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S-AMP: Approximate Message Passing for General Matrix Ensembles

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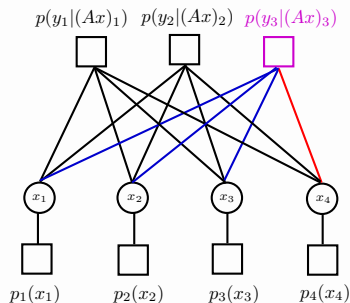
Motivation

- **Low complexity** and **near optimal** inference algorithms for
- linear observation models

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \epsilon, \quad \epsilon \sim \mathcal{N}(\epsilon|0, \sigma^2 \mathbf{I}), \quad \mathbf{x} \sim \prod_{k=1}^K p_k(x_k)$$

- \mathbf{A} : $N \times K$, known and **drawn from known matrix ensemble**
- $N, K \gg 1$

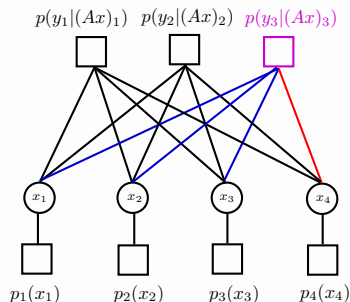
Loopy BP



$$m_{n \rightarrow k}(x_k) = \int p(y_n | (Ax)_n) \prod_{l \neq k} m_{l \rightarrow n}(x_l) dx_l$$

$$m_{l \rightarrow n}(x_l) \cong p(x_l) \prod_{m \neq n} m_{m \rightarrow l}(x_l)$$

Loopy BP



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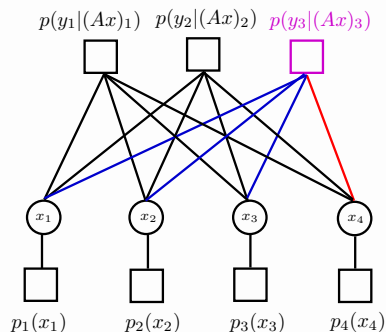
$$m_{l \rightarrow n}(x_l) \cong p(x_l) \prod_{m \neq n} m_{m \rightarrow l}(x_l)$$

- Local Cavity Argument:

$$h_{n, \setminus k} = \sum_{l \neq k} A_{nl} x_l, \quad x_l \sim m_{l \rightarrow n}(x_l)$$

- Due to CLT, $h_{n, \setminus k}$ is approximated by Gaussian.
- This leads Loop BP to Loopy EP

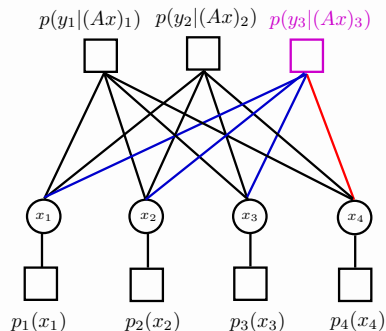
Loopy EP



$$m_{n \rightarrow k}(x_k) = \int p(y_n|(Ax)_n) \prod_{l \in \mathcal{K} \setminus k} m_{l \rightarrow n}(x_l) dx_l$$

$$m_{l \rightarrow n}(x_l) = \exp\left(-\frac{\lambda_{l \rightarrow n}}{2} x_l^2 + \gamma_{l \rightarrow n} x_l\right)$$

Loopy EP



Define

$$q_k(x_k) \cong p_k(x_k) \prod_{n \in \mathcal{N}} m_{n \rightarrow k}(x_k)$$

Let $\tilde{q}_k(x_k) \triangleq N(x_k | \mu_k, \sigma_k^2)$ such that

$$\mu_k = \mathbb{E}[x_k | q(x_k)]$$

$$\sigma_k^2 = \text{Var}[x_k | q(x_k)]$$

Then loopy EP update rule is

$$m_{n \rightarrow k}(x_k) = \int p(y_n | (Ax)_n) \prod_{l \in \mathcal{K} \setminus k} m_{l \rightarrow n}(x_l) dx_l$$

$$m_{l \rightarrow n}(x_l) = \exp\left(-\frac{\lambda_{l \rightarrow n}}{2} x_l^2 + \gamma_{l \rightarrow n} x_l\right)$$

$$m_{k \rightarrow n}(x_k) = \frac{\tilde{q}_k(x_k)}{\prod_{n \in \mathcal{N}} m_{n \rightarrow k}(x_k)}$$

Approximate message passing (AMP) [Donoho et al 2009]

- Assume A_{nk} zero mean-iid, $\overline{A_{nk}^2} = 1/N$, $N, K \rightarrow \infty$, $\alpha \equiv N/K$ finite
- Reduces the number of messages to $N + K$ means.

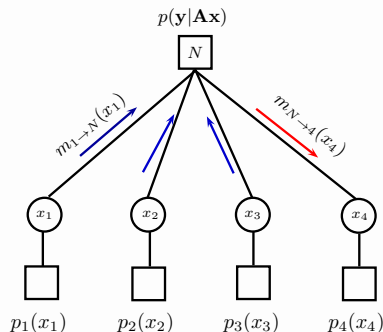
$$\begin{aligned}\boldsymbol{\mu}^{t+1} &= \eta_t (\mathbf{A}^T \mathbf{z}^t + \boldsymbol{\mu}^t) \\ \mathbf{z}^t &= \mathbf{y} - \mathbf{A} \boldsymbol{\mu}^t + \frac{1}{\alpha} \langle \eta'_{t-1}(\mathbf{A}^T \mathbf{z}^{t-1} + \boldsymbol{\mu}^{t-1}) \rangle \mathbf{z}^{t-1}\end{aligned}$$

with $\langle \mathbf{u} \rangle \triangleq \sum_{k=1}^K u_k / K$.

- $\eta_t(\kappa_k)$ and $\eta'_t(\kappa_k)/\tau$ are (in some cases) the mean and variance of (Krzakala et al 2012)

$$q_k(x_k) \cong p_k(x_k) \mathcal{N}(x_k | \kappa_k, 1/\tau)$$

- Return to τ when discussing EP and S-AMP



Define

$$q_k(x_k) \cong p_k(x_k) m_{N \rightarrow k}(x_k)$$

Let $\tilde{q}_k(x_k) = N(x_k | \mu_k, \sigma_k^2)$ such that

$$\mu_k = \mathbb{E}[x_k | q_k(x_k)]$$

$$\sigma_k^2 = \text{Var}[x_k | q_k(x_k)]$$

Then EP update rule is

$$m_{N \rightarrow k}(x_k) = \int p(y | \mathbf{A} \mathbf{x}) \prod_{l \in \mathcal{K} \setminus k} m_{l \rightarrow N}(x_l) dx_l$$

$$m_{k \rightarrow N}(x_k) = \exp\left(-\frac{\Lambda_{kk}}{2} x_k^2 + \gamma_k x_k\right)$$

$$m_{k \rightarrow N}(x_k) = \frac{\tilde{q}_k(x_k)}{m_{N \rightarrow k}(x_k)}$$

- generalizes AMP for arbitrary (**orthogonally invariant**) matrix ensembles.

$$\begin{aligned}\boldsymbol{\mu}^{t+1} &= \eta_t (\mathbf{A}^\dagger \mathbf{z}^t + \boldsymbol{\mu}^t) \\ \mathbf{z}^t &= \mathbf{y} - \mathbf{A} \boldsymbol{\mu}^t + \left(\mathbf{1} - \frac{\mathbf{1}}{S_{\mathbf{A}}^{t-1}} \right) \mathbf{z}^{t-1} \\ S_{\mathbf{A}}^{t-1} &\triangleq S_{\mathbf{A}} \left(- \langle \eta'_{t-1} (\mathbf{A}^\dagger \mathbf{z}^{t-1} + \boldsymbol{\mu}^{t-1}) \rangle \right)\end{aligned}$$

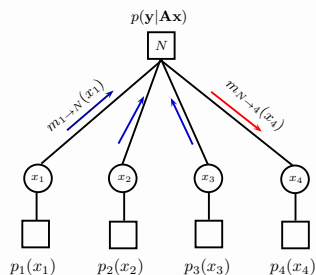
$S_{\mathbf{A}}$ denotes the S-transform (**in free probability theory**) of the limiting eigenvalue distribution (LED) of $\mathbf{A}^\dagger \mathbf{A}$.

- Indeed when the entries of \mathbf{A} be iid with zero mean variance $1/N$:

$$S_{\mathbf{A}}(\omega) = \frac{1}{1 + \omega/\alpha}$$

which yields AMP iteration steps.

EP \rightarrow S-AMP: Start with EP Update Rule



$$m_{N \rightarrow k}(x_k) = \int p(\mathbf{y} | \mathbf{A}\mathbf{x}) \prod_{l \in \mathcal{K} \setminus k} m_{l \rightarrow N}(x_l) dx_l$$

$$m_{k \rightarrow N}(x_k) = \exp\left(-\frac{\Lambda_{kk}}{2} x_k^2 + \gamma_k x_k\right)$$

$$q_k(x_k) \cong p_k(x_k) m_{N \rightarrow k}(x_k)$$

$$m_{N \rightarrow k}(x_k) = \tilde{q}_k(x_k) / m_{k \rightarrow N}(x_k)$$

- Let $\mathbf{J} = \mathbf{A}^\dagger \mathbf{A} / \sigma^2$ and $\boldsymbol{\theta} = \mathbf{A}^\dagger \mathbf{y} / \sigma^2$. Define

$$\boldsymbol{\Sigma} = (\boldsymbol{\Lambda} + \mathbf{J})^{-1} \quad \boldsymbol{\mu} = \boldsymbol{\Sigma}(\boldsymbol{\theta} + \boldsymbol{\gamma})$$

- Then we have

$$m_{N \rightarrow k}(x_k) = \exp\left\{-\frac{1}{2} \left(\frac{1}{\Sigma_{kk}} - \Lambda_{kk}\right) x_k^2 + \left(\frac{\mu_k}{\Sigma_{kk}} - \gamma_k\right) x_k\right\}$$

EP \rightarrow S-AMP: Use AMP Notations

- Let $\tau_k = \frac{1}{\Sigma_{kk}} - \Lambda_{kk}$ and $\kappa_k = (\frac{\mu_k}{\Sigma_{kk}} - \gamma_k)/\tau_k$.
- Hence we can write

$$m_{N \rightarrow k}(x_k) \cong \mathcal{N}(x_k | \kappa_k, 1/\tau_k)$$

- Write $q_k(x_k)$ in the form of

$$q_k(x_k) = \frac{p_k(x_k) \mathcal{N}(x_k | \kappa_k, 1/\tau_k)}{Z(\kappa_k, \tau_k)}$$

- Define

$$\eta(\kappa_k; \tau_k) \triangleq \kappa_k + \frac{1}{\tau_k} \frac{\partial \log Z(\kappa_k, \tau_k)}{\partial \kappa_k}$$
$$\eta'(\kappa_k; \tau_k) \triangleq \frac{\partial \eta(\kappa_k; \tau_k)}{\partial \kappa_k}$$

where $\eta(\kappa_k; \tau_k)$ and $\eta'(\kappa_k; \tau_k)/\tau_k$ are respectively the mean and the variance of $q_k(x_k)$ [Krzakala et.al. 2012].

EP \rightarrow S-AMP: Move to ADATAP [Opper and Winther 2001]

- Note that

$$\boldsymbol{\mu} = (\boldsymbol{\Lambda} + \boldsymbol{J})^{-1}(\boldsymbol{\gamma} + \boldsymbol{\theta}) \iff (\boldsymbol{\Lambda} + \boldsymbol{J})\boldsymbol{\mu} = \boldsymbol{\gamma} + \boldsymbol{\theta}$$

EP \rightarrow S-AMP: Move to ADATAP [Oppor and Winther 2001]

- Note that

$$\boldsymbol{\mu} = (\boldsymbol{\Lambda} + \mathbf{J})^{-1}(\boldsymbol{\gamma} + \boldsymbol{\theta}) \iff (\boldsymbol{\Lambda} + \mathbf{J})\boldsymbol{\mu} = \boldsymbol{\gamma} + \boldsymbol{\theta}$$

- Putting everything together leads EP to

$$\begin{aligned}\mu_k &= \eta(\kappa_k; \tau_k) \\ \kappa_k &= \frac{1}{\tau_k \sigma^2} \sum_{n \in \mathcal{N}} A_{nk} \left(y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l \right) + \mu_k \\ \tau_k &= \frac{1}{\Sigma_{kk}} - \Lambda_{kk}, \quad \Lambda_{kk} = \frac{\tau_k}{\eta'(\kappa_k; \tau_k)} - \tau_k\end{aligned}$$

exactly coincides ADATAP for the linear observation models.

EP \rightarrow S-AMP: Apply Adaptive Damping

Define

$$z_{n,k} \triangleq \frac{1}{\tau_k \sigma^2} \left(y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l \right).$$

Using this definition we “devise” the following identity:

$$z_{n,k} = y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l + (1 - \sigma^2 \tau_k) z_{n,k}.$$

EP \rightarrow S-AMP: Apply Adaptive Damping

Define

$$z_{n,k} \triangleq \frac{1}{\tau_k \sigma^2} \left(y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l \right).$$

Using this definition we “devise” the following identity:

$$z_{n,k} = y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l + (1 - \sigma^2 \tau_k) z_{n,k}.$$

Doing so leads to

$$\begin{aligned} \mu_k &= \eta \left(\sum_{n \in \mathcal{N}} A_{nk} z_{n,k} + \mu_k; \tau_k \right) \\ z_{n,k} &= y_n - \sum_{l \in \mathcal{K}} A_{nl} \mu_l + (1 - \sigma^2 \tau_k) z_{n,k} \\ \tau_k &= \frac{1}{\Sigma_{kk}} - \Lambda_{kk}, \quad \Lambda_{kk} = \frac{\tau_k}{\eta'(\kappa_k; \tau_k)} - \tau_k \end{aligned}$$

This equations can be thought as a finite size interpretation of AMP.

EP \rightarrow S-AMP: Invoke Self Averaging Ansatz [Opper and Winther 2001]

- We can recover self-averaging matrix ensembles $\tau_k \rightarrow \tau$:

$$\Sigma_{kk} = [(\mathbf{\Lambda} + \mathbf{J})^{-1}]_{kk} = \frac{\partial}{\partial \Lambda_{kk}} \ln \det(\mathbf{\Lambda} + \mathbf{J})$$

- by using

$$\frac{1}{K} \ln \det(\mathbf{\Lambda} + \mathbf{J}) \rightarrow \frac{1}{K} \mathbb{E}_J [\ln \det(\mathbf{\Lambda} + \mathbf{J})] \quad \text{for } K \rightarrow \infty$$

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- Doing so leads τ_k to τ that is the solution of

$$\sigma^2 \tau = \frac{1}{\sigma^2} \mathbf{R}_A \left(- \frac{\langle \eta'(\mathbf{A}^\dagger \mathbf{z} + \boldsymbol{\mu}; \tau) \rangle}{\sigma^2 \tau} \right)$$

\mathbf{R}_A is the R-transform (in free probability theory) of the LED of $\mathbf{A}^\dagger \mathbf{A}$ and

$$\mathbf{z} = \mathbf{y} - \mathbf{A}\boldsymbol{\mu} + (1 - \sigma^2 \tau) \mathbf{z}$$

EP \rightarrow S-AMP: Move to S-transform

- Recall that

$$\sigma^2 \tau = \mathbb{R}_{\mathbf{A}} \left(\frac{-\langle \eta'(\mathbf{A}^\dagger \mathbf{z} + \boldsymbol{\mu}; \tau) \rangle}{\sigma^2 \tau} \right)$$

- By invoking the fact [Haagerup and Larsen 2001]

$$S_{\mathbf{A}}(\omega) = \frac{1}{\mathbb{R}_{\mathbf{A}}(\omega S_{\mathbf{A}}(\omega))}$$

- we have

$$\sigma^2 \tau = \frac{1}{S_{\mathbf{A}}(-\langle \eta'(\mathbf{A}^\dagger \mathbf{z} + \boldsymbol{\mu}; \tau) \rangle)}$$

EP \rightarrow S-AMP: Move to S-transform

- Recall that

$$\sigma^2 \tau = \mathbb{R}_{\mathbf{A}} \left(\frac{-\langle \eta'(\mathbf{A}^\dagger \mathbf{z} + \boldsymbol{\mu}; \tau) \rangle}{\sigma^2 \tau} \right)$$

- By invoking the fact [Haagerup and Larsen 2001]

$$S_{\mathbf{A}}(\omega) = \frac{1}{\mathbb{R}_{\mathbf{A}}(\omega S_{\mathbf{A}}(\omega))}$$

- we have

$$\sigma^2 \tau = \frac{1}{S_{\mathbf{A}}(-\langle \eta'(\mathbf{A}^\dagger \mathbf{z} + \boldsymbol{\mu}; \tau) \rangle)}$$

- this completes the mapping at "fixed points":

$$\boldsymbol{\mu} = \eta(\mathbf{A}^\dagger \mathbf{z} + \boldsymbol{\mu}; \tau)$$

$$\mathbf{z} = \mathbf{y} - \mathbf{A}\boldsymbol{\mu} + \left(1 - \frac{1}{S_{\mathbf{A}}}\right)\mathbf{z}$$

$$S_{\mathbf{A}} = S_{\mathbf{A}}(-\langle \eta'(\mathbf{A}^\dagger \mathbf{z} + \boldsymbol{\mu}; \tau) \rangle)$$

What is S-AMP?

- In summary

$$\boldsymbol{\mu}^{t+1} = \eta_t(\mathbf{A}^\dagger \mathbf{z}^t + \boldsymbol{\mu}^t)$$

$$\mathbf{z}^t = \mathbf{y} - \mathbf{A}\boldsymbol{\mu}^t + \left(1 - \frac{1}{S_{\mathbf{A}}^{t-1}}\right) \mathbf{z}^{t-1}$$

$$S_{\mathbf{A}}^{t-1} \triangleq S_{\mathbf{A}^\dagger \mathbf{A}} \left(-\langle \eta'_{t-1}(\mathbf{A}^\dagger \mathbf{z}^{t-1} + \boldsymbol{\mu}^{t-1}) \rangle\right)$$

where $\eta_t(x^t) = \eta(x^t; \tau^t)$ and

$$\tau^t = \frac{1}{\sigma^2 S_{\mathbf{A}} \left(-\langle \eta'(\mathbf{A}^\dagger \mathbf{z}^t + \boldsymbol{\mu}^t; \tau^t) \rangle\right)}$$

- Oops, S-AMP includes a hard fixed point equation.
- As a matter of fact we don't know what is the best update rule for τ^t

A Variant of S-AMP

- By making analogy with the state evolution formula [Bayati and Montari 2011]

$$\begin{aligned}\boldsymbol{\mu}^{t+1} &= \eta(\mathbf{A}^\dagger \mathbf{z}^t + \boldsymbol{\mu}^t; \tilde{\tau}^t) \\ \mathbf{z}^t &= \mathbf{y} - \mathbf{A}\boldsymbol{\mu}^t + \left(1 - \frac{1}{S_{\mathbf{A}}^{t-1}}\right) \mathbf{z}^{t-1} \\ S_{\mathbf{A}}^{t-1} &\triangleq S_{\mathbf{A}} \left(-\langle \eta'(\mathbf{A}^\dagger \mathbf{z}^{t-1} + \boldsymbol{\mu}^{t-1}; \tilde{\tau}^{t-1}) \rangle\right)\end{aligned}$$

where $\tilde{\tau}^t$ is updated by using the solution

$$\tilde{\tau}^t = \frac{1}{\sigma^2 S_{\mathbf{A}} \left(-\frac{\tilde{\tau}^t}{\tilde{\tau}^{t-1}} \langle \eta'(\mathbf{A}^\dagger \mathbf{z}^{t-1} + \boldsymbol{\mu}^{t-1}; \tilde{\tau}^{t-1}) \rangle\right)}$$

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- i.e.

$$\tilde{\tau}^t = \frac{1}{\sigma^2} R_{\mathbf{A}} \left(-\frac{\langle \eta'(\mathbf{A}^\dagger \mathbf{z}^{t-1} + \boldsymbol{\mu}^{t-1}; \tilde{\tau}^{t-1}) \rangle}{\sigma^2 \tilde{\tau}^{t-1}} \right)$$

Application: Row Orthogonal Ensembles in Compressed Sensing

- A random row orthogonal ensemble defined as

$$\mathbf{A} = \alpha^{-\frac{1}{2}} \mathbf{P}_\alpha \mathbf{O}, \quad \alpha \leq 1$$

where \mathbf{P}_α is the $N \times K$ matrix with entries $[\mathbf{P}_\alpha]_{ij} = \delta_{ij}, \forall ij$.

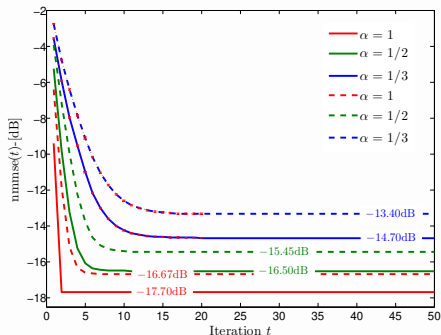
- In this case we have

$$S_{\mathbf{A}}(z) = \frac{1+z}{1+z/\alpha}$$

$$R_{\mathbf{A}}(z) = \frac{z - \alpha + \sqrt{(\alpha - z)^2 + 4\alpha^2 z}}{2\alpha z}$$

Simulation Results

- Let $p_k(x_k) = (1 - \rho)\delta(x_k) + \rho N(x_k|0, 1)$, with $\rho \in (0, 1)$.
- For the closed-forms of $\eta_t(\cdot)$ and $\eta'_t(\cdot)$, see [Krzakala et.al. 2012].



- S-AMP for the row orthogonal matrix ensemble (solid curves) and the iid zero-mean ensemble (dashed curves).
- Confidence intervals (CIs) are also shown for $\alpha = 1/3$.
- We set $\sigma^2 = -20$ dB and $\rho = 0.1$, and $K = 1200$.
- The numbers in the plot are the predictions of replica theory [Kabashima and Vekopera 2014]