Ruin probabilities under general investments and heavy-tailed claims

Filip Lindskog, KTH

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Joint work with Henrik Hult.

lindskog@kth.se, www.math.kth.se/∼lindskog/

On the one hand, an insurance company is required to measure and manage risks, and stay solvent with a very high probability.

On the other hand, the management wants a profitable insurance business and does not invest according to an investment strategy that minimizes the ruin probability.

Does it matter?

The risk of extreme claims cannot be avoided, but investments can make things better - or worse.

We want to understand and quantify the effects of different dynamic investment strategies, assets, and heavy-tailed claim sizes on the finite time horizon ruin probability.

Problem: not clear which stochastic processes govern the future values of investment assets. $\rightarrow$ avoid making specific parametric assumptions.

Consider an insurance company facing heavy-tailed claims and investing its capital in a combination of assets according to some investment strategy. The evolution of the capital is given by

$$X_t^\varepsilon = x + \int_0^t X_{s-}^\varepsilon \, dZ_s + \varepsilon Y_t, \quad t > 0,$$

where x is the initial capital, dZ_s is the instantaneous return on the investments at time s and εY_t is the accumulated premiums minus claims at time t.

We assume that x is a lot larger than typical claim sizes (= sizes of downward jumps of εY), that Z is a semimartingale and that Y is a Lévy process.

We assume that the Lévy measure ν of Y_1 has a regularly varying left tail, i.e.

$$\lim_{u\to\infty} \frac{\nu(-\infty, -\lambda u)}{\nu(-\infty, -u)} = \lambda^{-\alpha} \quad \text{for some } \alpha > 0 \text{ and all } \lambda > 0.$$

This means that a few large claims are responsible for a substantial part of the total claim amount.

If $[Z,Y]=0$ (but Z and Y not necessarily independent), then

$$X_t^\varepsilon = x + \int_0^t X_{s-}^\varepsilon dZ_s + \varepsilon Y_t$$

has a unique solution X^ε given by

$$X_t^\varepsilon = \mathcal{E}(Z)_t \left(x + \varepsilon \int_0^t \frac{dY_s}{\mathcal{E}(Z)_{s-}} \right),$$

where $\mathcal{E}(Z)$ is the Doléan-Dade exponential

$$\mathcal{E}(Z)_t = e^{Z_t - \frac{1}{2}[Z,Z]_t^c} \prod_{s\in(0,t]} (1+\Delta Z_s) e^{-\Delta Z_s}.$$

If $\inf_{t\in(0,1]} \Delta Z_t > -1$, then $\inf_{t\in[0,1]} \mathcal{E}(Z)_t > 0$ (ruin only due to bad investments is not possible) which implies that

$$\left\{ \inf_{t\in[0,1]} X_t^\varepsilon < 0 \right\} = \left\{ \inf_{t\in[0,1]} \int_0^t \frac{dY_s}{\mathcal{E}(Z)_{s-}} < -\frac{x}{\varepsilon} \right\}.$$

Without stronger assumptions on the processes Y and Z we cannot go any further unless we resort to asymptotic approximations (as $\varepsilon \to 0$ or equivalently as $x \to \infty$).

If the company invests prudently so that $\mathcal{E}(Z)_t$ stays sufficiently far away from 0 with sufficiently high probability, then

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}(\inf_{t \in [0,1]} X_t^\varepsilon < 0)}{\nu(-\infty, -1/\varepsilon)} = x^{-\alpha} \int_0^1 \mathbb{E}\mathcal{E}(Z)_t^{-\alpha} dt.$$

The precise condition is that

$$\mathbb{E} \sup_{t \in [0,1]} \mathcal{E}(Z)_t^{-\alpha-\delta} < \infty \quad \text{for some } \delta > 0.$$

The above result for the asymptotic decay of the ruin probability is for a fixed (possibly dynamic) investment strategy and a fixed set of assets.

We need a stronger result giving the asymptotic decay of the ruin probability for the ruin minimizing strategy.

Let's get more explicit: consider

• n assets with spot price processes S_t^k (positive semimartingales),

• a bank account giving instantaneous interest rate r_t (càdlàg, adapted), and

• an investment strategy $\pi = \{(\pi_t^0, \dots, \pi_t^n)\}_{t \geq 0}$ (càglàd, predictable), where π_t^k is the fraction of the capital invested in the kth asset and $\pi_t^0 = 1 - \pi_t^1 - \cdots - \pi_t^n$ is the fraction of the capital put on the bank account, at time t.

Then

$$Z_t^\pi = \int_0^t \pi_s^0 r_{s-} ds + \sum_{k=1}^n \int_0^t \pi_s^k \frac{dS_s^k}{S_{s-}^k},$$

Note: $\Delta S_t^k = S_t^k - S_{t-}^k > -S_{t-}^k$ so $\Delta Z_t^\pi = \sum_{k=1}^n \pi_t^k \Delta S_t^k / S_{t-}^k > -\sum_{k=1}^n \pi_t^k = -1$ if $\pi_t^k \geq 0$. Without short positions bad investments alone do not lead to ruin.

We would like to compute the ruin probability for the ruin minimizing strategy, i.e.

$$\inf_\pi \mathbb{P}\left(\inf_{t \in [0,1]} X_t^{\varepsilon,\pi} < 0 \right),$$

where ε is small and π belongs to some family Π of strategies.

Since this is not possible we hope that, for $c(\varepsilon) = 1/\nu(-\infty, -1/\varepsilon)$,

$$\inf_{\pi} c(\varepsilon)\, \mathbb{P}\left(\inf_{t\in[0,1]} X_t^{\varepsilon,\pi} < 0 \right) \approx \lim_{\varepsilon\to 0} \inf_{\pi} c(\varepsilon)\, \mathbb{P}\left(\inf_{t\in[0,1]} X_t^{\varepsilon,\pi} < 0 \right)$$

and show that (essentially uniform convergence in π as $\varepsilon \to 0$)

$$\begin{aligned} \lim_{\varepsilon\to 0} \inf_{\pi} c(\varepsilon)\, \mathbb{P}\left(\inf_{t\in[0,1]} X_t^{\varepsilon,\pi} < 0 \right) &= \inf_{\pi} \lim_{\varepsilon\to 0} c(\varepsilon)\, \mathbb{P}\left(\inf_{t\in[0,1]} X_t^{\varepsilon,\pi} < 0 \right) \\ &= \inf_{\pi} x^{-\alpha} \int_0^1 \mathbb{E}\mathcal{E}(Z^{\pi})_t^{-\alpha} dt, \end{aligned}$$

when π belongs to any family Π of investment strategies satisfying

$$\sup_{\pi\in\Pi} \mathbb{E} \sup_{t\in[0,1]} \mathcal{E}(Z^{\pi})_t^{-\alpha-\delta} < \infty \quad \text{for some } \delta > 0.$$

This condition - conservative investments - may rule out some assets for possible investment, and may rule out the possibility to take short positions - depending on the specification of the spot price processes S^k, $k = 1, \ldots, n$.

Next we look at a specific model for the investment assets' spot prices.

We check that (or impose conditions ensuring that)

$$\sup_{\pi\in\Pi} \mathbb{E} \sup_{t\in[0,1]} \mathcal{E}(Z^\pi)_t^{-\alpha-\delta} < \infty \quad \text{for some } \delta > 0.$$

Then we conclude that

$$\underbrace{\inf_\pi \mathbb{P}\left(\inf_{t\in[0,1]} X_t^{\varepsilon,\pi} < 0\right)}_{\text{impossible to compute}} \approx \nu(-\infty,-1/\varepsilon)x^{-\alpha} \underbrace{\inf_\pi \int_0^1 \mathbb{E}\mathcal{E}(Z^\pi)_t^{-\alpha} dt}_{\text{possible to compute}}$$

and we compute and compare

$$\inf_\pi \int_0^1 \mathbb{E}\mathcal{E}(Z^\pi)_t^{-\alpha} dt \quad \text{and} \quad \int_0^1 \mathbb{E}\mathcal{E}(Z^{\pi_0})_t^{-\alpha} dt$$

for some proposed strategy π_0 and determine the asymptotically ruin minimizing strategy π^*.

For càdlàg adapted processes μ and σ with $\inf_{t\in[0,1]} \sigma_t > 0$, let

$$S_t = S_0 + \int_0^t \mu_{s-} S_{s-} ds + \int_0^t \sigma_{s-} S_{s-} dB_s,$$
$$X_t^{\varepsilon,\pi} = x + \int_0^t (1-\pi_s) X_{s-}^{\varepsilon,\pi} r_{s-} ds + \int_0^t \pi_s X_{s-}^{\varepsilon,\pi} \frac{dS_s}{S_{s-}} + \varepsilon Y_t.$$

For conservative strategies (essentially that $\int_0^1 \pi_t^2 \sigma_t^2 dt$ has $\approx 2\alpha(1+\alpha)$ exponential moments) it holds that

$$\lim_{\varepsilon\to 0} \frac{\mathbb{P}(\inf_{t\in[0,1]} X_t^{\varepsilon,\pi} < 0)}{\nu(-\infty, -1/\varepsilon)} = \int_0^1 \mathbb{E}\mathcal{E}(Z^\pi)_t^{-\alpha} dt \geq \int_0^1 \mathbb{E}\mathcal{E}(Z^{\pi^*})_t^{-\alpha} dt,$$

where π_t^* is the conservative asymptotically ruin minimizing strategy

$$\pi_t^* = \frac{\mu_{t-} - r_{t-}}{(1+\alpha)\sigma_{t-}^2} \quad \text{if} \quad \mathbb{E}\exp\left\{\frac{2\alpha}{1+\alpha}\int_0^1 \left(\frac{\mu_t - r_t}{\sigma_t}\right)^2 dt\right\} < \infty.$$

Note: the non-asymptotic ruin minimizing strategy is impossible to compute.

For numerical comparisons let's look at the case when r, μ and σ are constants. In this case any bounded investment strategy π is a conservative strategy and

$$\lim_{\varepsilon\to 0}\frac{\mathbb{P}(\inf_{t\in[0,1]} X_t^{\varepsilon,\pi^*}<0)}{\nu(-\infty,-1/\varepsilon)} = x^{-\alpha}\frac{1-\exp\Big\{-\alpha r-\frac{\alpha}{2(1+\alpha)}\gamma^2\Big\}}{\alpha r+\frac{\alpha}{2(1+\alpha)}\gamma^2}, \quad \gamma=\frac{\mu-r}{\sigma},$$

$$\lim_{\varepsilon\to 0}\frac{\mathbb{P}(\inf_{t\in[0,1]} X_t^{\varepsilon,1}<0)}{\nu(-\infty,-1/\varepsilon)} = x^{-\alpha}\frac{1-\exp\Big\{-\alpha\mu+\frac{\alpha(1+\alpha)}{2}\sigma^2\Big\}}{\alpha\mu-\frac{\alpha(1+\alpha)}{2}\sigma^2}.$$

For $\pi_t\in[0,1]$ and typical values for α, r, μ and σ we find that, asymptotically, the ruin probability is not much higher than for the asymptotically ruin minimizing strategy.

For small values of α it is not necessary to focus on minimizing the ruin probability.

For large values of σ and α the increase in risk can be substantial.

However, for large αs the whole asymptotic analysis is questionable.

Illustration: the maximum ruin probability (for $\pi_t \in [0,1]$) divided by the minimum ruin probability, as $\varepsilon \to 0$.

$r = 0.05$ and $\alpha = 2$

$\mu \backslash \sigma$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.01	1.08	1.11	1.12	1.34	1.55	1.87	2.37	3.15
0.03	1.04	1.08	1.17	1.31	1.51	1.83	2.31	3.06
0.05	1.01	1.06	1.15	1.28	1.48	1.78	2.25	2.99
0.07	1.01	1.04	1.12	1.25	1.45	1.74	2.20	2.91
0.09	1.03	1.03	1.10	1.23	1.42	1.71	2.15	2.84

$r = 0.05$ and $\alpha = 8$

$\mu \backslash \sigma$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.01	1.45	2.61	8.70	62.5	1019	$3.7 \cdot 10^4$	$2.9 \cdot 10^6$	$5.0 \cdot 10^8$
0.03	1.30	2.35	7.73	54.7	883	$3.2 \cdot 10^4$	$2.5 \cdot 10^6$	$4.2 \cdot 10^8$
0.05	1.19	2.13	6.88	47.9	766	$2.7 \cdot 10^4$	$2.2 \cdot 10^6$	$3.6 \cdot 10^8$
0.07	1.11	1.95	6.16	42.1	665	$2.4 \cdot 10^4$	$1.9 \cdot 10^6$	$3.1 \cdot 10^8$
0.09	1.05	1.79	5.52	37.0	579	$2.1 \cdot 10^4$	$1.6 \cdot 10^6$	$2.7 \cdot 10^8$