# RUN-UP DISTRIBUTIONS OF WAVES BREAKING ON SLOPING WALLS

by

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## SYNOPSIS

Distributions of run-up are calculated by assigning to each individual wave in an irregular wave train a run-up value according to Hunt's formula. The use of this formula permits a normalization of the run-up in such a way that the run-up distributions are independent of slope angle, mean wave height and mean wave period. Expressions are derived for the probability density and the distribution function of the run-up and of the wave steepness for arbitrary joint distributions of wave height and period. Explicit results are obtained for wind waves by assuming wave height and period squared to be jointly Rayleigh distributed with arbitrary degree of correlation. Empirical data from the laboratory are discussed. These lend support to some of the main premises used and results obtained. A few field measurements of run-up distributions are presented; the Rayleigh distribution appears to fit these data.

LIST OF CONTENTS

# LIST OF SYMBOLS

| Symbol | Definition |
|---|---|
| $C$ | Chézy coefficient (eq. 2.7) |
| $c$ | coefficient in run-up formula (eq. 8.2) |
| $d$ | mean water depth |
| $e$ | base of natural logarithms |
| $E(k)$ | complete elliptic integral of the second kind of modulus $k$ |
| $f$ | probability density |
| $F$ | distribution function (cumulative probability) |
| ${}_2F_1$ | hypergeometric function |
| $g$ | acceleration due to gravity |
| $h$ | value assumed by $H'$ |
| $H$ | wave height: max. (crest) height minus min. (trough) height of the water surface between two successive zero up-crossings |
| $H_{1/3}$ | mean height of the highest 1/3 of the waves |
| $H'$ | normalized wave height: $H' = H/\overline{H}$ |
| $I_0$, $I_1$ | modified Bessel functions of the first kind of order zero resp. one |
| $J$ | Jacobian |
| $k$ | parameter of the bivariate Rayleigh distribution |
| $K_0$, $K_1$ | modified Bessel functions of the third kind of order zero resp. one |
| $l$ | value assumed by $L_0'$ |
| $L_0$ | wave length in deep water: $L_0 = gT^2/2\pi$ |
| $L_0'$ | normalized deep-water wave length: $L_0' = L_0/\overline{L}_0$ |
| $n$ | probability of exceedance, in % (subscript) |
| $r$ | value assumed by $R'$ |
| $R$ | run-up: max. height above mean water level reached by a wave which runs up a slope |
| $R_{1/3}$ | run-up of a wave with $H = H_{1/3}$ and $T = T_{1/3}$ |
| $R'$ | normalized run-up (eq. 3.3) |
| $s$ | value assumed by $S'$ |
| $S$ | steepness: $S = H/L_0$ |
| $S'$ | normalized steepness: $S' = H'/L_0'$ |

| Symbol | Definition |
|---|---|
| $T$ | wave period: time interval between two successive zero up-crossings of water surface |
| $T_{1/3}$ | mean period of the highest 1/3 of the waves |
| $\hat{T}$ | period of spectral component with maximum energy density in frequency domain |
| $U$ | magnitude of mean wind velocity |
| $y$ | a thickness of the uprush |
| $\alpha$ | angle of slope with respect to the horizontal |
| $\Gamma$ | gamma function |
| $\delta$ | Dirac's unit impulse "function" |
| $\epsilon$ | spectral width parameter |
| $\rho$ | coefficient of linear correlation between $H$ and $L_0$ |

Auxiliary constants and variables: $k_1$, $k_2$, $p$, $t$, $x$, $\dot{y}$, $z$, $\gamma$

A bar denotes average value.

# 1 INTRODUCTION

## 1.1 Background

The run-up of waves is often an important factor in the design of shore structures. Many studies have been made to determine it as a function of the characteristics of structure and waves, both from a theoretical and an empirical point of view. First the theoretical approach will be very briefly considered.

In all analytical theories for wave run-up known to the author (e.g., Pocklington, 1921; Miche, 1944) the fluid is assumed to be non-viscous, and, in most of these, the motion to be irrotational. These theories cannot be expected to be applicable in the case of waves breaking on a sloping structure, which is often the case to be considered. For an approximate analytical description of breaking waves in shallow water, the long-wave theory must be used, as was first done by Stoker (1948). A review of the method, including recent additions, has been given by Amein (1966). The run-up of breaking waves is found by numerical integration of the differential equations, generally by means of the method of characteristics. In most cases the propagation of a bore advancing into water at rest is dealt with. In a few cases the run-up of periodic waves has been calculated (Amein, 1966; Daubert and Warluzel, 1967). It seems likely that this procedure could also be used to calculate the run-up of irregular waves. These waves would then have to be simulated by suitably chosen boundary conditions. However, such a procedure would be rather laborious. In this paper a different approach is used, which, in contrast with the preceding methods, is not based on principles of fluid dynamics. It resembles the method used by Saville (1962), to be described in the following.

Saville assumes that the distribution of run-ups of an irregular wave train can be calculated by assigning to each individual wave the run-up value of a periodic wave train of corresponding height and period. This will henceforth be referred to as the "hypothesis of equivalency". A similar hypothesis has been widely used to compute distributions of wave forces on piles.

It should be noted that the hypothesis of equivalency does not necessarily imply that each individual wave causes a run-up equal to the run-up of the corresponding uniform wave train. In fact, it does not even imply that corresponding to each wave an identifiable run-up exists. The assumption is rather weaker, as it pertains to the distribution of wave heights and periods on the one hand, and of run-ups on the other hand. In other words, it pertains to averages of many values, rather than to individual values.

In order to apply the hypothesis of equivalency one has to know the run-up of periodic waves, and the joint distribution of the wave height ($H$) and period ($T$). Saville uses the run-up data obtained previously by him (1956), as published by the B.E.B. (1961), and the joint distribution of $H$ and $T$ proposed by Bretschneider (1959) for the case when these are stochastically independent. The resulting run-up distributions have to be calculated numerically for each combination of slope angle and wave steepness.

## 1.2 Outline of contents

The approach used in this paper is similar to the one used by Saville, as far as the hypothesis of equivalency is concerned. However, this hypothesis is elaborated differently. By considering only waves which break on the slope, a simple analytical expression can be used for the run-up of periodic waves (Hunt, 1959). This will be dealt with in chapter 2. The use of this expression obviates the need to compute the run-up distribution anew for each combination of wave steepness and slope angle. Furthermore, it permits the transformation from joint distribution of height and period, into the distribution of run-up, to be carried out analytically. This transformation is carried out first for an arbitrary distribution of $H$ and $T^2$ in chapter 3, and subsequently for specific distributions. This is done in chapter 4 for wind waves with a bivariate Rayleigh distribution of $H$ and $T^2$ and arbitrary degree of correlation, and in chapter 5 for swell with a negligible variation in period.

In order to be able to estimate the fraction of the waves which is breaking on the slope, the distribution of wave steepness is determined in the chapters 6 and 7. Exact solutions are obtained for waves with the same joint distribution of $H$ and $T^2$ as used in the chapters 4 and 5. In chapter 8 some empirical run-up distributions are presented and discussed. A brief summary and some conclusions and recommendations are given in the chapters 9 and 10.

It has been attempted throughout to state clearly which assumptions are being used in the various stages of the developments. Details of algebra and calculus have been omitted for the sake of brevity.

## 2 RUN-UP OF PERIODIC WAVES BREAKING ON A SLOPE

Numerous experiments have been carried out to determine the run-up of periodic waves. It appears from these experiments that there are considerable differences in the run-up of waves which break on the slope and those which do not break. For breaking waves for instance, the run-up increases with increasing slope angle, while the reverse is true for non-breaking waves. Whether or not the waves break on the slope depends largely on the slope angle ($\alpha$) and the wave steepness. Iribarren and Nogales (1949) give the following formula for the slope corresponding to a regime halfway between no breaking and complete breaking:

1sec

$$\tan \alpha_{cr} = \frac{8}{T}\sqrt{\frac{H}{2g}} \tag{2.1}$$

$T$ is the wave period, $H$ the wave height and $g$ the acceleration due to gravity. Substituting the following expression for the deep-water wave length

1sec 2sec

$$L_o = \frac{gT^2}{2\pi} \tag{2.2}$$

$\frac{10.1}{6.3} = 1.5$ 6

and rearranging gives

$H = 0.03$ $H$ 0.12

$$\left(\frac{H}{L_o}\right)_{cr} = 0.19 \tan^2 \alpha \tag{2.3}$$

Steepness

Breaking occurs when the given value of $H/L_o$ exceeds $(H/L_o)_{cr}$. This will mostly be the case if wind waves impinging on gently sloping walls are considered. For example, the steepest slope of coastal dikes in the Netherlands is 1:3. The corresponding critical steepness according to eq. 2.3 is about 0.02. The design waves have steepnesses well in excess of this value. This was the main reason for considering breaking waves only in this paper.

Hunt (1959) has given the following formula for the run-up of periodic waves breaking on a smooth slope:

Run-up

$$R = 2.3\, T\sqrt{H}\, \tan\alpha \qquad (2.4)$$

Hunt

mean water level

in the ft-sec system. $R$ is the height above M.W.L. reached by a wave which runs up the slope. Restoring dimensional homogeneity by substitution of $g = 32.2$ ft/sec$^2$, eq. 2.4 may be written as

$$R = 0.4\, T\sqrt{gH}\, \tan\alpha \qquad (2.5)$$

or as

$$R = \sqrt{HL_0}\, \tan\alpha \qquad (2.6)$$

Eq. 2.4 is based on measurements made at the Waterways Experiment Station in Vicksburg, Miss., and at the Beach Erosion Board in Washington, D.C. According to Hunt, the criterion of Iribarren and Nogales (eq. 2.3) is adequate to delineate the transition from breaking to no breaking.

Hunt's formula is purely empirical, and it is not known why the formula is as it is. It would be desirable to gain some insight into its structure. To this end the following interpretation is offered.

The formula applies to waves breaking on the slope. The initial velocity of the water particles in the tongue which runs up the slope must be of the same order of magnitude as the particle velocities in the breaking wave, i.e., $O(\sqrt{gH})$. The motion is periodic, with period $T$, and the run-up time is $O(T)$. If it is assumed that the shape of the velocity-time curve does not significantly depend on the characteristics of slope and waves, then the displacements along the slope are expected to be $O(T\sqrt{gH})$, and the vertical displacements, among which the run-up, to be $O(T\sqrt{gH}\sin\alpha)$. This agrees with eq. 2.5, except for a factor $\cos\alpha$, which is almost 1 for gentle slopes. The difference is unimportant, as the reasoning given above applies to gentle slopes only, and is stated in terms of orders of magnitude.

A run-up formula similar to Hunt's has been derived by Wagner (1968) from a differential equation describing the motion of the mass of water which runs up the slope:

$$\frac{R}{H \sin\alpha} = k_1 \sqrt{\frac{L}{H}\coth\frac{2\pi d}{L}}\left(1 - k_2\, \frac{\sqrt{HL\coth 2\pi d/L}}{3gC^2}\right) \qquad (2.7)$$

The coefficients $k_1$ and $k_2$ are unknown constants, $L$ is the wave-length in the depth $d$, $y$ is a thickness of the uprush, and $C$ is a Chezy-coefficient for the slope. No details are given about the differential equation or about the method of solving it.

For smooth slopes, the factor in parentheses is almost 1. The experimental value of $k_1$ for the median run-up averaged over the flume width is given as 0.971. If in addition the following relationship is substituted:

$$L_0 = L \coth \frac{2\pi d}{L} \qquad (2.8)$$

then eq. 2.7 becomes

$$R = 0.971 \sqrt{HL_0} \sin \alpha \qquad (2.9)$$

$$R = \sqrt{HL_0} \tan \alpha$$

which for not-too-steep slopes is almost the same as Hunt's formula.

## 3 RUN-UP DISTRIBUTION OF BREAKING WAVES WITH ARBITRARY JOINT DISTRIBUTION OF $H$ AND $L_0$.

### 3.1 Analytical solution

The derivations in this chapter are based on the hypothesis of equivalency, and on the hypothesis that Hunt's formula can be applied to the waves. The latter hypothesis implies that the run-up distribution is not significantly affected by the fact that not all the waves break on the slope. (The fraction which does not break is estimated in chapter 7.) Accordingly, the run-up distribution will be determined by assigning to each wave with height $H$ and deep-water length $L_0$ a run-up given by

$$R = \sqrt{HL_0} \tan \alpha \qquad (2.6)$$

It should be noted that $H$, $L_0$ and $\tan\alpha$ appear in product form only. This has the advantage that the run-up can be so normalized as to make the shape of its distribution independent of the characteristics of waves and slope. The variables will be normalized as follows:

$$H' \equiv \frac{H}{\overline{H}} , \tag{3.1}$$

$$L_o' \equiv \frac{L_o}{\overline{L_o}} \tag{3.2}$$

and

$$R' \equiv \frac{R}{\sqrt{\overline{H}\,\overline{L_o}}\tan\alpha} \tag{3.3}$$

in which a bar denotes average (expected) value. Substitution of eq. 2.6 into eq. 3.3 gives

$$R' = \sqrt{H' L_o'} \tag{3.4}$$

Thus, the normalized run-up distribution equals the distribution of $\sqrt{H'L_o'}$ , which in turn may be found by a transformation of the joint distribution of $H'$ and $L_o'$ , which is assumed to be known in the present context. This transformation may be carried out in a number of ways. The method presented here is believed to be rather straightforward. A more formal procedure will be used in chapter 6 for solving a similar problem.

Throughout this paper, stochastic variables will be denoted by capital letters, and particular values which they may assume by lower case letters, with omission of primes and indices.

Let $f(h,l)$ be the joint probability density of $H'$ and $L_o'$ , defined by

$$\mathrm{prob}[h < H' \leq h + dh \text{ and } l < L_o' \leq l + dl] = f(h,l)\,dh\,dl \tag{3.5}$$

Because $H'$ and $L_o'$ are positive quantities,

$$f(h,l) = 0 \quad \text{if} \quad h < 0 \quad \text{or} \quad l < 0 \tag{3.6}$$

The expression in the right-hand member of eq. 3.5 is called a probability element. The probability that $H'$ and $L_o'$ simultaneously assume values $h$ and $l$ in a certain area of the $h$-$l$ plane is determined by summing the corresponding probability elements, i.e. by integrating $f(h,l)$ over the area of the $h$-$l$ plane under consideration. This will be used in the following.

The cumulative probability of $R'$, denoted by $F(r)$, is the probability $[R' \leq r]$. Substitution of eq. 3.4 gives

$$F(r) = prob\left[\sqrt{H'L'_o} \leq r\right] \tag{3.7}$$

This probability in turn may be determined by integration of $f(h,l)$ with respect to $h$ and $l$ for all values thereof which fulfill the inequality $\sqrt{hl} \leq r$:

$$F(r) = \iint\limits_{\substack{\text{all } h,l \text{ for} \\ \text{which } \sqrt{hl} \leq r}} f(h,l)\, dh\, dl \tag{3.8}$$

From figure 1 and eq. 3.6, it can be seen that the integration of $f(h,l)$ must be carried out over the hatched area bordered by the hyperbola $hl = r^2$ and by the straight lines $h=0$ and $l=0$:

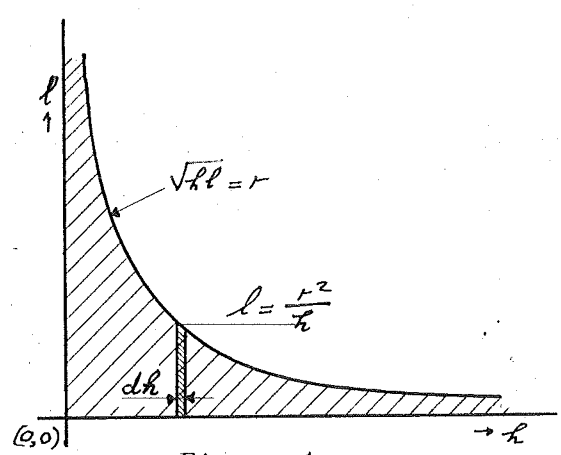



Figure 1

$$F(r) = \int_0^{\infty} dh \int_0^{\frac{r^2}{h}} f(h,l)\, dl \tag{3.9}$$

Differentiation with respect to $r$ yields the probability density of $R'$:

$$f(r) = \frac{dF(r)}{dr} \tag{3.10}$$

which gives

$$f(r) = 2r \int_0^{\infty} \frac{1}{h} f\left(h, \frac{r^2}{h}\right) dh \tag{3.11}$$

The eqs. 3.9 and 3.11 represent the formal solution to the problem of determining the run-up distribution from a known joint distribution of wave height and period, if Hunt's formula is applied to individual waves. These expressions are valid for arbitrary $f(h,l)$. In the next two chapters specific functions will be substituted for $f(h,l)$. Before this is done, however, a graphical method of estimating $F(r)$ from discrete data on wave heights and periods is presented.

## 3.2 Graphical solution for discrete data

In this section it will be assumed that a scatter diagram is available of wave height vs. period. The problem is to compute an estimate of the associated run-up distribution, if for individual waves the run-up is given by Hunt's formula:

$$R = 0.4\, T \sqrt{gH} \tan\alpha \tag{2.5}$$

A practical solution to this problem is suggested by the analytical derivation in the preceding section. On a transparant sheet of paper a family of curves is drawn, along each of which $T\sqrt{H}$ is constant. (On double-log paper such curves will be straight lines.) This sheet is used as an overlay over the given scatter diagram. By simply counting the total number of points of the scatter diagram between consecutive pairs of curves $T\sqrt{H}$ = constant, the number of anticipated run-ups in certain classes can be determined; the cumulative distribution can next be found by summation.

# 4 RUN-UP DISTRIBUTIONS OF WIND WAVES WITH A BIVARIATE RAYLEIGH DISTRIBUTION OF $H$ AND $L_o$.

## 4.1 The Bivariate Rayleigh distribution

It is generally known that wind wave heights are very nearly Rayleigh-distributed. According to Bretschneider (1959) the same is true for the periods squared of wind waves. Based on this observation, Bretschneider assumes that the joint distribution of heights and periods squared is some type of bivariate Rayleigh distribution.

However, such a distribution was unknown to him. Only in the cases when $H$ and $T^2$ are stochastically independent resp. 100% correlated was Bretschneider able to use the joint distribution, since in both cases it is completely determined by the marginal distributions.

Uhlenbeck (1943) and Rice (1944) in their work on signal statistics derive the joint probability density for two values of the envelope of a narrow band random noise signal, separated by a certain time interval. The resulting distribution may be called the bivariate Rayleigh distribution, although this is not done by Rice (who even in the one-dimensional case does not mention Rayleigh).

The probability density of two variables $H'$ and $L_o'$, which are jointly Rayleigh distributed, and of which the mean value is equal to one, is given by

$$f(h,l) = \frac{\pi^2}{4}\frac{hl}{1-k^2}\, e^{-\frac{\pi}{4}\frac{h^2+l^2}{1-k^2}}\, I_o\left(\frac{\pi}{2}\frac{k}{1-k^2}hl\right) \qquad (4.1)$$

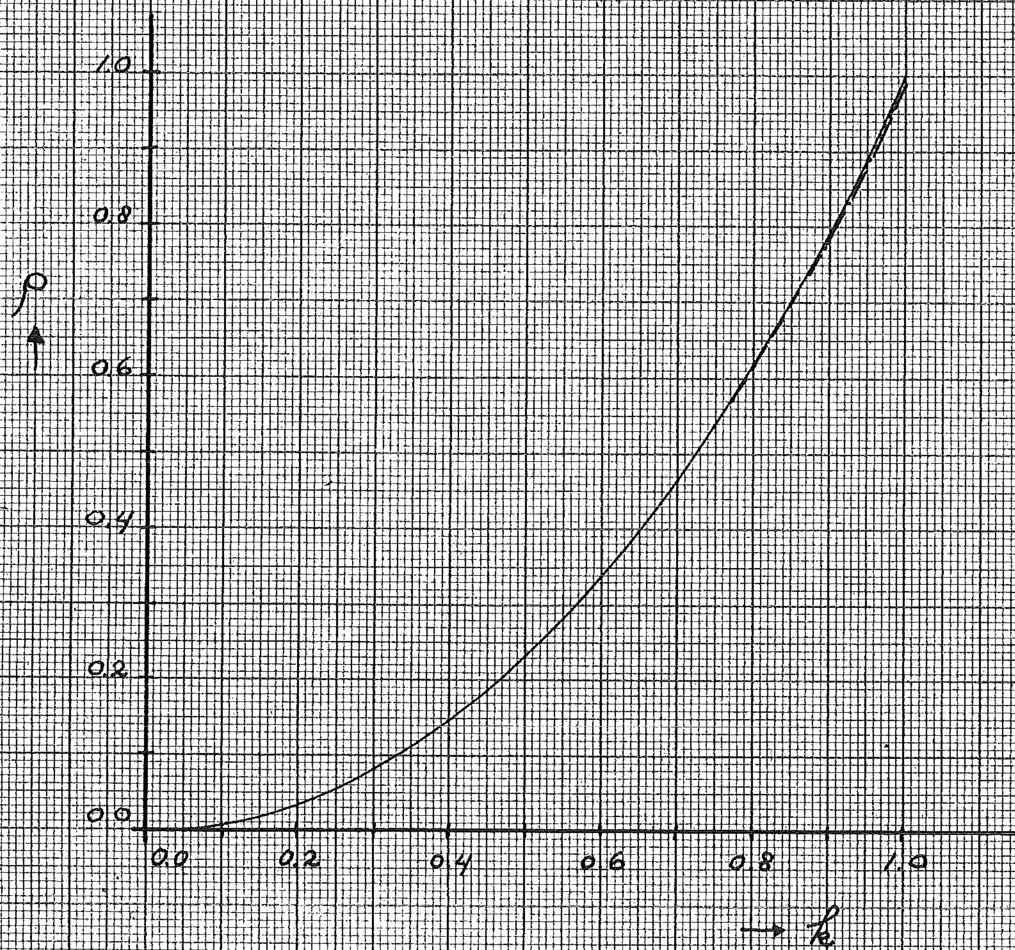
for $h$ and $l \geq 0$; $f(h,l) = 0$ for $h$ or $l < 0$. $I_o$ is the modified Bessel function of the first kind of order zero. The distribution has one parameter $k$, which fulfills the inequality $|k| \leq 1$. This parameter is not in general equal to $\rho$, the coefficient of linear correlation between $H'$ and $L_o'$, which is defined by

$$\rho = \frac{\overline{H'L_o'} - 1}{\sqrt{\overline{H'^2} - 1}\,\sqrt{\overline{L_o'^2} - 1}} \qquad (4.2)$$

The relationship between $\rho$ and $k$ has been dealt with elsewhere (Battjes, 1969). It is shown graphically in figure 2.

$H$
$L$

The application of eq. 4.1 to wind waves would imply that a certain fraction of the waves has a steepness exceeding the limit imposed by the process of breaking. This has been considered for the case $\rho = 0$ by Saville (1962), who found that the fraction of breaking waves is generally of the order of a few percent at most. The steepness distribution to be derived in the chapters 6 and 7 permits a similar check for $\rho \neq 0$, in which case the values were found to be even smaller than those given by Saville. The effect of a limiting steepness will not be taken into consideration in the following.




$$\rho = \frac{E(k) - \frac{1}{2}(1-k^2)K(k) - \frac{\pi}{4}}{1 - \frac{\pi}{4}} \qquad \text{(exact)}$$

$$k^2 = \frac{\rho}{a} - \frac{1}{16}\frac{\rho^2}{a^2} - \frac{1}{128}\frac{\rho^3}{a^3} \qquad \text{(series appr.)}$$

$$\rho = a\left(k^2 + \frac{k^4}{16} + \frac{k^6}{64}\right) \qquad (\;\text{"}\qquad\text{"}\;)$$

$$a = \frac{\pi}{16 - 4\pi}$$

Figure 2

## 4.2 Derivation of the run-up distribution

The probability density of $R'$ will now be determined by substitution of eq. 4.1 into eq. 3.11:

$$f(r) = \frac{\pi^2}{2} \frac{r^3}{1-k^2} I_0\left(\frac{\pi}{2} \frac{k}{1-k^2} r^2\right) \int_0^\infty \frac{1}{h} e^{-\frac{\pi}{4} \frac{h^2 + \frac{r^4}{h^2}}{1-k^2}} dh \qquad (4.3)$$

By substituting

$$t = \frac{\pi}{4} \frac{h^2}{1-k^2} \qquad (4.4)$$

and

$$x = \frac{\pi}{2} \frac{r^2}{1-k^2} \qquad (4.5)$$

the integral can be written as

$$\int_0^\infty e^{-t} \left( \frac{1}{2t} e^{-\frac{x^2}{4t}} \right) dt$$

This integral may be found in a table of Laplace transforms (Abramowitz and Stegun, 1965). It is equal to $K_0(x)$, where $K_0$ is the modified Bessel function of the third kind of order zero. Substitution in eq. 4.3 gives the probability density of the run-up as

Bessel functie 1ste soort nulde orde
Besselfunctie 3de soort nulde orde

$$f(r) = \frac{\pi^2}{2} \frac{r^3}{1-k^2} I_0\left(\frac{\pi}{2} \frac{k}{1-k^2} r^2\right) K_0\left(\frac{\pi}{2} \frac{r^2}{1-k^2}\right) , \qquad (4.6)$$

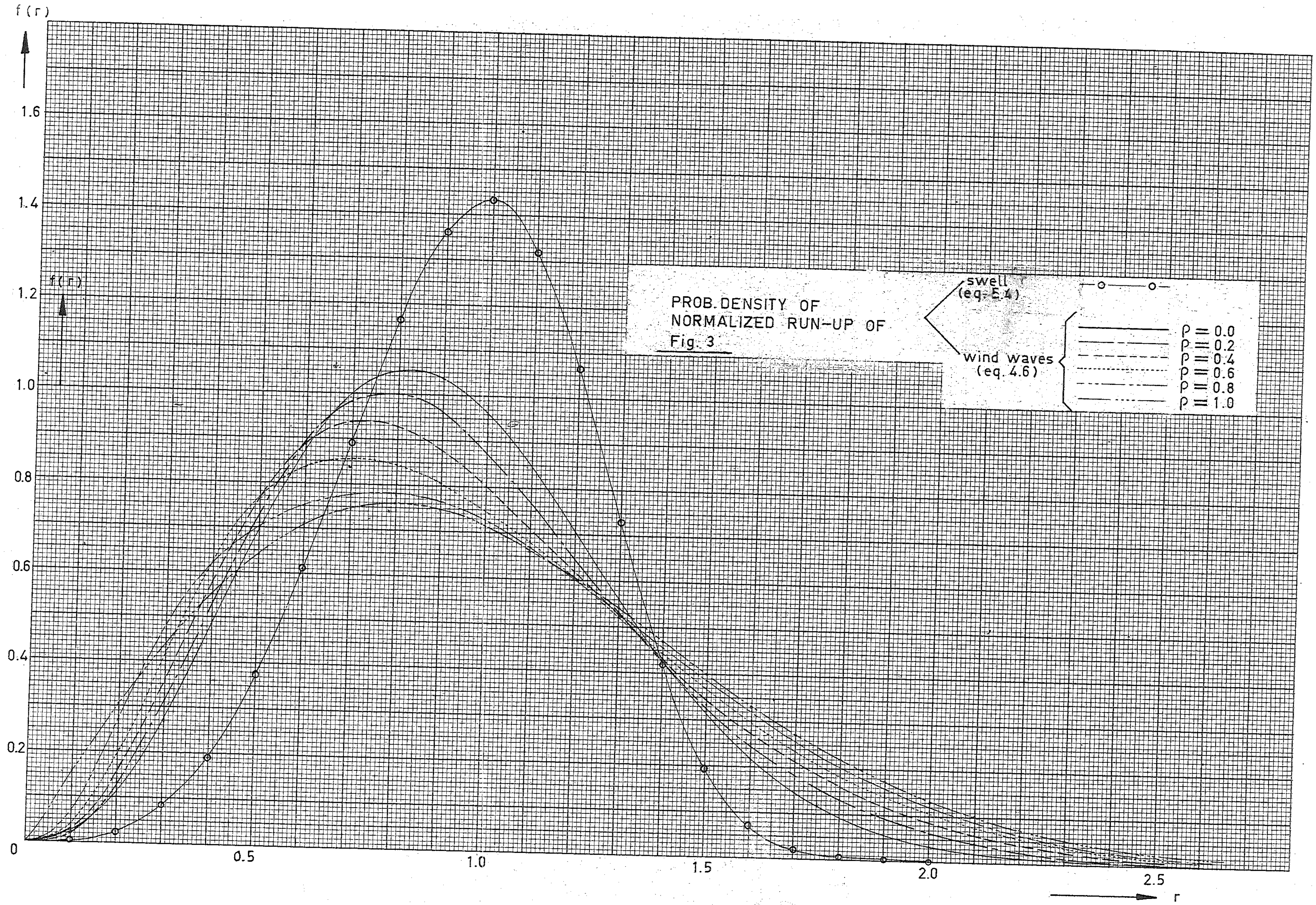
a graph of which is shown in figure 3 for six values of $k$ from 0 to 1, which have been so selected as to give equal increments of the coefficient of correlation, $\rho$.

The cumulative distribution may be found by integration of $f(r)$:

$$F(r) = \int_0^r f(r^*) dr^* \qquad (4.7)$$

In order to carry out the integration it is convenient to transform to the variable $x$ defined by eq. 4.5. Because $x$ is a single-valued function of $r$, and vice versa, the following relationship holds:

$$F\{x(r)\} = F\{r(x)\} \qquad (4.8)$$



f(r)
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
0.5
1.0
1.5
2.0
2.5
r
PROB. DENSITY OF
NORMALIZED RUN-UP OF
Fig. 3
swell
(eq. 5.4)
wind waves
(eq. 4.6)
ρ = 0.0
ρ = 0.2
ρ = 0.4
ρ = 0.6
ρ = 0.8
ρ = 1.0

Therefore,

$$f(x)\,dx = f(r)\,dr \qquad \text{or} \qquad f(x) = f(r)\frac{dr}{dx}\,, \tag{4.9}$$

substitution of which into eq. 4.6 gives

$$f(x) = (1-k^2)\,x\,I_0(kx)\,K_0(x) \tag{4.10}$$

The cumulative probability is

$$F(x) = (1-k^2)\int_0^x x^* I_0(kx^*)\,K_0(x^*)\,dx^* \tag{4.11}$$

The following formulas for derivatives of $I$ and $K$ hold:

$$\begin{aligned} K_0'(z) &= -K_1(z) & \qquad I_0'(z) &= I_1(z) \\ \{zK_1(z)\}' &= -zK_0(z) & \qquad \{zI_1(z)\}' &= zI_0(z) \end{aligned} \tag{4.12}$$

in which a prime denotes differentation with respect to $z$ (Abramowitz and Stegun, 1965). Using these relationships, eq. 4.11 may be integrated by parts, with the result

$$F(x) = 1 - x\,I_0(kx)\,K_1(x) - kx\,I_1(kx)\,K_0(x) \tag{4.13}$$

Substitution of eq. 4.5 in the right hand member of eq. 4.13 yields $F(r)$, which is depicted in figure 4 for selected values of $k$.

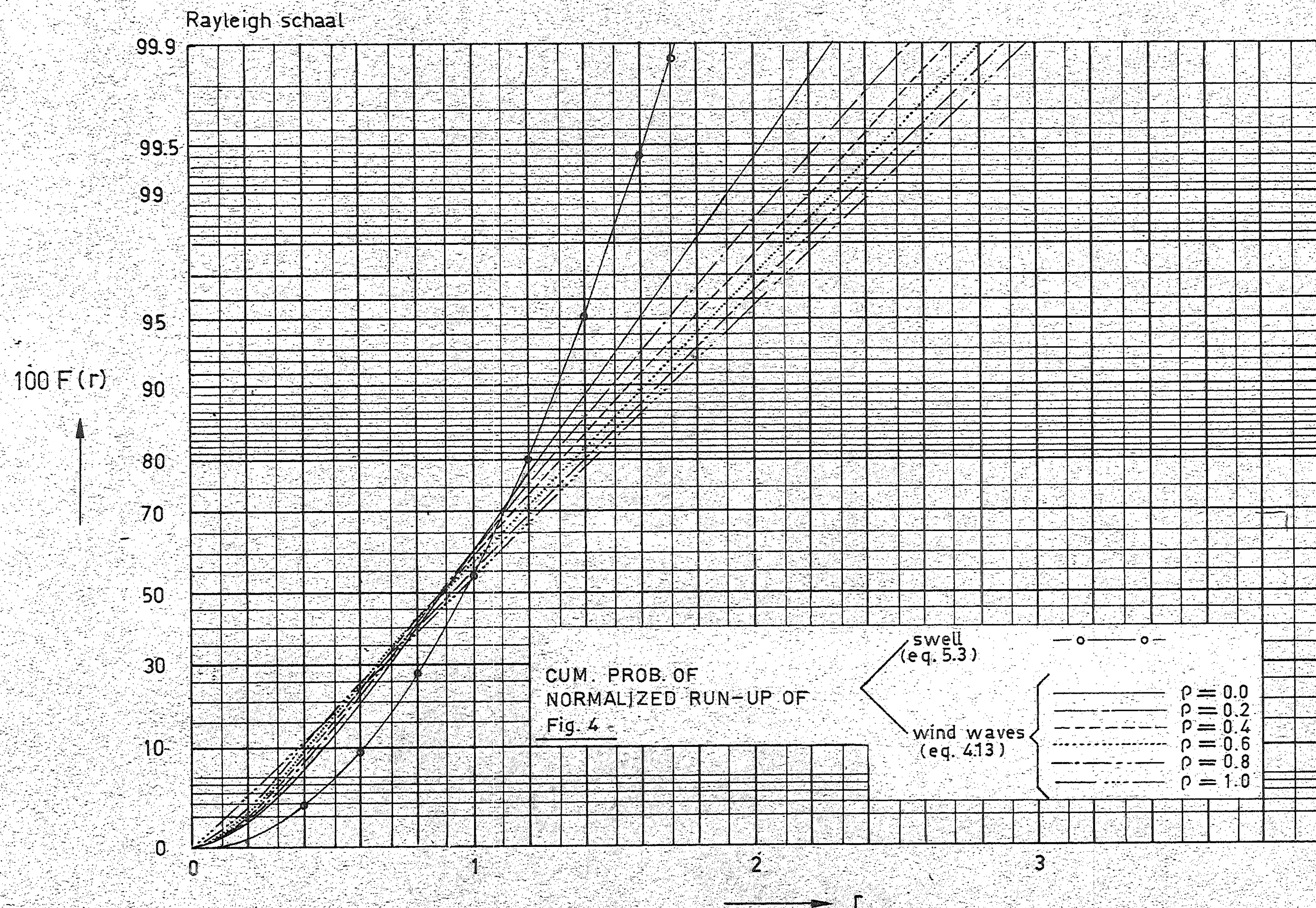
Eq. 4.13 becomes

$$F(r) = 1 - \frac{\pi r^2}{2} K_1\left(\frac{\pi r^2}{2}\right) \qquad \text{if } k = 0\,. \tag{4.14}$$

This corresponds to the case of zero correlation between $H'$ and $L_0'$, which, for the bivariate Rayleigh distribution, implies stochastic independence of $H'$ and $L_0'$.

When $k = 1$, $x$ is unbounded for all $r \neq 0$, as follows from eq. 4.5. The same is true for the Bessel functions in eq. 4.13. Therefore, eq. 4.13 cannot be used as it stands if $k = 1$. Using asymptotic expansions of the Bessel functions, it may be shown that

$k = 1$
$\rho = 0$



Rayleigh schaal
99.9
99.5
99
95
90
80
70
50
30
10
0
100 F (r)
0
1
2
3
4
r
swell
(eq. 5.3)
wind waves
(eq. 4.13)
ρ = 0.0
ρ = 0.2
ρ = 0.4
ρ = 0.6
ρ = 0.8
ρ = 1.0
CUM. PROB. OF
NORMALIZED RUN-UP OF


Fig. 4

$$F(r) \rightarrow 1 - \frac{1+k}{2\sqrt{k}} e^{-\frac{\pi}{2}\frac{r^2}{1+k}} \qquad \text{if } k \rightarrow 1 \tag{4.15}$$

so that

$$F(r) = 1 - e^{-\frac{\pi}{4}r^2} \qquad \text{if } k = 1 \tag{4.16}$$

This is the one-dimensional Rayleigh distribution, as expected: $k = 1$ corresponds to $\rho = 1$ (see figure 2), which means that $H'$ and $L_o'$ are linearly dependent. This in turn implies $H' = L_o'$ because they are identically distributed. Eq. 3.4 then becomes

$$R' = \sqrt{H'L_o'} = H' = L_o' \qquad \text{if } k = 1 \tag{4.17}$$

so that the distribution of $R'$ becomes equal to that of $H'$ or $L_o'$, i.e. equal to the Rayleigh distribution.

The <u>mean</u> of the normalized run-up can be found as a moment of the joint distribution of $H'$ and $L_o'$ :

$$\overline{R'} = \overline{(H'L_o')^{1/2}} = \int_0^\infty\int_0^\infty (h\,l)^{1/2} f(h,l)\,dh\,dl \tag{4.18}$$

Substituting eq. 4.1 and carrying out the integration yields

$$\overline{R'} = \frac{2}{\sqrt{\pi}}\left\{\Gamma\left(\frac{5}{4}\right)\right\}^2 {}_2F_1\left(-\frac{1}{4}, -\frac{1}{4}; 1; k^2\right) \tag{4.19}$$

in which ${}_2F_1$ is the hypergeometric function (Middleton, 1960). This expression reduces to

$$\overline{R'} = \frac{2}{\sqrt{\pi}}\left\{\Gamma\left(\frac{5}{4}\right)\right\}^2 \simeq 0.93 \qquad \text{if } k = 0 \tag{4.20}$$

and to

$$\overline{R'} = 1 \qquad \text{if } k = 1 \tag{4.21}$$

The associated cumulative probabilities are 0.535 and 0.544 respectively, i.e. approximately 0.54 in both cases. It is assumed that this value will also hold for other values of $k$.

An inspection of figures 3 and 4 shows that the width of the run-up distribution increases as $\rho$ increases. According to Bretschneider (1959), $\rho = 0$ occurs when the sea is fully developed, while $\rho = 1$ is considered to be a limiting value which is more nearly approached in a young sea.

At this stage it may be useful to revert briefly to non-normalized variables. As an example, the value exceeded by 2% of the run-ups will be considered for the limiting cases $\rho = 0$ and $\rho = 1$. Entering figure 4 with the value $\mathcal{F} = 0.98$ and reading the corresponding values of $r$ gives

$$\begin{aligned} R_2' &= 1.78 \quad , \quad \text{if } \rho = 0 \\ R_2' &= 2.23 \quad \quad \text{if } \rho = 1 \end{aligned} \tag{4.22}$$

Substitution of eq. 3.3 gives

$$\begin{aligned} R_2 &= 1.78\sqrt{\bar{H} L_o} \tan\alpha \quad \text{if } \rho = 0 \\ R_2 &= 2.23\sqrt{\bar{H} L_o} \tan\alpha \quad \text{if } \rho = 1 \end{aligned} \tag{4.23}$$

If $H$ and $T^2$ are Rayleigh distributed, the following relationships hold :

$$\bar{T} \approx 0.96\sqrt{\overline{T^2}} \tag{4.24}$$

and

$$\bar{H} \approx 0.63\, H_{1/3} \tag{4.25}$$

in which $H_{1/3}$ is the average of the highest 1/3 of the wave heights. Substitution of eqs. 2.2, 4.24 and 4.25 into eqs. 4.23 yields finally:

$$\begin{aligned} R_2 &= 0.60\, \bar{T}\sqrt{g H_{1/3}} \tan\alpha \quad \text{if } \rho = 0 \\ R_2 &= 0.75\, \bar{T}\sqrt{g H_{1/3}} \tan\alpha \quad \text{if } \rho = 1 \end{aligned} \tag{4.26}$$

## 4.3 Comparison with Saville's work

Saville (1962) gives run-up distribution curves which are based on the joint distribution of $H$ and $T^2$ given by Bretschneider (1959) for the case when the marginal distributions are of the Rayleigh type, and $H$ and $T^2$ are stochastically independent. This distribution is the same as given by eq. 4.1 if $\rho = 0$. The run-up distribution given by eq. 4.14 may therefore be compared with Saville's results. Such a comparison cannot be carried out directly because the run-ups have been normalized in different ways. The normalization factor used by Saville is $R_{1/3}$, the run-up of a periodic wave with height $H_{1/3}$ and period $T_{1/3}$, i.e. the mean period of the highest 1/3 of the waves, which in this case (zero correlation) equals $\bar{T}$. Values of $R_{1/3}/H_{1/3}$ were obtained from the run-up curves published by the B.E.B. (1961). These curves are based on the same data as used by Hunt. The factor used here is $\sqrt{\bar{H}\,\bar{L}_0}\,\tan\alpha$, i.e. the run-up, according to Hunt's formula, of a periodic wave with height $\bar{H}$ and deep-water length $\bar{L}_0$.

Two questions should be distinguished in comparing the run-up distributions:

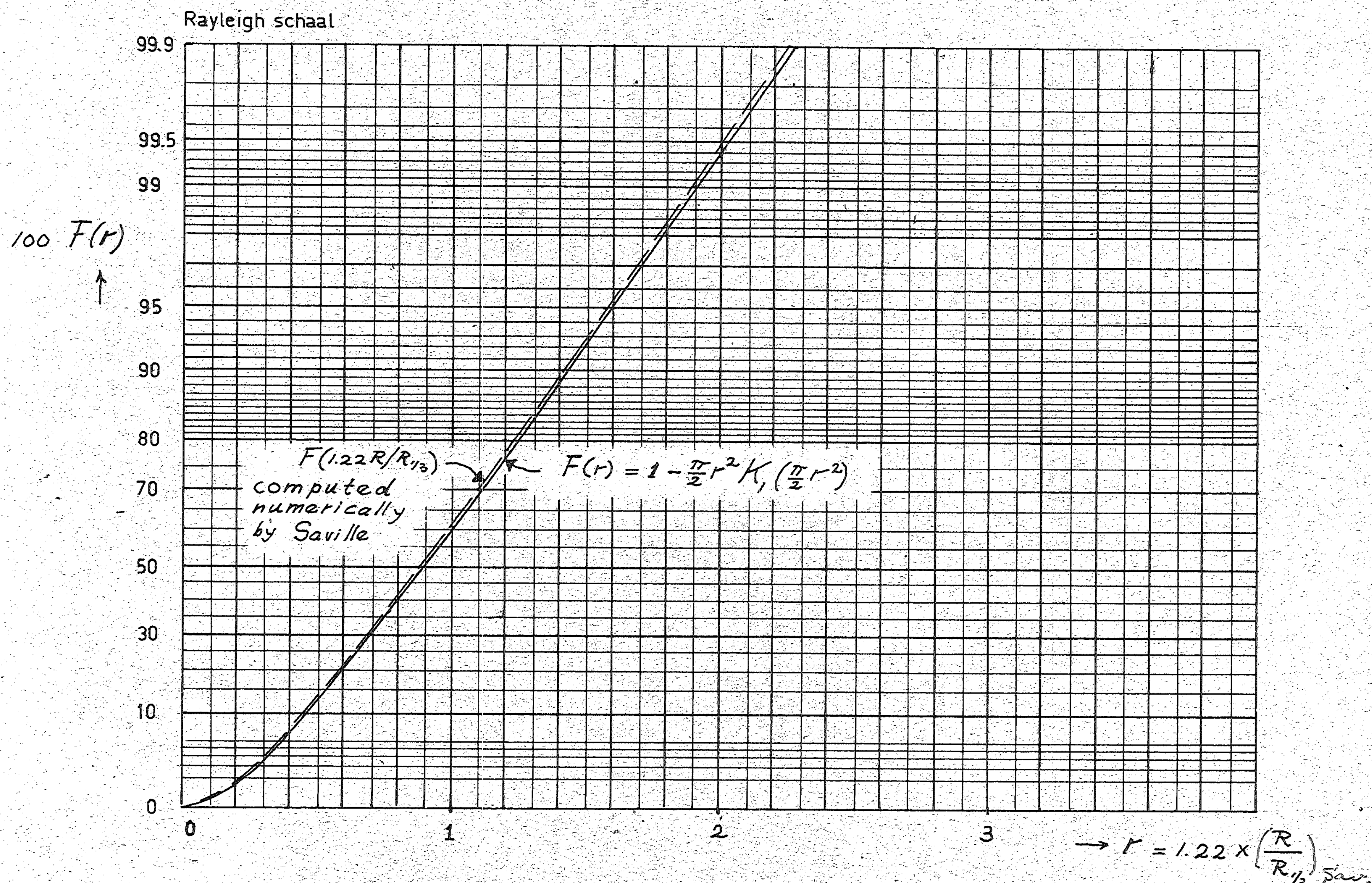
(a) What is the value of the normalization factor for given waves and slope?

(b) How is the normalized run-up distributed?

The first question in fact mutually compares Hunt's formula and the run-up curves used by Saville. Such a comparison is not relevant here. Both normalization factors will therefore be based on the same relationship between run-up, waves, and slope. Using Hunt's formula, the ratio of the normalization factors becomes:

$$\gamma \equiv \frac{\sqrt{H_{1/3} L_{0\,1/3}}\ \tan\alpha}{\sqrt{\bar{H}\,\bar{L}_0}\ \tan\alpha} = \sqrt{\frac{H_{1/3} * \bar{T}^2}{\bar{H} * \bar{T}^2}} \qquad (4.27)$$

Substitution of eqs. 4.24 and 4.25 gives $\gamma = 1.22$. With the use of this conversion factor, Saville's curve for $\tan\alpha = 1:6$ and $H_{1/3}/T_{1/3}^2 = 0.22$ ft/sec$^2$ has been plotted in figure 5. This curve appears to be in very close agreement with eq. 4.14, which is also shown in the figure. Saville's curves for the steeper slopes (not shown here) are somewhat different, predominantly in the range of low run-up values. This will be investigated further in chapter 7.



Rayleigh schaal
99.9
99.5
99
95
90
80
70
50
30
10
0
100 F(r)
0
1
2
3
F(1.22R/R1/3) computed numerically by Saville
F(r) = 1 − π/2 r² K1(π/2 r²)
r = 1.22 × (R/R1/3) Sav.


Figure 5

## 5 RUN-UP DISTRIBUTION OF SWELL

In this chapter waves will be considered of which the heights are Rayleigh-distributed, while the variation in periods is neglected. This model is approximately applicable to old swell. It represents a degenerate case of the general joint distribution of $H'$ and $L_o'$ , because $L_o' = 1$ with a probability of 100%. One can proceed formally as before by writing for the joint probability density

$$f(h, l) = f(h)\,\delta(l-1) \tag{5.1}$$

in which $f(h)$ is the marginal probability density of $H_s'$ and $\delta$ is Dirac's unit impulse "function", and by substituting this into eqs. 3.9 and 3.11. A more direct method is to revert to eq. 3.4:

$$R' = \sqrt{H' L_o'} \tag{3.4}$$

which in this case becomes

$$R' = \sqrt{H'} \tag{5.2}$$

It follows that $R'^2$ is Rayleigh-distributed (eq. 4.16), so that the distribution function of $R'$ can be written down at once: ~~using eq. 4.8:~~

$$F(r) = 1 - e^{-\frac{\pi}{4} r^4} \tag{5.3}$$

from which the probability density is found by differentation as

$$f(r) = \pi r^3 e^{-\frac{\pi}{4} r^4} \tag{5.4}$$

The mean value of the normalized run-up in this case is $\overline{R'} = (4/\pi)^{1/4}\,\Gamma(5/4) \simeq$ 0.96; the mean square value is of course 1. The functions given by eqs. 5.3 and 5.4 are depicted in figures 4 and 3, respectively.

From an inspection of all the curves in these figures it is clear that the shape of the run-up distribution varies more or less monotonically with the wave age. The widest spread (greatest variance) occurs with a very young sea, the smallest spread with an old swell, while the distribution corresponding to fully developed wind waves is approximately halfway between these extremes.

# 6 STEEPNESS DISTRIBUTION OF WAVES WITH ARBITRARY JOINT DISTRIBUTION OF $H$ AND $L_o$.

It has been pointed out in chapter 2 that Hunt's run-up formula is not applicable to waves of which the steepness $S \equiv H/L_o$ is less than $S_{cr}$ given by eq. 2.3. Like $H$ and $L_o$, $S$ is a stochastic variable. The distribution of this variable will be determined in order to be able to estimate the fraction of the waves to which Hunt's formula could be applied.

It is worth noting that the peak horizontal inertia wave force on a fixed body is proportional to $H/T^2$ if the wave is in deep water. The same is true for the vertical force in water of any depth. Thus, the distribution of these peak forces, if suitably normalized, is the same as the distribution of the normalized steepness, to be determined in the following.

The steepness $S$ will be normalized as follows:

$$S' \equiv \frac{S}{\bar{H}/\bar{L}_o} = \frac{H/L_o}{\bar{H}/\bar{L}_o} \tag{6.1}$$

or

$$S' = \frac{H'}{L_o'} \tag{6.2}$$

The distribution of $S'$ can be obtained from $f(h,l)$ by exactly the same procedure as was used in chapter 3 for the determination of $F(r)$. A more formal procedure is to use the standard technique for the transformation of one bivariate probability density into another. The wave steepness should be one of the two variables of which the joint probability density is to be found. The other one is a dummy variable which will be eliminated afterwards. It can be chosen at convenience.

Let $f(h,l)$ be transformed into $f_*(s,t)$, in which $s = \frac{h}{l}$ and the dummy variable $t$ is some function of $h$ and $l$. An asterisk is used to distinguish the two probability densities. The transformation is given by

$$f_*(s,t) = \frac{f(h,l)}{|J|} \tag{6.3}$$

in which $J$ is the Jacobian:

$$J = \frac{\partial(s,t)}{\partial(h,l)} = \begin{vmatrix} \frac{\partial s}{\partial h} & \frac{\partial s}{\partial l} \\ \frac{\partial t}{\partial h} & \frac{\partial t}{\partial l} \end{vmatrix} \tag{6.4}$$

(Cramer, 1966). Choosing $t = l$ gives $J = l^{-1}$. Therefore,

$$f_*(s, l) = l f(h, l) = l f(sl, l) \tag{6.5}$$

The probability density of $S'$ can now be found as a marginal probability density of $f_*(s, l)$:

$$f(s) = \int_0^\infty f_*(s, l)\, dl \tag{6.6}$$

or

$$f(s) = \int_0^\infty l f(sl, l)\, dl \tag{6.7}$$

Integration with respect to $s$ yields the cumulative probability:

$$F(s) = \int_0^s ds^* \int_0^\infty l f(s^* l, l)\, dl \tag{6.8}$$

which may be transformed into

$$F(s) = \int_0^\infty dl \int_0^{sl} f(h, l)\, dh \tag{6.9}$$

This would have been obtained at once with the procedure used in chapter 3.

# 7 STEEPNESS DISTRIBUTION OF WIND WAVES WITH A BIVARIATE RAYLEIGH DISTRIBUTION OF $H$ AND $L_0$.

## 7.1 Derivation of the distribution

Substitution of the bivariate Rayleigh probability density (eq. 4.1) into eq. 6.7 gives

$$f(s) = \frac{\pi^2}{4} \frac{s}{1-k^2} \int_0^\infty l^3 e^{-\frac{\pi}{4}\frac{1+s^2}{1-k^2} l^2} I_0\left(\frac{\pi}{2} \frac{ks}{1-k^2} l^2\right) dl \tag{7.1}$$

By substituting

$$y = \frac{\pi}{2} \frac{ks}{1-k^2} l^2 \tag{7.2}$$

and

$$p = \frac{1+s^2}{2ks} \tag{7.3}$$

eq. 7.1 becomes

$$f(s) = \frac{1-k^2}{2k^2 s} \int_0^{\pm\infty} y\, e^{-py} I_0(y)\, dy \qquad (7.4)$$

The sign of the upper limit of integration must equal the sign of $k$.

For large $|y|$, $I_0(y) \simeq e^{|y|}/\sqrt{2\pi|y|}$, so that the integral converges only if $|p| > 1$ (Watson, 1966). Eq. 7.3 may be rewritten as

$$|p| = \frac{\frac{1}{2}\left(\frac{1}{s} + s\right)}{|k|} \qquad (7.5)$$

The numerator is always $> 1$ except for $s = 1$, in which case its value is 1. The denumerator is at most 1. Thus, only if $|k| = s = 1$ does the integral fail to converge. This is to be expected, as $|k| = 1$ implies $H_0' = L_0'$ (see page 14), so that in that case $S'$ assumes the value 1 with a probability of 100%. The corresponding probability density is zero for all $s \neq 1$ and is unbounded for $s = 1$. It is described by Dirac's unit impulse function:

$$f(s) = \delta(s-1) \qquad \text{if } |k| = 1 \qquad (7.6)$$

The distribution function is the unit step function centered at $s = 1$.

If $|k| < 1$, then $|p| > 1$ for all $s$, and the integral in eq. 7.4 is bounded. It has been evaluated using section 13.2 from Watson (1966), with the result

$$f(s) = 2(1-k^2)\frac{s(1+s^2)}{\{s^4 + (2-4k^2)s^2 + 1\}^{3/2}} \qquad \text{if } |k| < 1 \qquad (7.7)$$

This reduces to

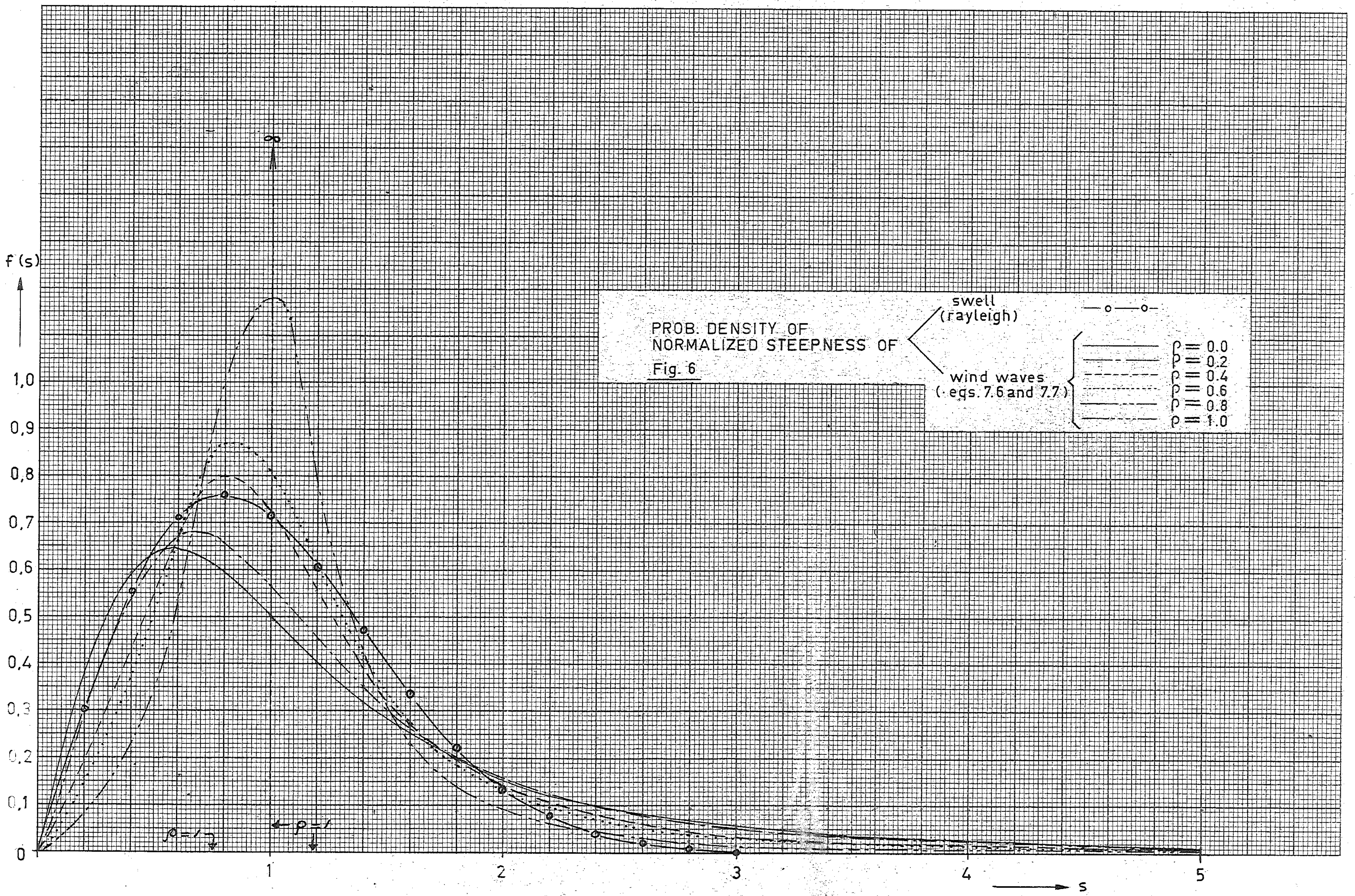
$$f(s) = \frac{2s}{(1+s^2)^2} \qquad \text{if } k = 0 \qquad (7.8)$$

$f(s)$ has been plotted in figure 6 for selected values of $k$.

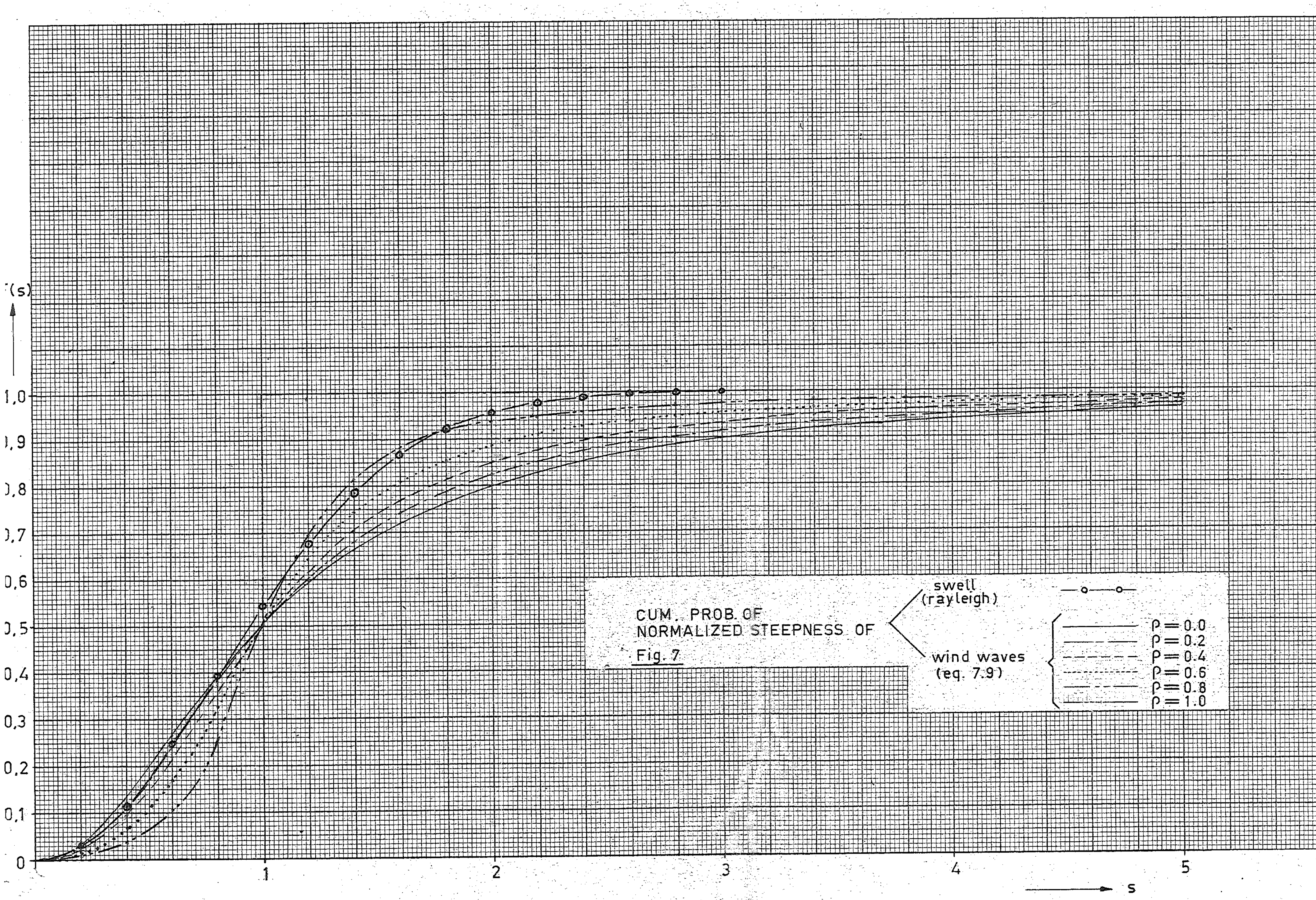
Integration of $f(s)$ given by eq. 7.7 yields the distribution function:

$$F(s) = \frac{s^2 - 1}{2\{s^4 + (2-4k^2)s^2 + 1\}^{1/2}} + \frac{1}{2} \qquad \text{for } |k| \leq 1 \qquad (7.9)$$

This equation is also valid for $|k| = 1$, in which case it represents the unit step function centered at $s = 1$.



PROB. DENSITY OF
NORMALIZED STEEPNESS OF
Fig. 6
swell
(rayleigh)
wind waves
( eqs. 7.6 and 7.7 )
ρ = 0.0
ρ = 0.2
ρ = 0.4
ρ = 0.6
ρ = 0.8
ρ = 1.0
f (s)
s
∞
ρ = 1
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1,0
1
2
3
4
5



F(s)
1,0
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
1
2
3
4
5
s
CUM. PROB. OF NORMALIZED STEEPNESS OF
swell (rayleigh)
wind waves (eq. 7.9)
ρ = 0.0
ρ = 0.2
ρ = 0.4
ρ = 0.6
ρ = 0.8
ρ = 1.0

Fig. 7

Eq. 7.9 reduces to

$$F(s) = \frac{s^2}{1+s^2} = \frac{1}{1+s^{-2}} \qquad \text{if } k = 0 \tag{7.10}$$

The co-cumulative probability (probability of exceedance) is

$$1 - F(s) = \frac{1}{1+s^2} \qquad \text{if } k = 0 \tag{7.11}$$

Graphs of $F(s)$ are given in figure 7, inspection of which shows that the distribution is not very sensitive to variations of $\rho$ if $\rho$ is small ($< 0.4$, for instance). This means that the simple expressions given by eqs. 7.10 and 7.11 can be used as an approximation in cases of low linear correlation between $H$ and $L_o$.

It is evident from figure 7 as well as from eq. 7.9 that the <u>median</u> of the steepness distribution does not depend on $\rho$ ; it is always 1.

The <u>mean</u> steepness has been evaluated as a moment of $f(h,l)$:

$$\overline{S'} = \overline{H'/L'_o} = \int_0^\infty\int_0^\infty (h/l)\, f(h,l)\, dh\, dl \tag{7.12}$$

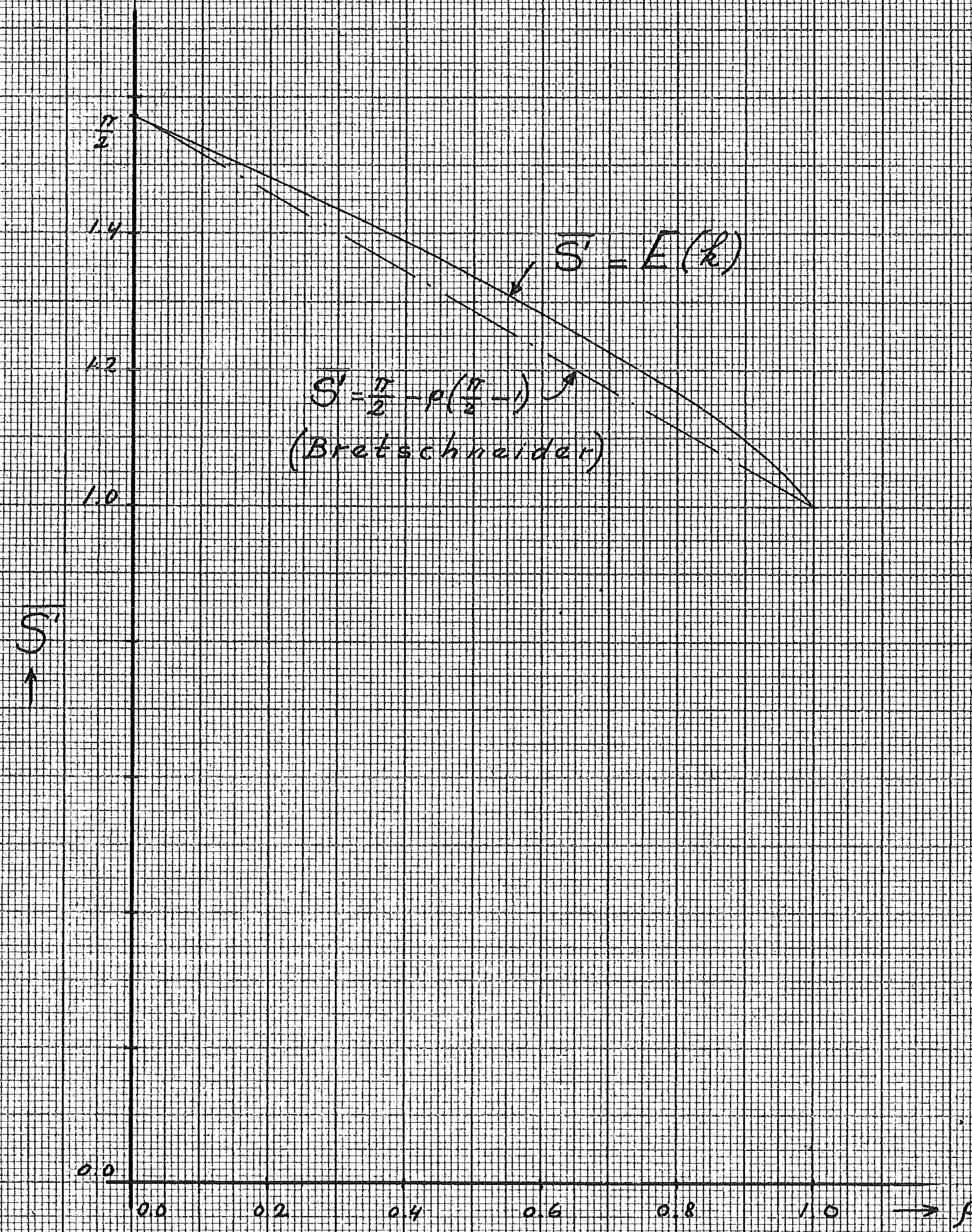
Substitution of the bivariate Rayleigh probability density (eq. 4.1) and evaluation of the integral gives

$$\overline{S'} = E(k), \tag{7.13}$$

the complete elliptic integral of the second kind of modulus $k$. Utilizing the relationship between $k$ and $\rho$ given in figure 2, $\overline{S'}$ has been plotted as a function of $\rho$ in figure 8.

The steepness distribution of swell having the distribution of $H'$ and $L'_o$ described in chapter 5 need not be considered separately: it is equal to the distribution of $H'$ (Rayleigh) because $L'_o$ is assumed to be constant. For purposes of comparison, it has also been plotted in the figures 6 and 7.

The distribution functions depicted in figure 7 may be used to estimate the fraction of the waves for which Hunt's formula would (not) be applicable. As an example, the values used by Saville in the computation of the curve shown in figure 5 will be taken: $\tan\alpha = 1:6$, zero correlation between $H$ and $T$, and $H_{1/3}/T_{1/3}^2 = 0.22$ ft/sec$^2$, which corresponds to $\overline{H}/\overline{L}_o = 0.025$. From eq. 2.3, $S_{cr} = 0.19(1/6)^2 = 0.0053$.




Mean normalised wave steepness vs. $\rho$

$$\rho = \frac{E(k) - \frac{1}{2}(1-k^2)K(k) - \pi/4}{1 - \pi/4}$$

Figure 8

Therefore, $S'_{cr} = 0.0053/0.025 = 0.21$. From figure 7 or from eq. 7.10 it can be seen that $F(0.21) = 0.04$ if $\rho = 0$, which implies that in this case Hunt's formula would be applicable to 96% of the waves. For a 1:3 slope and the same mean wave steepness this value drops to 59%. However, the curve computed by Saville for this case differs at most 10% from the curve for the 1:6 slope, or from eq. 4.14. This indicates that the effect of a few non-breaking waves is relatively weak.

## 7.2. Comparison with Bretschneider's and Saville's work

Bretschneider (1959) in his work on wave variability considered the wave steepness. He did not determine its distribution but only the mean value, with the result

$$\overline{S'} = \frac{\pi}{2} - \rho\left(\frac{\pi}{2} - 1\right) \tag{7.14}$$

which is also shown in figure 8. Only for $\rho = 0$ and $\rho = 1$ do Bretschneider's equation and eq. 7.13 give the same values. Bretschneider's derivation is based on knowledge of the marginal distributions of $H$ and $L_o$ alone (Rayleigh), and on the assumption that the mutual regressions of $H$ and $L_o$ would be linear. It has been shown elsewhere (Battjes, 1969) that the assumption of linear regression is incompatible with the assumption of a bivariate Rayleigh distribution, except in the limiting cases $\rho = 0$ and $\rho = 1$. Therefore, only in these limiting cases is exact agreement between eqs. 7.13 and 7.14 to be expected. For intermediate values of $\rho$ the difference is at most 5%.

Based on the $H$-$T$ distribution given by Bretschneider in the case $\rho = 0$, Saville (1962) calculated the wave steepness distribution numerically. His result has been compared with the exact solution given by eq. 7.10 above, which was derived on the same premises. The agreement appeared to be quite good in almost the whole range of the distribution. Only the extreme lower and upper tails were found to deviate noticeably from eq. 7.10.

# 8 EMPIRICAL RUN-UP DISTRIBUTIONS

## 8.1 Introduction

Before proceeding to the presentation and discussion of some empirical data, it is recalled that the derivation of the run-up distribution of wind waves has been based on the following premises:

I The run-up distribution can be determined by assigning to each individual wave a run-up equal to the run-up of a periodic wave train of corresponding height and period.

II The run-up of periodic waves breaking on the slope is given by Hunt's formula.

III The fact that not all the waves break on the slope has a negligible effect on the run-up distribution.

IV $H$ and $L_0$ have a bivariate Rayleigh distribution.

The second of these premises will in the present context be considered as an empirically established fact. The others are a priori considered to be hypothetical, and in need of checking. Hypothesis III is more nearly valid for the steeper waves and the gentler slopes, because the fraction of the waves which is not breaking is then very small.

In the following two sections, empirical data from the laboratory and the field will be presented. Only the laboratory data can be used to check some of the hypotheses I through IV, or combinations thereof. The field data will be presented for their intrinsic interest. Such data are almost nonexistent.

## 8.2 Laboratory data

In 1939 run-up experiments were carried out at the Delft Hydraulics Laboratory, in behalf of Zuiderzee Works, The Netherlands. The run-ups of irregular waves on various slopes were measured (Wassing, 1958). The waves were generated by a combination of wind and a bulkhead with a periodic motion. As a result, the model waves were not natural wind waves on a small scale. This is evidenced by the fact that the measured wave height distribution is much narrower than the Rayleigh distribution. The deviation from natural conditions is even greater for the wave periods, which in the model varied but very little. Because of this it is not useful to compare the measured run-up distributions with those derived in this paper on the basis of hypothesis IV. The measurements can, however, be used to check to a certain extent the validity of the hypotheses (I + II + III). This check will be carried out in two stages. First the magnitude of the median run-up ( $R_{50}$ ) is considered, and after that the shape of the run-up distribution, as given by the values $R_n/R_{50}$, in which $n$ is the probability of exceedance in % ( $n = 100(1-F)$ ).

Two series of measurements were made. In each series the characteristics of the incident waves were kept constant, and the run-up was measured on 7 different slopes, with tan $\alpha$ ranging from 0.1 to 0.4. The waterdepth (0.35 m), the mean wave period (1 sec), and the mean wave length (1.40 m) were the same in both series. The wave periods varied very little and will here be considered to be constant. The wave height was 0.10 m resp. 0.07 m. It is not clear from the original report how this height had been defined; for this reason it will here be called the nominal wave height.

According to the hypothesesto be tested, the ratio

$$R_{50}^{*} \equiv \frac{R_{50}}{0.4\, T \sqrt{g H_{nom}}\ \tan\alpha} \qquad (8.1)$$

should be constant. The value of the constant cannot be predicted because of the uncertainty with respect to $H_{nom}$. Experimental values are listed in Table 1.

| $H_{nom}$ (cm) → | 10 | 7 | 10 | 7 |
|---|---|---|---|---|
| tan $\alpha$ – | $R_{50}$ (cm) | $R_{50}$ (cm) | $R_{50}^{*}$ – | $R_{50}^{*}$ – |
| 0.1 | 4.7 | 3.7 | 1.17 | 1.11 |
| 0.15 | 6.9 | 5.7 | 1.15 | 1.14 |
| 0.2 | 9.3 | 8.1 | 1.16 | 1.21 |
| 0.25 | 11.8 | 9.3 | 1.18 | 1.12 |
| 0.286 | 15.4 | 13.4 | 1.35 | 1.40 |
| 0.333 | 15.8 | 13.2 | 1.19 | 1.19 |
| 0.4 | 17.5 | 15.4 | 1.09 | 1.15 |

Table 1

The agreement between $R^*_{50}$ values in two colums for the same value of $\tan\alpha$ (horizontally) is good, and confirms the assumed proportionality of $R$ and $\sqrt{H}$. The agreement between $R^*_{50}$ values within one column (vertically) is fairly good; this confirms the proportionality of $R$ and $\tan\alpha$. Only the two points for $\tan\alpha$ = 0,286 (= 1:3½) deviate considerably from the others. Apart from these two, all measured values of $R^*_{50}$ are grouped quite closely around the mean value 1,15, with a maximum deviation of approximately 5% only. This means that for these experiments the variation of the median run-up with wave height and slope angle is adequately expressed by Hunt's formula. Whether or not this is also the case for run-up values with a different probability of exceedance can be investigated by comparing $R_n/R_{50}$ with $\sqrt{H_n/H_{50}}$, which according to the hypotheses to be tested should be equal to each other for all $n$, because $T$ was assumed to be constant. Such a comparison has been given in figures 9[a] and 9[b] for $H_{nom}$ = 0,10 m and 0,07 m respectively for values of $n$ from 50 to 2. The plotted points represent values which have been obtained by averaging over the different slope angles in each series. The shape of the wave height distribution is also shown.

There appears to be a fair agreement between $R_n/R_{50}$ and $\sqrt{H_n/H_{50}}$ for the waves with $H_{nom}$ = 0,10 m (figure 9[a]) and a very good agreement for the waves with $H_{nom}$ = 0,07 m (figure 9[b]). This lends support to the hypotheses (I + II + III).

Van Oorschot and d'Angremond (1968) have carried out run-up experiments with irregular waves. Disposition of a programmed wave board enabled them to generate waves with prescribed energy density spectra. In addition, a wind with a mean velocity of up to 3 m/sec was blown over the water surface. One test was run in which the waves were generated entirely by wind, with a mean velocity of approximately 8 m/sec. The effect of the spectral shape on the wave run-up was the main object of the study, in particular the effect of the spectral width. The value of the spectral width parameter $\epsilon$, introduced by Cartwright and Longuet-Higgins, was 0,22 for the wind-generated waves and varied from 0,34 to 0,59 for the others. The widest experimental spectrum was chosen simi-

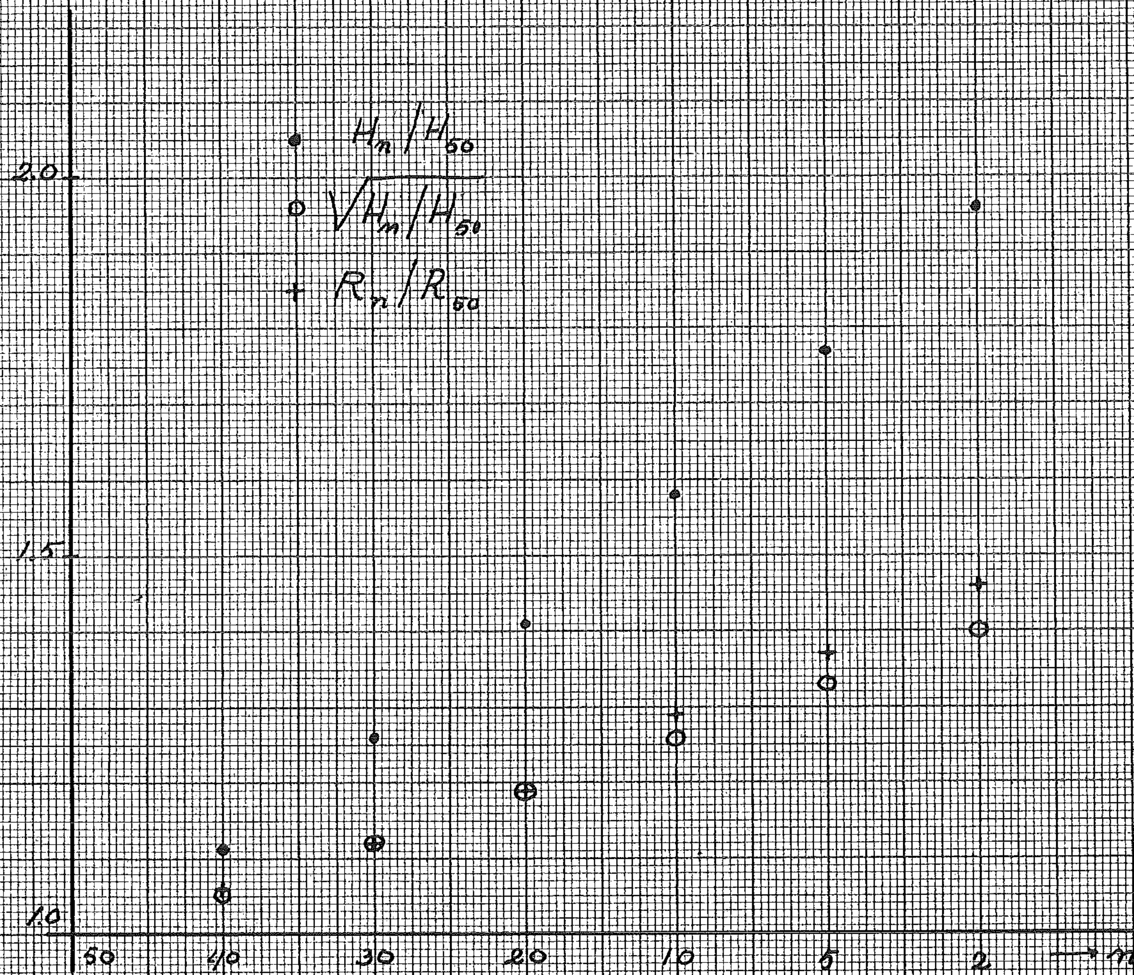



Figure 9a

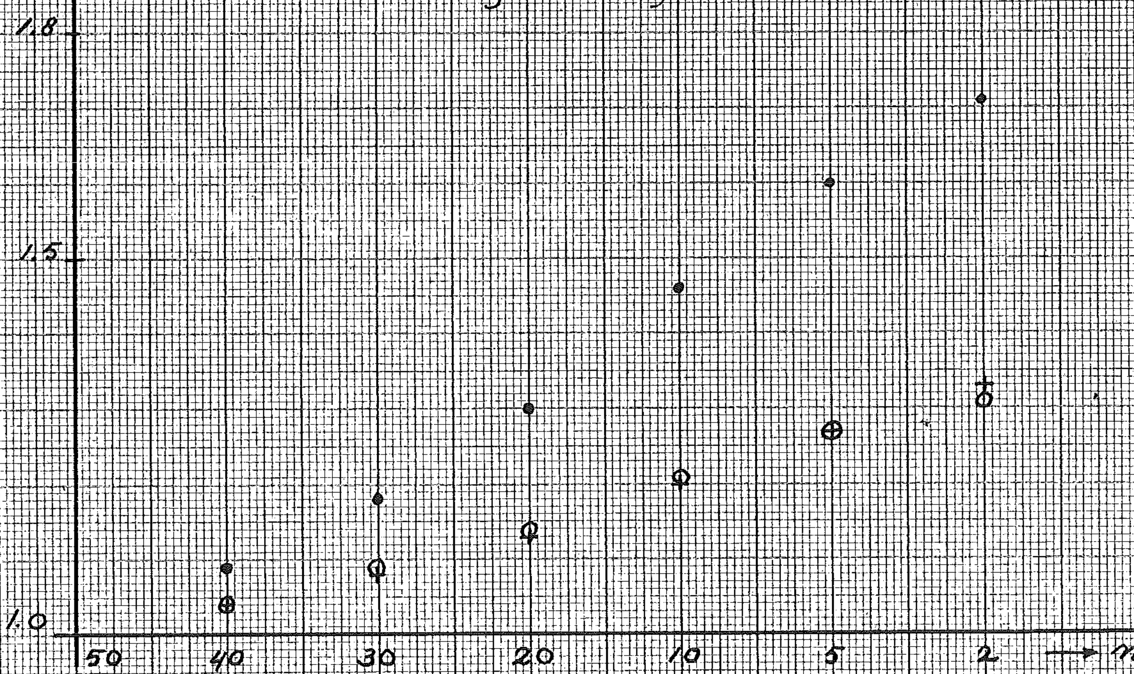



Figure 9b

lar to the widest spectra which had been measured with a wave pole in the North Sea off the Dutch coast in a depth of approximately 15 m.

The values of $\epsilon$ were computed after cutting off the high-frequency tail of each spectrum at the frequency where the energy density was 5% of the maximum value. As a result, the actual $\epsilon$-values are considerably underestimated. The computed values therefore have comparative significance only, rather than absolute significance. This is of no concern, however, as in this application there is no compelling reason to use just $\epsilon$ as a measure of the spectral width.

In addition to $\epsilon$, the following parameters were varied (the experimental range is given in parentheses): $\hat{T}$, the period of the spectral component with maximum energy density (0.71 sec - 1.64 sec); $H_{1/3}$, the significant wave height (3.7 cm - 13.6 cm); $H_{1/3}/g\hat{T}^2$, a wave steepness ($4.0 \times 10^{-3}$ - $12.2 \times 10^{-3}$); $d/g\hat{T}^2$, a relative waterdepth ($1.7 \times 10^{-2}$ - $8.1 \times 10^{-2}$); and $\tan\alpha$, the slope (1:4 and 1:6).

The most important results obtained by van Oorschot and d'Angremond, and the interpretations thereof in the context of this paper, can be summarised as follows:

- The effect on the run-up of wave height, wave period and slope angle is adequately expressed by a Hunt-type formula with a proportionality factor which depends on $n$ and $\epsilon$:

$$R_n = C_n(\epsilon)\, \hat{T} \sqrt{g H_{1/3}}\, \tan\alpha \tag{8.2}$$

This implies that the shape of the run-up distribution is significantly affected by $\epsilon$ only, not by either the wave steepness, the relative waterdepth or the slope angle. This is in agreement with hypotheses (I + II + III) if the additional assumption is made that the shape of the $H$-$T$-distribution is determined by the spectral shape.

- The width (spread) of the run-up distributions increases with $\epsilon$, i.e. with the width of the energy spectrum.
- The run-up distributions, computed on the basis of hypothesis I, the measured $H$-$T$-distribution, and the run-up data from the B.E.B. (1961), agreed fairly well with the measured distributions.

This proves that hypothesis I can lead to useful results.

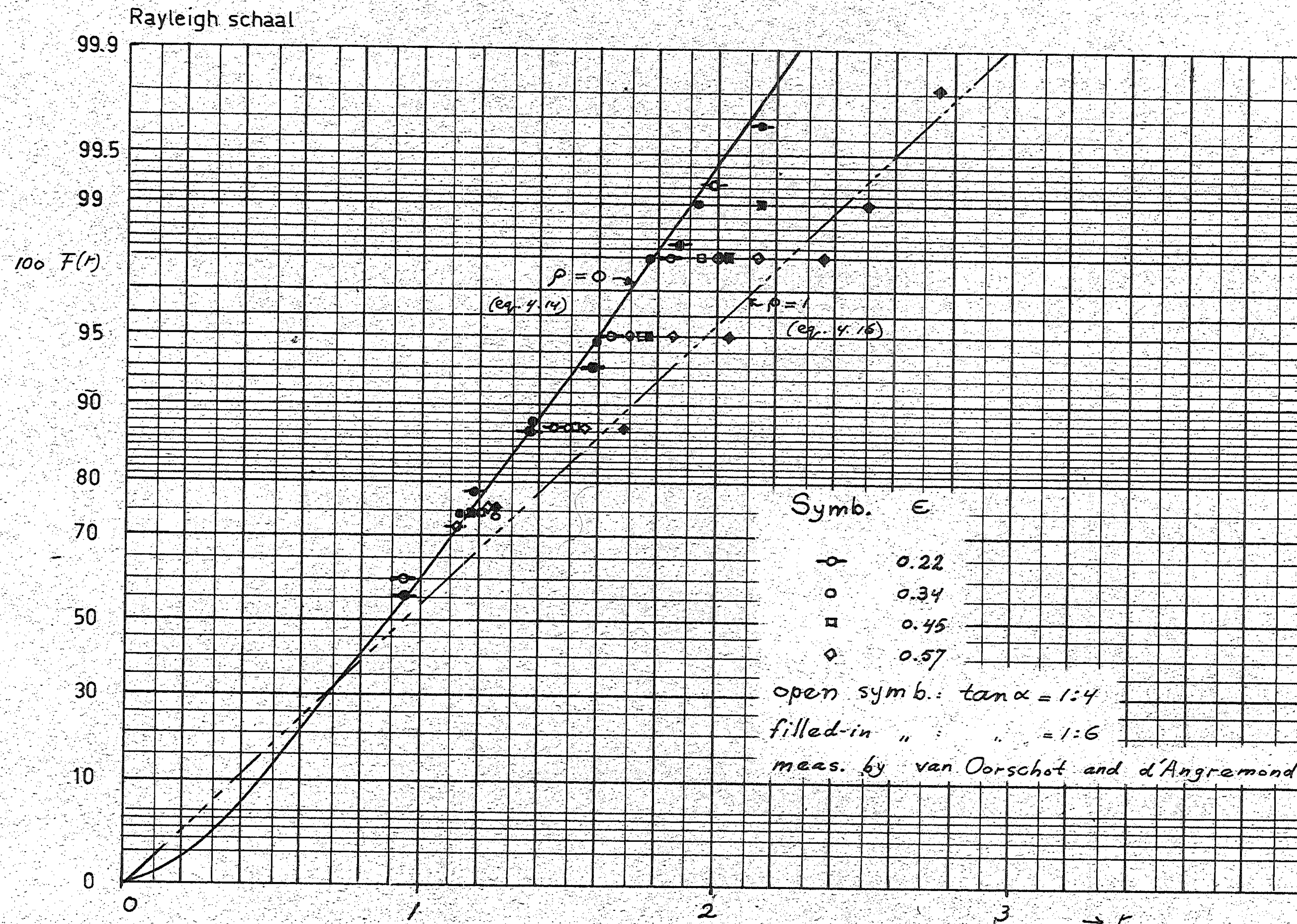

Rayleigh schaal
99.9
99.5
99
100 F(r)
95
90
80
70
50
30
10
0
0
1
2
3
→ r
ρ = 0
(eq. 4.14)
ρ = 1
(eq. 4.16)
Symb. ε
0.22
0.34
0.45
0.57
open symb.: tan α = 1:4
filled-in " : " = 1:6
meas. by van Oorschot and d'Angremond

Figure 10

Some run-up distributions measured by van Oorschot and d'Angremond have been replotted in figure 10. The corresponding $\epsilon$-values range from 0.22 (wind-generated waves) to 0.57, so that almost the entire experimental range of $\epsilon$ (0.22 - 0.59) is represented in the figure. The distributions given by eqs. 4.14 and 4.16 above, for the limiting cases $\rho = 0$ and $\rho = 1$, are also shown. It appears that the range of run-ups predicted on the basis of hypotheses I through IV agrees with the range of experimental run-ups of waves with spectra which varied from narrow to wide. Although this agreement does not prove each of the hypotheses correct, it does at least indicate that the end results are not unreasonable.

## 8.3 Field data

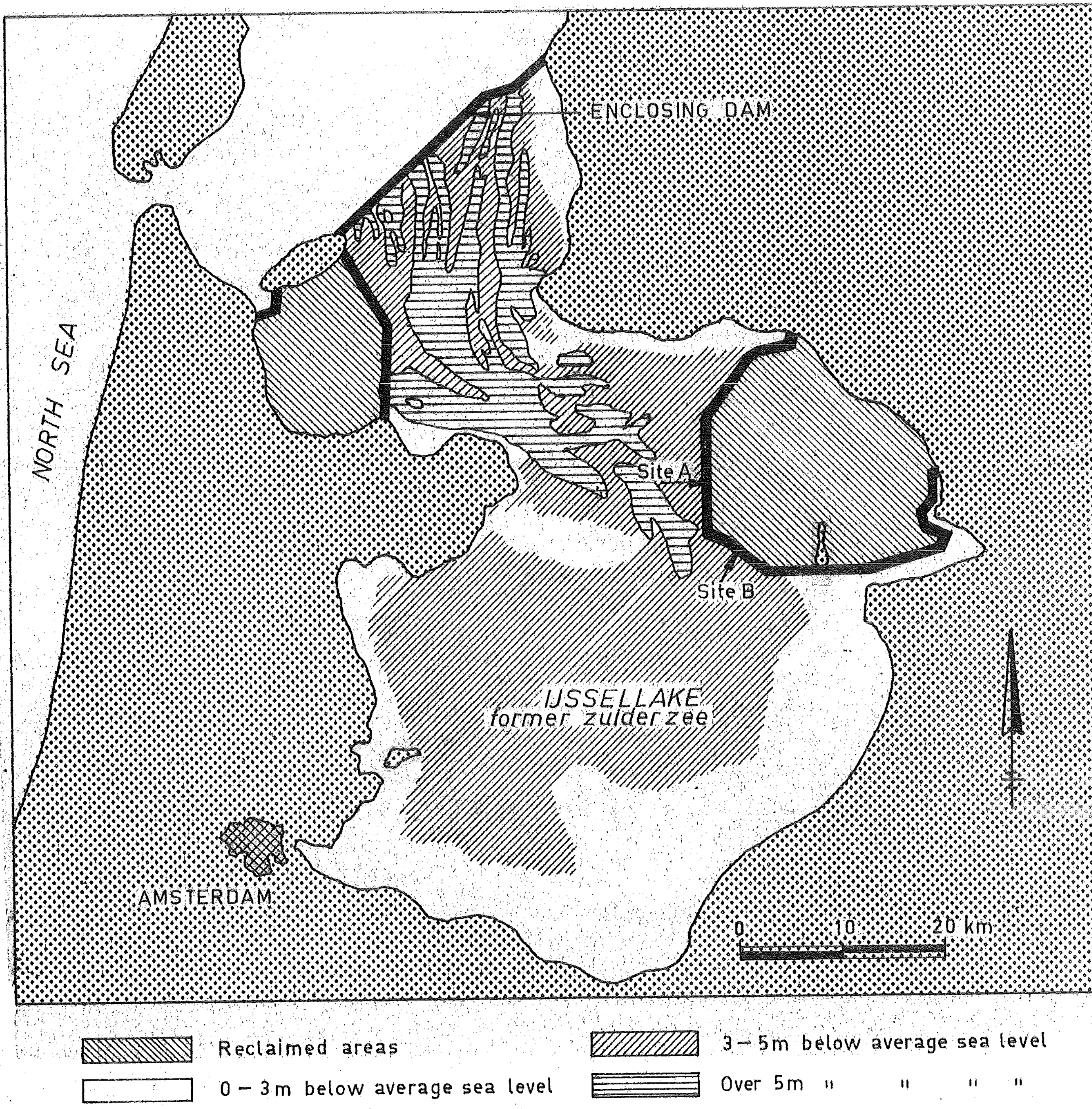
In this section some prototype run-up distributions will be presented which have been measured by Zuiderzee Works of the Netherlands. The author had access to the original data for the purpose of analysis.

The measurements stem from a time (1943-1944) when wave meters in the field were not available. It is therefore not possible to relate the magnitude of the run-ups to those of the incident waves. Only the shape of the run-up distribution will be dealt with.

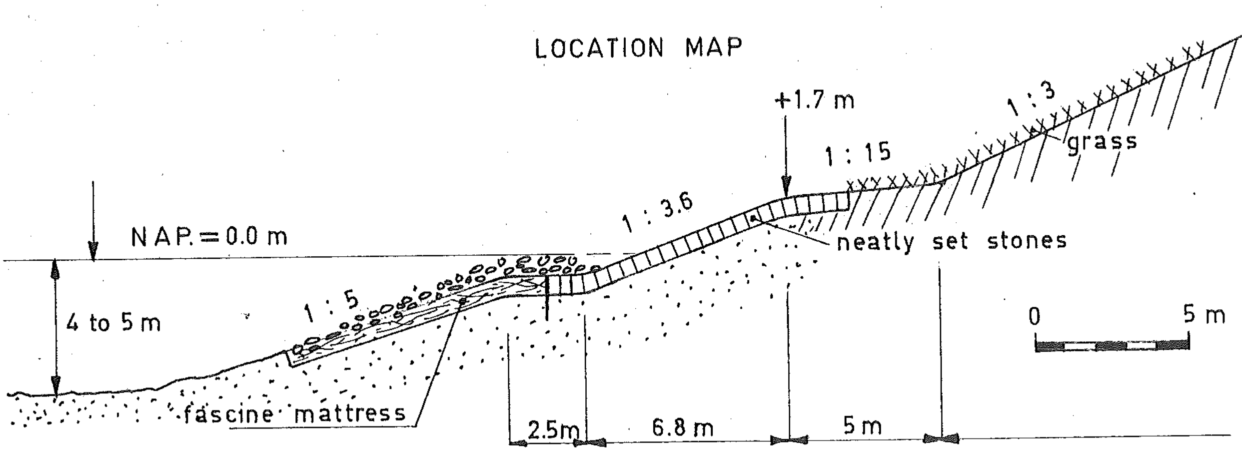
The measurements were carried out on a dike of the IJssellake (figure 11). This is essentially a tideless body of water with depths of approximately 5 m; the maximum fetch at the points of observation is approximately 50 km. Because of the short fetches and the absence of astronomical tide, a statistically steady state is more often approached, and longer lasting, than on a seacoast. This permits rather long measuring times, depending mainly on the duration of the local wind. The measurement time for the data to be presented was from 1 hr to 3 hr, which corresponds to approximately 1000 to 3000 observations per series.

The location of the two points of observation and a sketch of the dike cross-sections are shown in figure 11. The narrow berm which is present near mean lake level is for purposes of dike construction. The broader berm at an elevation of + 1.7 m has been installed for purposes of run-up reduction.

At the selected sites the dike facing was divided in a series




LOCATION MAP




TYPICAL CROSS SECTION

Fig. 11

of numbered areas of equal vertical increments (appr. 0.2 m). The number of run-ups reaching each numbered area was tallied manually.

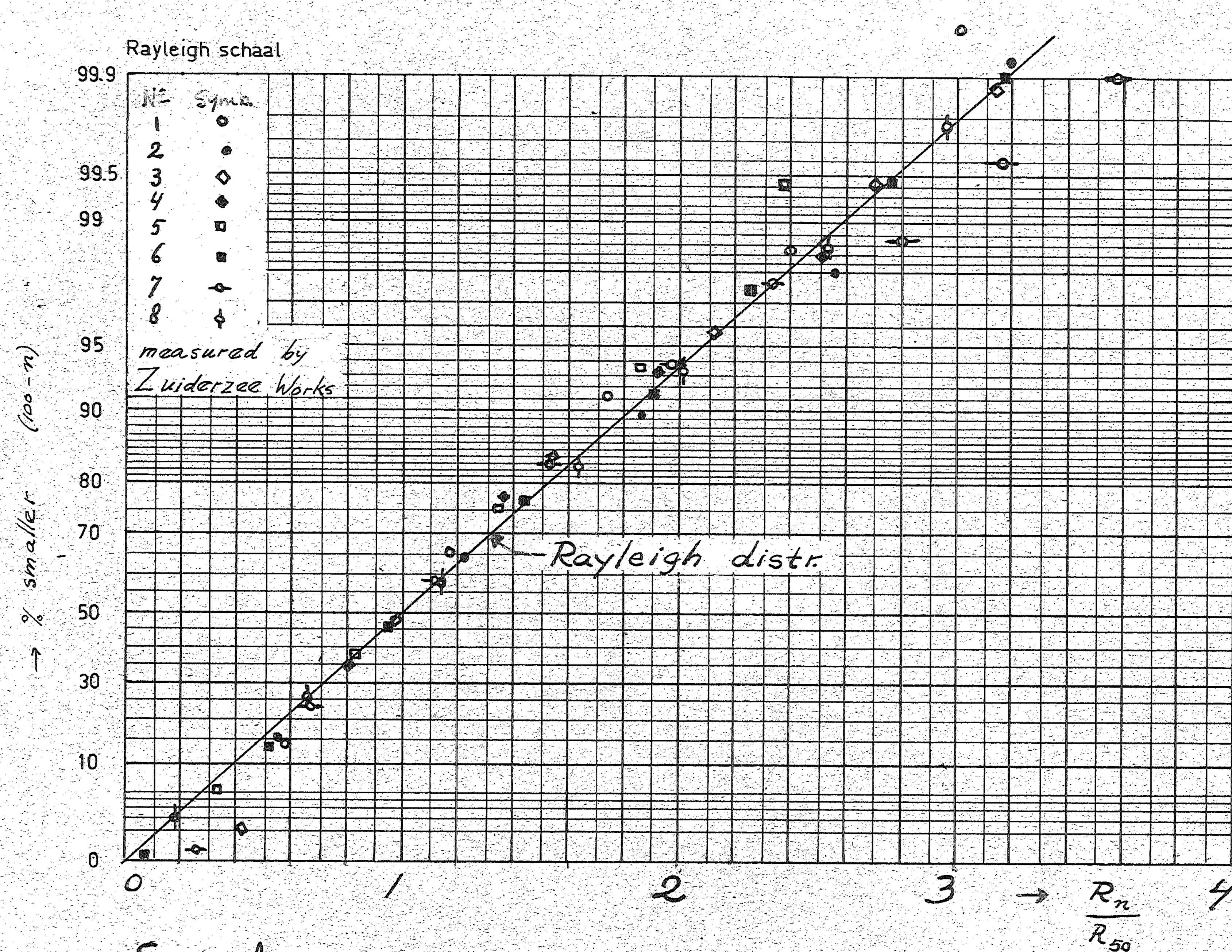
A summary of the environmental conditions during the measurements is given in table 2. $\mathcal{U}$ is the absolute value of the mean wind velocity, and M.W.L. is the mean water level, referred to the datum level NAP which is indicated in figure 11.

| no | date | wind velocity | | site A | | site B | |
|---|---|---|---|---|---|---|---|
| | | $\mathcal{U}$ (m/s) | direction | M.W.L. (m) | $R_{50}$ (m) | M.W.L. (m) | $R_{50}$ (m) |
| 1 | 9-8-43 | 13 | WNW-W | -0,12 | 0,34 | -0,18 | 0,24 |
| 2 | 30-8-43 | 15 - 18 | WSW-W | -0,10 | 0,31 | -0,05 | 0,38 |
| 3 | 15-9-43 | 13 | SW | -0,07 | 0,36 | -0,09 | 0,46 |
| 4 | 15-9-43 | 13 | WSW | -0,21 | 0,36 | -0,21 | 0,36 |
| 5 | 20-9-43 | 13 - 17 | SSW-S | -0,05 | 0,40 | -0,10 | 0,65 |
| 6 | 22-1-44 | 20 - 25 | SSW | +0,05 | 0,45 | +0,05 | 0,50 |
| 7 | 13-3-44 | 15 - 20[1] | WNW | -0,05 | 0,48 | -0,05 | 0,50 |
| 8 | 3-5-44 | 15 - 20 | WNW-W | +0,20 | 0,43 | - | - |
| 9 | 7-11-44 | 20 - 25 | W | +0,40 | 0,85 | +0,40 | 1,00 |
| 10 | 7-11-44 | 20 - 25 | W | +0,50 | 1,02 | +0,50 | 1,08 |

[1] very gusty; frequently more than 25 to 30 m/sec.

Table 2

Distributions of $R/R_{50}$ have been plotted on Rayleigh paper in the figures 12 and 13 for site A and Site B, respectively. Only those series have been included in which the run-ups did not reach the high berm. It is evident that almost all of these distributions are very well described by a Rayleigh distribution. Exception should be made for series no. 7, site A, in which case the data are spread much more than a Rayleigh distribution. This is possibly due to the fact that the wind on that occasion was particularly gusty, as indicated in table 1. This wind blew almost perpendicular to the dike at site A, where the anomalous distribution was measured.




Site A

Figure 12

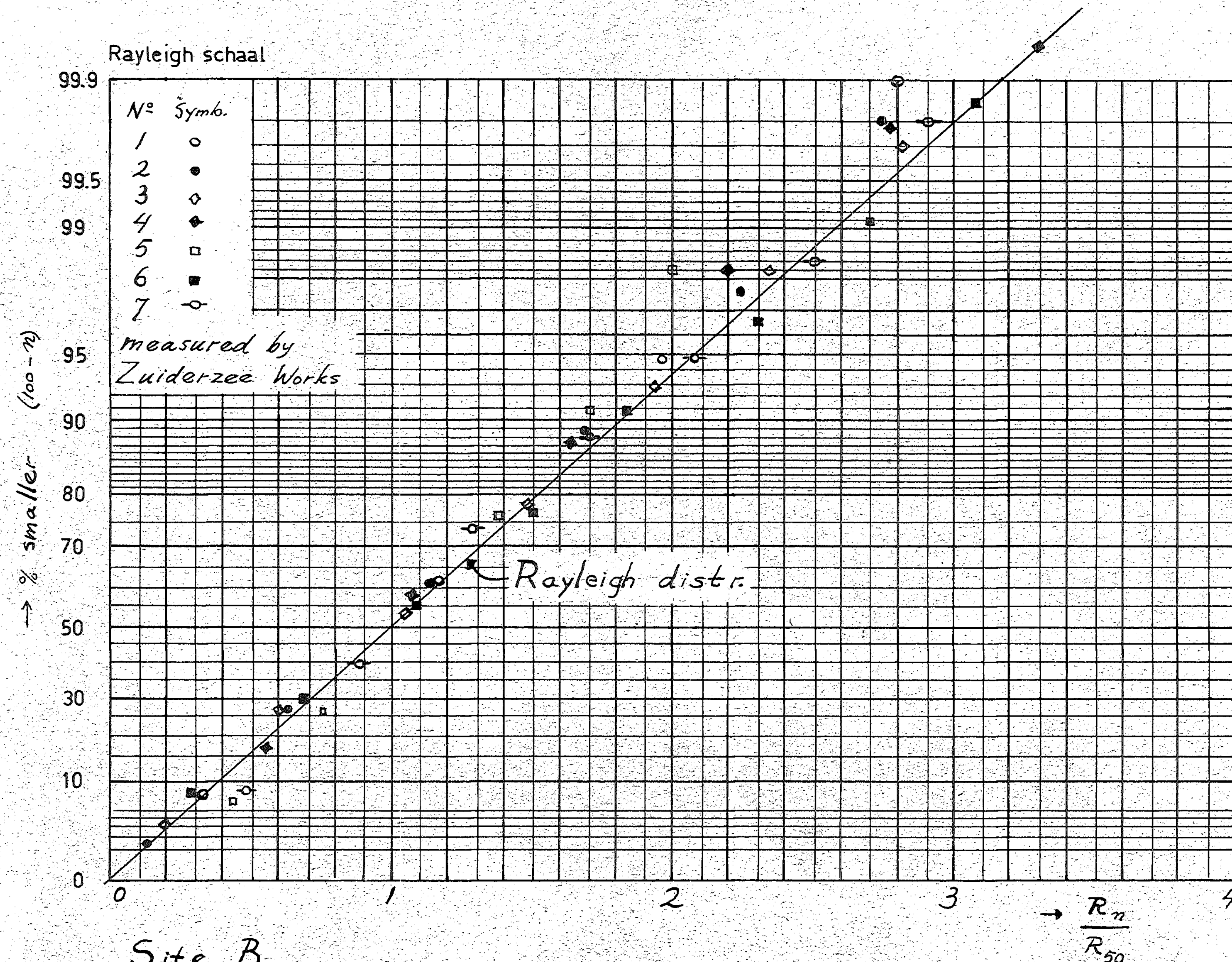
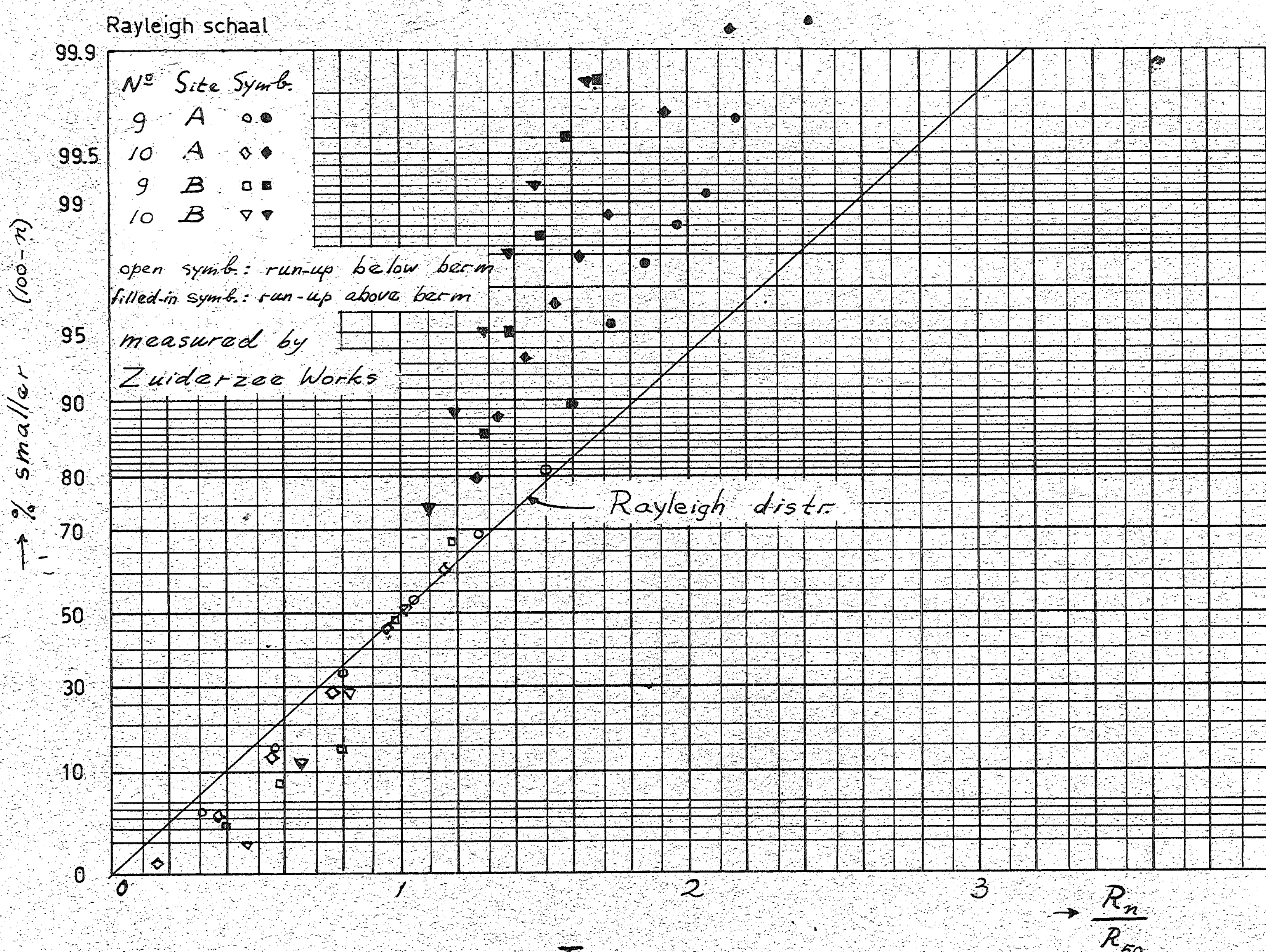

Rayleigh schaal
Nº Symb.
1
2
3
4
5
6
7
measured by
Zuiderzee Works
Rayleigh distr.
→ % smaller (100 - n)
99.9
99.5
99
95
90
80
70
50
30
10
0
0
1
2
3
4
→ $\frac{R_n}{R_{50}}$
Site B

Figure 13



Rayleigh schaal
99.9
99.5
99
95
90
80
70
50
30
10
0
→ % smaller (100 - n)
Nº Site Symb.
9 A
10 A
9 B
10 B
open symb.: run-up below berm
filled-in symb.: run-up above berm
measured by
Zuiderzee Works
Rayleigh distr.
0
1
2
3
→ $\frac{R_n}{R_{50}}$


Figure 14

In interpreting the distributions given in the figures 12 and 13, particularly the fact that they have almost the same shape, it should not be forgotten that the wave generating conditions, in terms of fetch, depth and mean wind velocity, did not differ greatly from one series to another. Another factor which has some effect is the berm at approximately M.W.L. This berm probably causes an increase of the spread in the run-up values, compared with the run-up on a plane slope, because the effect of a berm near M.W.L. is to reduce smaller run-ups more than larger run-ups. For this reason it is expected that the distribution of run-up on plane slopes will be narrower than the Rayleigh distribution, for the same wave conditions as prevailed in the measurements presented in figures 12 and 13.

In figure 14 some distributions are given in which an appreciable fraction of the run-ups exceeded the high berm. The effect of the berm on the run-up distribution is quite conspicuous. Primarily the run-ups above the berm are reduced, with the result that the total spread is less than in the case when the berm is not reached by the run-up.

## 9 SUMMARY AND CONCLUSIONS

Distributions of run-up of breaking waves are derived by assigning to individual waves in an irregular wave train a run-up value according to Hunt's formula. The potential validity of this approach is confirmed by comparison with laboratory data. Explicit expressions for the run-up are obtained for waves of which the heights and periods squared have a bivariate Rayleigh distribution. The extremes of this distribution for a value 1 resp. 0 of the correlation coefficient $\rho$ supposedly are limiting cases for a young sea resp. a fully developed sea. The assumption of a bivariate Rayleigh distribution has not in the present study been checked separately. However, the range of run-ups calculated on the basis of this assumption, with $\rho$ varying from 0 to 1, appears to agree with the range of experimental values for waves with energy density spectra ranging from narrow to wide.

Some prototype run-up distributions are presented. These had been measured on a dike of the IJssellake in the Netherlands. The variation of wave generating conditions in this lake is very limited. This is probably the reason why the run-up distributions are remarkably similar in shape; they are very well described by a Rayleigh distribution. The measured run-ups are affected by a berm near M.W.L. It is believed that the effect of such a berm is to increase the width of the distribution.

A general expression is derived for the distribution of steepness of waves with an arbitrary joint distribution of heights and periods squared. This expression is subsequently evaluated for the special case of a bivariate Rayleigh distribution. The resulting distribution function has a strikingly simple form when $\rho = 0$. This simple form may be used as an approximation to the actual distribution function for small $\rho$ (less than 0.4 for instance).

## 10 RECOMMENDATIONS

The following studies pertaining to the problems of variability of waves and run-up are recommended:

- A statistical study of wind wave records to check the hypothesis of a bivariate Rayleigh distribution for $H$ and $L_o$.
- Field measurements of run-up, preferably on a plane and fairly smooth slope. Needless to say, measurements of the incident waves and other environmental factors should be included.
- An exploratory study of the feasibility of numerical computation of run-up distributions.
- A study of the limiting wave run-up (resp. overtopping) which can be allowed in view of the stability of the structure. Very little is known about this problem, although it is of fundamental importance in the design of many types of shore structures.

## ACKNOWLEDGEMENTS

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