loop, indeed, can be computed using a fundamental string coming from infinity, with endpoints on the boundary at $t = \infty$. The string will minimize its energy by reaching $t = 0$ where the metric components $\sqrt{g_{xx}g_{tt}}$ have a minimum. All the relevant contribution to the energy between two external sources is then due to a string sitting at $t = 0$ and stretched in the x direction. The tension for confining strings $\alpha' T_s \sim h^{-1/2}$ is thus of the order $\varepsilon^{4/3}/g_s M \alpha'$. We see that the free parameter ε sets the scale of the confining theory.

Let us now discuss the chiral symmetry. We expect various phenomena:

- The chiral symmetry breaking. In quantum field theory, $U(1)_R$ is anomalous and broken to \mathbb{Z}_{2N} by instantonic effects. These non-perturbative effects in the field theory are already captured at the supergravity level. The existence of an anomaly can be detected with an UV computation in quantum field theory and therefore should be already visible in the UV region of the solution [64, 65]. In the KS solution the $U(1)_R$ symmetry acts as a rotation of the complex variables w_i ; on the metric it acts as a shift of the angle ψ . The UV form of the metric is invariant under such shift, but this is not the case for the RR two-form $C_{(2)} \sim -\frac{N\alpha'}{2}\psi \sin\theta d\theta \wedge d\phi$. In particular, the flux $\frac{1}{2\pi\alpha'} \int_{S^2} C_{(2)}$ varies by $-N\delta\psi$ under a shift of ψ . Since the flux is periodic with period 2π , the only allowed transformations are those with $\delta\psi = \frac{2\pi n}{N}$ [64]: the R-symmetry is then broken to the \mathbb{Z}_{2N} subgroup $\psi \rightarrow \psi + 2\pi n/N$ (ψ has period 4π). The spontaneous symmetry breaking $\mathbb{Z}_{2M} \rightarrow \mathbb{Z}_2$ is manifest in the full KS solution. Indeed, while the UV metric is $U(1)_R$ invariant (the metric of $T^{1,1}$), the full KS metric depends on ψ through $\cos\psi$ and $\sin\psi$, and has in fact only a \mathbb{Z}_2 invariance under $\psi \rightarrow \psi + 2\pi$. The breaking is also evident in eq. (4.19) that is not invariant under arbitrary phase shifts of the w_i , but only under $w_i \rightarrow -w_i$.
- Domain walls. In a theory with spontaneous breaking of \mathbb{Z}_{2N} and multiple vacua, we expect the existence of domain walls. In the string solution, they correspond to D5-branes wrapped on S^3 , located at $t = 0$ in order to minimize the energy. One can estimate the tension T_{DW} of the domain wall from the fact that the metric in the IR is approximately of the form $\mathbb{R}^7 \times S^3$ and the radius of the three-sphere goes as $\sqrt{e^{\Phi_0} N \alpha'}$. We have

$$T_{DW} \sim \frac{1}{\alpha'^3} \int_{S^3} e^{-\Phi} \sqrt{G} = \frac{e^{2\Phi_0} N^{3/2}}{\alpha'^{3/2}}. \quad (4.32)$$

Since a fundamental string can end on a D5-brane and the QCD string is a fundamental string, we see that a QCD string can end on a domain wall.

The glueball spectrum can be determined by solving the equation of motions of the bulk fields propagating in the background. The typical glueball masses can be estimated as $\varepsilon^{2/3}/g_s M \alpha'$.

4.5 The baryonic branch

The study of glueballs in the KS solution offers interesting surprises. As noticed in [33], one can find a massless glueball in the spectrum. This implies that the theory has no mass gap. Strictly speaking, indeed, the theory in the deep IR is not in the same class of universality of pure SYM $\mathcal{N} = 1$ with gauge group $SU(M)$. In fact, as suggested in [32, 8], the last step in the duality cascade, corresponding to the gauge group $SU(2M) \times SU(M)$ is on the baryonic branch. We expect that the $SU(2M)$ factor becomes strongly coupled while the $SU(M)$ factor, which is IR free, is still weakly coupled. Ignoring for the moment the $SU(M)$ factor, the $SU(2M)$ theory has a number of flavors equal to the number of colors. In this situation, we cannot simply apply a Seiberg duality [66]. Let us analyze in details the vacua of this theory. We can form mesonic operators $(N_{ij})_{\alpha}^{\beta}$ with $SU(2) \times SU(2)$ indices i, j and $SU(M)$ flavor indices, and baryonic operators that are $SU(2) \times SU(2)$ and flavor invariant

$$\mathcal{B} \sim \varepsilon_{\alpha_1 \alpha_2 \dots \alpha_{2M}} (A_1)_{\alpha_1}^{\alpha_1} (A_1)_{\alpha_2}^{\alpha_2} \dots (A_1)_{\alpha_M}^{\alpha_M} (A_2)_{\alpha_1}^{\alpha_{M+1}} (A_2)_{\alpha_2}^{\alpha_{M+2}} \dots (A_2)_{\alpha_M}^{\alpha_{2M}} \quad (4.33)$$

$$\bar{\mathcal{B}} \sim \varepsilon^{\alpha_1 \alpha_2 \dots \alpha_{2M}} (B_1)_{\alpha_1}^1 (B_1)_{\alpha_2}^2 \dots (B_1)_{\alpha_M}^M (B_2)_{\alpha_{M+1}}^1 (B_2)_{\alpha_{M+2}}^2 \dots (B_2)_{\alpha_{2M}}^M \quad (4.34)$$

The theory with number of flavors equal to the number of colors has a classical moduli space of vacua that can be parametrized by baryons and mesons subject to the classical constraint $Det N = \mathcal{B} \bar{\mathcal{B}}$. In the quantum theory this relation is corrected to [66]

$$Det N - \mathcal{B} \bar{\mathcal{B}} - \Lambda_{2M}^{4M} = 0 \quad (4.35)$$

where Λ_{2M} is the UV scale of the gauge group $SU(2M)$. We can write the superpotential of the $SU(2M) \times SU(M)$ theory in terms of the new variables as

$$W = \lambda Tr(N_{ij} N_{pq}) \varepsilon^{ip} \varepsilon^{jq} + X (Det N - \mathcal{B} \bar{\mathcal{B}} - \Lambda_{2M}^{4M}) \quad (4.36)$$

where X is a Lagrangian multiplier. We have two types of vacua: a mesonic branch where $\mathcal{B} = \bar{\mathcal{B}} = 0$ while $Det N = \Lambda_{2M}^{4M}$ and a baryonic branch where $X = N = 0$, $\mathcal{B} \bar{\mathcal{B}} = -\Lambda_{2M}^{4M}$. The two branches are disjoint and of complex dimension M and one, respectively. The fact that the global symmetry $SU(2) \times SU(2)$ is typically broken in a mesonic vacuum strongly suggests that the KS solution describes a point on the baryonic branch⁹. In such vacua, the $U(1)_B$ global symmetry ($A_i \rightarrow e^{i\alpha} A_i$, $B_j \rightarrow e^{-i\alpha} B_j$) is explicitly broken. The baryonic branch has complex dimension 1, and it can be parametrized by ζ :

$$\mathcal{B} = i\zeta \Lambda_{2M}^{2M}, \quad \bar{\mathcal{B}} = \frac{i}{\zeta} \Lambda_{2M}^{2M}. \quad (4.37)$$

Note that the $U(1)_B$ corresponds to changing ζ by a phase. The Goldstone boson associated to the spontaneous breaking of the $U(1)_B$ symmetry was identified in the supergravity dual as a massless pseudo-scalar bound state (glueball) in [33]. By supersymmetry the Goldstone boson is in a $\mathcal{N} = 1$

⁹It is expected that the supergravity description of the mesonic branch involve M physical branes moving on the deformed conifold.

chiral multiplet; hence there will be a massless scalar mode that must correspond to changing ζ by a positive real factor. This is a modulus of the theory whose expectation value induces a one parameter family of supersymmetric deformations.

If this interpretation is correct there must exist a one-parameter family of supersymmetric solutions of type IIB containing the KS solution as a special point. A linearized deformation of KS was indeed found in [33] and the full family in [34]. The family belongs to an ansatz for metrics with $\mathbf{R} \times S^2 \times S^3$ topology proposed by Papadopoulos and Tseytlin (PT) [62],

$$\begin{aligned} ds_E^2 &= e^{2A} dx_\mu dx_\mu + e^{6p-x} dt^2 + ds_\zeta^2 = \\ &= e^{2A} dx_\mu dx_\mu + \sum_i^6 E_i^2 \end{aligned} \quad (4.38)$$

where p , x and A are functions of the radial coordinate t only¹⁰. The vielbeins are defined by

$$\begin{aligned} E_1 &= e^{\frac{x+g}{2}} e_1 = e^{\frac{x+g}{2}} d\theta_1, \\ E_2 &= e^{\frac{x+g}{2}} e_2 = -e^{\frac{x+g}{2}} \sin\theta_1 d\phi_1, \\ E_3 &= e^{\frac{x-g}{2}} \tilde{\epsilon}_1 = e^{\frac{x-g}{2}} (\epsilon_1 - a(t)e_1), \\ E_4 &= e^{\frac{x-g}{2}} \tilde{\epsilon}_2 = e^{\frac{x-g}{2}} (\epsilon_2 - a(t)e_2), \\ E_5 &= e^{-3p-\frac{x}{2}} dt \\ E_6 &= e^{-3p-\frac{x}{2}} \tilde{\epsilon}_3 = e^{-3p-\frac{x}{2}} (\epsilon_3 + \cos\theta_1 d\phi_1) \end{aligned} \quad (4.39)$$

where g and a are also functions of the radial coordinate only. All the relevant quantities are defined in Section 4.3.

The fluxes of the PT ansatz are given by

$$\begin{aligned} H &= h_2 \tilde{\epsilon}_3 \wedge (\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2) + dt \wedge [h'_1 (\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) \\ &\quad + \chi' (e_1 \wedge e_2 - \epsilon_1 \wedge \epsilon_2) + h'_2 (\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)], \end{aligned} \quad (4.40)$$

$$\begin{aligned} F_3 &= P [\tilde{\epsilon}_3 \wedge (\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2 - b(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)) \\ &\quad + b' dt \wedge (\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2)], \end{aligned} \quad (4.41)$$

$$F_5 = \mathcal{F}_5 + * \mathcal{F}_5, \quad (4.42)$$

$$\mathcal{F}_5 = K e_1 \wedge e_2 \wedge \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3. \quad (4.43)$$

where h_1 , h_2 , b , χ and K are function of the coordinate t , and primes always denote derivatives with respect to t . The function K is related to h_1 , h_2 and b by $K(t) = Q + 2P[h_1(t) + b(t)h_2(t)]$, where the constants Q and P are proportional to the number of regular and fractional D3 branes respectively. In particular, $P = -M\alpha'/4$. The fluxes and the expression for K are chosen in such a way that they automatically satisfy the Bianchi identities. The topology of the ansatz is $\mathbf{R} \times S^2 \times S^3$, and it has $SU(2) \times SU(2)$ symmetry.

¹⁰We found it more convenient to use slightly different conventions than PT: our radial variable t is related to the PT one by $du = e^{-4p} dt$ and similarly the function A corresponds in PT to $2p - x + 2A$. We also define a new set of vielbeins E_i . We also use the ansatz for the metric in string frame, while PT work in Einstein frame.

The KS solution fits in the PT ansatz with

$$\begin{aligned}
A &= -\frac{1}{4} \ln h & a &= -\frac{1}{\cosh t} \\
e^{6p+2x} &= \frac{3}{2} (\coth t - t \operatorname{csch}^2 t) & e^g &= \tanh t \\
e^{2x} &= \frac{(\sinh t \cosh t - t)^{2/3}}{16} h & e^\phi &= e^{\phi_0}.
\end{aligned} \tag{4.44}$$

where h is given in formula (4.22), and

$$\begin{aligned}
h_1 &= \frac{1}{2} (\coth t - t \coth^2 t) e^{\phi_0} & b &= -\frac{t}{\sinh t} \\
h_2 &= -\frac{(-1 + t \coth t)}{2 \sinh t} e^{\phi_0} & \chi &= 0.
\end{aligned} \tag{4.45}$$

The PT ansatz accommodates both KS and MN solutions. Moreover there is an entire family of supersymmetric solutions interpolating between KS and MN [34]. The solution can be determined using the supersymmetry conditions found in [67] in terms of $SU(3)$ structures. The internal manifold is complex and it is a generalized Calabi-Yau, according to Hitchin's definition [68], since it has a never vanishing $(3,0)$ form conformally closed; it is not however Kahler. The condition that the manifold is complex gives a functional relation between a and g

$$\frac{e^{2g} + 1 + a^2}{2a} = -\cosh t \tag{4.46}$$

The other supersymmetry conditions give a pair of coupled first-order differential equations for the quantities a and $v = e^{6p+2x}$

$$\begin{aligned}
a' &= -\frac{\sqrt{-1 - a^2 - 2a \cosh t} (1 + a \cosh t) \operatorname{csch} t}{v} - \frac{a \sinh t (t + a \sinh t)}{t \cosh t - \sinh t}, \\
v' &= \frac{-3a \sinh t}{\sqrt{-1 - a^2 - 2a \cosh t}} \\
&+ v [-a^2 \cosh^3 t + 2at \coth t + a \cosh^2 t (2 - 4t \coth t) + \cosh t (1 + 2a^2 \\
&- (2 + a^2) t \coth t) + t \operatorname{csch} t] / [(1 + a^2 + 2a \cosh t) (t \cosh t - \sinh t)]
\end{aligned} \tag{4.47}$$

and a set of algebraic and differential equations that allow to determine all the other unknown functions in the metric and fluxes in terms of the quantity a ,

$$\begin{aligned}
b &= -\frac{t}{\sinh t} \\
h_1 &= h_2 \cosh t \\
h_2' &= -\frac{(t - a^2 t + 2at \cosh t + a^2 \sinh 2t)}{(1 + a^2 + 2a \cosh t) (-1 + t \coth t)} h_2 \\
\chi' &= \frac{ah_2 (1 + a \cosh t) (2t - \sinh 2t)}{(1 + a^2 + 2a \cosh t) (-1 + t \coth t)} \\
A' &= -\frac{(-1 + t \coth t) \operatorname{csch} t (-\cosh t + t \operatorname{csch} t)}{8} e^{-2x+2\phi}
\end{aligned}$$

$$\begin{aligned}\sin w &= -\frac{2e^{x-\phi}(1+a\cosh t)}{\sqrt{-1-a^2-2a\cosh t}(-1+t\coth t)} \\ \cos w &= \frac{2h_2\sinh t}{e^\phi(1-t\coth t)} = \eta e^\phi\end{aligned}\quad (4.48)$$

The previous equations can be solved analytically only by a perturbative expansion around KS. However, the existence of a regular second order solution suggests that there is indeed a one parameter family of KS deformations. This expectation can be confirmed by a numerical analysis. There is a family of regular solutions that can be parametrized by the constant appearing in the IR expansion for a

$$a = -1 + \xi t^2 + O(t^4). \quad (4.49)$$

ξ ranges in the interval $[1/6, 5/6]$, with $\xi = 1/2$ corresponding to KS. The solution depends on two parameters, ξ and the value of the dilaton at $t = \infty$, that are interpreted as the baryonic VEV and the value of the diagonal coupling constant in the dual gauge theory. The range $[1/2, 5/6]$ is related to $[1/6, 1/2]$ by a Z_2 symmetry that exchanges (θ_1, ϕ_1) with (θ_2, ϕ_2) and reverse sign to some of the type IIB forms; the KS solution is the symmetric point. All the arbitrary constants in the supersymmetry equations (except for an additive constant for the dilaton) are fixed in terms of ξ by requiring IR regularity and the absence of an asymptotically flat region in the UV. For all values $1/6 < \xi \leq 1/2$ the solution is asymptotic in the UV to the KS solution and the dilaton is bounded. More amazingly, by fixing the value of the dilaton at $t = 0$ we can obtain an interpolating flow between KS and MN. For $\xi \rightarrow 1/6$, indeed, the asymptotic suddenly changes, the dilaton blows up in the UV and the solution smoothly approaches MN.

The previous results require some explanation. The two-parameter family of solutions we have described contains the dual description of the baryonic branch of the KS theory. According to the standard philosophy of the AdS/CFT correspondence [69], different vacua of the same theory correspond to solutions with the same asymptotic behavior. The baryonic branch is then obtained by varying the IR parameter ξ while keeping fixed the value of the dilaton at $t = \infty$. The Z_2 symmetry of the supergravity solution is interpreted in the gauge theory as the symmetry that exchange \mathcal{B} with $\bar{\mathcal{B}}$; the KS solution is then interpreted as the point on the baryonic branch where $\mathcal{B} = \bar{\mathcal{B}} = i\Lambda_{2M}^2$. For the other values of ξ we obtain the full baryonic branch (4.37). This does not necessarily imply that the endpoint of the baryonic branch corresponds to the MN theory¹¹. As can be seen numerically [34], when $\xi \rightarrow 1/6$ with fixed asymptotic dilaton the radius of the three-sphere goes to zero, implying large curvatures in the solution. This means that far along the baryonic branch the solution becomes strongly coupled. On the other hand, it is still true that by varying both ξ and the asymptotic value of the dilaton we can connect KS and MN in a controllable way (at least for finite t). MN can be reached from KS by moving both in space of vacua and in the space of theories. This results certainly requires an interpretation.

As we saw, there is an additional massless mode in the IR dynamics of the KS theory beside the confined $SU(M)$ factor. The reader is already alerted that, in any event, these theories should only be considered as cousins of the real pure $\mathcal{N} = 1$ SYM. In the supergravity approximation, $g_s M \rightarrow \infty$ and so the cascade steps are not well separated and the additional massive fields of the original theory $SU(N+M) \times SU(M)$ are not decoupled. As usual, we can get a pure SYM

¹¹We thank Igor Klebanov and Anatoly Dymarski for pointing out this.

theory only beyond the supergravity regime. The supergravity solution is dual to a four-dimensional gauge theory with a large number of massive matter fields. The absence of mass gap motivates the question whether these theories are really connected to pure SYM theory. This connection necessarily involves a stringy limit and it is outside our investigative ability. However, by noticing that the stringy limit corresponds to separating the cascades, we may expect that by separating the scales Λ_{2M} and Λ_M of the two IR groups also the Goldstone boson will decouple. In this case, a full string version of the KS solution will reproduce pure $\mathcal{N} = 1$ SYM theory.

It would be quite interesting to see whether the KS solution can be generalized to the case of fractional branes on other toric singularities. In particular, since we know the metrics for $Y^{p,q}$ and $L^{p,q,r}$ it is mandatory to examine in details these cases. A first step in this direction was taken in [48], where a singular solution describing the UV behavior of the theory in the $Y^{p,q}$ case was found. If a regular solution exists, it would be necessarily more complicated than KS. It is known, in fact, that there are no CY deformations for the cones over $Y^{p,q}$ and smooth $L^{p,q,r}$. We cannot then obtain a supergravity solution in the simple class described in [63], with constant dilaton and a conformally CY metric. A solution would necessarily involve at least an $SU(3)$ structure, or even worse, since a solution would probably involve a non complex manifold. A more prosaic possibility is that the gauge theory has no supersymmetric vacuum at all. All these theories flow in the infrared to cases where at least one gauge group has number of flavors less than the number of colors. In this situation, we expect the generation of a non perturbative ADS superpotential that will destabilize the vacuum, like in SQCD. This situation seems to be generic in the cases of $Y^{p,q}$ and $L^{p,q,r}$ [70, 51]. It also seems that in explored cases where the CY singularity has no complex deformation, the corresponding gauge theory has no supersymmetric vacuum. It would be quite important to understand this experimental observation in more details and to look for cases of metrics admitting deformations.

5. A link to string compactifications

The relation between holography and the Randall-Sundrum (RS) models [71, 72] is known since the first days of the AdS/CFT correspondence [73, 74]. In taking the near horizon geometry we also create large warp factors that may serve as an explanation of the hierarchy of physical scales. In compactifications with fluxes is not difficult to exhibit local throats, i.e. region with a sensible warp factor. In particular, the KS solution is used as the prototype of such throats. We refer to other lectures in this school for a detailed discussion of the huge subject of string compactifications with fluxes and we briefly describe how the KS solution can be used for extra-dimensions phenomenology.

Warped solutions with a throat of the form

$$ds^2 = h^{-1/2}(r)dx_\mu dx^\mu + h^{1/2}(r)(dr^2 + r^2 ds_{(5)}^2) \quad (5.1)$$

are common in string theory. They can be generated by either a stack of branes or by using solutions with RR fluxes. The two pictures (branes versus fluxes) are dual to each other in the sense of the AdS/CFT correspondence. One can take many examples out of the AdS/CFT literature. In this context one usually consider non-compact solutions with a radial coordinate r . To obtain a

compact model, one truncates the metric at a certain UV scale r_{UV} and glues a compact manifold for $r > r_{UV}$.

As discussed in [75], the Einstein equations for a *compact* manifold without brane sources imply the vanishing of all fluxes and a constant warp factor. Fluxes and non-trivial warping can be introduced only in the presence of sources. Moreover, as shown in [75], some of these sources must have negative tension. This is not a great problem in string theory, where many consistent objects, for example orientifold planes, may have negative energy. There are further constraints from charge conservation. With a compact manifold, the total D3 charge must vanish. Integrating the $F_{(5)}$ equation of motion

$$dF_{(5)} = H_{(3)} \wedge F_{(3)} + sources \quad (5.2)$$

we get the condition

$$\int H_{(3)} \wedge F_{(3)} + Q_{D3}^{Sources} = 0 \quad (5.3)$$

Contributions to $Q_{D3}^{Sources}$ may come from physical D3 branes, orientifold planes O3 or induced D3 charges from wrapped branes.

Example 5.1 (the RSII model): If we choose $h(r) = R^4/r^4$ and the metric for the round five-sphere for $ds_{(5)}$, we obtain the product of $AdS_5 \times S^5$. The solution also contains N units of flux for the RR form $F_{(5)}$ along S^5 . This choice of warp factor corresponds to a maximally supersymmetric solution of string theory and it is equivalent to the RSII model [72]. The compact manifold glued for $r > r_{UV}$ corresponds to an explicit realization of the Plank brane of the RS scenario. We may choose, for example, the orientifold T^6/Z_2 [73]. Here $h(r) = R^4/r^4$ is obtained in the vicinity of a stack of N D3 branes. $H_{(3)} = F_{(3)} = 0$ and the positive D3 contribution to $Q_{D3}^{Sources}$ is compensated by the negative contribution of orientifold planes O3 (for N not too large), which also provide the negative tension required by Einstein equations.

Example 5.2 (the RSI model): The RSI model [71] is obtained by truncating the metric (5.1) at $r = r_0$ by the insertion of an IR brane. In contrast to the RSII model, the warp factor is now bounded above zero and has a minimal value that has been used to study the hierarchy problem. The IR brane can be replaced by any regular geometry that has a non-zero minimal warp factor, for example, the KS solution. The non-compact KS solution can be embedded in a genuine string compactification as explained in [75]. The most convenient way is to consider F-theory solutions that can develop a local conifold singularity. An explicit example is provided in [75]. In the compact solution, the R-R and NS-NS two-forms have integer fluxes along the S^3 cycle of the conifold (call it A) and along its Poincaré dual B, respectively:

$$\frac{1}{(2\pi)^2 \alpha'} \int_A F_{(3)} = M, \quad \frac{1}{(2\pi)^2 \alpha'} \int_B H_{(3)} = -K, \quad (5.4)$$

In order to avoid large curvature in the solution that would invalidate the supergravity approximation, the integers M and K must be large. The relation to RSI is pictured in Figure 6. This time, the negative contribution to $Q_{D3}^{Sources}$ and the negative tension come from the D7 branes of the F theory compactification. In particular, the total $Q_{D3}^{Sources}$ can be computed in purely geometrical terms using the Euler number of the fourfold X , so that relation (5.3) simplifies to

$$KM - \frac{\chi(X)}{24} = 0 \quad (5.5)$$

In the specific example chosen in [75], $\chi(X)/24 = 72$, but $\chi(X)/24$ can be much larger for known fourfolds.

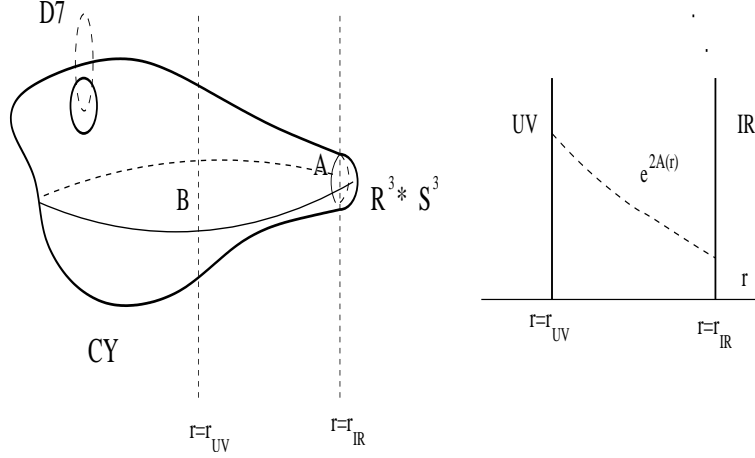


Figure 6: The CY compactification with a throat and its corresponding interpretation in terms of a simplified RSI model.

An important property of these compactifications with fluxes is that all the complex structure moduli of the Calabi-Yau are stabilized. The superpotential for type IIB compactifications indeed reads [76]

$$W = \int (F_{(3)} - \tau H_{(3)}) \wedge \Omega \tag{5.6}$$

The presence of the holomorphic three-form in this formula shows that the complex structure moduli will generically get a mass. We can study, as an example, the modulus of the deformed conifold. This can be identified with the parameter ϵ appearing in equation (4.19) and governing the volume of the three cycle. More precisely, by defining

$$z = \int_A \Omega \tag{5.7}$$

a standard geometrical result gives, for the conifold,

$$\int_B \Omega = \frac{z}{2\pi i} \log z + \text{holomorphic} \tag{5.8}$$

The minimization of the superpotential (5.6)

$$W \sim M \int_B \Omega - K\tau \int_A \Omega \quad \rightarrow \quad W' \sim \frac{M}{2\pi i} \log z - i \frac{K}{g_s} + \dots \tag{5.9}$$

gives

$$z \sim e^{-\frac{2\pi K}{M g_s}} \tag{5.10}$$

z determines the hierarchy of energy scales. Its value is exponentially small if $K/Mg_s \gg 1$ (thus justifying the neglected terms in the superpotential). This condition is easily obtained with reasonable values for the fluxes and the string coupling; in the [75] example with $KM = 72$ we can choose M and K of order 10 and $g_s \sim 0.1$. Recall that z is related to the parameter ϵ in the KS solution.

In the AdS/CFT context, this is a free parameter determining the scale of the IR confining gauge theory. We see that upon compactification the value of ϵ has been fixed. This can be understood by considering that, after compactification, the KS solution has a UV cut-off where $N = KM$. The theory will then undergo exactly K cascades. Since each cascade takes place on a ratio of energy scales of order $e^{2\pi/3g_s M}$, we see that the total hierarchy is indeed $z^{1/3} \sim e^{-2\pi K/3g_s M}$.

The Kahler moduli, on the other hand, are not stabilized by the superpotential (5.6). The scalar associated with the volume of the internal manifold, for instance, will always remain massless. Its stabilization is usually obtained with non-perturbative superpotentials (generated by gaugino condensations on wrapped D7 branes or similar phenomena). We refer to the lectures on string compactifications for more details. We just finish by noticing that the KS throat has been used for many interesting phenomenological or cosmological scenarios (we deliberately choose not to give full references to the huge literature on these subjects):

- The realization of metastable de-Sitter vacua in string theory [77]. This is obtained by introducing anti D3-branes in the IR part of the geometry. This simple idea opened an avenue to the literature on the *landscape* and the statistical study of string vacua.
- The realization of models for inflation [78].
- The resurrection of cosmic strings [79].

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