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RULED REAL HYPERSURFACES OF COMPLEX SPACE FORMS

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Dedicated to professor Yoshihiko Suyama on his 60th birthday

Abstract

Real hypersurfaces in non-flat complex space forms with integrable holomorphic distribution and symmetric ϕ-Ricci tensor which are ϕ-Einstein are ruled real hypersurfaces.

1. Introduction

Real hypersurfaces in complex space forms provide a rich class of CR-manifolds.

As is well known, there are no Einstein real hypersurfaces in non-flat complex space forms. Even worse, the non-existence of real hypersurfaces in non-flat complex space forms with parallel Ricci tensor is known (see [13]).

These results indicate that for differential geometric study on real hypersurfaces, one need to introduce curvature tensors of another type which are *compatible* to the induced almost contact structure and its associated CR-structure.

In [8], the first named author introduced a covariant tensor field S^* which involves geometric informations on interactions between almost contact structure and Ricci tensor. We call the tensor field S^*, the *ϕ-Ricci tensor*.

Some standard examples of real hypersurfaces are well characterized in terms of ϕ-Ricci tensor S^*. In fact, some model hypersurfaces (*homogeneous* and of *constant principal curvatures*) are characterized as *ϕ-Einstein hypersurfaces*. A real hypersurface is said to be a ϕ-Einstein real hypersurface if its ϕ-Ricci tensor is a constant multiple of the metric over the holomorphic distribution. In our previous papers [8], [9], we have classified ϕ-Einstein real hypersurfaces in complex projective and hyperbolic spaces on which the structure vector field is *principal*. As a corollary, we proved that all the ϕ-Einstein real hypersurfaces

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with principal structure vector field in non-flat complex space forms are homogeneous and of constant principal curvatures.

In complex projective space, all the homogeneous real hypersurfaces have principal structure vector field and constant principal curvatures. On the other hand, in complex hyperbolic space, homogeneity does not imply the property "principal structure vector field".

In fact, there exist homogeneous ruled real hypersurfaces in complex hyperbolic space with non-principal structure vector field (see *cf.* [4]. For the classification of homogeneous real hypersurfaces in complex hyperbolic space, we refer to [5, Theorem 4.4]). This fact actually shows that ruled real hypersurfaces play a important role in differential geometry of real hypersurfaces in complex space forms. Remarkably, as we have exhibited in [8], every ruled real hypersurface in non-flat complex space forms is ϕ-Einstein.

In this paper we continue our study on ϕ-Einstein real hypersurfaces. The purpose of this paper is to investigate ϕ-Einstein real hypersurfaces whose structure vector field is *non-principal*.

More precisely we prove the following classification result of ϕ-Einstein real hypersurfaces.

MAIN THEOREM. *Let M be a real hypersurface with symmetric ϕ-Ricci tensor in a complex space form $\tilde{M}_n(c)$ of constant holomorphic sectional curvature $4c \neq 0$ on which the holomorphic distribution $T^\circ M$ is integrable, then M is ϕ-Einstein if and only if M is locally congruent to a ruled real hypersurface of $\tilde{M}_n(c)$.*

To close Introduction, we emphasize that ϕ-Einstein property is very different from the so-called *pseudo-Einstein* property (For the precise definition, see Remark 4.2). Although pseudo-Einstein property is a generalization of Einstein condition, it is still a strong restriction for real hypersurfaces. In fact, the only pseudo-Einstein real hypersurfaces in complex hyperbolic space are horospheres, geodesic spheres and tubes over complex hyperbolic hyperplanes.

On the other hand, all the pseudo-Einstein real hypersurfaces, tubes over totally real and totally geodesic real hyperbolic space $H_n(\mathbf{R})$ as well as all ruled real hypersurfaces are ϕ-Einstein.

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Throughout this paper we denote by $\Gamma(E)$ the space of all smooth sections of a vector bundle E over a manifold M.

2. Preliminaries

2.1. A *complex space form* is a complete and connected Kähler manifold of constant holomorphic sectional curvature. A simply connected n-dimensional complex space form $\tilde{M}_n(c)$ of constant holomorphic sectional curvature $4c$ is holomorphically isometric to *complex projective space* $P_n(\mathbf{C})$, *complex Euclidean space* $\mathbf{C}^n$ or *complex hyperbolic space* $H_n(\mathbf{C})$, according as $c > 0$, $c = 0$ or $c < 0$. We denote the Kähler structure of $\tilde{M}_n(c)$ by $(J, \tilde{g})$. Here J is the almost complex structure and $\tilde{g}$ the Kähler metric, respectively.

Now let M be a real hypersurface of a non-flat ($c \neq 0$) complex space form $\tilde{M}_n(c)$ with induced Riemannian metric g.

Take a local unit normal vector filed N of M in $\tilde{M}_n(c)$. Then the Levi-Civita connections $\tilde{\nabla}$ of $(\tilde{M}_n(c), \tilde{g})$ and ∇ of (M, g) are related by the following *Gauss formula* and *Weingarten formula*:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad X, Y \in \Gamma(TM),$$
$$\tilde{\nabla}_X N = -AX, \quad X \in \Gamma(TM).$$

The linear endomorphism field A is called the *shape operator* of M derived from N.

An eigenvector X of the shape operator A is called a *principal curvature vector*. The corresponding eigenvalue λ of A is called a *principal curvature*. As is well known, the Kähler structure $(J, \tilde{g})$ of the ambient space induces an almost contact metric structure (ϕ, ξ, η, g) on M. In fact, the *structure vector field* ξ and its dual 1-form η are defined by

$$\eta(X) = g(\xi, X) = \tilde{g}(JX, N), \quad X \in TM.$$

The $(1,1)$-tensor field ϕ is defined by

$$g(\phi X, Y) = \tilde{g}(JX, Y), \quad X, Y \in TM.$$

One can easily check that this structure (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, it satisfies

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0. \tag{1}$$

It follows that

$$\nabla_X \xi = \phi AX.$$

Let $\tilde{R}$ and R be the Riemannian curvature tensors of $\tilde{M}_n(c)$ and M, respectively. From the expression of the curvature tensor $\tilde{R}$ of $\tilde{M}_n(c)$, we have the following *equations of Gauss and Codazzi*:

$$\begin{aligned} R(X, Y)Z = c(&g(Y, Z)X - g(X, Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z) \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$
$$(\nabla_X A)Y - (\nabla_Y A)X = c(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi).$$

The Ricci tensor S of (M, g) is defined by

$$S(X, Y) = \mathrm{trace}(Z \mapsto R(Z, X)Y), \quad X, Y \in TM.$$

By the Gauss equation, S is given by

$$S(X, Y) = c((2n+1)g(X, Y) - 3\eta(X)\eta(Y)) + hg(AX, Y) - g(A^2X, Y),$$

for all $X, Y \in TM$. Here h denotes the trace of A.

2.2. To close this section, we recall the following two fundamental results (See *eg.*, [13]).

LEMMA 2.1. *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

LEMMA 2.2. *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $AX = \lambda X$ for $X \perp \xi$, then we have $(2\lambda - \alpha)A\phi X = (\alpha\lambda + 2c)\phi X$.*

We refer to the reader [13] about general theory of differential geometry of real hypersurfaces in complex space forms.

3. ϕ-Einstein real hypersurfaces

3.1. Let M be a real hypersurface in $\tilde{M}_n(c)$. Then the *ϕ-Ricci tensor* S^* of M is defined by ([8]):

$$S^*(X, Y) = \frac{1}{2}\,\mathrm{trace}(Z \mapsto R(X, \phi Y)\phi Z).$$

Then the Gauss equation implies that

$$S^*(X, Y) = 2cn(g(X, Y) - \eta(X)\eta(Y)) - g(\phi A\phi AX, Y), \tag{2}$$

for all $X, Y \in TM$.

The *ϕ-Ricci operator* Q^* is the linear endomorphism field associated to S^*;

$$S^*(X, Y) = g(Q^*X, Y), \quad X, Y \in TM.$$

The trace ρ^* of Q^* is called the *ϕ-scalar curvature* of M.

One can easily to check that

$$S^*(X, \xi) = 0 \tag{3}$$

$$S^*(\xi, Y) = -\eta(A\phi A\phi Y), \tag{4}$$

for any $X, Y \in TM$, and

$$S^*(\phi X, \phi Y) = S^*(Y, X) + \eta(A\phi A\phi X)\eta(Y),$$

for any $X, Y \in TM$.

3.2. It should be remarked that S^* is not symmetric, in general. Now we take symmetric part Sym S^* and the alternate part Alt S^* of ϕ-Ricci tensor S^* of M;

$$\mathrm{Sym}\, S^*(X, Y) = \frac{1}{2}(S^*(X, Y) + S^*(Y, X)),$$

$$\mathrm{Alt}\, S^*(X, Y) = \frac{1}{2}(S^*(X, Y) - S^*(Y, X)),$$

for any $X, Y \in TM$.

Direct computation using (2) shows that

$$\mathrm{Sym}\, S^*(X, Y) = 2cn(g(X, Y) - \eta(X)\eta(Y)) - \frac{1}{2}g((\phi A\phi A + A\phi A\phi)X, Y), \tag{5}$$

$$\mathrm{Alt}\, S^*(X, Y) = \frac{1}{2}g((A\phi A\phi - \phi A\phi A)X, Y). \tag{6}$$

Lemma 3.1. *The ϕ-Ricci tensor S^* of a real hypersurface in $\tilde{M}_n(c)$ is symmetric if and only if $(A\phi)^2 = (\phi A)^2$.*

3.3. Let $T^\circ M$ be a distribution defined by a subspace

$$T_x^\circ M = \{X \in T_xM : X \perp \xi_x\}$$

in the tangent space T_xM. The formulas (1) imply that the distribution $T^\circ M$ is invariant under ϕ. The distribution $T^\circ M$ is called the *holomorphic distribution* of M. If the ϕ-Ricci tensor is a constant multiple of the Riemannian metric over the holomorphic distribution, *i.e.*,

$$S^*(X, Y) = \frac{\rho^*}{2(n-1)}g(X, Y)$$

for $X, Y \in T^\circ M$ on M, then M is said to be a *ϕ-Einstein* real hypersurface.

Remark 3.1. The ϕ-Ricci tensor S^* is also called *Ricci $*$-tensor*. In our previous works [8]–[9], we used the terminology Ricci $*$-tensor.

Lemma 3.2 ([8]). *A real hypersurface M is ϕ-Einstein if and only if its ϕ-Ricci tensor S^* satisfies the following equation*:

$$S^*(X, Y) = \frac{\rho^*}{2(n-1)}(g(X, Y) - \eta(X)\eta(Y)) - \eta(X)\eta(A\phi A\phi Y), \tag{7}$$

for any $X, Y \in TM$.

LEMMA 3.3 ([9]). *If ξ is a principal curvature vector field, then S^* is symmetric.*

The converse of this lemma does not hold. As we will see later, ruled real hypersurfaces provide counterexamples. In fact, ruled real hypersurfaces are ϕ-Einstein real hypersurfaces with symmetric S^* but on which ξ is non-principal.

In our previous paper [8], ϕ-Einstein real hypersurfaces in $\tilde{M}_n(c)$, $(c \neq 0)$ with principal structure vector filed ξ are classified. In the next section we shall study ϕ-Einstein real hypersurfaces on which ξ is *non-principal*.

4. Proof of Main Theorem

4.1. We start this section with recalling fundamental properties of ruled real hypersurfaces.

Take a regular curve γ in $\tilde{M}_n(c)$ with tangent vector field γ'. At each point of γ, there is a unique complex projective or hyperbolic hyperplane cutting γ so as to be orthogonal not only γ' but also to $J\gamma'$. The union of these hyperplanes is called a *ruled real hypersurface*.

PROPOSITION 4.1 ([13], [8]). *Ruled real hypersurfaces have the following properties.*

(i) *The holomorphic distribution $T^\circ M$ is integrable,*
(ii) *The structure vector ξ is not principal,*
(iii) *M is ϕ-Einstein with ϕ-scalar curvature $\rho^* = 4cn(n-1)$,*
(iv) *The shape operator A satisfies the following formulas;*

$$A\xi = \mu\xi + \nu U, \quad AX = 0, \tag{8}$$

$$|U| = 1, \quad U \perp \xi, \quad \nu \neq 0, \quad X \perp \xi, \quad X \perp U. \tag{9}$$

In particular A satisfies $A\phi A = 0$.

For more details on ruled real hypersurfaces, we refer to [4], [11], [12] and references therein.

4.2. Next, we prepare two characterizations of integrability of $T^\circ M$ for our use.

LEMMA 4.2. *Let M be a real hypersurface in $\tilde{M}_n(c)$ with $c \neq 0$. Then the following three statements are mutually equivalent:*

(i) *The holomorphic distribution $T^\circ M$ is integrable,*
(ii) *$g((\phi A + A\phi)X, Y) = 0$ for all $X, Y \in T^\circ M$,*
(iii) *$\phi AX = -A\phi X + \eta(X)\phi A\xi + \eta(A\phi X)\xi$ for all $X \in TM$.*

Proof. The equivalence of the first and second items have been proved by M. Kimura and S. Maeda [12]. Thus we only need to check the equivalence of the second and third items. The case (iii $\Rightarrow$ ii) is clear.

(ii $\Rightarrow$ iii) Take any tangent vectors $X, Y \in TM$ and inserting $W_1 = X - \eta(X)\xi$ and $W_2 = Y - \eta(Y)\xi$ into the equation $g((\phi A + A\phi)W_1, W_2) = 0$, we obtain

$$g(\phi AX + A\phi X - \eta(X)\phi A\xi - \eta(A\phi X)\xi, Y) = 0. \qquad \square$$

THEOREM 4.3. *If the holomorphic distribution is integrable, then ξ can not be a principal vector field.*

Proof. Assume that ξ is principal and let α be its corresponding principal curvature. Then by Lemma 2.2,

$$(2\lambda - \alpha)A\phi X = (\lambda\alpha + 2c)\phi X \tag{10}$$

for *any* principal vector field $X \perp \xi$. Here λ is the principal curvature corresponding to X. If $\alpha = 2\lambda$, (10) implies that $\lambda^2 + c = 0$. Hereafter we assume that $\alpha \neq 2\lambda$. Applying η to both hand sides of (10), we have

$$\eta(A\phi X) = 0. \tag{11}$$

Next, by the integrability condition (Lemma 4.2-(iii)) of $T^\circ M$ together with (11), we have

$$\lambda\phi X = \phi AX = -A\phi X.$$

Inserting this into (10), we get $\lambda^2 + c = 0$. Clearly, in case $c > 0$, there are no such λ. This is a contradiction. Hence ξ is principal.

Next, in the case $c < 0$, $\lambda = \pm\sqrt{-c}$. In this case, the real hypersurface in complex hyperbolic space has *constant principal curvatures*, say α and $\lambda = \pm\sqrt{-c}$. Such real hypersurfaces are completely classified by Berndt [1]. More precisely, real hypersurfaces in complex hyperbolic space $H_n(\mathbf{C})$ with principal ξ and constant principal curvature are locally holomorphically congruent to one of the following model spaces:

(N) Horospheres,

($A1$) Geodesic spheres and tubes over totally geodesic complex hyperbolic hyperplanes

($A2$) Tubes over totally geodesic $H_k(\mathbf{C})$, where $1 < k < n-2$.

(B) Tubes over totally real and totally geodesic real hyperbolic space $H_n(\mathbf{R})$.

Among these real hypersurfaces, horospheres and type $A1$ hypersurfaces have non-integrable holomorphic distributions. In fact the holomorphic distributions of these hypersurfaces are *contact structure*. See [2]–[3, §4.3–4.4]. Now we check principal curvatures of real hypersurfaces of type $A2$ and B (see [1], [13, Section 3]).

(i) The type $A2$ hypersurfaces in $H_n(\mathbf{C})$ have three distinct principal curvatures: $\lambda_1 = \dfrac{1}{r}\tanh u$ of multiplicity $2p$, $\lambda_2 = \dfrac{1}{r}\coth u$ of multiplicity $2q$, and $\alpha = \frac{2}{r} \coth 2u$ of multiplicity 1, where $r = 1/\sqrt{-c}$, $p > 0$, $q > 0$, and $p + q = n - 1$.

(ii) The type B real hypersurfaces in $H_n(\mathbf{C})$ have three principal curvatures, namely, $\lambda_1 = \frac{1}{r} \coth u$ of multiplicity $n-1$, $\lambda_2 = \frac{1}{r} \tanh u$ of multiplicity $n-1$, and $\alpha = \frac{2}{r} \tanh 2u$ of multiplicity 1. Here $r = 1/\sqrt{-c}$. These principal curvatures are distinct unless $\coth u = \sqrt{3}$, in which case λ_1 and α coincide to make a principal curvature of multiplicity n.

From these informations, one can see that on type $A2$ or B real hypersurfaces, the principal curvatures λ_1 and λ_2 can not take the values $\pm\sqrt{-c} = \pm 1/r$. Hence we arrive at the conclusion, ξ can not be principal. □

Remark 4.1. Let M be a real hypersurface in $\tilde{M}_n(c)$, $c \neq 0$. Then the rank of the shape operator A is greater than or equal to 2 at some points (see [13, Proposition 2.14]).

4.3. Hereafter we restrict our attention to real hypersurfaces on which ξ is non-principal.

PROPOSITION 4.4. *Let M be a real hypersurface with symmetric ϕ-Ricci tensor in $\tilde{M}_n(c)$, $c \neq 0$. Assume that ξ is non-principal and express $A\xi$ as*

$$A\xi = \mu\xi + \nu U, \quad |U| = 1, \quad U \perp \xi, \quad \nu \neq 0. \tag{12}$$

Then $A\phi U = 0$.

Proof. Since S^* is symmetric, we have $A\phi A\phi = \phi A\phi A$ by Lemma 3.1. From (12), we obtain

$$\phi A\phi A\xi = \phi A\phi(\mu\xi + \nu U) = \nu\phi A\phi U.$$

On the other hand, we notice that $A\phi A\phi\xi = 0$. Hence $\phi A\phi U = 0$, because $\nu \neq 0$. Next, applying ϕ to the formula $\phi A\phi U = 0$,

$$\phi^2 A\phi U = -A\phi U + \eta(A\phi U)\xi.$$

Here we notice that $\eta(A\phi U) = 0$, in fact,

$$\eta(A\phi U) = g(\xi, A\phi U) = g(A\xi, \phi U) = g(\mu\xi + \nu U, \phi U) = 0.$$

Thus we obtain $A\phi U = 0$. □

LEMMA 4.5. *Let M be a real hypersurface with symmetric ϕ-Ricci tensor in $\tilde{M}_n(c)$, $c \neq 0$. If $T^\circ M$ is integrable, then $AU = \nu\xi$.*

Proof. Since $T^\circ M$ is integrable, Lemma 4.2 yields that

$$\phi AY = -A\phi Y + \eta(Y)\phi A\xi + \eta(A\phi Y)\xi, \quad Y \in TM. \tag{13}$$

Choose $Y = U$ in (13), we obtain

$$\phi AU = -A\phi U + \eta(A\phi U)\xi = 0.$$

Here we used a fact $A\phi U = 0$, since S^* is symmetric. Thus we get $\phi AU = 0$. By computing $0 = \phi^2 AU$, we can deduce that $AU = \nu\xi$. □

Lemma 4.6. *Let M be a ϕ-Einstein real hypersurface with symmetric ϕ-Ricci tensor in $\tilde{M}_n(c)$, $c \neq 0$. If the structure vector field is non-principal, then $A\phi A = 0$ and the ϕ-scalar curvature is $\rho^* = 4cn(n-1)$.*

Proof. Comparing (2) and (7), we obtain (*cf.* [8]):

$$\begin{aligned}&(4cn(n-1) - \rho^*)\phi X + 2(n-1)\\ &\quad \times \{A\phi AX - \eta(A\phi AX)\xi - \eta(X)A\phi A\xi + \eta(X)\eta(A\phi A\xi)\xi\} = 0.\end{aligned}$$

Using the expression $A\xi = \mu\xi + \nu U$ and the fact $A\phi U = 0$,

$$A\phi A\xi = A\phi(\mu\xi + \nu U) = 0.$$

Moreover we have $\eta(A\phi AX) = 0$. In fact,

$$\eta(A\phi AX) = g(\xi, A\phi AX) = -g(A\phi A\xi, X) = -g(\nu A\phi U, X) = 0.$$

Hence we get

$$A\phi AX = \frac{\rho^* - 4cn(n-1)}{2(n-1)}\phi X, \quad X \in TM.$$

In particular, if we choose $X = \phi U$ in this equation, we have

$$0 = A\phi(A\phi U) = \frac{\rho^* - 4cn(n-1)}{2(n-1)}\phi^2 U = -\frac{\rho^* - 4cn(n-1)}{2(n-1)}U.$$

This implies that $\rho^* = 4cn(n-1)$ and hence $A\phi A = 0$. □

Lemma 4.7. *Let M be a real hypersurface with symmetric ϕ-Ricci tensor in $\tilde{M}_n(c)$ with $c \neq 0$. If $T^\circ M$ is integrable and M is ϕ-Einstein, then $A^2 X = 0$ for all $X \in TM$ such that $X \perp \xi$ and $X \perp U$.*

Proof. Take $X \in T^\circ M$ such that $X \perp U$ and choose $Y = AX$ in (13), then

$$\phi A^2 X = -A\phi AX + \eta(AX)\phi A\xi + \eta(A\phi AX)\xi.$$

Then the preceding Lemma 4.6 implies that $\phi A^2 X = \eta(AX)\phi A\xi$. Applying ϕ to both hand sides of this equation, we have

$$\phi^2 A^2 X = \eta(AX)\phi^2 A\xi.$$

Direct computation of both hand sides yields

$$A^2X = \eta(A^2X)\xi + \eta(AX)(A\xi - \eta(A\xi)\xi).$$

By using the expression $A\xi = \mu\xi + \nu U$, we obtain

$$\eta(AX) = g(\mu\xi + \nu U, X) = 0,$$
$$\eta(A^2X) = g((\mu^2 + \nu^2)\xi + \mu\nu U, X) = 0.$$

Hence $A^2X = 0$. □

4.4. Now we prove our main theorem.

THEOREM 4.8. *Let M be a real hypersurface with symmetric ϕ-Ricci tensor in $\tilde{M}_n(c)$, $(c \neq 0)$ on which the holomorphic distribution $T^\circ M$ is integrable, then M is ϕ-Einstein if and only if M is locally congruent to a ruled real hypersurface of $\tilde{M}_n(c)$.*

Proof. $(\Rightarrow)$ As we have seen in Proposition 4.1, every ruled real hypersurface M in non-flat complex space form $\tilde{M}_n(c)$ is ϕ-Einstein with integrable holomorphic distribution.

$(\Leftarrow)$ Let M be a ϕ-Einstein real hypersurface with integrable holomorphic distribution $T^\circ M$ and local unit normal vector field N. We only need to show that integral manifolds of $T^\circ M$ are totally geodesic in $\tilde{M}_n(c)$.

By Theorem 4.3, the structure vector field ξ of M is non-principal. Thus by virtue of Proposition 4.4, there exists a local unit vector field U and functions μ and $\nu \neq 0$ such that

$$A\xi = \mu\xi + \nu U, \quad U \perp \xi.$$

Moreover U satisfies $A\phi U = 0$. Next, by Lemma 4.5, we have $AU = \nu\xi$. Lemma 4.6 implies that $A\phi A = 0$.

Now let L be the leaf (maximal integral manifold) of $T^\circ M$. Then the normal bundle of L in $\tilde{M}_n(c)$ is spanned by $\xi = -JN$ and N. The Gauss-Weingarten formulas of L in $\tilde{M}_n(c)$ are given by

$$\tilde{\nabla}_V W = \nabla^L_V W + \mathrm{h}(V, W)\xi + \mathrm{k}(V, W)N,$$
$$\tilde{\nabla}_V N = -A^L_N V + \tau(V)\xi, \quad \tilde{\nabla}_V \xi = -A^L_\xi V + \tilde{\tau}(V)N$$

for all sections $V, W \in \Gamma(TL)$. Here ∇^L is the induced connection of L. Comparing these equations with Gauss-Weingarten formulas of M in $\tilde{M}_n(c)$, we obtain

$$AV = A^L_N V - \tau(V)\xi, \quad A^L_N V = AV - \eta(AV)\xi, \quad V \in \Gamma(TL),$$
$$A^L_\xi V = -\phi AV, \quad \tilde{\tau}(V) = -\tau(V) = \eta(AV),$$
$$\mathrm{k}(V, W) = g(A^L_N V, W), \quad \mathrm{h}(V, W) = g(A^L_\xi V, W), \quad V, W \in \Gamma(TL).$$

From these equations, we have

$$A_\xi^L = -\phi A_N^L.$$

For any $X \in \Gamma(TL)$ which is orthogonal to U, we have

$$\eta(AX) = g(\xi, AX) = g(A\xi, X) = \mu\eta(X) + \nu g(U, X) = 0.$$

Hence

$$A_N^L X = AX, \quad X \in \Gamma(TL), \quad X \perp U.$$

Next, since $AU = \nu\xi$, we get $A_N^L U = 0$.

For any $V \in \Gamma(TL)$,

$$(A_N^L)^2 V = A_N^L(AV - \eta(AV)\xi) = A^2 V - \eta(AV)A\xi - \eta(A^2V)\xi + \eta(AV)\eta(A\xi)\xi.$$

If we choose $V = X \in \Gamma(TL)$ which is orthogonal to U, then by Lemma 4.7, we get $(A_N^L)^2 X = 0$. Here we used a fact $\eta(AX) = g(X, A\xi) = 0$ again.

Hence $(A_N^L)^2 = 0$ and hence $A_N^L = 0$ on $TL = T^\circ M$ because $A^L U = 0$. Since $A_\xi^L = -\phi A_N^L$, we conclude that $A_N^L = A_\xi^L = 0$. Thus we get $\mathrm{h} = \mathrm{k} = 0$. Namely the leaf L is totally geodesic in $\tilde{M}_n(c)$. This completes the proof. □

Remark 4.2. A real hypersurface M in $\tilde{M}_n(c)$ is said to be *pseudo-Einstein* (or *η-Einstein*) if there exist real constants α and β such that $S = \alpha g + \beta\eta \otimes \eta$. Ruled real hypersurfaces in $\tilde{M}_n(c)$, $c \neq 0$ are not pseudo-Einstein. More generally, it is known that every pseudo-Einstein real hypersurface in $\tilde{M}_n(c)$ with $c \neq 0$ and $n \geq 2$ has principal ξ (see *eg.*, [13, p. 271] for $n > 2$ and [6], [7] for $n = 2$). Moreover, in $H_n(\mathbf{C})$, real hypersurfaces of type B in $H_n(\mathbf{C})$ are ϕ-Einstein but not pseudo-Einstein.

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