

# *S*-duality of boundary conditions in $\mathcal{N} = 4$ super Yang–Mills theory

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## **Abstract**

By analyzing brane configurations in detail, and extracting general lessons, we develop methods for analyzing *S*-duality of supersymmetric boundary conditions in  $\mathcal{N} = 4$  super Yang–Mills theory. In the process, we find that *S*-duality of boundary conditions is closely related to mirror symmetry of three-dimensional gauge theories, and we analyze the IR behavior of large classes of quiver gauge theories.

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## 1 Introduction

In a recent paper [1], we have described half-BPS boundary conditions in  $\mathcal{N} = 4$  super Yang–Mills theory with gauge group  $G$ . The general classification of boundary conditions is rather elaborate and depends on a triple  $(\rho, H, \mathfrak{B})$ . The gauge group  $G$  is explicitly broken near the boundary to a subgroup  $H$ . Part of the symmetry breaking involves a choice of homomorphism  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{g}$  from the Lie algebra of  $SU(2)$  to that of  $G$ . Finally,  $\mathfrak{B}$  is a boundary field theory with  $H$  symmetry. Because of the explicit symmetry breaking, the gauge fields on the boundary are valued in  $\mathfrak{h}$ , the Lie algebra of  $H$ , and can be naturally coupled to  $\mathfrak{B}$ .

In this brief summary, we have omitted the role of the four-dimensional theta-angle, which adds an extra layer of structure as explained in [1, 2]. In the present paper, we take the theta-angle to vanish until Section 8.

Our goal in the present paper is to understand the action of electric–magnetic duality on this class of boundary conditions. To gain experience, we begin with concrete examples. In Section 2, we review boundary conditions in  $U(n)$  gauge theory that can be constructed using D3-branes, D5-branes, and NS5-branes of Type IIB superstring theory. Because our topic turns out to be closely related to IR dynamics in three dimensions, we also re-examine the behavior of purely three-dimensional theories constructed from those ingredients. We describe an important and mirror symmetric class of three-dimensional theories, and we use “monopole operators” to learn something about their IR dynamics. Monopole operators were discussed qualitatively in relation to supersymmetric gauge dynamics in [3, 4] and the formalism we use was developed in [5–7]. In Section 3, we analyze in detail  $S$ -duality for boundary conditions constructed from branes. This involves many interesting details but also some general lessons. One important lesson is that  $S$ -duality of four-dimensional boundary conditions is closely related to mirror symmetry of three-dimensional gauge theories [8]. A second important lesson is that a certain class of superconformal field

theories plays an important role. The most basic of these is a certain self-mirror theory that we call  $T(SU(n))$  that appears when one applies  $S$ -duality to Dirichlet boundary conditions. (For  $n = 2$ ,  $T(SU(n))$  coincides with one of the main examples in [8].)

In Section 4, we attempt to extract the important lessons from our investigation of branes and formulate some general statements that are valid for any compact gauge group  $G$ . The key step is to generalize  $T(SU(n))$  and its close cousins, which we do using Janus domain walls [9–13]. We describe the key properties of the theories  $T_{\rho\nu}^{\rho}(G)$  that we construct this way, and show in general how they can be used to construct the  $S$ -dual of a given boundary condition.

In Section 5, we return to three dimensions and analyze some important properties of quivers with orthogonal and symplectic gauge groups. We use the results in Section 6 to analyze  $S$ -duality of boundary conditions in  $U(n)$  gauge theory constructed with orientifold and orbifold fiveplanes. These give tractable and interesting illustrations of some of the general ideas of this paper. In Section 7, by using brane with O3 planes, we extend many of our results to the case that the gauge group is  $SO(n)$  or  $Sp(n)$ . Among other things, we describe quiver constructions of  $T(SO(n))$  and  $T(Sp(n))$ .

Up to this point, our analysis concerns the basic electric–magnetic duality operation  $S : \tau \rightarrow -1/\tau$ , rather than the complete duality group  $SL(2, \mathbb{Z})$ . Indeed, for most half-BPS boundary conditions, only the action of  $S$  can be defined. In Section 8, we incorporate the gauge theory  $\theta$ -angle, and describe the action of  $SL(2, \mathbb{Z})$  on those half-BPS boundary conditions that admit such an action. As an application, we give a quiver-like description of the low-energy effective field theory that describes the interaction of D3-branes with a  $(p, q)$ -fivebrane. This description uses Chern–Simons couplings with  $\mathcal{N} = 4$  supersymmetry. Finding such a description has been a longstanding problem.

We will often refer to the three-dimensional theory  $\mathfrak{B}$  that is part of the definition of a supersymmetric boundary condition as a boundary superconformal field theory or SCFT since the conformally invariant case tends to be particularly interesting. Moreover, once one understands  $S$ -duality of conformally invariant boundary conditions, one can understand the general case by following the duality under relevant perturbation. Focusing on the IR limit has another important advantage. The brane configurations that we will use for our explicit examples are most tractable if one is free to make standard rearrangements of the fivebranes. The justification for these rearrangements is that they involve deformations that are irrelevant in the IR.

## 2 Brane constructions for unitary groups

Rather than attempt an abstract explanation from the beginning, we will start this paper by considering the case  $G = U(n)$ , where everything can be described concretely via manipulations of branes. In the present section, we describe the necessary facts about brane constructions of boundary conditions, and we describe some facts about dynamics of three-dimensional supersymmetric gauge theories that will also be important. In Section 3, we use these facts together with standard brane manipulations to gain a fairly detailed understanding of the  $S$ -duality of boundary conditions for  $G = U(n)$ .

### 2.1 Brane construction of boundary conditions

Supersymmetric boundary conditions of any kind inevitably break the  $R$ -symmetry group of  $\mathcal{N} = 4$  super Yang–Mills theory to a subgroup. For half-BPS boundary conditions, we can be more specific. The full  $R$ -symmetry group, which is  $SO(6)_R$  (or its cover  $SU(4)_R$ ), is broken to a subgroup<sup>1</sup>  $SO(3) \times SO(3)$  (or its cover  $SO(4)$ ). Under this subgroup, the six adjoint-valued scalar fields of  $\mathcal{N} = 4$  super Yang–Mills theory split up into two groups of three scalar fields, say  $\vec{X}$  and  $\vec{Y}$ , which are rotated respectively, by the two factors of  $SO(3) \times SO(3)$ . We call these factors  $SO(3)_X$  and  $SO(3)_Y$ .

Since the idea of a boundary condition determined by a triple  $(\rho, H, \mathfrak{B})$  (as summarized in the Introduction) is daunting at first sight, we will begin by using a concrete and familiar brane construction to build half-BPS boundary conditions and study their  $S$ -duality. The construction [14, 15] involves branes in 10-dimensional Minkowski spacetime with coordinates  $x^0, x^1, \dots, x^9$ . We make use of three types of brane: D3-branes with worldvolume spanned by  $x^0, x^1, x^2, x^3$ , D5-branes with worldvolume spanned by  $x^0, x^1, x^2$  together with  $x^4, x^5, x^6$ , and NS5-branes with worldvolume spanned by  $x^0, x^1, x^2$  together with  $x^7, x^8, x^9$ . Thus all branes share the directions  $x^0, x^1, x^2$ . The D3-branes are semi-infinite in the  $x^3$  direction, being supported on the region  $x^3 \geq 0$ , with a boundary at  $x^3 = 0$ . We also write  $y$  for  $x^3$ . The fivebranes are located at specified values of  $y$  (such as  $y = 0$ ) and are used to provide boundary conditions (or couplings to matter systems) for the D3-branes.

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<sup>1</sup>For example, in the conformally invariant case, a half-BPS boundary condition breaks the conformal group  $PSU(4|4)$  to  $OSp(4|4)$ , whose  $R$ -symmetry subgroup is  $SO(4) \cong SO(3) \times SO(3)$ .

In the gauge theory on the D3-branes, fluctuations in  $x^4, x^5, x^6$  correspond to the scalar fields  $\vec{X}$  of  $\mathcal{N} = 4$  super Yang–Mills theory and fluctuations in  $x^7, x^8, x^9$  correspond to the scalar fields  $\vec{Y}$ . Brane configurations of the type just summarized are useful in studying three-dimensional mirror symmetry and the methods used in that context will be very helpful in what follows.

Let us recall from [1, Section 2.5.1], the boundary conditions obtained from such a brane configuration. In the example sketched in figure 1, there are eight D3-branes, so the four-dimensional gauge group is  $U(8)$ . Reading the figure from right to left, the first three D3-branes terminate on a D5-brane. At this point,  $\vec{X}$  develops a rank 3 pole, reducing the gauge symmetry from  $U(8)$  to  $U(5)$ . This pole is governed by Nahm’s equations and represents the way the D3-branes flare out into a fuzzy funnel that joins the D5-brane [16, 17]. A single D3-brane ends on each of the next two D5-branes, reducing the rank of the gauge group without a further pole. To the left of the D5-branes, the D3-brane gauge group is reduced to  $U(3)$ . The symbol  $\oplus$  then represents a further system of NS5-branes and D5-branes that describes a three-dimensional matter system coupled to the  $U(3)$  gauge fields.

There are many possible choices of this further system. Some illustrative examples are shown in figure 2. In figure 2(a), the additional system consists of a single NS5-brane, and the  $U(3)$  gauge fields simply obey Neumann boundary conditions. In figure 2(b), a D5-brane has been added. As

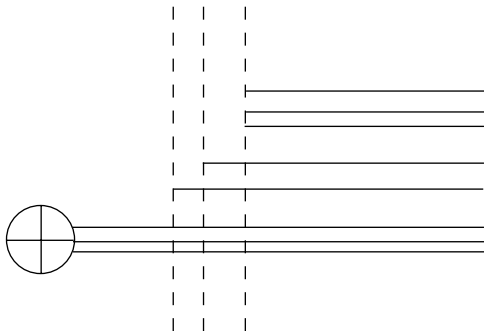


Figure 1: A brane configuration that determines a half-BPS boundary condition in  $\mathcal{N} = 4$  super Yang–Mills theory. Here and later, horizontal solid lines designate D3-branes spanning directions 0123; vertical dotted lines designate D5-branes spanning directions 012456. In this example, there are eight D3-branes and the gauge group is  $U(8)$ . The symbol  $\oplus$  denotes a further fivebrane system, of which some possible examples are sketched in figure 2.



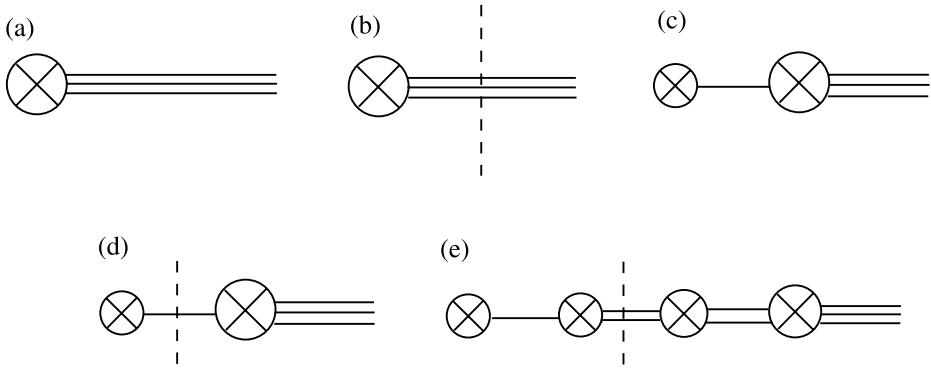


Figure 2: Some brane configurations, any one of which can correspond to the symbol  $\oplus$  on the left of figure 1. Here and later, the symbol  $\otimes$  represents an NS5-brane spanning directions 012789. In (a), three D3-branes end on a single NS5-brane. This leads to Neumann boundary conditions in  $U(3)$  gauge theory. In (b), the D3-branes intersect a D5-brane before terminating on a single NS5-brane. This leads (in the limit that all fivebrane separations in the  $y = x^3$  direction are taken to zero) to Neumann boundary conditions with a fundamental hypermultiplet supported on the boundary. The hypermultiplet comes from the brane intersection. In (c), (d), and (e), there is more than one NS5-brane. This leads to Neumann boundary conditions modified by coupling to a non-trivial boundary SCFT, as described in the text.

a result, the  $U(3)$  gauge fields couple to a hypermultiplet in the fundamental representation (or more briefly a fundamental hypermultiplet) that is supported in codimension 1.

Figure 2(c) requires a more detailed explanation. Two NS5-branes are separated by a distance  $L$  in the  $x^3$  direction. The worldvolume theory for the D3-brane in the slab between the two NS5-branes is a  $U(1)$  gauge theory with Neumann boundary conditions on the two ends. If the four-dimensional gauge coupling  $g_{4d}$  is sufficiently small, the Kaluza–Klein scale on the slab, which is  $1/L$ , is much larger than the scale set by the three-dimensional gauge coupling, which is  $g_{3d}^2 = g_{4d}^2/L$ . The result is that at sufficiently low energy the worldvolume theory reduces to a three-dimensional gauge theory. The 3–3 strings stretched across the NS5-branes give a single bifundamental hypermultiplet coupled to both this three-dimensional  $U(1)$  gauge theory and to the bulk gauge theory on the half line. The  $U(3)$  gauge symmetry of the semi-infinite D3-branes is a global symmetry from the point of view of the three-dimensional gauge theory. The three-dimensional theory is actually a theory of a  $U(1)$  vector multiplet coupled to three hypermultiplets of charge 1 and in the fundamental representation of a  $U(3)$  global symmetry. In

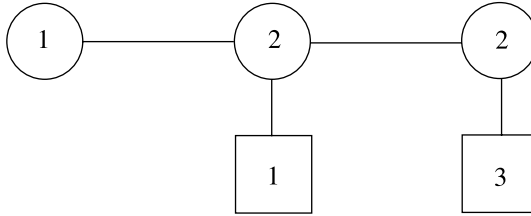


Figure 3: A quiver such as this one gives a convenient way to summarize the construction of a gauge theory with suitable gauge group and matter representation. A circle containing an integer  $n$  represents a  $U(n)$  factor in the gauge group. The gauge group is the product of such factors, one for each circle. A line joining two circles labeled by  $n$  and  $m$  represents a bifundamental hypermultiplet, that is a collection of hypermultiplets transforming under  $U(n) \times U(m)$  as  $(\mathbf{n}, \bar{\mathbf{m}}) \oplus (\bar{\mathbf{n}}, \mathbf{m})$ . Finally, if a circle labeled by  $n$  is linked to square labeled  $p$ , this means that there are  $p$  fundamental hypermultiplets of  $U(n)$ . For every square labeled by  $p$ , there is a  $U(p)$  global symmetry acting on the corresponding hypermultiplets. The specific quiver drawn here represents the boundary SCFT that arises from the brane configuration of figure 2(e).

the infrared, the three-dimensional gauge coupling becomes large. If we simply turn off the four-dimensional gauge coupling on the semi-infinite D3-branes of figure 2(c), the IR flow gives a purely three-dimensional SCFT. Turning back on the four-dimensional gauge coupling, we get a combined system consisting of four-dimensional gauge fields on a half-space coupled to a boundary SCFT. The boundary theory is coupled to the bulk gauge fields by gauging its  $U(3)$  global symmetry.

Figure 2(d) is a small modification of (c): each extra D5-brane inserted between the two NS5-branes, with no D3-branes ending on it, adds a fundamental hypermultiplet coupled to the three-dimensional gauge theory. So here (with one extra D5-brane in this example) we get a boundary SCFT that has  $U(4)$  global symmetry, of which a  $U(3)$  subgroup is coupled to bulk gauge fields. Finally, in general, if the brane system  $\oplus$  consists of several displaced NS5-branes with a variable number of D3-brane segments stretched between them, and extra D5-branes with no D3-branes ending on them, as in figure 2(e), then at low energies, the worldvolume theory consists of a certain linear quiver of three-dimensional  $U(n_i)$  gauge theories with fundamental matter possibly coupled to each of the nodes. The quiver for this example is sketched in figure 3. If the brane multiplicities obey certain inequalities, which will be explained, then the quiver system flows in the IR to an SCFT with  $\mathcal{N} = 4$  supersymmetry, and the overall system can be described by gauge fields on a half-space coupled to this SCFT.

In all cases, we have slightly separated the various branes in the  $x^3$  direction to avoid ambiguities and to make possible a description by gauge theory. However, the intent is always to consider a limit in which the brane separations  $L_i$  are taken to zero and the brane configuration determines a boundary condition for the four-dimensional gauge fields. The boundary conditions obtained this way are special cases of the general definition in [1], involving a triple  $(\rho, H, \mathfrak{B})$ . Here  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{g}$  is an embedding of the Lie algebra of  $SU(2)$  in that of  $G$ ,  $H$  is a subgroup of  $G$  that commutes with  $SU(2)$ , and  $\mathfrak{B}$  is a boundary superconformal field theory with  $H$  symmetry. For the brane constructions that we have described,  $G$  is a unitary group  $U(N)$  for some  $N$ ;  $\rho$  is arbitrary;  $\mathfrak{B}$  is constructed from a quiver gauge theory; and  $H$ , which is a subgroup of  $G$  of the form  $U(M)$  for some  $M$ , is a global symmetry group acting at one end of the quiver.

## 2.2 Ordering of branes

We are not interested in studying brane configurations for their own sake, but as a tool for generating boundary conditions and studying the action of  $S$ -duality. For this purpose, it turns out that it suffices to consider a certain subset of brane configurations. As we will see, other configurations can be reduced to this subset by moving branes along the lines of [15].

In the brane constructions that we have described, the brane separations  $L_i$  are irrelevant in the infrared, but the specific ordering of the fivebranes along  $y = x^3$  is quite important. We have ordered the fivebranes in figures 1 and 2 in a way that makes the field theory interpretation of the boundary condition understandable and the infrared limit simple. This depends on two constraints that will be described here.

To state these constraints, one important concept is the net number of D3-branes ending on a fivebrane. We define this number to be the number of D3-branes ending on the fivebrane from the right, minus the number ending from the left.

### 2.2.1 The first constraint

Our first constraint is that any D5-brane on which a net non-zero number of threebranes ends is to the right of all NS5-branes. This constraint has been incorporated in figures 1 and 2. The D5-branes that are shown explicitly in figure 1 are to the right of the  $\oplus$  symbol, in which any NS5-branes are hidden. In expanding out the  $\oplus$  symbol in figure 2, there may be additional D5-branes, but the net number of D3-branes ending on any one of them is zero.

This constraint ensures that the brane configurations of figure 2 have an interpretation in gauge theory. If the net number of D3-branes ending on any D5-brane is zero, it is possible (in one phase of the theory) to detach all D3-branes from D5-branes and let D3-branes end on NS5-branes only. A moveable D3-brane connecting two NS5-branes is described by a vector multiplet of a gauge theory. The branch of the moduli space of vacua in which all moveable D3-branes end on NS5-branes is the Coulomb branch of this gauge theory. Thus our condition implies that the configuration labeled  $\oplus$  can be interpreted in gauge theory. (In fact, it is a quiver gauge theory, as we have already noted.) The IR limit of this gauge theory is the boundary SCFT  $\mathfrak{B}$  that is part of the definition of our boundary condition. In this gauge theory, each vector multiplet comes from a D3-brane segment that is of finite extent in the  $y$  direction, so it is reasonable to hope that one can extract the zero modes of all vector multiplets and reduce to a purely three-dimensional gauge theory before taking the infrared limit. (For this actually to be true depends on the additional constraint of Section 2.2.2.)

The gauge theory associated with the  $\oplus$  symbol has a global symmetry group  $H$  that couples to four-dimensional gauge fields. In general,  $H$  is not the gauge group  $G$  of the bulk four-dimensional gauge theory, but a subgroup. For instance, in figure 1,  $H = U(3)$ , where 3 is the number of D3-branes near the  $\oplus$  symbol. This results from the fact that, as one comes in from the right in figure 1, some of the D3-branes terminate on D5-branes before interacting with the gauge theory hidden in the  $\oplus$  symbol. D3-branes ending on D5-branes can have moduli (if there is more than one D5-brane involved, as in figure 1) since D3-brane segments that join two D5-branes can break away and move in the  $\vec{X}$  direction. Modes resulting from motion of such D3-brane segments are hypermultiplets (rather than vector multiplets). The resulting hypermultiplet moduli space can be described, as we have explained in detail in [1], by Nahm's equations  $d\vec{X}/dy + [\vec{X}, \vec{X}] = 0$ . These equations cannot be readily described in terms of purely three-dimensional gauge dynamics, since they describe precisely the  $y$ -dependence of  $\vec{X}$ .

To summarize then the first constraint, it says that as one approaches the boundary from the right, one first encounters the part of the construction (the Nahm pole  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{g}$  and the reduction from  $G$  to a subgroup  $H$ ) that is not naturally expressed in terms of three-dimensional field theory. Then one meets, compressed to the symbol  $\oplus$ , the construction via three-dimensional gauge theory of a three-dimensional boundary theory  $\mathfrak{B}$ . It only makes sense to describe  $\mathfrak{B}$  once  $H$  is known (since  $\mathfrak{B}$  must have  $H$  symmetry, not  $G$  symmetry), so it is convenient to encounter  $\rho$  and the reduction to  $H$  "first."

There is no loss in imposing this first constraint, since given a second constraint that we describe next, it can always be implemented without changing the infrared physics by moving branes, as we will see in Section 2.3.

### 2.2.2 The second constraint

The second constraint that we want can be succinctly stated in terms of a certain “linking number” invariant that was defined in [15]. The linking number of a fivebrane is the D3-brane charge measured at infinity on that fivebrane. Since a D3-brane ending on a fivebrane is a magnetic source for the  $U(1)$  gauge field on the fivebrane, the D3-brane charge on a fivebrane can be computed by integrating the  $U(1)$  field strength over a two-sphere at infinity. The reason that the linking number is important is that, since it can be measured at infinity along a brane, it is invariant under the sort of brane manipulations that are needed to understand  $S$ -duality.

Concretely, the linking number of a fivebrane is the number of fivebranes of the opposite kind to the left of the given fivebrane, plus the net number of D3-branes ending on this fivebrane on the right.<sup>2</sup> The constraint that we want on the brane ordering is that for each kind of fivebrane — NS or D — the linking numbers are non-decreasing from left to right. An example is given in figure 4.

Let us first discuss what this constraint means for D5-branes. First consider a D5-brane that is not to the right of all NS5-branes. The net number of D3-branes ending on such a D5-brane is zero (by our first constraint), so its linking number is just the number of NS5-branes to its left. The number of NS5-branes to one’s left can only increase (or remain constant) as one moves to the right along the chain. So for such D5-branes, the linking numbers are automatically non-decreasing.

Hence for D5-branes, the linking number constraint only says something non-trivial for those D5-branes that are to the right of all NS5-branes — for example, the ones drawn explicitly in figure 1 and the two on the right in figure 4(a). Since all such D5-branes have the same number of NS5-branes to their left, the constraint is that the net number of D3-branes ending on a D5-brane is non-decreasing as one moves to the right. This constraint is satisfied in both examples.

The meaning of the constraint was explained in Section 3.5 of [1]. To get a boundary condition, we must take the limit that the brane separations  $L_i$  are all taken to zero. The behavior in this limit of the moduli

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<sup>2</sup>This definition differs by an inessential constant from the definition used in [15].

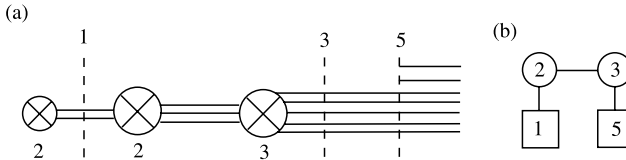


Figure 4: (a) A configuration of three D5-branes and three NS5-branes in  $U(7)$  gauge theory. Each fivebrane has a linking number, defined as the number of fivebranes of the opposite kind that are to the left of the given fivebrane, plus the net number of threebranes ending on the right of the given fivebrane. In the figure, the linking number of a D5-brane (or an NS5-brane) is given by the integer that is written just above (or below) the brane in question. This configuration has been chosen so that the linking numbers of fivebranes of a given type are non-decreasing if one reads the figure from left to right. (b) In the boundary condition derived from (a), a  $U(5)$  subgroup of the gauge group is coupled at the boundary to an SCFT with  $U(5)$  symmetry. This SCFT can be obtained as the infrared limit of the three-dimensional gauge theory associated with the quiver indicated here (together with a free fundamental hypermultiplet from interaction with the D5-brane of linking number 3).

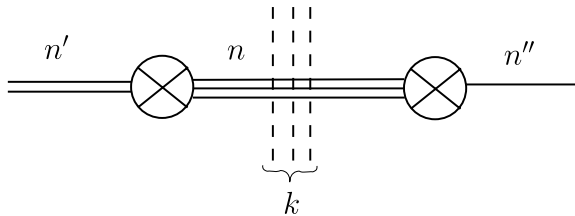


Figure 5: Two NS5-branes with  $k$  D5-branes between them. To the left of the NS5-branes, between them, and to their right, there are respectively  $n'$ ,  $n$ , and  $n''$  D3-branes. Here  $n', n, n''$  equal 2, 3, and 1, respectively.

space of solutions of Nahm's equations associated to a brane configuration is most simple if the D5 linking numbers are non-decreasing. Moreover, the interesting boundary conditions all arise from configurations of this type. If the D5-branes are not arranged in order of increasing (or at least non-decreasing) linking number, then the moduli space of solutions of Nahm's equations contain extra hypermultiplets that decouple as  $L_i \rightarrow 0$ . For our goal of studying boundary conditions, it does not add anything to consider brane configurations that generate such decoupled hypermultiplets.

$S$ -duality suggests that similarly, the Coulomb branch will contain vector multiplets that decouple for  $L_i \rightarrow 0$  unless the NS5-branes are arranged with non-decreasing linking number. To see what the condition means, let

us examine in detail the condition that the linking number for a pair of successive NS5-branes is non-decreasing. We write  $n', n, n''$  for the number of D3-branes to the left of the two NS5-branes, between them, and to their right (figure 5). There may also be D5-branes between the two NS5-branes, but if so (as the net number of D3-branes ending on such a D5-brane is required to vanish) the number of D3-branes does not jump in crossing them. Let  $k$  be the number of such D5-branes and let  $t$  be the number of D5-branes to the left of both NS5-branes drawn in the picture. The linking numbers  $\ell_L$  and  $\ell_R$  of the left and right NS5-brane in figure 5 are  $\ell_L = t + n - n'$ ,  $\ell_R = t + k + n'' - n$ . The condition that  $\ell_R \geq \ell_L$  therefore gives

$$n' + n'' + k \geq 2n. \quad (2.1)$$

This condition has a simple interpretation in gauge theory. The  $n$  D3-branes between the two NS5-branes support a  $U(n)$  gauge theory. This  $U(n)$  gauge theory is coupled to  $k$  fundamental hypermultiplets that arise from D3–D5 intersections. In addition, the interactions of the D3-branes that meet at the two NS5-branes in the picture give bifundamental hypermultiplets of  $U(n') \times U(n)$  and  $U(n) \times U(n'')$ . From the point of view of the  $U(n)$  theory, these are  $n' + n''$  fundamental hypermultiplets.

Altogether, the  $U(n)$  theory therefore interacts with a total of  $n_f = n' + n'' + k$  fundamental hypermultiplets, and the condition of equation (2.1) is equivalent to

$$n_f \geq 2n. \quad (2.2)$$

This condition on the matter fields in a three-dimensional  $U(n)$  gauge theory with  $\mathcal{N} = 4$  supersymmetry is similar to conditions encountered in [8].

The most direct interpretation of (2.2) is that it is the condition under which complete Higgsing is possible; that is, it is the condition under which there exists a vacuum in which the hypermultiplets have expectation values, the scalars in the vector multiplets do not, and the gauge symmetry is completely broken. Consider an  $\mathcal{N} = 4$  theory with gauge group  $U(n)$  and  $n_f$  fundamental hypermultiplets. Viewing the  $\mathcal{N} = 4$  theory as an  $\mathcal{N} = 2$  theory, the hypermultiplets consist of an  $n \times n_f$  matrix  $A$  and an  $n_f \times n$  matrix  $B$ . The scalars in the vector multiplet are an  $n \times n$  matrix  $\phi$ . The superpotential is  $\text{Tr } \phi AB$ , so the condition for a critical point of the superpotential with  $\phi = 0$  is

$$AB = 0. \quad (2.3)$$

For  $n_f = 2n$ , we can satisfy this condition and completely break the gauge symmetry with

$$A = \begin{pmatrix} M & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ N \end{pmatrix}, \quad (2.4)$$

where  $M, N$ , and  $0$  are all  $n \times n$  blocks and  $M$  and  $N$  are generic.

For  $n_f > 2n$ , we simply add more rows and columns of zeroes to  $A$  and  $B$ . However, for  $n_f < 2n$ , it is not possible to completely break the gauge symmetry while also satisfying (2.3) and the  $D$ -term condition  $AA^\dagger = B^\dagger B$ . That latter condition implies that  $A$  and  $B$  have the same rank  $r$ . Equation (2.3) implies that  $2r \leq n_f$  so  $r < n$  if  $n_f < 2n$ . The equation  $AA^\dagger = B^\dagger B$  says that  $B$  and  $A^\dagger$  have the same kernel. The dimension of the kernel is  $n - r$ , so if  $r < n$ , the kernel is non-empty and complete Higgsing has not occurred.

When complete Higgsing is possible, the three-dimensional gauge theory has a critical point at the intersection of the Coulomb and Higgs branches at which all vector multiplets are strongly coupled. In our context, this gives the SCFT  $\mathfrak{B}$  that is part of the boundary conditions. When complete Higgsing is not possible, some vector multiplets remain free in the IR. For example, if  $n_f = 0$ , then the Coulomb branch is smooth and all vector multiplets are free in the IR limit. Since we are interested in boundary SCFT's rather than in brane configurations, we are not interested in considering brane configurations whose Coulomb branch has degrees of freedom that decouple in the IR.

Some brane configurations that do not obey our constraints are also understandable, but it is not necessary to consider them. A more precise description of the kind of infrared limit that we want in our study of boundary conditions is given in Section 2.4, along with another interpretation of the condition  $n_f \geq 2n_c$ .

### 2.3 $S$ -duality

$S$ -duality of a brane configuration can be defined in a purely formal way. We simply replace NS5-branes with D5-branes, and vice versa. We also exchange  $\vec{X}$  with  $\vec{Y}$ , or equivalently, we make a spatial rotation transforming  $x^4, x^5, x^6$  into  $x^7, x^8, x^9$ , and vice versa. The combined operation maps the class of configurations that we have been considering back to itself.

The only problem is that after this transformation, the branes are incorrectly ordered; the first constraint of Section 2.2 is not obeyed. The second constraint, which states that linking numbers are non-decreasing for five-branes of both types, remains valid.

However, a reader familiar with three-dimensional mirror symmetry may guess what to do. Brane configurations of the type considered here can



be manipulated without changing their infrared limit by moving D5-branes in the  $x^3$  direction. The D5-branes may cross NS5-branes, but every time a D5-brane is moved across an NS5-brane, a D3-brane segment stretched between them will be created or destroyed in such a way that the linking numbers remain constant. By judiciously moving the D5-branes, we can get back to a configuration that obeys the constraints.

Though it is also possible to move NS5-branes, this does not add anything. To make NS5-branes cross each other, or D5-branes cross each other, complicates the analysis of the infrared limit. The simplest type of brane motion is to move D5-branes relative to NS5-branes without changing their ordering. This is enough to restore the first constraint on ordering of branes, so it is the only operation we need to consider.

We move any D5-brane whose linking number  $l$  is smaller than the total number of NS5-branes to the interval between the  $l$ th and  $l + 1$ th NS5-branes. Then the net number of D3-branes ending on it will be zero. Any other D5-brane can be pushed to the right of all NS5-branes, and an appropriate number of D3-branes will end on it from the right to give the right linking number. Once this has been done, our constraints are obeyed. The number of D3-brane segments between each consecutive pair of NS5-branes is uniquely determined by the linking numbers of the NS5-branes.

There is only one catch: we need to show that the number of D3-brane segments in the mirror configuration always turns out to be positive. For concreteness, let us denote the linking numbers of the  $P$  NS5-branes as  $\ell_i$ ,  $i = 1, \dots, P$  and of the  $Q$  D5-branes as  $\tilde{\ell}_a$ ,  $a = 1, \dots, Q$ . Let  $n_i$  be the number of D3-branes ending on the  $i$ th NS5-brane on the right. By definition,

$$\ell_i = n_i - n_{i-1} + \#\{a | \tilde{\ell}_a < i\}. \quad (2.5)$$

We can invert this relation as

$$n_i = \sum_{j=1}^i \ell_j - \sum_{a | \tilde{\ell}_a < i} (i - \tilde{\ell}_a) \quad (2.6)$$

In particular, the number of D3-branes just to the right of the rightmost NS5-brane in the original configuration is

$$n_P = \sum_{j=1}^P \ell_j - \sum_{a | \tilde{\ell}_a < P} (P - \tilde{\ell}_a). \quad (2.7)$$

Moving farther to the right, the number of D3-branes will further increase when crossing the remaining D5-branes. At each D5 brane of linking number

$\tilde{\ell}$ , the number of D3-branes will increase by  $\tilde{\ell} - P$ . The final number of semi-infinite D3-branes is

$$n = \sum_{j=1}^P \ell_j - \sum_{a=1}^Q (P - \tilde{\ell}_a) = \sum_{j=1}^P \ell_j + \sum_{a=1}^Q \tilde{\ell}_a - PQ. \quad (2.8)$$

We can assume that in this original configuration, the numbers  $n_i$  of D3-brane segments are strictly positive; otherwise, the system would break into decoupled subsystems which we would study separately. Hence we have a simple inequality

$$\sum_{j=1}^i \ell_j > \sum_{a|\tilde{\ell}_a < i} (i - \tilde{\ell}_a). \quad (2.9)$$

The sum on the right-hand side starts from  $a = 1$  and ends at some  $a = b$ . We can actually take  $b$  to be unconstrained: if we lower  $b$  we are omitting some positive terms from the sum; if we increase  $b$ , we include some extra non-positive terms in the sum. Hence we can consider the more symmetric inequality

$$\sum_{j=1}^i \ell_j + \sum_{a=1}^b \tilde{\ell}_a > bi. \quad (2.10)$$

$S$ -duality exchanges the two kinds of linking numbers  $\ell$  and  $\tilde{\ell}$ . This leaves the collection of inequalities (2.10) unaffected. In particular, the dual number of D3-branes

$$\tilde{n}_b = \sum_{a=1}^b \tilde{\ell}_a - \sum_{j|\ell_j < b} (b - \ell_j) \quad (2.11)$$

is positive. Moreover, the number of semi-infinite D3-branes  $n$  also had a symmetric expression (2.8) in  $\ell_j$  and  $\tilde{\ell}_a$ . Thus, the class of brane configurations that obey our constraints is closed under  $S$ -duality. Starting with such a configuration and applying  $S$ -duality, there is a unique way to move D5-branes to put it back in the desired form.

We are interested in applying this result both for boundary conditions in four dimensions and for purely three-dimensional configurations, where no semi-infinite D3-branes are present. In the above inequalities the two cases differ only by whether  $n$  is positive or zero. In case  $n = 0$ , the linking numbers obey  $0 < \ell_i < Q$  and  $0 < \tilde{\ell}_a < P$ , ensuring that the original and  $S$ -dual configurations have no D5-branes to the left or right of all NS5-branes. For the purely three-dimensional configurations, we have proved that every linear quiver with  $n_f \geq 2n_c$  at each node has a mirror of the same kind.

In Section 3.5, we will consider a few examples of domain walls, generated as configurations of branes with semi-infinite D3-branes on both sides. In that case the linking numbers are still non-decreasing from left to right, but they are not necessarily positive. The inequalities above apply to such configurations with very minor modifications.

## 2.4 Quivers: good, bad, and ugly

In our study of  $S$ -duality of boundary conditions, we will need some understanding of the infrared dynamics of three-dimensional gauge theories defined by linear quivers of unitary groups. The information we need can be extracted by supplementing what is visible classically with the properties of monopole operators [5–7].

We will make extensive use of the operators constructed in those papers, but our point of view is slightly different. We do not want to assume properties of the infrared theory, so we define monopole operators at short distances, using the fact that three-dimensional gauge theory is ultraviolet-free, and then we see what deductions we can make about the infrared behavior.

The definition of a monopole operator in three dimensions is analogous to the definition of an 't Hooft operator in four dimensions. The definition is based on a codimension three magnetic monopole singularity of gauge fields, leading to a local operator in three dimensions, or a line operator in four dimensions.

Like disorder operators in statistical mechanics, monopole operators are most easily defined by giving a recipe to calculate in the presence of a monopole operator. If the gauge group is  $G = U(1)$ , to compute in the presence of a charge  $a$  monopole operator inserted at a point  $x = x_0$  in  $\mathbb{R}^3$ , we perform the path integral over a space of fields with a suitable Dirac monopole singularity:

$$F = \frac{a}{2} \star d \frac{1}{|\vec{x} - \vec{x}_0|}. \quad (2.12)$$

$a$  must be an integer. In supersymmetric gauge theory, to define a BPS monopole operator, we pick one of the scalar fields  $\vec{Y}$  of the vector multiplet, say  $Y_3$ , and require that it also should have a singularity compatible with the Bogomolny equations  $dY_3 = \star F$ . The choice of  $Y_3$  is determined by the choice of a unit vector  $\vec{n}$  in three-space, and accordingly we will denote the resulting monopole operator of charge  $a$  as  $\mathcal{O}_{\vec{n}}(a)$ .

The special case of free  $U(1)$  gauge theory (with  $\mathcal{N} = 4$  supersymmetry in three dimensions) is illuminating. The monopole operator  $\mathcal{O}_{\vec{n}}(a)$  can

be written as  $\exp(a(Y_3 + i\phi)/e^2)$ , or equivalently as  $\exp(a(\vec{n} \cdot \vec{Y} + i\phi)/e^2)$ , where  $\phi$  is the dual photon. There is such an operator for any specified choice of  $\vec{n}$ , with no reason to treat  $\vec{n}$  as a collective coordinate. The choice of a particular  $\vec{n}$  breaks  $SO(3)_Y$  to  $SO(2)_Y$ . This enables us to define an  $\mathcal{N} = 2$  algebra<sup>3</sup> whose global  $R$ -symmetry acts on monopole operators via  $SO(2)_Y$ . The operator  $\mathcal{O}_{\vec{n}}(a)$  is a chiral operator (the lowest component of a chiral multiplet) from the point of view of this  $\mathcal{N} = 2$  algebra. This multiplet has the unusual property that its  $R$ -charge is zero, since  $\exp(a(\vec{n} \cdot \vec{Y} + i\phi)/e^2)$  is certainly invariant under rotations around  $\vec{n}$ . There are two important comments to make about this:

(1) If we want to place the operator  $\mathcal{O}_{\vec{n}}(a)$  in a multiplet of the microscopic global  $\mathcal{N} = 4$  supersymmetry, we have to let  $\vec{n}$  vary. When we do this, we get an infinite-dimensional  $\mathcal{N} = 4$  multiplet, since the operator  $\mathcal{O}_{\vec{n}}(a)$  has no simple dependence on  $\vec{n}$ .

(2) In conformal field theory, the dimension of a chiral operator is at least the  $R$ -charge, so  $R$ -charges of chiral operators other than the identity are positive. The fact that this theory has a chiral operator of  $R$ -charge zero means that, although it has global  $\mathcal{N} = 4$  supersymmetry, it cannot be given the structure of a superconformal field theory in which the  $R$ -symmetry is the microscopic  $SO(3)_X \times SO(3)_Y$  symmetry. (This can also be shown more directly by observing that the field  $Y_1 + iY_2$  is a chiral operator with  $R$ -charge 1 and dimension 1/2.) After dualizing the photon, the model describes four free scalars and four free spinors, so it can be given an  $\mathcal{N} = 4$  superconformal structure, but the  $R$ -symmetry is not the one that one sees in the ultraviolet.<sup>4</sup>

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<sup>3</sup>The supercharges of three-dimensional  $\mathcal{N} = 4$  supersymmetry transform as (2, 2) under  $SO(3)_X \times SO(3)_Y$ . The  $R$ -symmetry of an  $\mathcal{N} = 2$  subalgebra is actually a diagonal subgroup of  $SO(2)_X \times SO(2)_Y$ , where  $SO(2)_X$  is an arbitrarily chosen  $SO(2)$  subgroup of  $SO(3)_X$ .  $SO(3)_X$  leaves invariant the monopole operators of interest (since it acts trivially on the fields whose singularity characterizes them), as a result of which the  $R$ -symmetry acts on monopole operators via  $SO(2)_Y$ .

<sup>4</sup>A necessary condition [18, 19] for an  $\mathcal{N} = 4$  superconformal structure is that, near some chosen vacuum, the Coulomb branch should look like a tri-Sasakian cone. This means that the metric must be conical, with a scaling symmetry generated by a vector field  $V$  that obeys  $D_\nu V^\mu = \delta_\nu^\mu$ , and moreover the generators of the  $SO(3)$   $R$ -symmetry must be the vector fields  $W^k = I^k V$ , where  $I^k$ ,  $k = 1, 2, 3$ , are the three complex structures. For a free vector multiplet, the Coulomb branch is a smooth manifold  $\mathbb{R}^3 \times S^1$ , so it looks conical near any point, but the microscopic  $SO(3)_Y$   $R$ -symmetry (which acts by rotating  $\mathbb{R}^3$ ) does not have the required form, regardless of which vacuum we choose in taking the infrared limit.

### 2.4.1 $U(1)$ examples with hypermultiplets

The next step, still with gauge group  $U(1)$ , is to add hypermultiplets of charges  $q_1, \dots, q_s$ . The  $q_i$  are all non-negative integers. (A hypermultiplet contains fields of equal and opposite charge, and by convention we take the positive sign for the hypermultiplet charge.) The monopole operator of charge  $a$  is defined in the same way, and is still a chiral operator, but as shown in [6], it now carries a non-zero  $R$ -charge

$$q_R = \frac{1}{2} \sum_{i=1}^s |aq_i|, \quad (2.13)$$

due to an asymmetry in the fermionic spectrum.<sup>5</sup>

Since the monopole operators  $\mathcal{O}_{\vec{n}}(a)$  now have positive  $R$ -charges, it is conceivable that such a theory might flow to an infrared critical point in which the  $R$ -symmetry is the microscopic  $SO(3)_X \times SO(3)_Y$ . We will refer to any  $\mathcal{N} = 4$  critical point with this  $R$ -symmetry as a standard critical point.

Since (by definition of a Higgs or Coulomb branch)  $SO(3)_Y$  acts trivially on a Higgs branch of vacua and  $SO(3)_X$  acts trivially on a Coulomb branch, a vacuum at the intersection of the two branches is automatically  $SO(3)_X \times SO(3)_Y$ -invariant. So this is a candidate for the locale of a standard critical point. A necessary condition for this [18, 19] is that the Higgs and Coulomb branches must both be tri-Sasakian cones near their intersection, with the microscopic  $SO(3)_X$  and  $SO(3)_Y$   $R$ -symmetries entering in the tri-Sasakian structures. In  $\mathcal{N} = 4$  supersymmetric gauge theories (without bare masses, FI terms, or Chern–Simons couplings) the Higgs branch always has the appropriate tri-Sasakian structure. Mirror symmetry sometimes makes it possible to show that the Coulomb branch also has the right structure, near its intersection with the Higgs branch. For instance, this is true for models derived from linear quivers with  $n_f \geq 2n_c$  at every node, since we have shown in Section 2.3 that such models have mirrors of the same type. It is reasonable to expect that when both branches have the appropriate structure near their intersection, a standard critical point does indeed arise at this intersection.

If a  $U(1)$  theory coupled to hypermultiplets flows in the infrared to a standard critical point, the structure of the superconformal algebra  $OSp(4|4)$

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<sup>5</sup>The computation in [6] is justified by using a large  $n_f$  limit to suppress fluctuations. Here, we use the fact that the gauge theory is free in the ultraviolet to justify the computation for all  $n_f$ . The Dirac equation in an external field is conformally invariant, justifying the conformal mapping to  $\mathbb{R} \times S^2$  that is used in [6].

implies that in this limit the operators  $\mathcal{O}_{\vec{n}}(a)$  transform in a finite-dimensional multiplet, even though this is not true in the ultraviolet. Indeed, in the infrared theory, the operator  $\mathcal{O}_{\vec{n}}(a)$  has dimension  $q_R$ , and is part of an irreducible  $\mathfrak{so}(3)_Y$  representation of dimension  $2q_R + 1$ . (In an appropriate formalism,  $\mathcal{O}_{\vec{n}}(a)$  varies with  $\vec{n}$  as a holomorphic section of the line bundle  $\mathbb{O}(q_R) \rightarrow \mathbb{C}\mathbb{P}^1$ . The relevant multiplet is an  $\mathbb{O}(q_R)$  multiplet, in the language of [20, 21].) Otherwise,  $\mathcal{O}_{\vec{n}}(a)$  would be related by repeated action of raising or lowering operators in  $\mathfrak{so}(3)_Y$  to an operator of dimension  $q_R$  but with  $R$ -charge greater than  $q_R$  in absolute value. This would contradict unitarity of the IR fixed point.

Let us consider a few special cases. If there is only a single hypermultiplet of charge 1, the  $R$ -charge is  $q_R = |a|/2$ . Setting  $a = \pm 1$ , the basic monopole operators have  $R$ -charge  $\pm 1/2$ . In three dimensions, unitarity implies that a chiral superfield of  $R$ -charge and dimension  $1/2$  is actually part of a free hypermultiplet. In the present case, we actually have a twisted hypermultiplet, in the sense that it transforms non-trivially under  $SO(3)_Y$  and trivially under  $SO(3)_X$ , like the scalar fields  $\vec{Y}$  of the vector multiplet and in contrast to the bosonic fields of an ordinary electrically charged hypermultiplet. The existence of this field shows [6] that the Coulomb branch of the  $U(1)$  theory with a single charge 1 hypermultiplet is equivalent to  $\mathbb{R}^4$ , parametrized by a free twisted hypermultiplet.

A free hypermultiplet has a global symmetry group  $SU(2)$ , commuting with the superconformal group. Let us try to find this symmetry. It helps to know that in a theory with  $\mathcal{N} = 4$  superconformal symmetry, a conserved current  $J$  appears in a multiplet whose lowest component  $\mu$  is an  $\mathcal{N} = 2$  chiral superfield with dimension and  $R$ -charge 1. (In free field theory,  $\mu$  is the moment map for the symmetry associated with the conserved current  $J$ .)

One symmetry of the Coulomb branch of this  $U(1)$  theory with one hypermultiplet is visible classically. This is the group  $U(1)_\phi$  of translations of the dual photon,  $\phi \rightarrow \phi + \text{constant}$ . To find additional symmetries, we need monopole operators of  $R$ -charge 1. Such operators arise precisely for  $a = \pm 2$ , and their presence extends the classical symmetry of the Coulomb branch from  $U(1)$  to  $SU(2)$ .

Under shifts of the dual photon, the monopole operator  $\mathcal{O}_{\vec{n}}(a) = \exp(a(Y_3 + i\phi)/e^2)$  transforms by a phase that is clearly proportional to  $a$ . The conserved currents associated with operators  $\mathcal{O}_{\vec{n}}(\pm 2)$  therefore do not commute with  $U(1)_\phi$ . Together they generate a three-dimensional Lie algebra which must be  $SU(2)$ . This is the expected symmetry of the Coulomb branch.

The fact that the hypermultiplet fields arise for  $a = \pm 1$  and the symmetry currents for  $a = \pm 2$  has a simple interpretation: the hypermultiplet fields transform in the two-dimensional representation of  $SU(2)$ , whose weights are one-half the non-zero weights of the adjoint representation.

For our next example, consider the case of two hypermultiplets of charges  $q_1 = q_2 = 1$ . The  $R$ -charge of a monopole operator is  $q_R = |a|$ . The smallest possible value is 1, for  $a = \pm 1$ . Again, the existence of these chiral operators of  $q_R = 1$  means that if the theory flows to a standard IR critical point, the classical  $U(1)$  symmetry of the Coulomb branch is extended to  $SU(2)$ . In contrast to the case of  $n_f = 1$ , where the Coulomb branch has an  $SU(2)$  symmetry that follows from something more fundamental (existence of a free twisted hypermultiplet), for  $n_f = 2$  the  $SU(2)$  symmetry of the Coulomb branch seems to be an irreducible statement.

The  $U(1)$  theory with  $n_f = 2$  hypermultiplets of charge 1 has a Higgs branch<sup>6</sup> and is believed to flow to a standard critical point. Indeed, this model was one of the original examples of three-dimensional mirror symmetry [8]. (For an explanation of its mirror symmetry, see figure 10.) The Higgs branch of the model has a classical  $SU(2)$  global symmetry, rotating the two hypermultiplets. Mirror symmetry exchanging the Higgs and Coulomb branches implies that the Coulomb branch must also have an  $SU(2)$  symmetry in the infrared, as we have just argued in another way.

The IR critical point of  $U(1)$  coupled to two hypermultiplets will turn out to be an important example for understanding  $S$ -duality of boundary conditions. We will call it  $T(SU(2))$ . As we have just seen, this model has  $SU(2) \times SU(2)$  global symmetry, with one factor acting on the Higgs branch and on the Coulomb branch. (The group that acts faithfully is really  $SO(3) \times SO(3)$ .)

Continuing in this vein, consider  $U(1)$  coupled to  $n_f > 2$  hypermultiplets of charge 1. Monopole operators have  $R$ -charges  $|an_f|/2$ . As these numbers are greater than 1 (for  $|a| \geq 1$ ), the Coulomb branch has no symmetries beyond its classical  $U(1)$  symmetry. Likewise, there are no free hypermultiplets.

What happens if we couple  $U(1)$  gauge theory to hypermultiplets of charge greater than 1? The only case that leads to a monopole operator of  $|q_R| \leq 1$  is the case that we couple to one hypermultiplet of charge 2, leading for  $a = \pm 1$  to  $|q_R| = 1$ . This theory is simply an orbifold of the  $n_f = 1$  theory. The orbifolding operation is a shift in the dual photon, changing the

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<sup>6</sup>This is the  $n_c = 1$  case of what is described in equation (2.4).

quantum of charge. In the orbifolded theory, the Coulomb branch is  $\mathbb{R}^4/\mathbb{Z}_2$  rather than  $\mathbb{R}^4$ . The monopole operators of  $q_R = 1/2$  are projected out, but those of  $q_R = 1$  persist.

Returning to the case of  $n_f$  hypermultiplets of charge 1, let us summarize some properties of these models:

(1) For  $n_f \geq 2$ , we get what we will call “good” theories. They have Higgs branches, and flow to standard IR critical points at the intersection of the Higgs and Coulomb branches. These are non-Gaussian critical points, as is clear from the singularity of the moduli space of vacua, and no more elementary description of them is evident. (They have equally good mirror descriptions.) The borderline case  $n_f = 2$  is what we will call a “balanced” theory, with  $n_f = 2n_c$  (here  $n_c = 1$ ). In the balanced theory, the classical  $U(1)$  symmetry of the Coulomb branch is extended in the IR to  $SU(2)$ .

(2) The theory with  $n_f = 1$ , which has no Higgs branch, still flows in the IR to a standard critical point. However, this critical point is Gaussian and has a more economical description in terms of a free twisted hypermultiplet. We regard the  $U(1)$  theory with  $n_f = 1$  as an “ugly” description of a theory that actually is Gaussian.

(3) Finally, the theory with  $n_f = 0$  is “bad” in that, because it has chiral operators of  $R$ -charge 0, it cannot flow in the IR to a standard critical point.

### 2.4.2 Monopole operators in non-abelian gauge theory

To describe monopole operators in a three-dimensional theory with any gauge group  $G$ , we first pick a homomorphism  $\rho : \mathfrak{u}(1) \rightarrow \mathfrak{g}$ .  $\rho$  plays the role of the monopole charge  $a$  in the  $U(1)$  case. Then we modify the ansatz (2.12) for the singularity characterizing the monopole operator to

$$F = \frac{\rho(1)}{2} \star d \frac{1}{|\vec{x} - \vec{x}_0|}, \quad (2.14)$$

where 1 is a generator of  $\mathfrak{u}(1)$ , and  $\rho(1)$  is its image in  $\mathfrak{g}$ . After requiring that  $X_3$  should have a singularity compatible with the Bogomolny equations, we arrive at the definition of a monopole operator in the non-abelian case.

The  $R$ -charges of these operators were computed in [6]. Let  $h_i$  and  $v_i$  be the charges of vector multiplets and hypermultiplets under the  $U(1)$  subgroup of  $G$  that is defined by  $\rho$ . (The quantities  $h_i$  correspond to  $aq_i$  in the notation we used for  $G = U(1)$ .) The  $R$ -charge of the monopole operator



defined by  $\rho$  is then

$$q_R = \frac{1}{2} \left( \sum_i |h_i| - \sum_j |v_j| \right). \tag{2.15}$$

The fact that vector multiplets and hypermultiplets make contributions to  $q_R$  of equal magnitude and opposite sign is clear if one considers the special case that the hypermultiplets and vector multiplets transform in the same representation of  $G$ . This gives a theory with enhanced supersymmetry ( $\mathcal{N} = 8$  in three dimensions) and  $R$ -symmetry. The unbroken  $SO(2)_Y$  of the monopole operator is extended to  $SO(6)_Y$ . The monopole operator for given  $\rho$  now furnishes a one-dimensional representation of  $SO(6)_Y$ , and as this group has no non-trivial one-dimensional representation,  $q_R$  must vanish, along with its  $SO(6)_Y$  completion.

To orient ourselves to the implications of (2.15), we will consider a basic example:  $U(n_c)$  gauge theory with  $n_f$  fundamental hypermultiplets. We define  $\rho : \mathfrak{u}(1) \rightarrow \mathfrak{u}(n_c)$  by giving a diagonal matrix of integers  $a_1, a_2, \dots, a_n$ . (The  $a_i$  are only defined up to permutation.) With a little group theory, we find that

$$q_R = \frac{n_f}{2} \sum_{i=1}^{n_c} |a_i| - \sum_{1 \leq i < j \leq n_c} |a_i - a_j|. \tag{2.16}$$

Alternatively, this can be written as

$$q_R = \frac{n_f - 2n_c + 2}{2} \sum_{i=1}^{n_c} |a_i| + \sum_{1 \leq i < j \leq n_c} (|a_i| + |a_j| - |a_i - a_j|). \tag{2.17}$$

This formula is useful, since  $|a_i| + |a_j| - |a_i - a_j| \geq 0$  for all  $i, j$ .

We can draw the following conclusions:

(1) First consider the “good” theories with Higgs branches, the ones with  $n_f \geq 2n_c$ . If  $n_f > 2n_c$ , then  $|q_R| > 1$  for all monopole operators. This follows immediately from (2.17). There are no free hypermultiplets and no enhanced symmetries of the Coulomb branch. The last good theory is the “balanced” theory with  $n_f = 2n_c$ . In this theory, the monopole operators in which the  $a_i$  are  $(\pm 1, 0, \dots, 0)$  have  $q_R = 1$ . So, just as we saw earlier for  $n_c = 1$ , the classical  $U(1)$  symmetry of the Coulomb branch is extended to  $SU(2)$ .

(2) Now consider the next case,  $n_f = 2n_c - 1$ , which we consider “ugly.” Here we get  $q_R = 1/2$  if the  $a_i$  are  $(\pm 1, 0, \dots, 0)$ , so again there is a free

twisted hypermultiplet. In addition,  $q_R = 1$  arises from  $(\pm 2, 0, \dots, 0)$  and also from  $(1, -1, 0, \dots, 0)$ . (The first of these we have already seen for  $n_c = 1$ .) Combining this together, the Coulomb branch has a *four*-dimensional group of symmetries. By looking at how the operators transform under  $U(1)_\phi$ , one can see that the group is  $SU(2) \times U(1)$ .

What does this mean? The existence of a free twisted hypermultiplet is consistent with the fact that, since  $n_f < 2n_c$ , the model cannot be completely Higgsed. Instead, adapting the logic of equation (2.4), one finds that it can be Higgsed to  $U(1)$ , so it has a branch of vacua of the form  $\mathcal{C} \times \mathcal{H}$ , where  $\mathcal{C}$  parametrizes a  $U(1)$  vector multiplet and  $\mathcal{H}$  parametrizes the expectation values of the hypermultiplets. The factor  $\mathcal{C}$  is associated with the free twisted hypermultiplet. The  $U(n_c)$  theory with  $2n_c - 1$  flavors must be equivalent in its standard IR fixed point to a theory of a free twisted hypermultiplet times some other theory that can be completely Higgsed and whose Higgs branch is  $\mathcal{H}$ . To find this second theory, move on the Coulomb branch of the  $U(n_c)$  theory to the locus where  $U(n_c)$  is broken to  $U(1) \times U(n_c - 1)$  and the  $U(1)$ -invariant hypermultiplets are massless. It is then possible to give expectation values to those hypermultiplets. The result is a component of the moduli space of vacua that is of the form  $\mathcal{C} \times \mathcal{H}$ , where  $\mathcal{C}$  has hyper-Kähler dimension 1, and  $\mathcal{H}$  is the Higgs branch of a  $U(n_c - 1)$  theory with  $n_f = 2n_c - 1$ .

(3) Finally, the models with  $n_f = 2n_c - 2$  have monopole operators of  $q_R = 0$ , and those of  $n_f < 2n_c - 2$  have monopole operators of negative  $q_R$ . So these “bad” models do not have standard IR critical points.

### 2.4.3 Quiver theories

We now consider a gauge theory derived from a general linear quiver with  $P - 1$  nodes.<sup>7</sup> At the  $i$ th node there is a  $U(n_i)$  gauge theory, coupled to  $m_i$  fundamental hypermultiplets. There is also a bifundamental hypermultiplet of  $U(n_i) \times U(n_j)$  if and only if  $j = i \pm 1$ .

We define a “good” quiver to be one for which  $n_f \geq 2n_c$  at each node. Explicitly, this means that the quantities

$$e_i = m_i + n_{i-1} + n_{i+1} - 2n_i \tag{2.18}$$

are non-negative. We call  $e_i$  the “excess” at the  $i$ th node, and say that a node is “balanced” if it has zero excess. Our first goal is to show that in

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<sup>7</sup>A linear array of  $P - 1$  nodes can be viewed as the Dynkin diagram of the Lie group  $A_{P-1}$ . In Section 5.4, we will obtain results for Dynkin diagrams of type D that are analogous to what we will explain here for type A. We also will give partial results for quivers of type  $E_6, E_7$ , and  $E_8$ .

a gauge theory derived from a “good” quiver, every monopole operator has  $q_R \geq 1$ . We also want to determine for every good quiver precisely which monopole operators have  $q_R = 1$ . This will give us the symmetry of the Coulomb branch.

Consider a monopole operator whose magnetic charges at the  $i$ th node are integers  $a_{i,k}$ , for  $1 \leq k \leq n_i$ . The  $R$ -charge  $q_R$  of the monopole operator receives a contribution

$$\frac{m_i}{2} \sum_{k=1}^{n_i} |a_{i,k}| \tag{2.19}$$

from the  $m_i$  flavors at each node, a contribution

$$-\frac{1}{2} \sum_{k=1}^{n_i} \sum_{t=1}^{n_i} |a_{i,k} - a_{i,t}| \tag{2.20}$$

from the vector multiplets at that node, and a contribution

$$\frac{1}{2} \sum_{k=1}^{n_i} \sum_{t=1}^{n_{i+1}} |a_{i,k} - a_{i+1,t}| \tag{2.21}$$

from the bifundamental matter between the nodes  $i$  and  $i + 1$ .

If we plug the definition of  $e_i$  into the  $R$ -charge formula, and make substitutions like  $n_i = \sum_{k=1}^{n_i} 1$  judiciously, we find that  $q_R = \sum_i (\Delta_i + A_i + B_i)$ , where

$$\Delta_i = \frac{e_i}{2} \sum_{k=1}^{n_i} |a_{i,k}| \tag{2.22}$$

is non-negative, and the other contributions are

$$\begin{aligned} A_i &= \frac{1}{2} \sum_{k=1}^{n_i} \sum_{t=1}^{n_i} (|a_{i,k}| + |a_{i,t}| - |a_{i,k} - a_{i,t}|) \\ B_i &= -\frac{1}{2} \sum_{k=1}^{n_i} \sum_{t=1}^{n_{i+1}} (|a_{i,k}| + |a_{i+1,t}| - |a_{i,k} - a_{i+1,t}|). \end{aligned} \tag{2.23}$$

Each term in these sums is of the form  $|x| + |y| - |x - y|$ , and is zero if  $x$  and  $y$  have opposite signs. If  $x$  and  $y$  are both of the same sign,  $|x| + |y| - |x - y| = 2 \min(|x|, |y|)$ .

These formulas make clear that the total  $R$ -charge is the sum of a contribution from those charges  $a_{i,k}$  that are positive, plus a contribution from those charges  $a_{i,k}$  that are negative. The contribution from positive charges

can be computed by setting the negative charges to zero, and vice versa. Without essential loss of generality, we are then free to consider a configuration with non-negative charges only. A further simplification is as follows. If all  $a_{i,k} = 0$  for some node  $i$ , we can erase that node from the quiver and add its rank  $n_i$  to the number of flavors of the neighboring nodes without changing  $q_R$ . We can then treat each disconnected component of the reduced quiver separately. This means that we can restrict the analysis to monopoles with some non-zero charge at each node.

Let us also order the charges at each node in a non-decreasing fashion  $a_{i,k+1} \geq a_{i,k}$ . Then we can write

$$A_i = \frac{1}{2} \sum_{k=1}^{n_i} \sum_{t=1}^{n_i} 2 \min(a_{i,k}, a_{i,t}) = \sum_{k=1}^{n_i} a_{i,k}(2n_i - 2k + 1) \tag{2.24}$$

and similarly

$$B_i = -\frac{1}{2} \sum_{k=1}^{n_i} \sum_{t=1}^{n_{i+1}} 2 \min(a_{i,k}, a_{i+1,t}) \tag{2.25}$$

Now we want a lower bound

$$B_i \geq -\sum_{k=1}^{n_i} a_{i,k}(n_i - k) - \sum_{t=1}^{n_{i+1}} a_{i+1,t}(n_{i+1} - t + 1), \tag{2.26}$$

which will give a lower bound on  $q_R$ . We can get a lower bound on  $B_i$  by replacing  $\min(a_{i,k}, a_{i+1,t})$  with either  $a_{i,k}$  or  $a_{i+1,t}$ . We pick  $a_{i,k}$  if  $t > n_{i+1} - n_i + k$ , which for given  $k$  happens for at most  $n_i - k$  values of  $t$ ; and we pick  $a_{i+1,t}$  if  $t \leq n_{i+1} - n_i + k$ , which for given  $t$  happens for at most  $n_{i+1} - t + 1$  values of  $k$ . Adding up the possibilities now yields the lower bound (2.26).

The formula (2.24) and lower bound (2.26) lead to a very useful result when we sum over  $i$ . Everything cancels except for contributions from the ends of the chain, and we get simply

$$\sum_{i=1}^{P-1} (A_i + B_i) \geq \sum_{k=1}^{n_1} a_{1,k}(n_1 - k + 1) + \sum_{k=1}^{n_{P-1}} a_{P-1,k}(n_{P-1} - k). \tag{2.27}$$

So

$$q_R \geq \sum_{i=1}^{P-1} \frac{e_i}{2} \sum_{k=1}^{n_i} a_{i,k} + \sum_{k=1}^{n_1} a_{1,k}(n_1 - k + 1) + \sum_{k=1}^{n_{P-1}} a_{P-1,k}(n_{P-1} - k). \tag{2.28}$$

In particular, since  $e_i \geq 0$  and we have performed this computation in a sub-quiver in which  $a_{1,k} \neq 0$  for some  $k$ , we get the desired result that  $q_R \geq 1$ . The inequality  $q_R \geq 1$  actually holds separately for the contributions to  $q_R$  from positive monopole charges  $a_{i,k}$  as well as the contribution from negative ones. So a monopole operator of  $q_R = 1$  has all  $a_{i,k}$  non-negative or all  $a_{i,k}$  non-positive.

We further see that in order to get  $q_R = 1$ , we must have all  $a_{i,k} = 0$  unless the  $i$ th node is balanced, that is, unless  $e_i = 0$ . So to study monopole operators of  $q_R = 1$ , we can restrict ourselves to quivers in which every node is balanced. Moreover, we can assume that the set of nodes with non-zero charge is connected. Otherwise, each component would contribute at least 1 to  $q_R$ .

So now we want to consider a connected linear quiver with every node balanced. We want to determine exactly which monopole operators with non-zero charge at each node have  $q_R = 1$ . For this, a slight generalization of the above inequalities is useful. Obviously, the inequality (2.26) has a “mirror image,”

$$B_i \geq - \sum_{k=1}^{n_i} a_{i,k}(n_i - k + 1) - \sum_{t=1}^{n_{i+1}} a_{i+1,t}(n_{i+1} - t). \tag{2.29}$$

Now after picking a node  $s$ , to get a lower bound on  $\sum_i B_i$ , we use the mirror image inequality for  $i < s$  and the original one for  $i \geq s$ . The result is a slightly modified inequality for  $q_R$ :

$$q_R \geq \sum_{i=1}^{P-1} \frac{e_i}{2} \sum_{k=1}^{n_i} a_{i,k} + \sum_{k=1}^{n_1} a_{1,k}(n_1 - k) + \sum_{k=1}^{n_s} a_{s,k} + \sum_{k=1}^{n_{P-1}} a_{P-1,k}(n_{P-1} - k). \tag{2.30}$$

Now we see to get  $q_R = 1$ , it must be that at each value of  $s$  in the reduced quiver, the monopole charges take the form  $a_{s,k} = (0, 0, \dots, 0, 1)$ .

The complete set of monopole operators with  $q_R = 1$  is therefore easy to describe. Let  $b_i = \sum_k a_{i,k}$ . A monopole operator of  $q_R = 1$  is completely determined by the  $b_i$ . The reduced quiver is supported in a range  $i_0 \leq i \leq i_1$  for some  $i_0, i_1$ . The  $b_i$  (and  $a_{i,k}$ ) vanish outside this range, and in the range the  $b_i$  are all 1, or all  $-1$ .

We want to argue that for a linear quiver with  $P - 1$  consecutive balanced nodes, the classical symmetries of the Coulomb branch combine with the monopole operators to generate an  $SU(P)$  symmetry group. The classical

symmetries of the Coulomb branch, acting by translations of dual photons, are an abelian group that we will call  $U(1)_\phi^{P-1}$ .

We identify the group  $U(1)_\phi^{P-1}$  with the maximal torus of  $SU(P)$ . Its action on a monopole operator can be read off from the charges  $b_i$ . The simple roots of  $SU(P)$  correspond to monopole operators with only a single  $b_i$  equal to 1, and the rest vanishing. The other monopole operators of  $q_R = 1$  furnish the other roots of  $SU(P)$ .

So a string of  $P - 1$  balanced nodes in any quiver with a standard IR limit leads to an  $SU(P)$  symmetry of the Coulomb branch. More generally, the symmetry of the Coulomb branch for any good linear quiver is as follows. Every unbalanced node with  $e_i > 0$  contributes a  $U(1)$  factor. Every sequence of  $P - 1$  balanced nodes contributes a factor  $SU(P)$ .

These results can also be obtained from mirror symmetry. To have  $P - 1$  successive balanced nodes in a linear quiver means that  $P$  consecutive NS5-branes have the same linking number. In the mirror, the  $P$  dual D5-branes, since they have the same linking number, are located in the same D3-brane segment. They can be taken to be located at the same point in space and therefore generate a  $U(P)$  symmetry, of which the center may act trivially, depending on the details of the quiver. We factor  $U(P)$  as  $SU(P) \times U(1)$ . Every cluster of D5-branes at the same location gives a  $U(1)$  factor in the symmetry group (one overall diagonal  $U(1)$  decouples), and in addition every cluster of  $P > 1$  D5-branes gives an  $SU(P)$  factor.

Now we would like to analyze “ugly” and “bad” quivers. We call a quiver ugly if the smallest value of  $q_R$  is  $1/2$ , so that there can be a standard infrared limit, but it must have free twisted hypermultiplets. And we call a quiver bad if there are monopole operators of  $q_R \leq 0$ , so that a standard infrared limit is not possible. For example, any quiver with a node of  $e_i \leq -2$  is bad, since there exist monopole operators with charges only at that node and  $q_R \leq 0$ . So an ugly quiver has  $e_i \geq -1$  for all nodes. An ugly linear quiver has at least one of the  $e_i$  equal to  $-1$ . We call a node with  $e_i = -1$  minimally unbalanced. We will mainly be concerned with linear quivers with a single minimally unbalanced node, and  $e_i \geq 0$  for all other nodes. These quivers turn out to be always ugly.<sup>8</sup>

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<sup>8</sup>Quivers with more than one minimally unbalanced node can be either ugly or bad. For example, a linear quiver with two nodes of  $e = -1$  connected by a chain of balanced nodes is bad, since one can explicitly exhibit a monopole operator of  $q_R = 0$ . A linear quiver in which all nodes have  $e_i \geq -1$  and every two nodes of  $e = -1$  are separated by a node of  $e > 0$  is ugly. This can be shown by further use of the inequalities (2.26) and (2.29).

Let the minimally unbalanced node of such a quiver be at position  $s_0$ . Consider the inequality (2.30) for  $s = s_0$ :

$$q_R \geq \sum_{i \neq s_0} \frac{e_i}{2} \sum_{k=1}^{n_i} a_{i,k} + \sum_{k=1}^{n_1} a_{1,k}(n_1 - k) + \frac{1}{2} \sum_{k=1}^{n_{s_0}} a_{s_0,k} + \sum_{k=1}^{n_{P-1}} a_{P-1,k}(n_{P-1} - k). \tag{2.31}$$

$q_R$  may be  $1/2$  only if  $a_{s_0,k}$  is of the usual form  $(0, 0, \dots, 0, 1)$  and any node with  $e_i > 0$  has charge zero. Suppose that there are  $P_1 - 1$  consecutive balanced nodes to the left of  $s_0$  and  $P_2 - 1$  consecutive balanced nodes on the right. To get  $q_R = 1/2$ , the inequality (2.30) for each balanced node forces the charges at that node to be of the usual form  $a_{i,k} = (0, 0, \dots, 0, b_i)$  with  $b_i = 0, 1$ . The nodes with non-zero monopole charge must form a connected set for the same reason as before (or we will get  $q_R \geq 3/2$ ). There are  $P_1 P_2$  such monopoles, associated with all possible reduced quivers supported in a range  $i_0 \leq i \leq i_1$  for some  $i_0, i_1$  with  $i_0 \leq s_0 \leq i_1$ . All these monopoles actually have  $q_R = 1/2$ .

There are several ways to construct monopoles with  $q_R = 1$ . The ones with zero charge at the minimally unbalanced node combine with the classical symmetries at the balanced nodes to give a  $SU(P_1) \times SU(P_2)$  symmetry group. The  $P_1 P_2$  monopoles of charge  $q_R = 1/2$  carry the weights of a bifundamental representation of this  $SU(P_1) \times SU(P_2)$  symmetry group. As the monopoles of charge  $q_R = 1/2$  are expected to flow to free twisted hypermultiplets in the infrared, we expect a full  $Sp(2P_1 P_2)$  symmetry group acting only on them. Apart from this, there may be symmetries which act trivially on the free twisted hypermultiplets. Indeed many more monopoles of charge  $q_R = 1$  can be found which have non-zero charges at the unbalanced node. The full analysis of the symmetry group is complicated and depends on the  $n_i$ .

### 3 S-duality for $U(n)$ boundary conditions

We will now use the brane constructions reviewed and analyzed in the last section to study  $S$ -duality of boundary conditions in  $U(n)$  gauge theory. Before considering examples, we make a few general remarks.

The duality transformation  $S : \tau \rightarrow -1/\tau$  exchanges D5-branes and NS5-branes. To ensure that the class of brane configurations we consider is  $S$ -invariant (rather than being mapped by  $S$  to a different but equivalent class), we accompany  $S$  with a rotation that exchanges  $\vec{X}$  and  $\vec{Y}$ . The

$R$ -symmetries of Higgs and Coulomb branches are  $SO(3)_X$  and  $SO(3)_Y$ , respectively, so  $S$  exchanges Higgs and Coulomb branches.

The analyses of  $S$ -duality of brane configurations will almost always depend on the freedom to move D5-branes in the  $y$ -direction. The precise positions of the D5-branes are irrelevant in, but only in, the infrared limit. For this reason, brane methods are natural for studying  $S$ -duality of infrared critical points.

Focusing on critical points is not a real limitation. Once one establishes  $S$ -duality between two conformally invariant boundary conditions, one can expect to follow the duality after turning on relevant operators on the boundary.

The IR limits that are naturally studied via branes are standard IR limits in the sense of Section 2.4 — superconformal critical points at which the  $R$ -symmetry is the one seen in the ultraviolet. The ultraviolet  $R$ -symmetry is the  $R$ -symmetry that is visible in a brane configuration and whose behavior under  $S$ -duality is known. However, some constructions depend on global symmetries (as opposed to  $R$ -symmetries) that only exist in the IR limit.

Some final remarks mainly concern notation. We will generically write  $\mathcal{B}$  for a boundary condition, and  $\mathfrak{B}$  for a boundary SCFT. An important special case is that  $\mathcal{B}$  might be constructed from Neumann boundary conditions coupled to some boundary SCFT  $\mathfrak{B}$ . Then we say that  $\mathcal{B}$  is the boundary condition associated to  $\mathfrak{B}$ . (It is not true that every boundary condition is associated in this way to a boundary theory, since other ingredients — Nahm poles and reduction of gauge symmetry — can also enter.) If  $G$  is a compact group, we write  $G^\vee$  for its dual group. If  $\mathcal{B}$  is a boundary condition in  $G$  gauge theory, then the  $S$ -dual of  $\mathcal{B}$  is a boundary condition in  $G^\vee$  gauge theory; we denote this  $S$ -dual as  $\mathcal{B}^\vee$ . If  $\mathcal{B}^\vee$  is obtained by coupling Neumann boundary conditions to a boundary SCFT, then we denote this SCFT as  $\mathfrak{B}^\vee$ .

An important point is that  $\mathfrak{B}^\vee$  is *not* the  $S$ -dual of  $\mathfrak{B}$ . Such a statement would not even make sense, since  $S$ -duality is an operation on four-dimensional field theories, while  $\mathfrak{B}$  and  $\mathfrak{B}^\vee$  (when they exist) are three-dimensional SCFTs. There is, however, an operation of mirror symmetry for three-dimensional SCFTs [8] that is closely related to four-dimensional  $S$ -duality. The mirror of a three-dimensional SCFT  $\mathfrak{B}$  is a theory that we will call  $\tilde{\mathfrak{B}}$ , obtained by exchanging the Higgs and Coulomb branches of  $\mathfrak{B}$ . It turns out that when  $\mathfrak{B}^\vee$  exists, it is possible to construct its mirror  $\tilde{\mathfrak{B}}^\vee$  directly from  $\mathfrak{B}$ . Explaining this will be one of our main goals. But first we will work out a number of examples.



### 3.1 $U(1)$ examples

We begin with the case of a single D3-brane. This means that the bulk gauge group is  $G = U(1)$ , and that electric–magnetic duality in bulk can be explicitly understood. It maps the field strength  $F$  to a multiple of  $\star F$ , where  $\star$  is the Hodge star.

#### 3.1.1 Dirichlet and Neumann

The simplest statement of all is that Dirichlet boundary conditions are dual to Neumann boundary conditions. Dirichlet boundary conditions assert that  $F_{\mu\nu} = 0$  on the boundary, where  $\mu, \nu = 0, 1, 2$  are tangent to the boundary, or more succinctly  $F| = 0$ . Neumann boundary conditions assert that  $F_{\mu 3} = 0$  on the boundary, or more succinctly  $\star F| = 0$ . These two conditions are exchanged via  $F \leftrightarrow \star F$ .

The corresponding brane picture is simple (figure 6). Dirichlet boundary conditions arise for a D3-brane ending on a D5-brane, and Neumann boundary conditions arise for a D3-brane ending on an NS5-brane. For a direct path integral explanation of the duality of Dirichlet and Neumann for  $U(1)$ , see Section 4.4.

What we have just described involves practically the only example of a half-BPS boundary condition  $\mathcal{B}$  in  $U(1)$  gauge theory that is *not* associated with coupling to a boundary SCFT  $\mathfrak{B}$ . For the abelian group  $G = U(1)$ , there is no room for a Nahm pole. A reduction of gauge symmetry would necessarily reduce  $U(1)$  to a finite group. The Dirichlet boundary conditions that we have just analyzed are the ones that reduce  $U(1)$  to the trivial subgroup consisting only of the identity. Reduction to a finite subgroup is equivalent locally to Dirichlet (and our concerns in this paper are purely local). So in our further study of  $U(1)$  gauge theory, we can assume that

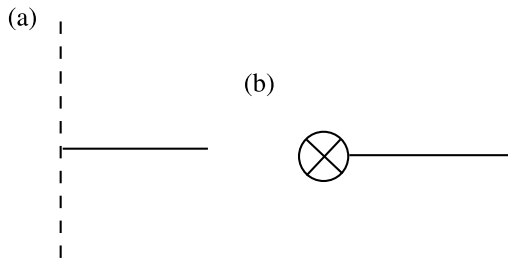


Figure 6: The most basic boundary conditions in  $U(1)$  gauge theory are Dirichlet and Neumann. These arise for a single D3-brane ending on a D5-brane, as in (a), or an NS5-brane, as in (b).

the boundary condition  $\mathcal{B}$  and its  $S$ -dual  $\mathcal{B}^\vee$  are associated to boundary SCFT's  $\mathfrak{B}$  and  $\mathfrak{B}^\vee$ .

### 3.1.2 Coupling to a boundary hypermultiplet

The next example is more interesting. We consider a  $U(1)$  gauge field with Neumann boundary conditions coupled to a charged hypermultiplet on the boundary. We take the hypermultiplet to have charge 1 as that is the only value that we can conveniently get from branes. As in figure 7, we realize this boundary condition by letting a D3-brane pass through a D5-brane at  $y = L$  and end on an NS5-brane at  $y = 0$ . The boundary condition arises in the limit  $L \rightarrow 0$ , but the advantages of starting with  $L > 0$  will become clear, especially when we get to the non-abelian case.

In figure 7(a), the D5-brane and NS5-brane both have linking number 1. So the dual will be precisely the same configuration. To see in more detail how this comes about, we first make naive  $S$ -duality, turning the D5-brane into an NS5-brane and vice versa, while exchanging  $\vec{X}$  and  $\vec{Y}$ . This gives the configuration of figure 7(b). A D3-brane passes through the NS5-brane and ends on the D5-brane. Then we move the D5-brane to the right of the NS5-brane. When it crosses the NS5-brane, a new D3 segment connecting the two fivebranes is created, according to [15]. So we get back to the configuration of figure 7(a).

Thus, a boundary condition  $\mathcal{B}$  consisting of Neumann boundary conditions coupled to a charge 1 hypermultiplet is self-dual. One might naively argue that this is so simply because  $\mathcal{B}$  was constructed from one D5-brane and one NS5-brane, which are exchanged by duality. But the crucial point is really that the linking numbers were the same. This will be particularly clear when we get to non-abelian examples.

To elucidate the physical meaning of the  $S$ -duality of  $\mathcal{B}$ , it is useful to move away from the conformal fixed point by giving expectation values to the scalar fields  $\vec{X}$  or  $\vec{Y}$  along the D3-brane. In the original configuration of figure 7(a), if we give an expectation value to  $\vec{Y}$ , this causes the D3-brane to move along the NS5-brane so that it no longer intersects the D5-brane. Hence, the electrically charged hypermultiplet acquires a mass. Therefore,  $S$ -duality implies that if we give an expectation value to  $\vec{X}$ , we should see a massive hypermultiplet of magnetic charge 1.

If we give an expectation value to  $\vec{X}$ , then the brane configuration of figure 7(a) is deformed to the configuration sketched in (c). The D3-brane splits at the D5-brane; the two fivebranes are connected by a finite D5-brane

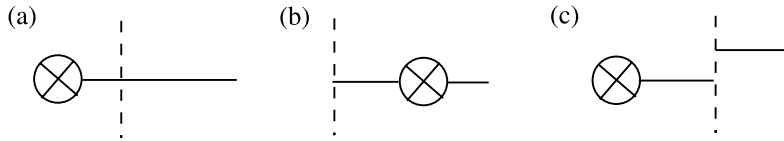


Figure 7: (a) A brane configuration, made from a D3-brane interacting with one D5-brane and one NS5-brane, that leads to  $U(1)$  gauge theory coupled to a boundary hypermultiplet of charge 1. This boundary condition is self-dual, because the fivebranes both have linking number 1. (b) A naive application of  $S$ -duality turns the configuration of (a) into the one sketched here. Upon moving the D5-brane to the right of the NS5-brane, we get back to (a), an outcome that is ensured because the two fivebranes have equal linking numbers. (c) Starting with (a) and displacing the D3-brane at infinity in the  $\vec{X}$  direction, the D3-brane “breaks” and we arrive at the configuration indicated here.

segment. From a field theory point of view, what is happening is the following. The charged hypermultiplet at  $y = 0$  has a hyper-Kähler moment map  $\vec{\mu}$ . The boundary condition for a vector multiplet with Neumann boundary conditions coupled to boundary hypermultiplets is

$$\vec{X}(0) + \vec{\mu} = 0 \tag{3.1}$$

according to equation (2.33) of [1]. Therefore, when we give  $\vec{X}$  an expectation value, we force a Higgsing of the  $U(1)$  gauge symmetry, so as to get  $\vec{\mu} \neq 0$ . This Higgsing occurs only on the boundary, because that is where the charged fields are. A spontaneously broken  $U(1)$  gauge theory in two space dimensions has vortices; in the field of a vortex, there is a unit of magnetic flux integrated over a spatial section of the boundary. By flux conservation in abelian gauge theory, the magnetic flux measured at spatial infinity is the same, and therefore in  $3 + 1$  dimensions this configuration looks like a magnetic monopole of charge 1, localized at the boundary.

Supersymmetry is broken if we give expectation values to both  $\vec{X}$  and  $\vec{Y}$ , because turning on  $\vec{Y}$  gives a mass to the boundary hypermultiplet, and turning on  $\vec{X}$  forces this hypermultiplet to have an expectation value. From the point of view of branes, for a supersymmetric configuration, the D3-brane must end on one of the fivebranes, and so must be located at  $\vec{X} = 0$  or  $\vec{Y} = 0$ .

### 3.1.3 Two boundary hypermultiplets

Our next example will be a  $U(1)$  theory coupled to two boundary hypermultiplets of charge 1. The brane configuration and the steps in understanding

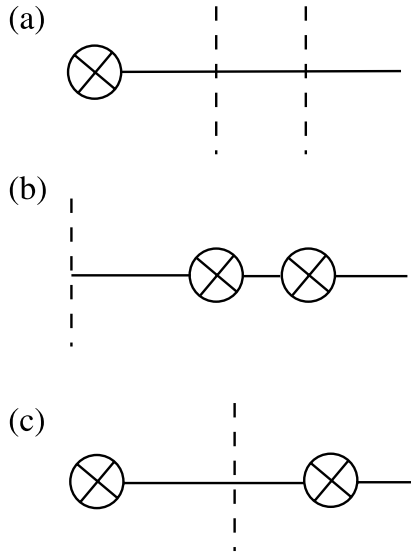


Figure 8: (a) A brane configuration that leads to a  $U(1)$  theory coupled to two boundary hypermultiplets. (b) The naive  $S$ -dual. (c) A standard form of the  $S$ -dual after some rearrangement.

its  $S$ -duality are sketched in figure 8. Each fivebrane has linking number 1. Starting with configuration (a), the naive  $S$ -dual is (b), and a brane rearrangement that preserves the linking numbers brings us to (c), which is more easily interpreted.

This gives us our first example in which the  $S$ -dual  $\mathcal{B}^\vee$  of a boundary condition  $\mathcal{B}$  involves a coupling of four-dimensional gauge theory to a non-trivial three-dimensional boundary SCFT  $\mathfrak{B}^\vee$ . In figure 8(c), there is a  $U(1)$  gauge theory in the D3-brane segment connecting the two NS5-branes. From a three-dimensional point of view, the  $U(1)$  gauge theory is coupled to two hypermultiplets of charge 1. One comes from the intersection of the D3-brane with the D5-brane and one from the interaction with the semi-infinite D3-brane on the right. The second hypermultiplet is also charged under the bulk  $U(1)$  gauge theory.  $U(1)$  gauge theory with two hypermultiplets was discussed in Section 2.4.1. It has a standard IR limit consisting of a non-trivial SCFT, which we have called  $T(SU(2))$ . So the dual of  $U(1)$  theory coupled to two boundary hypermultiplets is  $U(1)$  theory coupled to  $\mathfrak{B}^\vee = T(SU(2))$ . To make this statement precise, one must describe how  $U(1)$  is coupled to  $T(SU(2))$ . The relevant coupling is simply derived from the fact that before the infrared flow, one of the two hypermultiplets of figure 8(c) has charge 1 under the bulk gauge group and one is neutral.

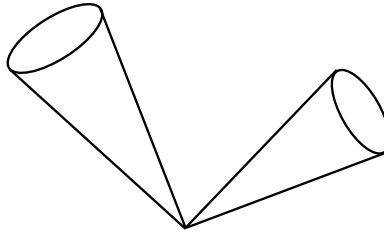


Figure 9: The conformal field theory  $T(SU(2))$  can be realized by the brane configuration of (a). All fivebranes have linking number 1, which implies the self-mirror property. Naive  $S$ -duality leads to (b), and brane rearrangement leads back to the mirror theory (a).

The theory  $T(SU(2))$ , which will play an important role in this paper, has<sup>9</sup> Higgs and Coulomb branches that are both of hyper-Kähler dimension 1. And there is a mirror symmetry [8] that exchanges the two branches. As summarized in figure 9, the mirror symmetry can be established by the same sort of brane manipulations that we are using to analyze boundary conditions. Some key aspects of this mirror symmetry picture are as follows. The Higgs branch has an  $SU(2)$  flavor symmetry that is obvious classically; the Coulomb branch, as described in Section 2.4.1, has a manifest  $U(1)$  global symmetry (translations of the dual photon) that is extended to  $SU(2)$  in the infrared. So the global symmetry is really  $SU(2) \times SU(2)$ , with one factor acting on each branch. In the absence of hypermultiplet bare masses and Fayet–Iliopoulos (FI) parameters, each branch is a copy of  $\mathbb{R}^4/\mathbb{Z}_2$ , with an  $A_1$  singularity at the origin; the two branches meet at their common singularity, which is a superconformal critical point (figure 10). If FI parameters are turned on, the Coulomb branch disappears and the singularity of the Higgs branch is resolved. If bare masses are turned on, the Higgs branch disappears and the singularity of the Coulomb branch is resolved. Mirror symmetry exchanges the two sets of parameters.

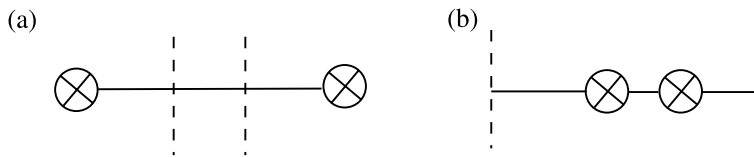


Figure 10: In a superconformal field theory with  $\mathcal{N} = 4$  supersymmetry, the Higgs and Coulomb branches are both conical, and meet at their common origin. In general, there may also be mixed Higgs–Coulomb branches.

<sup>9</sup>The dimension of the Higgs branch is the difference between the numbers of hypermultiplets and vector multiplets; the dimension of the Coulomb branch is the rank of the gauge group.

Going back to figure 8(a), even when the vacuum is fixed at infinity by requiring that  $\vec{X}$  and  $\vec{Y}$  vanish, the theory still has a moduli space of vacua that depends on the hypermultiplet expectation values. From the point of view of the brane picture, this is so because the D3-brane segment between the two D5-branes is free to move in the  $\vec{X}$  direction. In the description by  $U(1)$  gauge theory coupled to two boundary hypermultiplets, we can reason as follows. In this description, supersymmetry requires that  $\vec{X}$  is independent of  $y$ , and so  $\vec{X}(0)$  equals the value of  $\vec{X}$  at infinity. (In non-abelian gauge theory,  $\vec{X}$  would obey Nahm's equations, as described in [1], but for  $G = U(1)$ , these equations merely say that  $\vec{X}$  is constant.) In addition, we have the boundary condition  $\vec{X}(0) + \vec{\mu} = 0$ , where  $\vec{\mu}$  is the moment map of the fundamental hypermultiplets. After also dividing by  $G$  the space  $Z \cong \mathbb{R}^8$  that parametrizes the two hypermultiplets, the moduli space of vacua is the hyper-Kähler quotient  $Z//U(1)$ . This quotient is the  $A_1$  singularity  $\mathbb{R}^4/\mathbb{Z}_2$  if  $\vec{X}(0) = 0$ ; otherwise, it is a resolution of this singularity. If we take  $\vec{Y}$  to be non-zero at infinity, the D3-brane is displaced in the  $\vec{Y}$  direction, the hypermultiplets become massive, and this branch of vacua disappears.

Of course, these statements must have an analog in the  $S$ -dual theory of figure 8(c). First we might look for a Higgs branch in the  $S$ -dual theory. Just as in the last paragraph, such a Higgs branch would come by taking the hyper-Kähler quotient by  $G = U(1)$  of the Higgs branch of the conformal field theory  $T(SU(2))$ . However, the Higgs branch of that SCFT is  $\mathbb{R}^4/\mathbb{Z}_2$ , and its hyper-Kähler quotient by  $U(1)$  is trivial. On the other hand, the Coulomb branch of the SCFT does not couple directly to the bulk gauge theory and survives the coupling of the  $U(1)$  symmetry of the SCFT to the bulk gauge fields. This is visible in figure 8(c); the D3-brane segment that connects the two NS5-branes is free to move independently of the semi-infinite D3-brane. The Coulomb branch disappears if we give an expectation value to  $\vec{X}$  at infinity, because those are FI parameters that force the hypermultiplets to have expectation values. (An expectation value of  $\vec{X}$  causes a brane reconnection similar to that in figure 8(c), and the brane modulus disappears.) An expectation value for  $\vec{Y}$  at infinity gives a hypermultiplet bare mass and modifies the geometry of the Coulomb branch. These statements are  $S$ -dual to the statements in the last paragraph.

### 3.1.4 Generalization

A natural generalization of the previous example is to introduce  $n$  boundary hypermultiplets of charge 1. We represent this via a single D3-brane passing through  $n$  D5-branes and ending on an NS5-brane (figure 11(a)). All linking numbers equal 1. The  $S$ -dual brane configuration has  $n$  NS5-branes and one D5-brane. Since the D5-brane has linking number 1, in the dual

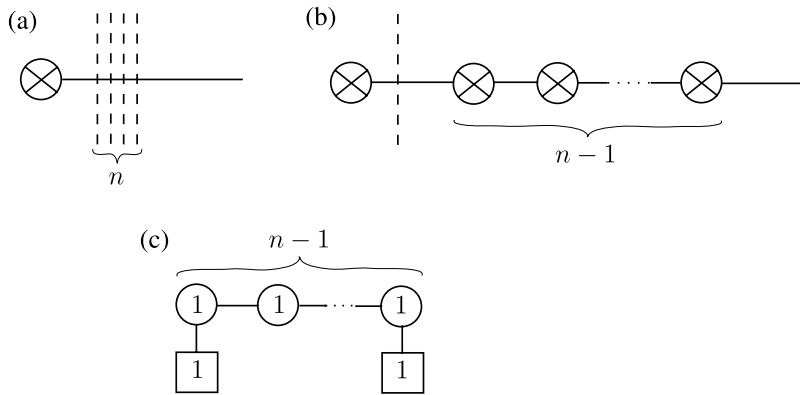


Figure 11: (a) A brane configuration representing a  $U(1)$  theory with  $n$  boundary hypermultiplets of charge 1. All linking numbers are 1. (b) The  $S$ -dual configuration, after a standard rearrangement. All linking numbers are again 1. (c) The quiver corresponding to the SCFT that arises from the infrared limit of the dual configuration. The hypermultiplet at one end comes from the D3–D5 intersection, and the hypermultiplet at the other end comes from the interaction of the rightmost D3-brane segment with the semi-infinite D3-brane.

configuration (figure 11(b)), it is to the right of just one NS5-brane. The first NS5-brane has linking number 1, and no D5-branes to the left; hence a single D3-brane must end on it. All the other NS5-branes have a D5-brane to the left; hence to make the linking numbers equal 1, no net D3-branes end on them.

In the infrared limit, the dual configuration corresponds to  $U(1)$  gauge theory coupled to a boundary SCFT  $\mathfrak{B}^\vee$  that is the IR limit of a well-known quiver, shown in figure 11(c). (This is sometimes called a quiver of type  $A_{n-1}$ .) It has a Higgs branch of hyper-Kähler dimension 1 and a Coulomb branch of hyper-Kähler dimension  $n - 1$ . The Higgs branch can be computed classically, and is the  $A_{n-1}$  singularity  $\mathbb{R}^4/\mathbb{Z}_n$ , according to a well-known result of Kronheimer. The Coulomb branch is most easily determined by mirror symmetry. In fact, the theory  $\mathfrak{B}^\vee$  is part of one of the original examples of a mirror pair in three dimensions [8]. Its mirror is another theory  $\tilde{\mathfrak{B}}^\vee$  that arises from the infrared limit of  $U(1)$  coupled to  $n$  hypermultiplets of charge 1. The Higgs branch of  $\tilde{\mathfrak{B}}^\vee$  and therefore Coulomb branch of  $\mathfrak{B}^\vee$  have an  $SU(n)$  flavor symmetry, which is  $S$ -dual to the flavor symmetry of the  $n$  hypermultiplets in the original boundary condition. (In fact, it can be shown that as a complex manifold, this Higgs or Coulomb branch is isomorphic to the minimal nilpotent orbit of  $SL(n, \mathbb{C})$ .)

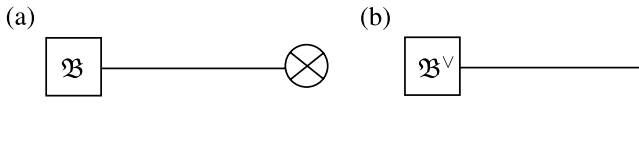


Figure 12: (a) This four-dimensional picture, with Neumann boundary conditions coupled to a three-dimensional theory  $\mathfrak{B}$  at one end and pure Neumann boundary conditions at the other end, flows in the IR to the three-dimensional theory  $\mathfrak{B}'$ . (b) The  $S$ -dual configuration has Dirichlet boundary conditions on the right and on the left a coupling to a boundary theory  $\mathfrak{B}^\vee$  that provides the boundary condition that is  $S$ -dual to the one determined by  $\mathfrak{B}$ . Dirichlet boundary conditions have the effect of turning off the gauge couplings on the interval, and the infrared limit is simply the three-dimensional theory  $\mathfrak{B}^\vee$ . Hence  $\mathfrak{B}'$  is mirror to  $\mathfrak{B}^\vee$ .

The reader may notice a resemblance between the three-dimensional theory  $\tilde{\mathfrak{B}}^\vee$  and the original boundary condition that we started with. They involve the same  $n$  hypermultiplets coupled to four-dimensional or three-dimensional gauge fields. Explaining this resemblance will lead to one of the main ideas of this paper.

To formalize what is going on, we start with a boundary condition in four-dimensional  $U(1)$  gauge theory obtained by coupling to some boundary theory  $\mathfrak{B}$ . In our example,  $\mathfrak{B}$  is the theory of  $n$  free hypermultiplets.

$\mathfrak{B}$  has a  $U(1)$  global symmetry that is used in coupling it to four-dimensional gauge fields. By means of the same  $U(1)$  global symmetry, we could instead couple  $\mathfrak{B}$  to *three*-dimensional gauge fields. If the coupled theory has a standard IR limit, this limit is a new three-dimensional theory  $\mathfrak{B}'$ . We claim that in fact,  $\mathfrak{B}'$  is the mirror of  $\mathfrak{B}^\vee$ , the SCFT that defines the  $S$ -dual boundary condition  $\mathcal{B}^\vee$ .

This can be deduced as follows. The three-dimensional theory  $\mathfrak{B}'$  can be obtained from the boundary condition  $\mathcal{B}$  by taking the D3-brane that generates the  $U(1)$  gauge symmetry to be of finite extent in the  $y = x^3$  direction. We take boundary conditions  $\mathcal{B}$  at  $y = 0$ , and Neumann boundary conditions at  $y = L$  (figure 12(a)). At low energies or for  $L \rightarrow 0$ , the gauge fields become effectively three-dimensional and we get the three-dimensional theory  $\mathfrak{B}'$ .

On the other hand, we can apply  $S$ -duality to this configuration. The boundary condition  $\mathcal{B}$  is replaced by its  $S$ -dual  $\mathcal{B}^\vee$ , and the Neumann boundary conditions at  $y = L$  are replaced by Dirichlet (figure 12(b)).



Because of the Dirichlet boundary conditions at one end, the gauge fields on the segment are massive. So although the theory  $\mathfrak{B}^\vee$  appears to have been coupled to gauge fields, at low energies it is effectively ungauged. Hence the configuration of figure 12(b) just leads to the three-dimensional theory  $\mathfrak{B}^\vee$ .

The *S*-duality operation that relates figures 12(a) and (b) is mirror symmetry from a three-dimensional point of view, since this operation exchanges  $\vec{X}$  and  $\vec{Y}$  and therefore exchanges Higgs and Coulomb branches. So the conclusion is that the relation between  $\mathfrak{B}$  and  $\mathfrak{B}^\vee$  is that  $\mathfrak{B}'$  is the mirror of  $\mathfrak{B}^\vee$ .

In the case that  $\mathfrak{B}$  and  $\mathfrak{B}^\vee$  are realized by branes, the brane manipulations that relate them are equivalent, after adding an extra fivebrane at  $y = L$  (to give Dirichlet or Neumann boundary conditions in figure 12), to the brane manipulations used to show mirror symmetry between  $\mathfrak{B}'$  and  $\mathfrak{B}^\vee$ .

Now let us assess what we actually learn from this construction. To understand *S*-duality of a boundary condition  $\mathfrak{B}$ , what we really want is to describe the boundary SCFT  $\mathfrak{B}^\vee$  associated to the dual boundary condition  $\mathfrak{B}^\vee$ . By coupling  $\mathfrak{B}$  to a three-dimensional  $U(1)$  gauge field  $B$ , we have an explicit way to construct  $\tilde{\mathfrak{B}}^\vee$ , the mirror of  $\mathfrak{B}^\vee$ , starting directly with  $\mathfrak{B}$ . From an abstract point of view, constructing the mirror  $\tilde{\mathfrak{B}}^\vee$  is just as good as constructing  $\mathfrak{B}^\vee$ . We simply declare that  $\mathfrak{B}^\vee$  is the same as  $\tilde{\mathfrak{B}}^\vee$  but with Higgs and Coulomb branches exchanged.

To make this answer useful, we want to make concrete how to couple a four-dimensional  $U(1)$  gauge theory, with gauge field  $A$ , to  $\mathfrak{B}^\vee$ . We must couple  $A$  to the  $U(1)$  symmetry of the Higgs branch of  $\mathfrak{B}^\vee$ , but by definition the Higgs branch of  $\mathfrak{B}^\vee$  is the Coulomb branch of  $\tilde{\mathfrak{B}}^\vee$ . The  $U(1)$  isometry of the Coulomb branch of  $\tilde{\mathfrak{B}}^\vee$  comes from a shift in the scalar dual to the three-dimensional gauge field  $B$ . The conserved current generating this symmetry is simply  $J = \star dB$ , and the coupling  $A_\mu J^\mu$  of another gauge field  $A$  to this current is simply a Chern–Simons-like coupling  $A \wedge dB$ . In the present case, since  $A$  is a four-dimensional gauge field, while  $B$  is defined only on the boundary, the appropriate interaction is a coupling between  $B$  and the restriction of  $A$  to the boundary:

$$\frac{1}{2\pi} \int_{y=0} A \wedge dB. \tag{3.2}$$

This coupling has an extension that has  $\mathcal{N} = 4$  supersymmetry in the three-dimensional sense.

To give this description of the dual boundary theory  $\mathfrak{B}^\vee$ , we did not have to come to grips with mirror symmetry in any serious way. The reason

for this was that in abelian gauge theory, the symmetries of the Coulomb branch are visible classically and one can write the explicit classical coupling (3.2). Non-abelian gauge theory is completely different; there is a procedure similar to what we have just described for finding the mirror of  $\mathfrak{B}^\vee$ , but the relevant symmetries of its Coulomb branch are hard to see without coming to grips with mirror symmetry.

With some care about the definitions, the exchange of Dirichlet and Neumann boundary conditions can also be understood as a special case of the above procedure. We will say more about this in the non-abelian case, where the exceptions are more numerous.

As a final comment, we want to explain why the above description of the action of  $S$ -duality on boundary conditions is consistent with the fact that  $S^2$  should be the identity. The operation that goes from  $\mathfrak{B}$  to  $\mathfrak{B}'$  can be described as follows.  $\mathfrak{B}$  by definition is a three-dimensional theory with a  $U(1)$  symmetry, generated by a current that we may call  $J$ . To define  $\mathfrak{B}'$ , we gauge the  $U(1)$  symmetry of  $\mathfrak{B}$  by coupling  $J$  to a  $U(1)$  gauge field  $B$ . This gives a new theory  $\mathfrak{B}'$  with a new conserved current  $J' = \star dB$  and therefore also a  $U(1)$  symmetry. The operation that goes from  $(\mathfrak{B}, J)$  to  $(\mathfrak{B}', J')$  has been considered before [22]; its square is indeed the identity.

### 3.2 $U(2)$ examples

For  $G = U(2)$ , we can construct three kinds of examples from branes.  $U(2)$  may be completely broken by the interaction with D5-branes, it may be broken to a  $U(1)$  subgroup that then couples to a boundary theory, or it may couple directly to a boundary theory. We will analyze all three cases.

#### 3.2.1 Complete breaking by D5-branes

As reviewed in [1], a configuration of two D3-branes ending on the same D5-brane (figure 13(a)) leads to a boundary condition for  $U(2)$  gauge theory in which the scalar fields  $\vec{X}(y)$  have a singularity at  $y = 0$  of the form  $\vec{X} \sim \vec{t}/y$ , where  $\vec{t}$  are the images of standard  $\mathfrak{su}(2)$  generators under the obvious embedding  $\rho: \mathfrak{su}(2) \rightarrow \mathfrak{u}(2)$ . This breaks  $U(2)$  to its center  $U(1)$  and the  $U(1)$  gauge fields obey Dirichlet boundary conditions. The  $S$ -dual of this corresponds to Neumann boundary conditions (figure 13(b)), with two D3-branes ending on a single NS5-brane. Both of these boundary conditions respect the factorization of the  $U(2)$  theory as a local product of  $SU(2)$  and  $U(1)$  theories. The boundary conditions for  $U(1)$  are the usual Dirichlet/Neumann pair, while for  $SU(2)$ , the boundary condition on one side is determined by the Nahm pole, and on the other side is Neumann.

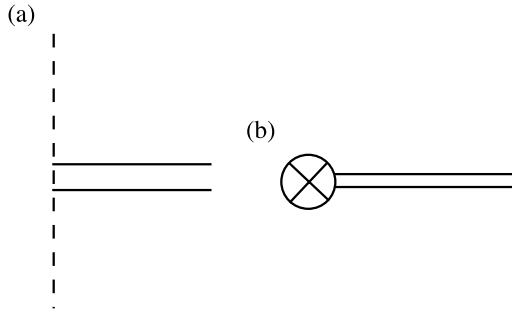


Figure 13: (a) Two D3-branes ending on a single D5-brane, leading to a boundary condition involving a pole in  $\vec{X}$ . (b) The dual configuration: two D3-branes ending on a single NS5-brane, leading to Neumann boundary conditions.

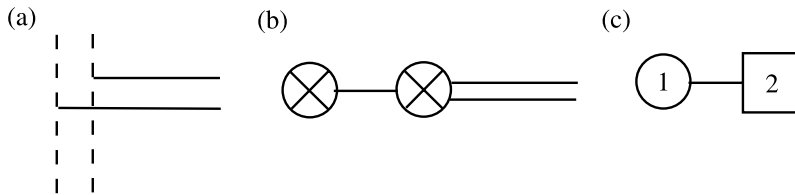


Figure 14: (a) Two D3-branes ending on two D5-branes. (b) The  $S$ -dual configuration, which describes  $U(2)$  gauge theory coupled to a non-trivial boundary SCFT. To establish the  $S$ -duality, one observes that in each picture, all linking numbers are 1. (c) The boundary SCFT is  $T(SU(2))$ , the infrared limit of  $U(1)$  coupled to two charged hypermultiplets (which in (b) arise from interaction of the D3-brane segment with the semi-infinite D3-branes).

To get Dirichlet boundary conditions for the full  $U(2)$  group, we need to consider a configuration in which two D3-branes end on two D5-branes (figure 14(a)). As usual, we displace the D5-branes slightly in the  $y$  direction, and then the  $S$ -dual configuration is straightforward to describe (figure 14(b)). It does not consist of standard Neumann boundary conditions for  $U(2)$  gauge theory, which we already encountered in the last paragraph. Rather, we get Neumann boundary conditions modified by coupling to a certain boundary SCFT. The relevant SCFT can be read off from the figure. It is the infrared limit of  $U(1)$  coupled to two charge 1 hypermultiplets, and thus is actually the self-mirror theory  $T(SU(2))$  that we have already encountered more than once.

Several remarks are in order:

(1) The flavor symmetry of  $U(1)$  with two hypermultiplets of charge 1 is  $SU(2)$ , not  $U(2)$  (a  $U(1)$  flavor rotation is equivalent to a gauge transformation). Accordingly, the flavor symmetry of the Higgs branch of  $T(SU(2))$  is  $SU(2)$  rather than  $U(2)$ , as is also clear from the fact that this branch is  $\mathbb{R}^4/\mathbb{Z}_2$ . As the theory is self-mirror,  $SU(2)$  is also the symmetry of the Coulomb branch, which is also  $\mathbb{R}^4/\mathbb{Z}_2$ , so the full global symmetry is  $SU(2) \times SU(2)$ . So the  $U(1)$  part of  $U(2) = SU(2) \times U(1)$  does not couple to  $T(SU(2))$  and in the above construction it simply obeys Neumann boundary conditions. The coupling of the  $SU(2)$  gauge theory to  $T(SU(2))$  removes one of the  $SU(2)$  symmetries of that theory. The other one matches with the global symmetry of  $SU(2)$  gauge theory with Dirichlet boundary conditions.

(2) We recall from [1] that the  $U(2)$  gauge theory with Dirichlet boundary conditions, even after one fixes  $\vec{X}$  and  $\vec{Y}$  to vanish at infinity, has a moduli space of vacua given by the space of solutions of Nahm's equations. Moreover, this space is the nilpotent cone of  $SL(2, \mathbb{C})$ , or equivalently is the  $A_1$  singularity  $\mathbb{R}^4/\mathbb{Z}_2$ . In the present construction, that moduli space matches the Coulomb branch of the boundary theory  $T(SU(2))$ . Before  $S$ -duality,  $\mathbb{R}^4/\mathbb{Z}_2$  is the moduli space of solutions of Nahm's equations, and after  $S$ -duality, it is the Coulomb branch of the boundary theory.

### 3.2.2 Breaking to $U(1)$

To construct a boundary condition  $\mathcal{B}$  that reduces the gauge group from  $U(2)$  to  $U(1)$ , we end one D3-brane on a D5-brane and let the second D3-brane end on a brane system which defines Neumann boundary conditions, possibly coupled to a boundary theory  $\mathfrak{B}_{U(1)}$  with  $U(1)$  symmetry. In figure 15, we denote this generic boundary system by the symbol  $\oplus$ , and we write  $\oplus'$  for its  $S$ -dual. We assume that  $\oplus'$  is  $U(1)$  coupled to some boundary theory  $\mathfrak{B}_{U(1)}^\vee$  with  $U(1)$  symmetry.<sup>10</sup> To get the  $S$ -dual of  $\mathcal{B}$ , we apply  $S$ -duality to the whole picture, arriving at figure 15(b). Now the boundary theory is a composite, which we will call  $\mathfrak{B}_{U(2)}^\vee$ ; it is obtained (figure 15(c)) by coupling a three-dimensional  $U(1)$  gauge theory both to  $\mathfrak{B}_{U(1)}^\vee$  and to two hypermultiplets of charge 1, which also have a  $U(2)$  flavor symmetry. In this discussion, we did not really need to know that  $\oplus$  can be realized by branes; it could be any boundary condition in  $U(1)$  gauge theory, with  $\oplus'$  as the dual boundary condition.

<sup>10</sup> As explained at the end of Section 3.1.1, this is so except in one case: if  $\oplus$  corresponds to pure Neumann boundary conditions with no boundary matter fields, then its dual is Dirichlet. We will discuss this exceptional case in Section 3.2.3.

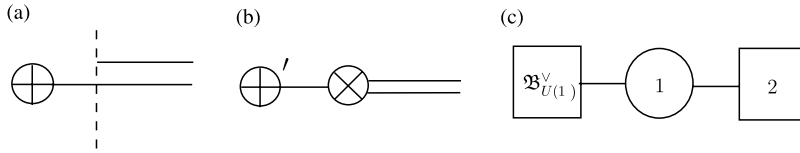


Figure 15: (a) Two D3-branes, one of which ends on a D5-brane, while the second then ends with Neumann boundary conditions coupled to some matter system. This boundary condition is indicated by the symbol  $\oplus$ . (b) The dual configuration;  $\oplus'$  is the  $S$ -dual of the boundary condition  $\oplus$  in  $U(1)$  gauge theory. (c) A schematic representation of the dual boundary condition as a quiver.  $\mathfrak{B}_{U(1)}^{\vee}$  is the boundary theory associated with the boundary condition  $\oplus'$  in  $U(1)$  gauge theory.

As in our analysis of  $U(1)$  theories, a convenient way to study the theory  $\mathfrak{B}_{U(2)}^{\vee}$  is to ungauged the  $U(2)$  symmetry, by terminating the two D3-branes on the right on a pair of D5-branes, giving Dirichlet boundary conditions (figure 16(a)). Since we have ungauged the symmetry, the infrared limit of this configuration is simply the three-dimensional theory  $\mathfrak{B}_{U(2)}^{\vee}$ . The  $S$ -dual configuration is shown in figure 16(b) and after a brane rearrangement to obey our rules, we arrive at figure 16(c). The infrared limit of this configuration gives the mirror  $\tilde{\mathfrak{B}}_{U(2)}^{\vee}$  of  $\mathfrak{B}_{U(2)}^{\vee}$ . It is again a composite theory, sketched in figure 16(d). A simple interpretation of this theory is that we have coupled a  $U(1)$  gauge field  $C$  to  $\mathfrak{B}_{U(1)}$  (the original boundary theory associated with the configuration  $\oplus$ ) and to a  $U(1)$  subgroup of the flavor symmetry of  $T(SU(2))$ . The Coulomb branch of the composite theory has a  $U(1)$  global symmetry (translations of the scalar field dual to  $C$ ) and an  $SU(2)$  global symmetry (acting on the Coulomb branch of  $T(SU(2))$ ). These combine to the  $U(2)$  symmetry of  $\mathfrak{B}_{U(2)}^{\vee}$ .

The theory  $\tilde{\mathfrak{B}}_{U(2)}^{\vee}$  is explicitly constructed and the desired boundary condition is constructed from its mirror  $\mathfrak{B}_{U(2)}^{\vee}$ . We could simply define  $\mathfrak{B}_{U(2)}^{\vee}$  as  $\tilde{\mathfrak{B}}_{U(2)}^{\vee}$  with the Higgs and Coulomb branches exchanged. However, to explicitly construct a boundary condition in  $U(2)$  gauge theory, we need to be able to see the  $U(2)$  currents that act on the Coulomb branch of  $\mathfrak{B}_{U(2)}^{\vee}$ . Unfortunately, in the construction of  $\mathfrak{B}_{U(2)}^{\vee}$ , only the currents of a Cartan subalgebra are realized as classical symmetries; the other currents are monopole or vortex operators [6, 7], as described in Section 2.4. Though later in the paper, this will be a problem in making some of the constructions explicit, in the present example, we can circumvent the problem with a trick. In figure 16(d), we have gauged a  $U(1)$  symmetry of the Higgs branch of  $T(SU(2))$ . Since  $T(SU(2))$  is self-mirror, we could equivalently have gauged

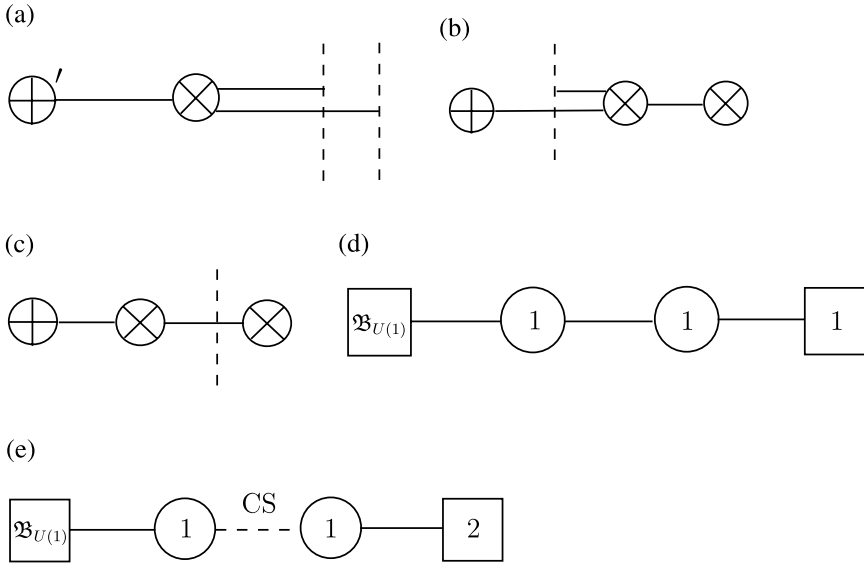


Figure 16: (a) The three-dimensional theory  $\mathfrak{B}_{U(2)}^\vee$  can be recovered by ungauging the gauge symmetry with the help of Neumann boundary conditions. (b) The naive  $S$ -dual configuration. (c) A well-ordered rearrangement; the D5-brane has been moved to the right to satisfy the usual conditions. (d) A quiver that schematically represents the configuration of (c); its infrared limit is the mirror of  $\mathfrak{B}_{U(2)}^\vee$ . The symbol  $\oplus$  has been replaced by a coupling to the corresponding boundary theory  $\mathfrak{B}_{U(1)}$ . (e) A variant of the quiver that makes the global symmetries visible. The dotted line indicates a Chern–Simons coupling between the two  $U(1)$  theories represented by the circles. (This coupling is the supersymmetric completion of equation 3.2.)

a  $U(1)$  symmetry of its Coulomb branch. This we can do explicitly, since the abelian symmetries of the Coulomb branch are visible classically, and when we do that, we are left with the visible  $SU(2)$  symmetries of the Higgs branch. The resulting description of  $\mathfrak{B}_{U(2)}^\vee$  is sketched in figure 16(e).

For  $G = U(1)$ , we had an analogous description of the dual boundary condition involving mirror symmetry. There was no problem because the symmetries of the Coulomb branch were visible classically.

### 3.2.3 Full $U(2)$ gauge symmetry at the boundary

The most basic boundary condition with full  $U(2)$  gauge symmetry at the boundary is Neumann. We have already encountered this boundary

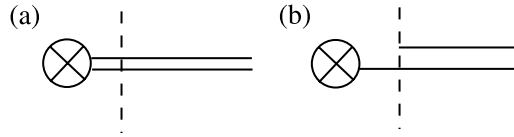


Figure 17: (a) This configuration leads to  $U(2)$  gauge theory coupled to a fundamental hypermultiplet at the boundary. (b) The  $S$ -dual configuration, in which one D3-brane ends on a D5-brane, reducing the gauge symmetry to  $U(1)$ , and then the second ends on an NS5-brane, leading to Neumann boundary conditions.

condition as the dual of the Nahm pole. We want to explore what happens if we add matter at the boundary.

As in our treatment of  $G = U(1)$ , we begin by adding a fundamental hypermultiplet at the boundary. The  $S$ -dual can be found in the standard way (figure 17) and turns out to be the exceptional case mentioned in footnote 10 of a theory in which the gauge symmetry is reduced to  $U(1)$  at the boundary and the boundary theory  $\mathfrak{B}_{U(1)}^\vee$  is trivial (so that the  $U(1)$  vector multiplet simply obeys Neumann boundary conditions).

In figure 16(d), we gave a general recipe for analyzing the  $S$ -dual of any  $U(2)$  boundary condition that involves reduction of gauge symmetry to  $U(1)$  and coupling to a *non-trivial* boundary theory  $\mathfrak{B}_{U(1)}$  with  $U(1)$  symmetry. (The role of non-triviality is to avoid the exceptional behavior mentioned in footnote 10.) Now that we know what the  $S$ -dual is for the case that  $\mathfrak{B}_{U(1)}$  is trivial, it is interesting to compare the answer to what we would get from the general recipe. When  $\mathfrak{B}_{U(1)}$  is trivial, figure 16(d) reduces to an ugly quiver theory in the sense of Section 2.4, and as explained there, it is equivalent to a theory of two free twisted hypermultiplets, which parametrize its Coulomb branch. These free twisted hypermultiplets have  $U(2)$  (and in fact  $Sp(4)$ ) symmetry, and can be identified with the fundamental hypermultiplets that we started with in figure 17(a). So, though it involves grappling with an ugly quiver, the general recipe works even in the exceptional case that does not quite fit the original discussion.

Another exceptional case is  $U(2)$  coupled to two fundamental hypermultiplets at the boundary (figure 18). The  $S$ -dual configuration, shown in (b), corresponds to Neumann boundary conditions for  $U(2)$  coupled to a boundary SCFT  $\mathfrak{B}^\vee$  that is a product of two factors. One factor is  $T(SU(2))$ , and the other consists of a fundamental hypermultiplet.

Before completing the analysis of those two exceptional cases, let us look at a more generic case, such as a boundary coupling to  $k > 2$  fundamental

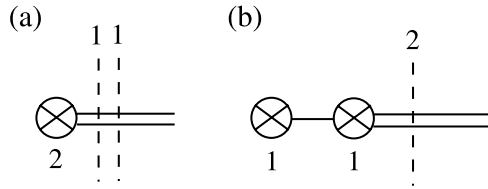


Figure 18: (a)  $U(2)$  gauge theory with two fundamental hypermultiplets at the boundary. Linking numbers are indicated. (b) The dual configuration. The boundary SCFT is the decoupled sum of  $T(SU(2))$  coming from the left of the picture, and a fundamental hypermultiplet of the bulk  $U(2)$ , coming from the D3-D5 intersection.

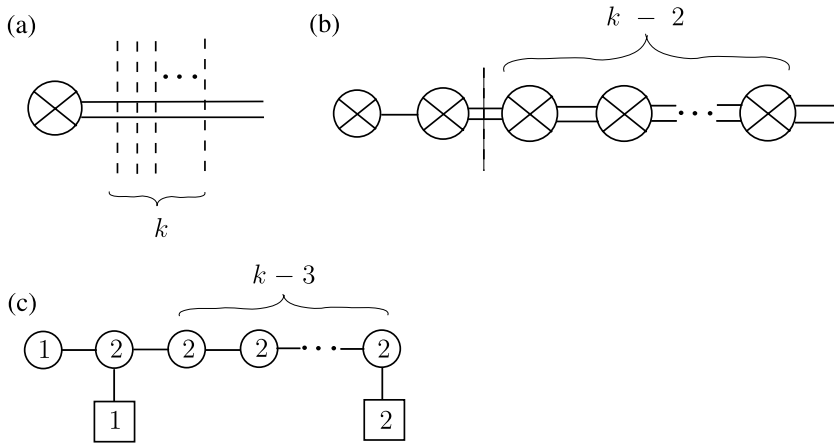


Figure 19: (a)  $U(2)$  gauge theory coupled to  $k$  fundamental hypermultiplets at the boundary. (b) The dual configuration, which is related to a boundary SCFT  $\mathfrak{B}^\vee$ . (c) The quiver corresponding to  $\mathfrak{B}^\vee$ .

hypermultiplets (figure 19). The dual boundary condition can be found in the usual way and corresponds to a boundary SCFT  $\mathfrak{B}^\vee$  that can be represented by the quiver of figure 19(c).

The mirror to  $\mathfrak{B}^\vee$  can also be found in the familiar way. Starting with  $\mathfrak{B}^\vee$  coupled to bulk  $U(2)$  gauge fields (as in figure 19(b)), we ungauged the  $U(2)$  gauge symmetry by introducing a second boundary with Dirichlet boundary conditions (figure 20(a)), and take the  $S$ -dual (figure 20(b)). We arrive at a remarkably simple quiver description of the mirror  $\tilde{\mathfrak{B}}^\vee$  (figure 20(c)). It has a very simple interpretation: it is described by a three-dimensional  $U(2)$  gauge theory coupled to the original theory  $\mathfrak{B}$  of  $k$  free hypermultiplets, and to  $T(SU(2))$ . (The node labeled 2 in the quiver represents a  $U(2)$  gauge field which couples to a bifundamental hypermultiplet of  $U(2) \times U(1)$  — part of the definition of  $T(SU(2))$  — and to  $k$  fundamental hypermultiplets.)



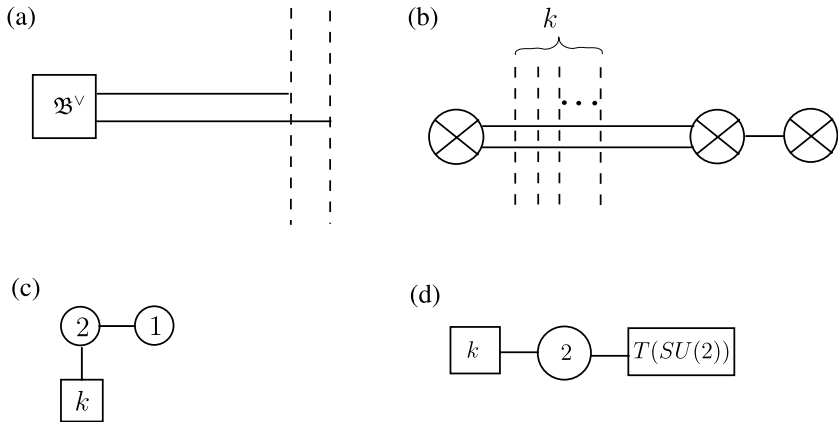


Figure 20: To find the mirror to the dual boundary condition  $\mathfrak{B}^\vee$ , we ungauged the symmetry, using Dirichlet boundary conditions (a), then take the  $S$ -dual, which is depicted in (b) after a brane rearrangement, and can be represented as a quiver (c). This quiver describes the mirror of  $\mathfrak{B}^\vee$ . It is the mirror of the quiver in figure 19(c), as one can verify by computing the linking numbers. As is shown schematically in (d), this quiver couples  $U(2)$  gauge theory to the product of  $T(SU(2))$  and  $k$  hypermultiplets.

So there is a simple general prescription to go from the starting theory  $\mathfrak{B}$  to  $\tilde{\mathfrak{B}}^\vee$ , the mirror of the desired boundary theory  $\mathfrak{B}^\vee$ . We simply couple  $\mathfrak{B}$  to  $T(SU(2))$  via  $U(2)$  gauge fields, as in figure 20(c). We call the theory made this way the composite gauge theory and denote it as  $\mathfrak{B} \times_{U(2)} T(SU(2))$ :

$$\tilde{\mathfrak{B}}^\vee = \mathfrak{B} \times_{U(2)} T(SU(2)). \tag{3.3}$$

Then the “answer”  $\mathfrak{B}^\vee$ , which determines the  $S$ -dual boundary condition, is the mirror of  $\tilde{\mathfrak{B}}^\vee$ . The only trouble with this answer is that to understand  $\mathfrak{B}^\vee$  as a theory with  $U(2)$  global symmetry (so that we can couple it to  $U(2)$  gauge fields), we need to be able to see the  $U(2)$  symmetries of the Coulomb branch of  $\mathfrak{B}^\vee$ . As usual, only a Cartan subalgebra is visible classically. (In any three-dimensional gauge theory in which the center of the gauge group has rank  $r$ , a  $U(1)^r$  symmetry of the Coulomb branch is visible classically, acting by shifts of the dual photons. In the present example,  $r = 2$  as the relevant quiver has two nodes. For more on this, see Section 2.4.)

If we could find a realization of  $T(SU(2))$  with manifest  $SU(2) \times SU(2)$  global symmetry, then using this in figure 20(c) would give an explicit way to construct the dual boundary condition. This is not available at the moment. However, if the original boundary theory  $\mathfrak{B}$  is constructed via branes, then

the usual  $D$ -brane manipulations give a construction of the mirror to  $\mathfrak{B}^\vee$  with the relevant symmetries visible, as in figure 19(c) for the case that the starting theory consists of  $k$  fundamental hypermultiplets.

As long as  $k \geq 3$ , the quiver that we have arrived at in figure 20(c) obeys the constraint  $n_f \geq 2n_c$  at each node and is a good quiver in the sense of Section 2.4. This is the condition under which a quiver gauge theory gives the most economical possible description of whatever infrared critical point it describes. For  $k = 2$ , it is an ugly quiver with one minimally unbalanced node and therefore describes free twisted hypermultiplets, times an additional SCFT. This is reflected in the fact that for  $k = 2$ , the  $S$ -dual configuration of figure 18(b) involves coupling to a free fundamental hypermultiplet, times  $T(SU(2))$ . (The appearance of  $T(SU(2))$  can be argued by further analysis of the ugly quiver, but we will omit this.)

For  $k < 2$ , the quiver is a bad one, with operators of  $q_R \leq 0$  according to Section 2.4. Not coincidentally, this is also the case that the  $S$ -dual boundary condition has reduced gauge symmetry, possibly with a Nahm pole. When this is so, our derivation of the quiver is not valid; the starting point of this derivation was to assume that the dual boundary condition has full  $U(2)$  gauge symmetry. Indeed, the first step in the derivation was to ungauged the  $U(2)$  symmetry by means of Dirichlet boundary conditions. But for  $k < 2$ , the dual boundary condition has reduced gauge symmetry, as we have seen.

In Sections 3.4 and 4.3.1, we will give another derivation of the recipe using  $T(SU(2))$  which works for any starting boundary condition. The details will be more subtle. The main point is that as we flow to the infrared, the  $U(2)$  isometries of  $\mathfrak{B} \times_{U(2)} T(SU(2))$  may be spontaneously broken.<sup>11</sup> As a result, the  $S$ -dual  $U(2)$  gauge symmetry will be Higgsed at the boundary, and reduced to a subgroup in the infrared. Moreover, if the  $U(2)$  isometry at the boundary is spontaneously broken, the hyper-Kähler moment maps of the isometry group may acquire non-zero expectation values, which are dimensionful. The boundary conditions  $\vec{X}| + \vec{\mu} = 0$  will then force the scalar fields  $\vec{X}$  to acquire non-zero boundary values which diverge as one flows to the infrared, leading to a Nahm pole.

As a preview of this, we will discuss this symmetry breaking scenario for our bad quivers with  $k < 2$ . These quivers have no Higgs branch (not even a mixed Coulomb–Higgs branch). They each have a Coulomb branch,

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<sup>11</sup>Such spontaneous breaking cannot occur in a theory that has a standard infrared critical point, as such a point is always invariant under continuous global symmetries. The coupling of a conserved current to a Goldstone boson would violate conformal invariance.

of hyper-Kähler dimension 3. For the question of whether the  $U(2)$  global symmetries of the Coulomb branch are spontaneously broken to make any sense, these symmetries have to be present. This is the case only when one takes the strong coupling limit on the  $U(1)$  node of the quiver, so as to generate  $T(SU(2))$ . So we take that limit, and then analyze the  $U(2)$  gauge dynamics at the remaining node. We claim that this  $U(2)$  gauge dynamics leads to a spontaneous breaking of the  $U(2)$  global symmetry of the Coulomb branch. If the claim is correct, it is best to analyze the gauge dynamics at finite gauge coupling, since a spontaneously broken global symmetry is simply lost in an IR limit. The Coulomb branch of the bad quivers with finite  $U(2)$  gauge coupling can be analyzed by Nahm's equations, as explained in figure 21, and the expected breaking of global symmetries does occur.

Indeed, the bad quiver gauge theories with  $k = 0$  and  $k = 1$  can be represented by the brane configurations of figure 21(a) and (c). Their Coulomb branches are most conveniently determined by solving Nahm's equations in the  $S$ -dual brane configurations of figure 21(b) and (d). In solving Nahm's equations, we require  $\vec{X}(y)$  to have a Nahm pole at  $y = 0$ , where two D3-branes end on a single D5-brane. In the limit of strong coupling on the  $U(1)$  node,  $\vec{X}(y)$  obeys Dirichlet boundary conditions at the other endpoint  $y = L$ . The existence of the pole at  $y = 0$  makes it impossible for  $\vec{X}$  to vanish at  $y = L$ , so the  $U(2)$  global symmetry is spontaneously broken. For  $k = 0$ , the Nahm pole breaks the symmetry to its center, but for  $k = 1$ , because  $\vec{X}$  jumps in crossing the NS5-brane (see Section 3.9.3 of [1], at the end of which the relevant solution is described),  $\vec{X}(L)$  can have rank 1, giving a solution invariant under a non-central  $U(1)$  subgroup of  $U(2)$  (consisting of matrices of the form  $\text{diag}(*, 1)$ ). This is indeed the unbroken symmetry of the  $S$ -dual boundary condition for  $k = 1$ , as we recall from figure 17. For  $k \geq 2$ ,  $\vec{X}$  would jump in crossing two successive NS5-branes, and can vanish at  $y = L$ , leaving the global symmetry unbroken, despite the Nahm pole at  $y = 0$ .

### 3.3 $U(n)$ examples

Examples with  $n$  D3-branes leading to  $U(n)$  gauge theory can be studied in much the same way, but the details are richer.

#### 3.3.1 The dual of Dirichlet boundary conditions

Our first example in  $U(n)$  gauge theory will be a simple Dirichlet boundary condition. This is realized by ending each of  $n$  D3-branes on a distinct D5-brane. The steps to the  $S$ -dual configuration are shown in figure 22. Each D5-brane has linking number 1. The  $S$ -dual boundary condition couples the

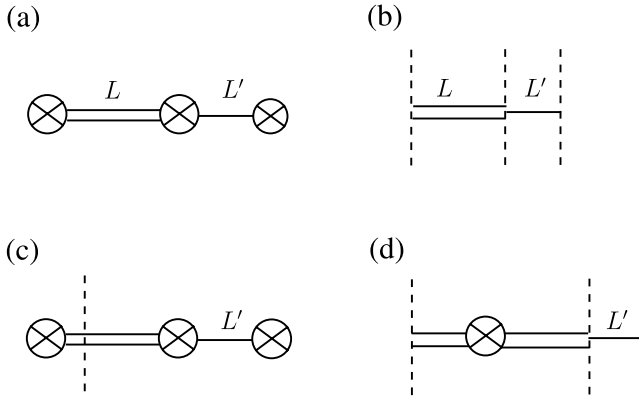


Figure 21: (a) The  $k = 0$  case of the bad quiver discussed in the text can be studied via this brane configuration. The full  $U(2)$  symmetry of the Coulomb branch arises when  $L' \rightarrow 0$  with fixed  $L$ . (b) The  $S$ -dual of (a). The expected  $U(2)$  global symmetry arises when  $L' \rightarrow 0$  and the two D5-branes are coincident. For  $L' \rightarrow 0$ , the desired Coulomb branch of (a) is the moduli space of solutions of Nahm's equations on an interval, with a rank two Nahm pole at  $y = 0$  (where two D3-branes end on one D5-brane) and Dirichlet boundary conditions at  $y = L$  (where two D3-branes end on two D5-branes). The existence of a Nahm pole at  $y = 0$  means that  $\vec{X}$  cannot vanish at  $y = L$ , and breaks the global  $U(2)$  symmetry (which acts on  $\vec{X}$  by conjugation) to its center. (c) The  $k = 1$  bad quiver corresponds to this brane configuration. (d) The Coulomb branch in (c), for  $L' \rightarrow 0$ , can again be obtained by solving Nahm's equations with a Nahm pole in the left and Dirichlet boundary conditions on the right. But now  $\vec{X}$  can jump in crossing the NS5-brane, leaving unbroken a  $U(1)$  subgroup of  $U(2)$  consisting of elements of the form  $\text{diag}(*, 1)$ .

$U(n)$  gauge fields to a special boundary theory  $T(SU(n))$  which generalizes our friend  $T(SU(2))$ .  $T(SU(n))$  is given by the IR limit of the quiver gauge theory in figure 22(c).

Like  $T(SU(2))$ , the theory  $T(SU(n))$  is self-mirror. To show this, we observe that  $T(SU(n))$  can be derived from the purely three-dimensional configuration shown in figure 23(a). The linking numbers of the D5-branes are all  $n - 1$ , while the linking numbers of the NS5-branes are all 1. As a consequence, the mirror is given by the same brane configuration written backwards, as in figure 23(b). This establishes the mirror symmetry. Finally, by separating all the NS5-branes from the D5-branes, it is possible to make the self-duality of the configuration more manifest as in figure 23(c), at the

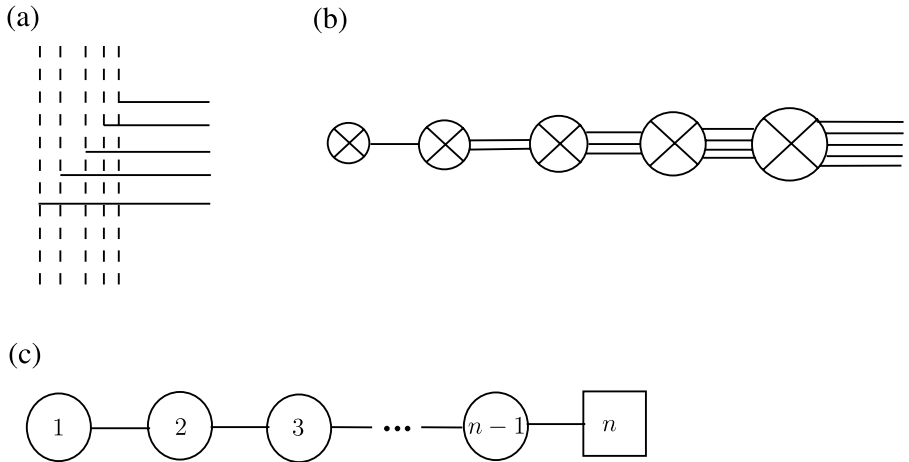


Figure 22: (a) Dirichlet boundary conditions in  $U(n)$  gauge theory arise from  $n$  D3-branes ending one at a time on  $n$  D5-branes, as sketched here for  $n = 5$ . (b) The  $S$ -dual is obtained by simply converting D5-branes to NS5-branes. There is no need for any rearrangement. (c) This leads to Neumann boundary conditions coupled to the SCFT that is the IR limit of this quiver.

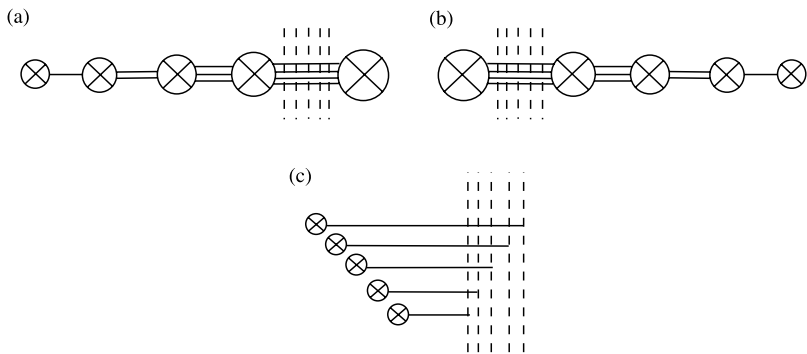


Figure 23: (a) A brane configuration bounded in the  $y$ -direction that leads to the theory  $T(SU(n))$ , sketched here for  $n = 5$ . D5-branes all have linking number  $n - 1$  and NS5-branes all have linking number 1. (b) The mirror is the same quiver written backwards. (c) The self-mirror property becomes manifest if we move all NS5-branes to the left and all D5-branes to the right. To make the picture visible, the NS5-branes and D5-branes have been displaced; a  $U(n) \times U(n)$  global symmetry (whose center acts trivially) appears when they become coincident. But this configuration has no interpretation in gauge theory.

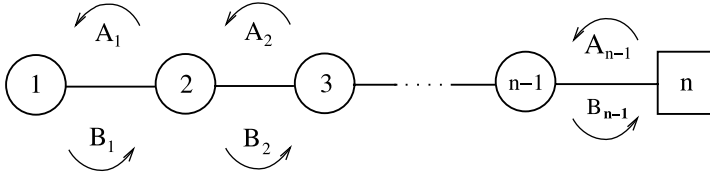


Figure 24: The quiver that leads to  $T(SU(n))$ , with chiral multiplets labeled.

price of obscuring the physical content of the theory, since this configuration has no gauge theory interpretation.

Because of the self-mirror property of  $T(SU(n))$ , its Higgs and Coulomb branches are isomorphic. By  $S$ -duality, the Coulomb branch of  $T(SU(n))$  is the same as the moduli space of solutions of the Nahm equations on the half line  $y \geq 0$  with Dirichlet boundary conditions at  $y = 0$  and prescribed behavior of  $\vec{X}$  at infinity. This moduli space is a hyper-Kähler manifold first introduced by Kronheimer [23]. In any one of its complex structures, the manifold is isomorphic to the nilpotent cone  $\mathcal{N}$  of  $SL(n, \mathbb{C})$ .  $\mathcal{N}$  is defined as the subspace of nilpotent elements of the  $SL(n, \mathbb{C})$  Lie algebra. It is a complex symplectic manifold of dimension  $n(n - 1)$ . For an explanation of these matters, and many other topics that are relevant here (including Nahm’s equations and Slodowy slices), see Section 3 of [1].

It is instructive to see directly that the Higgs branch of the quiver in figure 22(c) coincides with  $\mathcal{N}$  as a complex manifold. Picking a specific complex structure, each hypermultiplet splits into a pair of chiral multiplets in conjugate representations of the gauge group. Labeling these chiral multiplets as in figure 24, they consist of  $i \times (i + 1)$  matrices  $A_i$ ,  $i = 1, \dots, n - 1$ , transforming as  $\bar{\mathbf{i}} \otimes (\mathbf{i} + \mathbf{1})$  under  $U(i) \times U(i + 1)$ , and  $(i + 1) \times i$  matrices  $B_i$  transforming in the conjugate representation. Here  $U(i)$ ,  $i < n$  are the gauge groups at the nodes of the quiver, and  $U(n)$  is the global symmetry group acting at the end of the quiver. The center of  $U(n)$  acts trivially, so the flavor symmetry of the quiver theory is really  $SU(n)$ . The composite field  $M = B_{n-1}A_{n-1}$  is an  $n \times n$  matrix of rank at most  $n - 1$ . The traceless part of  $M$  is the complex moment map for the  $SU(n)$  flavor symmetry, and will parametrize the Higgs branch.

Using the  $F$ -term constraints

$$\begin{aligned} A_{i+1}B_{i+1} &= B_iA_i, \quad i = 1, \dots, n - 2, \\ A_1B_1 &= 0, \end{aligned} \tag{3.4}$$

we can compute

$$M^2 = B_{n-1}A_{n-1}B_{n-1}A_{n-1} = B_{n-1}B_{n-2}A_{n-2}A_{n-1} \tag{3.5}$$

and discover that  $M^2$  has rank at most  $n - 2$ . Iteratively,

$$M^a = \prod_{i=n-1}^{n-a} B_i \prod_{j=n-a}^{n-1} A_j, \tag{3.6}$$

and has rank at most  $n - a$ . Finally,  $M^n = 0$ , so  $M$  is nilpotent. Any nilpotent  $n \times n$  matrix will satisfy the rank constraints  $\text{rk } M^a \leq n - a$ , and it is not hard to show that for any nilpotent  $M$ , a set of matrices  $A_i, B_i$ , unique up to gauge transformations, can be found satisfying the above conditions. So the Higgs branch coincides with the nilpotent cone  $\mathcal{N}$ .

We can also introduce  $n - 1$  possible FI terms  $t_i$  for the center of the gauge group. This deforms the  $F$ -term constraints to

$$\begin{aligned} A_{i+1}B_{i+1} &= B_iA_i + t_{i+1}, & i = 1, \dots, n - 2 \\ A_1B_1 &= t_1. \end{aligned} \tag{3.7}$$

The above analysis changes only slightly. Starting with

$$M^2 = B_{n-1}A_{n-1}B_{n-1}A_{n-1} = B_{n-1}B_{n-2}A_{n-2}A_{n-1} + t_{n-1}M, \tag{3.8}$$

we derive iteratively

$$M(M - t_{n-1})(M - t_{n-1} - t_{n-2}) \cdots \left( M - \sum_{i=n-a}^{n-1} t_i \right) = \prod_{i=n-1}^{n-a} B_i \prod_{j=n-a}^{n-1} A_j. \tag{3.9}$$

Finally,

$$M(M - t_{n-1})(M - t_{n-1} - t_{n-2}) \cdots \left( M - \sum_{i=1}^{n-1} t_i \right) = 0 \tag{3.10}$$

expresses the characteristic polynomial of  $M$  in terms of the FI parameters. The Higgs branch as a complex manifold is the set of Lie algebra elements with the eigenvalues indicated by the characteristic polynomial.

In the brane realization in figure 23(c), the FI parameters of the quiver gauge theory correspond to the positions of the NS5-branes in the  $\vec{X}$  directions. The positions of the D5-branes in the  $\vec{Y}$  directions correspond to

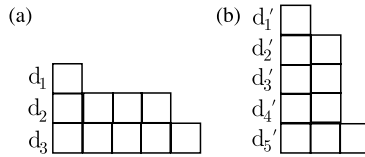


Figure 25: (a) A Young diagram with  $i$ th row of length  $d_i$ . (b) The dual Young diagram, with rows and columns exchanged.

mass parameters for the fundamental hypermultiplets of the quiver. In the original  $D5$  brane boundary condition, the positions of the D3-branes at infinity in the  $\vec{X}$  direction are the FI parameters, as explained in Section 3 of [1], where it is also shown that turning on those parameters deforms the nilpotent cone to a more generic conjugacy class.

### 3.3.2 The Dual of a Nahm Pole

Dirichlet boundary conditions can be generalized by requiring a pole for the scalar fields  $\vec{X}$  at the boundary. Supersymmetry requires that the residue of the pole should be the images of a standard set of  $\mathfrak{su}(2)$  generators  $\vec{t}$  under a homomorphism  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(n)$ .

Let us recall that for every positive integer  $n$ , the Lie algebra  $\mathfrak{su}(2)$  has up to isomorphism a unique irreducible representation of that dimension. Therefore, the choice of  $\rho$  is determined by a decomposition  $n = \sum_{i=1}^k d_i$ , where the  $d_i$  are positive integers that we can assume to be arranged in non-decreasing order. The information contained in this decomposition is conveniently displayed in a Young diagram whose  $i$ th row, counting from top to bottom, has length  $d_i$ , as in figure 25(a).

The choice of  $\rho$  has another interpretation. The image under  $\rho$  of the raising operator in the  $\mathfrak{su}(2)$  Lie algebra is a nilpotent element  $\rho_+$  of  $\mathfrak{gl}(n)$ , the complexification of  $\mathfrak{u}(n)$ . It is the direct sum of nilpotent Jordan blocks of dimension  $d_i$ . The existence of a Jordan canonical form for every matrix means that any nilpotent element of  $\mathfrak{gl}(n)$  is conjugate to this form for some  $d_i$ . Thus, the choice of  $\rho$  is equivalent to the choice of a nilpotent conjugacy class in the complexified Lie algebra. (According to the Jacobson–Morozov Theorem, this has an analog for every semi-simple Lie algebra, not just for  $\mathfrak{u}(n)$ .) One more remark will be helpful. There is a natural duality of Young diagrams in which they are reflected along the main diagonal (figure 25(b)). This gives a duality operation on homomorphisms  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(n)$ . We write  $\rho_D$  for the dual of  $\rho$  in this sense.

We pause to give a few examples (figure 26) because dual pairs will enter our story momentarily, though not in a symmetrical way. The regular or



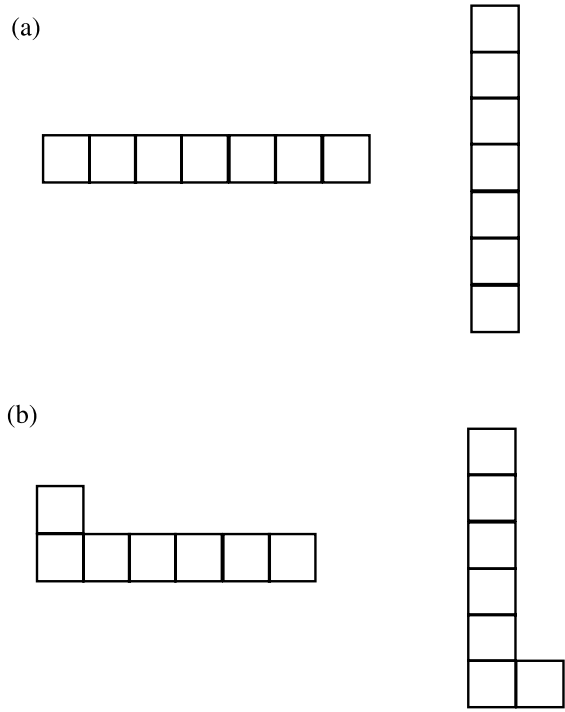


Figure 26: (a) The Young diagram associated with the regular or irreducible representation  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(n)$  (left), and its dual (right), which is associated with the trivial representation. (b) The Young diagram associated with the subregular representation, which corresponds to the decomposition  $n = (n - 1) + 1$ , and its dual, which corresponds to  $n = 2 + 1 + 1 + \dots + 1$ .

irreducible representation  $\rho$ , which corresponds to the case  $k = 1$  and  $d = n$ , is dual to the trivial representation  $\rho = 0$ , with  $k = n$  and all  $d_i = 1$ . If  $\rho$  is the regular representation, then  $\rho_+$  is known as a regular nilpotent element; if  $\rho = 0$ , then  $\rho_+ = 0$ . Similarly, the subregular representation, corresponding to the decomposition  $n = 1 + (n - 1)$ , is dual to the decomposition  $n = 1 + 1 + \dots + 1 + 2$ , related to the most obvious embedding of  $SU(2)$  in  $SU(n)$ . The nilpotent element corresponding to the subregular representation is called a subregular nilpotent element, and the nilpotent element corresponding to the dual representation is called a minimal nilpotent. For more background on these subjects and many other matters that will appear below, see Section 3 of [1].

An embedding which decomposes the fundamental representation of  $U(n)$  via  $n = \sum_{i=1}^k d_i$  can be realized by a boundary condition in which  $n$  D3-branes end on  $k$  D5-branes (figure 27(a)). Numbering the D5-branes from left to right, we take  $d_i$  D3-branes to end on the  $i$ th D5-brane. The linking

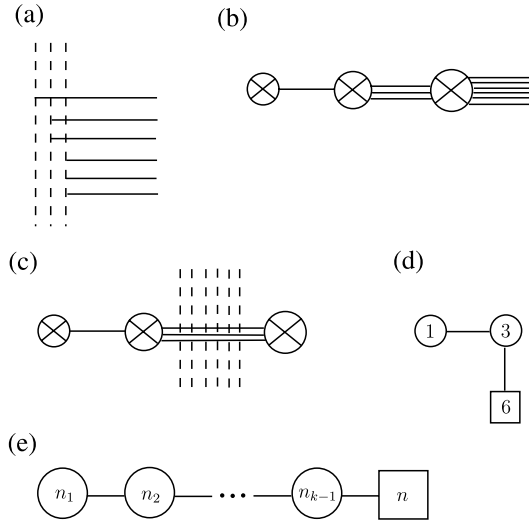


Figure 27: (a) Numbering the  $k$  D5-branes in this picture from left to right,  $d_i$  D3-branes end on the  $i$ th D5-brane. In this example,  $k = 3$  and the  $d_i$  are 1, 2, 3. (b) The  $S$ -dual configuration, and (c) a macroscopically three-dimensional configuration that leads to the same boundary SCFT. (d) The quiver gauge theory that flows in the IR to this boundary SCFT. The gauge group is  $U(n_1) \times U(n_2)$ , where  $n_1 = d_1 = 1$  and  $n_2 = d_1 + d_2 = 3$ . (e) The general quiver of this type, representing the theory  $T_\rho(SU(n))$ .

numbers of the D5-branes are then equal to the  $d_i$ . Since we take the  $d_i$  to be non-increasing, the D5-branes can be separated in the  $y$ -direction without introducing extra degrees of freedom, as we have done in the figure.

In figure 27(d) and (e), we indicate the quiver gauge theory which will flow in the infrared to the SCFT that determines the dual boundary condition. It is a linear quiver of  $k - 1$  unitary groups of rank  $n_i = \sum_{j=1}^i d_j$ . We will denote this SCFT as  $T_\rho(SU(n))$ . If  $\rho$  is trivial, then  $T_\rho(SU(n))$  is the same as  $T(SU(n))$ .

The Higgs branch of this quiver has an  $SU(n)$  flavor symmetry, and can be analyzed in the same fashion as the Higgs branch for the quiver of  $T(SU(n))$ . The matrices  $A_i$  and  $B_i$  can be defined as before, but now have size  $n_i \times n_{i+1}$  and  $n_{i+1} \times n_i$ , respectively. We can define again  $M = B_{k-1}A_{k-1}$ , whose traceless part is the moment map for the  $SU(n)$  action. The  $F$ -term constraints are

$$A_{i+1}B_{i+1} = B_iA_i + t_{i+1}, \quad i = 1, \dots, k - 2, \tag{3.11}$$

$$A_1B_1 = t_1. \tag{3.12}$$

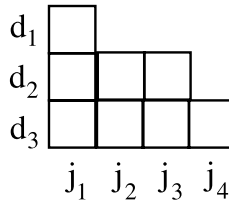


Figure 28: The Young diagram associated to  $\rho$ . The lengths of the rows are the sizes  $d_i$  of the Jordan blocks of  $\rho_+$  (arranged from top to bottom), and the heights of the columns are the sizes  $j_a$  of the Jordan blocks of  $M$  (arranged from left to right). The formula  $n_{k-i} = \sum_{j=1}^{k-i} d_j = \sum_{a|j_a \geq i} (j_a - i)$  asserts that the number of blocks above the bottom  $i$  rows can be computed by summing over either rows or columns.

The same proof as before shows that  $M$  is nilpotent, but now the rank of  $M^i$  is at most  $n_{k-i}$ . This constraint is not satisfied by an arbitrary nilpotent matrix. If we consider a nilpotent matrix  $M$  with Jordan blocks of sizes  $j_1, j_2, \dots, j_t$  (which we arrange in non-increasing order), then the rank of  $M^i$  is  $\sum_{a|j_a \geq i} (j_a - i)$ . An  $M$  which saturates the rank constraints will have  $\sum_{a|j_a \geq i} (j_a - i) = n_{k-i}$ . The orbit of nilpotent matrices with this Jordan structure is a dense open subset of the Higgs moduli space  $\mathcal{H}$ .  $\mathcal{H}$  is actually the *closure* of this orbit, since it also includes points at which the rank of  $M^i$  is lower for some  $i$ .

We started with one nilpotent element  $\rho_+$ , and we have arrived at another nilpotent element  $M$ . The relation between them appears obscure, but it has a simple combinatoric interpretation, as in figure 28:  $j_a$  is the height of the  $a$ th column in the Young diagram of  $\rho$ , reading from left to right. Indeed, the number of boxes above the bottom  $i$  rows is  $n_{k-i} = \sum_{j=1}^{k-i} d_j$ , and the formula  $n_{k-i} = \sum_{a|j_a \geq i} (j_a - i)$  gives a way to compute this number by summing over columns. This means that, letting  $\rho_D$  denote the dual of the  $\mathfrak{su}(2)$  embedding in the sense of figure 25(b), the Higgs branch of the SCFT  $T_\rho(SU(n))$  is the closure of the orbit of the nilpotent element  $\rho_{D+}$ .

To understand the Coulomb branch of  $T_\rho(SU(n))$ , we need to build the mirror quiver, which describes the mirror theory that we will call  $T^\rho(SU(n))$ . The linking numbers of the NS5-branes in the brane realization of  $T_\rho(SU(n))$  in figure 27 coincide with the  $d_i$ , while the linking numbers of the  $n$  D5-branes are all  $k - 1$ . The mirror quiver has  $n - 1$  nodes, and the  $i$ th fundamental flavor sits at the node number  $d_i$  from left to right. The ranks of the gauge groups are computed from the linking numbers  $k - 1$  of the NS5-branes. For example, the rank of the leftmost gauge group is  $k - 1$

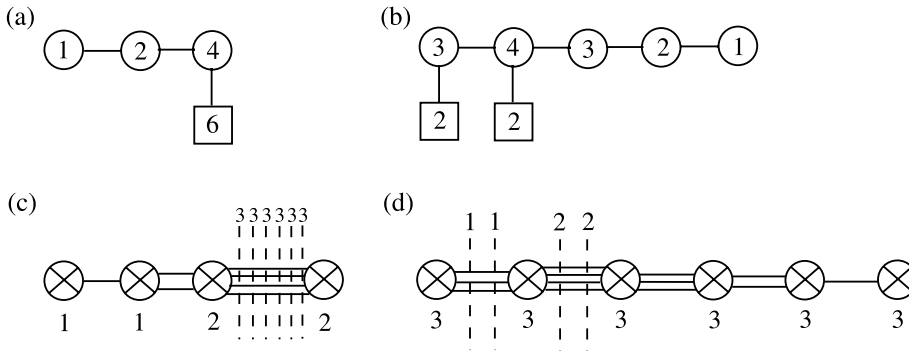


Figure 29: (a) The quiver representing  $T_\rho(SU(n))$  for  $n = 6$  and a particular  $\rho$ . (b) The mirror quiver. It contains a chain of five balanced nodes, leading to an  $SU(6)$  global symmetry of the Coulomb branch, which matches the flavor symmetry of the Higgs branch in (a). The mirror symmetry between the two quivers in (a) and (b) arises by comparing the two brane configurations of (c) and (d), in which linking numbers are indicated.

and the rank of the rightmost gauge group is 1. For an illustration of this procedure, see figure 29.

Unfortunately, we do not know how to describe the Higgs moduli space of this mirror quiver directly by solving the  $F$ -term constraints. According to  $S$ -duality, this Higgs moduli space should be the same as the moduli space of solutions of the Nahm equations with the original boundary condition determined by  $\rho$  and with  $\vec{X} \rightarrow 0$  at infinity. As explained in Section 3 of [1], this moduli space is isomorphic as a complex manifold to the Slodowy slice  $\mathcal{S}_\rho$  transverse to the nilpotent orbit associated with  $\rho$  intersected with the nilpotent cone  $\mathcal{N}$ . An indirect argument given by Nakajima [24] using the ADHM transform of instantons confirms the identity between the Higgs moduli space of  $T^\rho(SU(n))$  with this intersection  $\mathcal{S}_\rho \cap \mathcal{N}$ .

### 3.3.3 A simple application of $S$ -duality for Nahm poles

Consider a three-dimensional SCFT built from a general good linear quiver gauge theory, realized on a set of D3-brane segments stretched between NS5-branes and interacting with D5-branes. The linking numbers are non-decreasing from left to right, and there are no semi-infinite D3-branes (figure 30). A very natural rearrangement of the fivebranes is to push all the D5-branes to the left of all the NS5-branes. The D5-branes can be brought together to define a boundary condition at  $y = 0$  given by a certain  $SU(2)$  embedding  $\rho$ . The sizes of the irreducible blocks of  $\rho$  coincide with the linking numbers of the D5-branes. Similarly, the NS5-branes can be brought together to give a boundary condition at  $y = L$ , which is  $S$ -dual to the

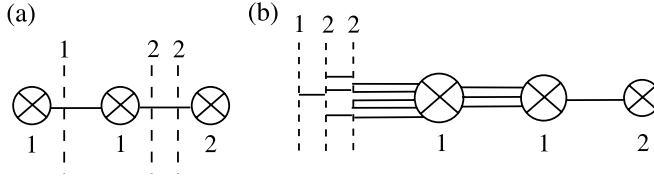


Figure 30: (a) A typical good linear quiver related to a three-dimensional gauge theory. All D3-brane segments end on NS5-branes and all linking numbers are non-decreasing from left to right. (b) Moving all D5-branes to the left, so that all NS5-branes are on the right, we get an IR equivalent configuration that treats D5-branes and NS5-branes symmetrically, but has no direct interpretation in gauge theory. The D5 configuration on the left determines an  $\mathfrak{su}(2)$  embedding  $\rho$ , and the NS5 configuration on the right determines the  $S$ -dual of another  $\mathfrak{su}(2)$  embedding  $\rho'$ . The example given here is self-mirror, since in (b), exchange of the two types of fivebrane is equivalent to a reflection  $y \leftrightarrow -y$ .

boundary condition given by another  $SU(2)$  embedding  $\rho'$ . The sizes of the irreducible blocks of  $\rho'$  can be read off from the linking numbers of the NS5-branes.

Although this last configuration has no direct relation to a three-dimensional gauge theory, it is certainly a four-dimensional  $U(n)$  gauge theory on the segment  $0 < y < L$  with boundary conditions associated with  $\rho$  and  $\rho'$ . Here  $n$  is the number of D3-brane segments in the space between the D5's and NS5's. The Higgs moduli space of vacua of this theory is described by Nahm's equations on the segment with appropriate conditions at the two ends. At  $y = 0$ ,  $\vec{X}$  must have a pole with residue determined by  $\rho$ . The appropriate condition at  $y = L$  is that  $\vec{X}$  must equal the moment map  $\vec{\mu}$  of  $T_{\rho'}$ . It is now straightforward to describe this moduli space as a complex manifold. Because of the boundary condition at  $y = 0$ ,  $\mathcal{X}(L) = X_1(L) + iX_2(L)$  lies in the Slodowy slice  $\mathcal{S}_\rho$  transverse to the nilpotent orbit  $\mathcal{O}_\rho$  associated to  $\rho$ . On the other hand, the complex moment map of the Higgs branch of  $T_{\rho'}$  takes values in the closure of the dual orbit  $\mathcal{O}_{\rho'_D}$  related to  $\rho'_D$ . The boundary condition at  $y = L$  therefore gives the intersection  $\mathcal{S}_\rho \cap \mathcal{O}_{\rho'_D}$ . This intersection is the Higgs branch. Reciprocally, the Coulomb branch is the intersection  $\mathcal{S}_{\rho'} \cap \mathcal{O}_{\rho_D}$ .

We denote an SCFT constructed this way using two homomorphisms  $\rho, \rho' : \mathfrak{su}(2) \rightarrow \mathfrak{u}(n)$  as  $T_{\rho'}^\rho(SU(n))$ . Clearly  $T(SU(n))$ ,  $T_\rho(SU(n))$ , and  $T^\rho(SU(n))$  are all examples of this class, with  $\rho$  and/or  $\rho'$  taken to be trivial. The mirror of  $T_{\rho'}^\rho$  is  $T_\rho^{\rho'}$ .

One limitation of what we have said is that starting from a suitable quiver gauge theory, we identified a pair  $\rho, \rho'$ , but we gave no indication of what pairs can be produced this way. Alternatively, for any given pair, we can generate a three-dimensional theory as the infrared limit of the four-dimensional  $U(n)$  gauge theory on a segment with boundary conditions given by that pair, as in figure 30(b). But this theory may spontaneously break supersymmetry.

For example (assuming  $n > 1$ ), if  $\rho$  is regular and  $\rho'$  is non-zero (which implies that  $\rho'_D$  is not regular), the Slodowy slice transverse to  $\rho$  does not intersect the orbit of  $\rho'_D$ , so supersymmetry is broken. If  $\rho$  and  $\rho'$  are both regular (so that  $\rho'_D = 0$ ), this statement is equivalent to the mysterious “ $s$ -rule” from [15]. A borderline example is  $\rho = \rho'_D$ : the Slodowy slice transverse to  $\rho$  intersects the orbit of  $\rho'_D = \rho$  in a single point. The theory has a single massive supersymmetric vacuum. In this case,  $T_{\rho'}^{\rho}$  is a trivial SCFT.

### 3.3.4 Reductions of the gauge symmetry to $U(1)$

Just as for  $U(2)$ , it is instructive to consider configurations in which the gauge group at the boundary is reduced from  $U(n)$  to  $U(1)$ . There are several distinct ways to do that; the partial Dirichlet boundary conditions that implement the reduction from  $U(n)$  to  $U(1)$  may be accompanied by a Nahm pole. Brane configurations do not allow us a generic choice of a  $U(1)$  subgroup of  $U(n)$ . The gauge symmetry at the boundary will have to be carried by a single D3-brane and so will correspond to the first factor  $U(1)$  of a subgroup  $U(1) \times U(n-1) \subset U(n)$ .

An example in which the  $U(1)$  subgroup has pure Neumann boundary conditions is shown in figure 31(a). In this example, the reduction in gauge symmetry is accomplished by ending the other  $n-1$  D3-branes on a single D5-brane, whose linking number is  $n$ . As the NS5-brane in the figure has linking number 1, in the dual configuration it will appear as a D5-brane just to the right of a single NS5-brane of linking number  $n$ , with no D3-brane ending on it (figure 31(b)). The dual boundary condition then will have full  $U(n)$  symmetry at the boundary, coupled to a single fundamental hypermultiplet.

It is interesting to compare this to the result of the usual ungauging strategy, by ending the D3-branes in the dual configuration on a Dirichlet boundary condition at  $y = L$  and taking the  $S$ -dual and IR limit. The resulting brane configuration is shown in figures 31(c) and (d) before and after rearrangement.

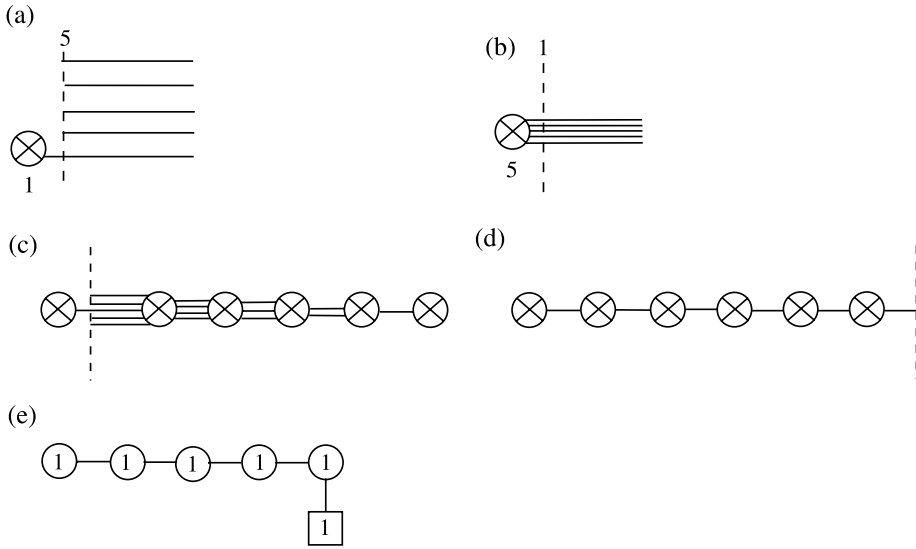


Figure 31: (a) Reduction of  $U(n)$  gauge theory to  $U(1)$  by a subregular Nahm pole (shown for  $n = 5$ ). The  $U(1)$  obeys Neumann boundary conditions. (b) The  $S$ -dual boundary condition, found by the usual brane procedure. It arises by coupling to a SCFT  $\mathfrak{B}^\vee$ , which in this case consists of a free hypermultiplet. (c) As usual, one can try to find  $\mathfrak{B}^\vee$  by the ungauging procedure. Alternatively, the mirror  $\tilde{\mathfrak{B}}^\vee$  of  $\mathfrak{B}^\vee$  can be found by coupling the original boundary condition to the dual of Dirichlet, represented here by the brane configuration related to  $T(SU(n))$ . (d) A more convenient rearrangement of (c), which is equivalent to the quiver in (e). All nodes but the leftmost one are balanced, leading to an  $SU(n)$  symmetry of the Coulomb branch. The leftmost node is minimally unbalanced, so the Coulomb branch describes twisted hypermultiplets in the fundamental representation of  $SU(n)$ , in agreement with (b).

The final result is that the mirror  $\tilde{\mathfrak{B}}^\vee$  of the SCFT that defines the  $S$ -dual boundary conditions is given by the linear quiver gauge theory of figure 31(e). In the language of Section 2.4, this is an ugly quiver with a single minimally unbalanced node (at the extreme left) and a chain of  $n - 1$  balanced nodes. The balanced nodes generate a  $U(n)$  symmetry of the Coulomb branch, and because of the minimally unbalanced node, there are free twisted hypermultiplets in the fundamental representation of  $U(n)$ . Counting dimensions, we see that the Coulomb branch should have hyper-Kähler dimension  $n$  and the Higgs branch is absent. So in fact, this quiver theory is precisely equivalent in the IR to a free theory of  $n$  twisted hypermultiplets. Thus, the  $S$ -dual of the original boundary condition is equivalent to the coupling to a free SCFT  $\mathfrak{B}^\vee$  consisting of  $n$  free hypermultiplets in the

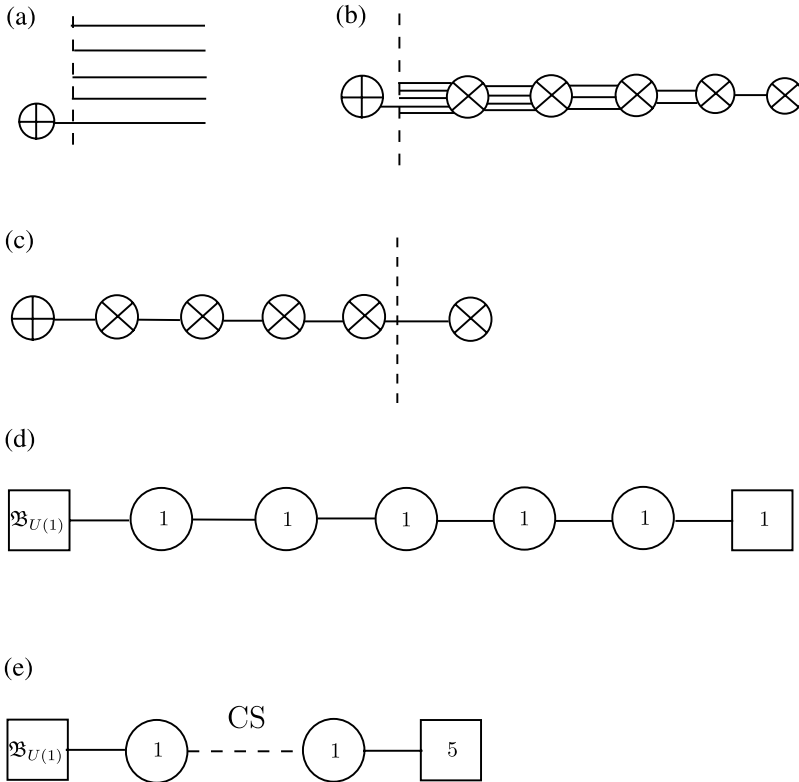


Figure 32: (a) The analog of figure 31(a) with Neumann for the unbroken  $U(1)$  altered by coupling to a boundary SCFT  $\mathfrak{B}_{U(1)}$ , represented by a brane system  $\oplus$ . The gauge group is here  $G = U(5)$ . Assuming that the  $S$ -dual boundary condition is obtained by coupling to an SCFT  $\mathfrak{B}^\vee$ , the mirror  $\tilde{\mathfrak{B}}^\vee$  to  $\mathfrak{B}^\vee$  can be found via the usual steps of (b), (c), and (d). (e) depicts a more exotic realization, analogous to figure 16(e) for  $U(2)$ , that makes the  $SU(5)$  flavor symmetry manifest.

fundamental representation. This is in perfect accord with what we found from a direct brane construction in figure 31(b).

It is straightforward to generalize this to the case that the  $U(1)$  gauge field at the boundary is not simply free but is coupled to some SCFT  $\mathfrak{B}_{U(1)}$  coupled to the  $U(1)$  gauge group at the boundary. The final result from the ungauging procedure for  $\tilde{\mathfrak{B}}^\vee$  (the mirror of the SCFT  $\mathfrak{B}^\vee$  that defines the  $S$ -dual boundary condition) is depicted in figure 32(d): it is given by the same chain of  $n$   $U(1)$  groups coupled to  $\mathfrak{B}_{U(1)}$  at one end. As the reader may suspect, this result is a simple generalization of the proposal involving  $T(SU(2))$  in the  $n = 2$  case.



The chain of  $n - 1$   $U(1)$  gauge groups on the right of figure 32(d) is something we have seen before, in figure 11(c). It is the mirror of a  $U(1)$  gauge theory coupled to  $n$  hypermultiplets of charge 1.

This has another interpretation. In the examples considered here, we reduce the gauge symmetry from  $U(n)$  to  $U(1)$  by letting  $n - 1$  D3-branes end on a single D5-brane, or equivalently by means of a subregular Nahm pole  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{u}(n)$ . For this choice of  $\rho$ , the theory  $T_\rho(SU(n))$  is the IR limit of  $U(1)$  coupled to  $n$  hypermultiplets of charge 1. Its mirror  $T^\rho(SU(n))$  is the IR limit of the quiver theory of figure 11(c).

We can therefore reformulate the prescription of figure 32(d) in a more intrinsic fashion: the mirror  $\tilde{\mathfrak{B}}^\vee$  to the SCFT  $\mathfrak{B}^\vee$  that defines the dual boundary condition is built by coupling a three-dimensional  $U(1)$  gauge theory to the product of the input SCFT  $\mathfrak{B}_{U(1)}$  and the theory  $T^\rho(SU(n))$ , and then flowing to the IR. In brief,  $\tilde{\mathfrak{B}}^\vee = \mathfrak{B}_{U(1)} \times_{U(1)} T^\rho(SU(n))$ .

Thus, at least for this class of examples, if we start with a boundary condition that contains a Nahm pole  $\rho$ , then  $T^\rho(SU(n))$  plays the same role that  $T(SU(2))$  played, in the absence of the Nahm pole, in Section 3.2. In Section 4.3, we will give a more systematic explanation of this. We did not see this role of  $T^\rho(SU(n))$  in Section 3.2, because if the gauge group is  $U(2)$ , the only way to get a Nahm pole is to end both D3-branes together on a single D5-brane, leaving no analog of the input SCFT  $\mathfrak{B}_{U(1)}$ .

The importance of using  $T^\rho(SU(n))$  rather than  $T(SU(n))$  in constructing  $\tilde{\mathfrak{B}}^\vee$  is that, although we can see the  $T(SU(n))$  brane configuration on the right of figure 31(c), this configuration does not have a gauge theory interpretation; instead, figure 31(d) does have such an interpretation, but involves  $T^\rho(SU(n))$ .

As in figure 16(e), we can alternatively use the CS-like coupling to the  $U(1)$  symmetry of the Coulomb branch of the mirror  $T_\rho(SU(n))$  to make the  $U(n)$  flavor symmetry manifest. We depict this in figure 32(e).

### 3.3.5 Reductions of the gauge symmetry to $U(2)$

For further practice, we will consider the case that the gauge symmetry at the boundary is reduced to  $U(2)$  by ending  $n - 2$  D3-branes on a single D5-brane.

As the first example, we will let the two remaining D3-branes end on two NS5-branes, as in figure 33(a). The corresponding theory  $\mathfrak{B}_{U(2)}$  coupled to the  $U(2)$  gauge symmetry at the boundary is  $T(SU(2))$ . Keeping track of the

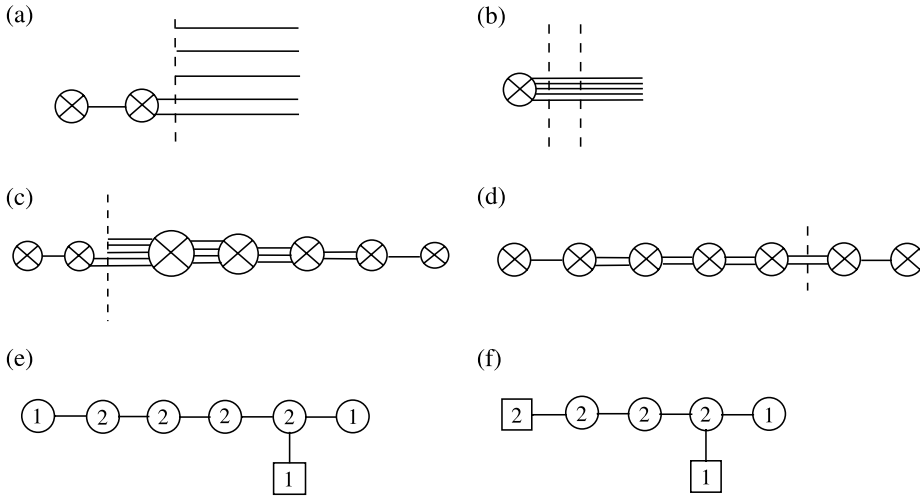


Figure 33: (a) Reduction of  $U(n)$  gauge theory (in this figure,  $n = 5$ ) to  $U(2)$  by a Nahm pole; the remaining two D3-branes end on two NS5-branes. (b) The dual has a single NS5-brane of linking number  $n$ , with two D5-branes to its right. (c) If one terminates the D3-branes of (b) at  $y = L$  using Dirichlet boundary conditions, and applies  $S$ -duality, one arrives here. (d) The result of a standard brane rearrangement applied to (c). (e) The quiver representing the configuration of (d). This is an ugly quiver, in the language of Section 2.4.3, as the second node from the left has  $e = -1$ . (f) The relevant  $T^\rho(SU(n))$  is associated with this quiver.

linking numbers, we construct the  $S$ -dual configuration of figure 33(b). The dual boundary condition consists of a pair of fundamental hypermultiplets coupled to the  $U(n)$  gauge theory. Let us see how the standard ungauging prescription reproduces such a result in an “uneconomical” fashion.

In the dual description of (b), if we end the D3-branes at  $y = L$  with Dirichlet boundary conditions and take the  $S$ -dual, we obtain the configuration in figure 33(c), with the initial boundary condition on the left and  $T(SU(n))$  (realized via branes) on the right. Rearranging the branes to get something with a gauge theory interpretation, and keeping track of the linking numbers, we arrive at figure 33(d), from which we get in figure 33(e) a quiver whose IR limit should be the mirror  $\mathfrak{B}^\vee$  of the theory  $\mathfrak{B}^\vee$  that defines the dual boundary condition.

That quiver is interpreted directly as the coupling of the input SCFT  $\mathfrak{B}_{U(2)} = T(SU(2))$  to a quiver that represents  $T^\rho(SU(n))$  for a certain  $\rho$ . Differently put, the relevant theory is a three-dimensional  $U(2)$  gauge theory coupled to the product of  $T(SU(2))$  and  $T^\rho(SU(n))$ , both of which (for

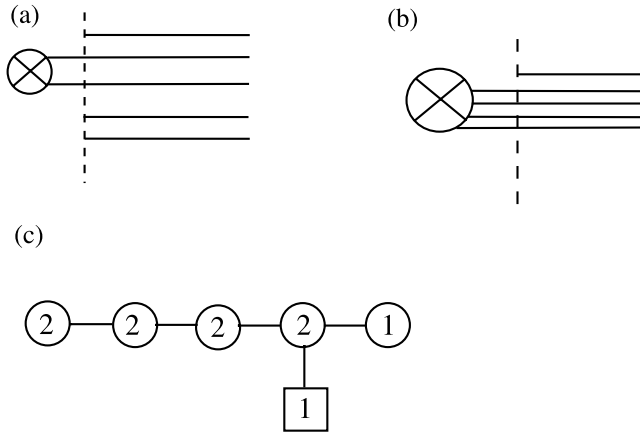


Figure 34: (a)  $n - 2$  D3-branes end on a single D5-brane, and the remaining ones then end on a single NS5-brane. (b) The dual brane configuration, as determined from the linking numbers. (c) The dual quiver, constructed in the same way as figure 33(e). It describes a  $U(2)$  gauge theory (acting at the left-most node of the quiver) coupled to  $T^\rho(SU(n))$ , for the choices of  $n$  and  $\rho$  that were made in (a). This is a bad quiver, in the language of Section 2.4.3, as the leftmost node has  $e = -2$ .

the relevant  $\rho$ ) have  $SU(2)$  global symmetry. This is what we call the composite gauge theory and denote  $\mathfrak{B}_{U(2)} \times_{U(2)} T^\rho(SU(n))$ . The relevant  $\rho$  is associated with the decomposition  $n = (n - 2) + 1 + 1$ , corresponding to the Nahm pole that we started with in 33(a).

This result is quite like what we found in Section 3.3.4: the theory  $T^\rho(SU(n))$  plays the same role in the presence of a Nahm pole that  $T(SU(n))$  plays without one. As usual, we expect this prescription to work well for every choice of the input theory  $\mathfrak{B}_{U(2)}$ , as long as the dual boundary condition has full  $U(n)$  gauge symmetry.

Finally, another interesting example is given by taking  $\mathfrak{B}_{U(2)}$  to be trivial. In terms of branes, we do this by ending the two leftmost D3-branes on a single NS5-brane, as in figure 34(a), so as to get a pure Neumann boundary condition for the surviving gauge group  $U(2)$ . In the dual brane configuration in figure 34(b), which is found as usual by matching the linking numbers, the gauge group is reduced to  $U(n - 1)$ , which obeys Neumann conditions at the boundary. The reduction of the dual gauge group invalidates our simple derivation of the mirror  $\tilde{\mathfrak{B}}^\vee$  of the SCFT  $\mathfrak{B}^\vee$  living at the  $S$ -dual boundary condition. If we nevertheless follow this recipe, by coupling the initial boundary condition to  $T(SU(n))$  and moving branes to get

something with a gauge theory configuration, we arrive at the quiver of figure 34(c). (This can be obtained from the quiver of figure 33(e) by deleting the leftmost node, which in that example generates  $\mathfrak{B}_{U(2)} = T(SU(2))$ ). In the present case, we take  $\mathfrak{B}_{U(2)}$  to be trivial so we replace the leftmost node of figure 33(e) by nothing.) This is a bad quiver, in the sense of Section 2.4, as the leftmost node has  $e = -2$ . By reasoning similar to that of figure 21, one can show that the symmetries of the Coulomb branch of this bad quiver gauge theory are spontaneously broken. As we will explain more fully later, this is related by  $S$ -duality to the fact that the dual boundary condition has reduced gauge symmetry.

### 3.3.6 Full gauge symmetry with matter at the boundary

Now we consider boundary conditions with full  $U(n)$  symmetry coupled to a boundary theory  $\mathfrak{B}$ . A simple example is to use D5-branes to generate  $k$  fundamental hypermultiplets at the boundary. We will get a result similar to what we found in Section 3.2 for  $n = 2$ : if the number  $k$  of fundamental hypermultiplets at the boundary is large enough, the dual boundary condition is described by coupling to a dual SCFT  $\mathfrak{B}^\vee$  whose mirror  $\tilde{\mathfrak{B}}^\vee$  admits a construction in terms of  $T(SU(n))$ . This standard construction will give an economical description of  $\mathfrak{B}^\vee$  if  $k > n$  and a less economical one if  $k = n$ . Conversely, if  $k < n$ , the  $S$ -dual boundary condition is not obtained by coupling to an SCFT; rather, the gauge group at the boundary is reduced to  $U(k)$ , possibly in the presence of a Nahm pole.

The brane manipulations and quivers for  $k > n$  are shown in figures 35 and 36. In figure 35, we construct a quiver for the SCFT  $\mathfrak{B}^\vee$  that appears in the dual boundary condition, and in figure 36, we construct the mirror quiver associated to the mirror SCFT  $\tilde{\mathfrak{B}}^\vee$ . This mirror quiver has a simple interpretation as  $T(SU(4))$  coupled to the original boundary condition ( $k = 6$  fundamental hypermultiplets) by a three-dimensional  $U(4)$  gauge theory.

In general, for any  $k$  and  $n$ , the original boundary condition has a single NS5-brane of linking number  $n$  and  $k$  D5-branes of linking number 1. The dual configuration has a single D5-brane of linking number  $n$  and  $k$  NS5-branes of linking number 1. If  $k > n$ , the dual configuration has the D5-brane to the left of the  $n$ th NS5-brane. This ensures that the dual consists of an SCFT  $\mathfrak{B}^\vee$  coupled to Neumann boundary conditions. This is the condition under which the usual ungauging procedure is a good way to determine the dual boundary condition. To extract  $\mathfrak{B}^\vee$ , we follow the familiar steps. First we end the D3-branes with Dirichlet boundary conditions. Applying  $S$ -duality, we get a representation of the mirror  $\tilde{\mathfrak{B}}^\vee$  of  $\mathfrak{B}^\vee$  in terms of  $T(SU(n))$  coupled to the original SCFT  $\mathfrak{B}$  (which in the present example consists of  $k$  fundamental hypermultiplets), as in figure 37. This strategy,

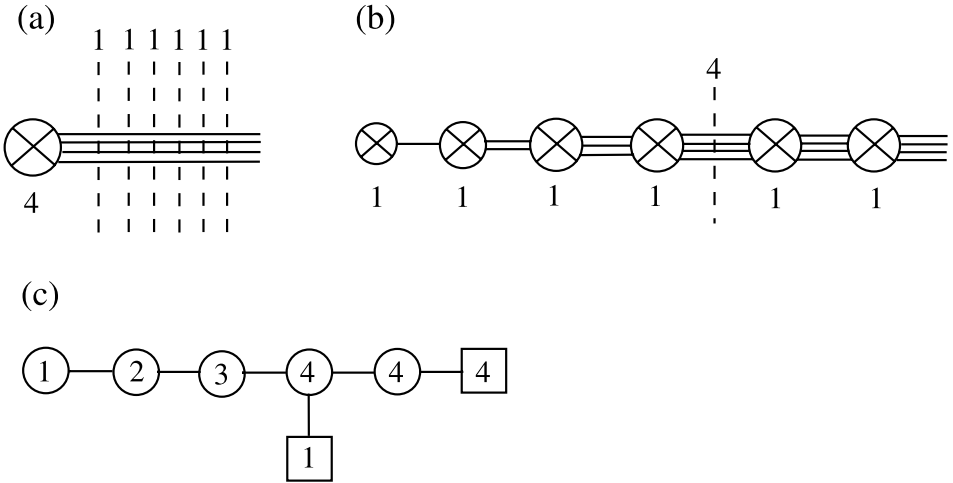


Figure 35: (a)  $U(n)$  gauge theory coupled to  $k$  fundamental hypermultiplets at the boundary, as shown here for  $n = 4$  and  $k = 6$ . Linking numbers have been labeled. (b) The dual brane configuration, as determined from the linking numbers. (c) The quiver that represents the dual brane configuration. All nodes are balanced; the chain of five consecutive balanced nodes leads to an  $SU(6)$  global symmetry of the Coulomb branch, matching the global symmetry of the six coincident fivebranes in (a).

which accounts for the result of figure 36, should work whenever the dual boundary condition has the full  $U(n)$  gauge symmetry at the boundary.

If  $k > n$ , the quiver description of  $\tilde{\mathfrak{B}}^\vee$  by  $T(SU(n))$  coupled to the original boundary conditions satisfies our constraints from Section 2.2, and gives a straightforward description of  $\tilde{\mathfrak{B}}^\vee$ .

If  $k = n$ , the dual brane configuration has a D5-brane of linking number  $n$  sitting to the right of  $n$  NS branes. The resulting SCFT  $\mathfrak{B}^\vee$  consists of two factors: a single fundamental hypermultiplet (from the D3–D5 intersection) and  $T(SU(n))$ . In this case, the quiver of figure 37 that arises from the ungauging procedure is ugly, accounting for the existence of the free hypermultiplets.

If  $k = n - 1$ , the D5-brane in the dual configuration is to the right of  $n - 1$  NS5-branes but has linking number  $n$ ; hence one D3-brane will end on it and the dual boundary condition involves a reduction of the gauge group down to  $U(n - 1)$  together with a coupling to  $T(SU(n - 1))$ . For smaller values of  $k$ , a Nahm pole appears in the dual boundary condition, as  $n - k$  D3-branes end on the single D5-brane. In these cases, the ungauging recipe leads to a bad quiver.

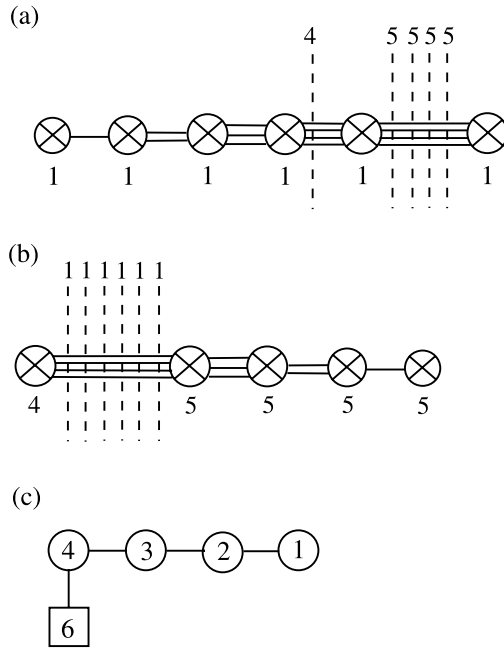


Figure 36: (a) The “flavors” coming from interaction with semi-infinite D3-branes on the right of figure 35(b) have been replaced by flavors coming from intersection with D5-branes. This will facilitate a purely three-dimensional construction of the relevant SCFT. Linking numbers are labeled. (b) The mirror arrangement, found from the linking numbers. (c) The quiver corresponding to (b). The original boundary condition (six fundamental hypermultiplets of  $U(4)$ ) has been coupled to  $T(SU(4))$  by gauging their common  $U(4)$  symmetry, which acts at the leftmost node. The other three nodes are balanced, giving the Coulomb branch an  $SU(4)$  global symmetry.

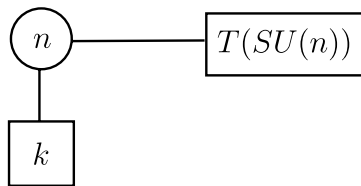


Figure 37: A schematic representation of the dual boundary condition for  $U(n)$  coupled to  $k$  fundamental hypermultiplets. A  $U(n)$  gauge theory is coupled both to  $T(SU(n))$  and to the hypermultiplets. The IR limit gives a three-dimensional SCFT  $\tilde{\mathfrak{B}}^\vee$  that is the mirror of the SCFT  $\mathfrak{B}^\vee$  that defines the dual boundary condition.

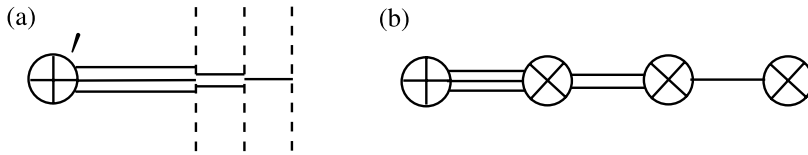


Figure 38: (a) Let  $\oplus$  symbolize an arbitrary given boundary condition, with dual  $\oplus'$ , and assume that  $\oplus'$  is defined by coupling to an SCFT  $\mathfrak{B}^\vee$ . Then  $\mathfrak{B}^\vee$  can be found by ungauging, that is by terminating the four-dimensional gauge theory at  $y = L$  using Dirichlet boundary conditions. This is achieved, for  $G = U(n)$ , by letting  $n$  D3-branes end on  $n$  D5-branes, as shown here for  $n = 3$ . (b) The mirror of  $\mathfrak{B}^\vee$  is represented by the  $S$ -dual configuration. Here  $\oplus'$ , which we do not know, is replaced by the given boundary condition denoted  $\oplus$ , and the Dirichlet boundary conditions are replaced by a coupling to  $T(SU(n))$ .

We have accumulated by now enough examples to guess how to build the dual of essentially any boundary conditions as long as the dual gauge symmetry is unbroken, in terms of coupling of  $T^\rho(SU(n))$  to  $\mathfrak{B}$ . We will now give a general derivation of this fact.

### 3.4 A general recipe

Let  $\mathcal{B}$  be any half-BPS boundary condition in  $U(n)$  gauge theory, possibly but not necessarily constructed by coupling Neumann boundary conditions to a boundary SCFT  $\mathfrak{B}$ . Let us further assume that the  $S$ -dual boundary condition  $\mathcal{B}^\vee$  has full gauge symmetry, and so is obtained by coupling Neumann boundary conditions to a boundary theory  $\mathfrak{B}^\vee$ . We have gained enough experience by now to formulate a general recipe for construction of  $\mathfrak{B}^\vee$ .

Let us first explain what we regard as a satisfactory answer.  $\mathfrak{B}^\vee$  is supposed to be a three-dimensional conformal field theory, and we are satisfied if we can give a purely three-dimensional description of it. Roughly speaking, if we can do so, this means that we have reduced the understanding of  $S$ -duality of boundary conditions in four-dimensional gauge theory to a problem only involving the boundary.

We begin in a simple and by now familiar fashion (figure 38).  $\mathfrak{B}^\vee$ , if it exists, is obtained from the  $S$ -dual boundary condition  $\mathcal{B}^\vee$  by terminating the gauge theory at  $y = L$  with a Dirichlet boundary condition and then flowing to the IR. The  $S$ -dual of this is a gauge theory on a slab  $\mathbb{R}^3 \times I$  ( $I$  is the interval  $0 \leq y \leq L$ ) with the original boundary condition  $\mathcal{B}$  at one

boundary, and a coupling to  $T(SU(n))$ , the dual of the Dirichlet boundary condition, at the other.

This gives a description of  $\tilde{\mathfrak{B}}^\vee$ , the mirror of  $\mathfrak{B}^\vee$ , in terms of four-dimensional gauge theory on  $\mathbb{R}^3 \times I$ , with specified boundary conditions at the two ends. To get a satisfactory answer, we should reduce this to gauge theory on  $\mathbb{R}^3$ . If the original boundary condition  $\mathcal{B}$  has no Nahm pole, there is no problem in doing this. Suppose that  $\mathcal{B}$  is defined by reducing the gauge group  $G = U(n)$  to some subgroup  $H$  and then coupling to an SCFT  $\mathfrak{B}_H$  with  $H$  symmetry. Then the low-energy limit can be obtained by simply restricting all four-dimensional fields to their zero modes in the fourth direction — the modes that are independent of  $y$ . These modes make up the vector multiplet of  $H$  gauge symmetry in three dimensions. So at low energies, the four-dimensional configuration of figure (38(b)) merely reduces to a three-dimensional gauge theory with gauge group  $H$  coupled to the product  $\mathfrak{B}_H \times T(SU(n))$ . (In the figure,  $H = G = U(3)$ .) We denote the theory obtained by gauging the diagonal  $H$  symmetry of  $\mathfrak{B}_H \times T(SU(n))$  as  $\mathfrak{B}_H \times_H T(SU(n))$  and we call it the composite gauge theory. This gives the required three-dimensional description of  $\tilde{\mathfrak{B}}^\vee$ .

If the starting boundary condition  $\mathcal{B}$  has a Nahm pole  $\rho$  (figure 39), we begin as before and represent  $\tilde{\mathfrak{B}}^\vee$  in exactly the same way in terms of four-dimensional gauge theory on  $\mathbb{R}^3 \times I$  (figure 39(b)). This gives a gauge theory in *four* dimensions, with three non-compact dimensions, whose IR limit is the desired three-dimensional theory  $\tilde{\mathfrak{B}}^\vee$ . Before declaring success, we are supposed to reduce this to a *three*-dimensional description. The Nahm pole on the left of figure (39(b)) forces the field  $\vec{X}$  to be  $y$ -dependent. So we cannot extract a three-dimensional description by simply taking all four-dimensional fields to be independent of  $y$ .

Instead, before trying to extract a low-energy limit, we take the D5-branes that create the Nahm pole in the original boundary condition  $\mathcal{B}$ , as well as those that reduce the gauge symmetry,<sup>12</sup> and move them to the right (figure 39(c)), positioning them in the usual way among the NS5-branes, so that no net D3-branes end on any of these D5-branes.

At this stage, we have gauge fields of a subgroup  $H$  of  $G$ , coupled on the left to some boundary theory  $\mathfrak{B}_H$  (represented in figure 39(c) as  $\oplus$ ) and on the right to a system of D5-branes and NS5-branes that are shown

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<sup>12</sup>We recall that a D5-brane on which several D3-branes end creates a Nahm pole, while one on which a single D3-brane ends reduces the gauge symmetry. Both kinds of D5-brane are depicted in figure 1, while for simplicity only a D5-brane creating a Nahm pole is depicted in figure 39.



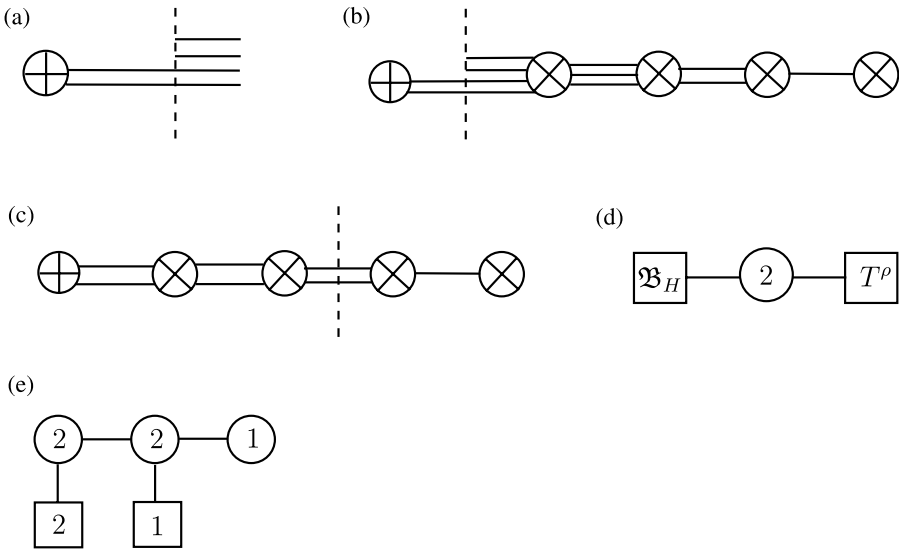


Figure 39: (a) A boundary condition with a Nahm pole or reduction of the gauge group. (b) A direct application of the ungauging procedure leads to this result, with branes ordered incorrectly. (c) A D5-brane has been moved to the right to restore correct ordering. Now the IR limit is straightforward; for example, if  $\oplus$  is constructed from properly ordered fivebranes, then this configuration corresponds to a good quiver. (d) The result can be described as a generalized quiver, with  $H = U(2)$  coupled both to  $\mathfrak{B}_H$  and to another theory, which in the example given can be represented by the quiver in (e) and is in fact our friend  $T^\rho(SU(n))$ , where here  $n = 4$  and  $\rho$  corresponds to the decomposition  $4 = 2 + 1 + 1$ .  $H$  acts at the leftmost node in (e).

explicitly in the figure. Collapsing all the separations among these branes, we arrive at a boundary SCFT that for the moment we call  $\mathfrak{B}^\rho$ . A three-dimensional description of the desired SCFT  $\widetilde{\mathfrak{B}}^\vee$  is now at hand; it is what we will call  $\mathfrak{B}_H \times_H \mathfrak{B}^\rho$ , the low-energy limit of three-dimensional  $H$  gauge theory coupled to  $\mathfrak{B}_H \times \mathfrak{B}^\rho$ .

But in fact,  $\mathfrak{B}^\rho$  is our friend  $T^\rho(SU(n))$ . The D3-branes that produce Nahm poles depend only on  $\rho$  and are present in the description of  $T^\rho(SU(n))$  that treats the two kinds of fivebrane symmetrically (see the example in figure 30(b)). The process of moving them to the right to get a well-ordered arrangement is the key step in getting a gauge theory interpretation of  $T^\rho(SU(n))$  (such as comes from figure 30(a)). If  $H$  is trivial, so that the symbol  $\oplus$  and the D3-branes ending on it are absent in figure 39(b) (the leftmost D3-branes would then all end on D5-branes), then figure 39(b) would simply be the definition of  $T^\rho(SU(n))$  for a particular  $\rho$ .

In general,  $\mathfrak{B}^\rho$  depends only on  $\rho$  and so coincides with  $T^\rho(SU(n))$ . If  $H$  is non-trivial, this means that some of the global symmetries of  $T^\rho(SU(n))$  have been gauged and also coupled to another theory  $\mathfrak{B}_H$ . This produces what we call the composite gauge theory and denote  $\mathfrak{B}_H \times_H T^\rho(SU(n))$ . So

$$\tilde{\mathfrak{B}}^\vee = \mathfrak{B}_H \times_H T^\rho(SU(n)). \quad (3.13)$$

This is the single most important conclusion of the present paper.

In this construction, the  $SU(n)$  isometries of the Coulomb branch of  $T^\rho(SU(n))$  survive in the IR as symmetries of the full theory  $\tilde{\mathfrak{B}}^\vee$ . This allows  $\tilde{\mathfrak{B}}^\vee$  to be coupled to the dual bulk gauge theory and produce the boundary condition  $\mathcal{B}^\vee$  dual to  $\mathcal{B}$ . The dual gauge group has to be coupled to the Coulomb branch of  $\tilde{\mathfrak{B}}^\vee$ , rather than the Higgs branch, as would be more standard, because  $\tilde{\mathfrak{B}}^\vee$  is the mirror of the theory  $\mathfrak{B}^\vee$  that more directly defines the dual boundary condition. As usual, to make the result useful in practice, it is very helpful to have a representation of  $\tilde{\mathfrak{B}}^\vee$  that makes its Coulomb branch symmetries manifest. This is what we sometimes get from mirror symmetry.

Our construction produces a  $U(n)$ -invariant SCFT  $\mathfrak{B}^\vee$  that defines the dual boundary condition whenever it exists, as the mirror of a standard IR limit of the composite  $H$  gauge theory. However, it may be that the dual boundary condition  $\mathcal{B}^\vee$  involves a reduction of the gauge symmetry, possibly with a Nahm pole; if so,  $\mathfrak{B}^\vee$  does not exist so the ungauging procedure cannot construct it. One would like a criterion for determining, given  $\mathcal{B}$ , whether the dual boundary condition  $\mathcal{B}^\vee$  has reduced gauge symmetry.

In fact, this will occur precisely when the “ungauging” configuration of figure 38(a) admits three-dimensional chiral operators of zero or negative  $R$ -charge  $q_R$ . (Here as in Section 2.4 we mean chiral operators for an  $\mathcal{N} = 2$  subalgebra of  $\mathcal{N} = 4$ .) Dually, the composite  $H$  gauge theory will also have such bad chiral operators and will not have a standard IR limit. The problematical chiral operator in figure 38(a) will be constructed from Wilson line operators stretched between the two boundaries. So dually, the chiral operators of the composite gauge theory are constructed from stretched ’t Hooft operators; they are the monopole operators reviewed in Section 2.4.

If the boundary condition  $\mathcal{B}^\vee$  comes from coupling  $U(n)$  gauge fields to an SCFT  $\mathfrak{B}^\vee$ , then chiral operators of  $\mathfrak{B}^\vee$  have strictly positive  $q_R$ , and coupling to four-dimensional gauge fields on  $\mathbb{R}^3 \times I$  with Dirichlet boundary conditions on the right does not change this fact. The situation is different if  $\mathcal{B}^\vee$  involves a reduction of the gauge symmetry or a Nahm pole. In this case, we will construct chiral operators of  $q_R \leq 0$  using supersymmetric Wilson

operators that end on the boundaries of  $\mathbb{R}^3 \times I$ . As in Section 2.4, we select a unit vector  $\vec{n}$  and form a linear combination of scalar fields  $X_3 = \vec{n} \cdot \vec{X}$ . And we let  $A_y$  be the component of the gauge field  $A$  in the  $I$  direction. Then the combination  $\mathcal{A} = A_y + iX_3$  is a chiral superfield for a suitable  $\mathcal{N} = 2$  subalgebra of  $\mathcal{N} = 4$ . Let  $S = p \times I$ , with  $p$  a point in  $\mathbb{R}^3$ . Now we consider the path-ordered exponential

$$W(p) = P \exp \int_S \mathcal{A}. \tag{3.14}$$

Any gauge-invariant matrix elements of  $W(p)$  are chiral superfields with  $q_R = 0$ . Whether there are gauge-invariant matrix elements depends on the boundary conditions. Since  $S$  stretches from  $y = 0$  to  $y = L$ , the pertinent boundary conditions are at  $y = L$  and  $y = 0$ . At  $y = L$ , we have Dirichlet boundary conditions and gauge transformations are trivial. If in addition the boundary condition  $\mathcal{B}^\vee$  has reduced gauge symmetry without a Nahm pole, then  $W(p)$  has gauge-invariant matrix elements in some representation of  $G$ .

If the dual boundary condition  $\mathcal{B}^\vee$  involves a Nahm pole associated with a homomorphism  $\rho' : \mathfrak{su}(2) \rightarrow G$ , then a similar construction actually gives operators of  $q_R < 0$ . In this case, we have to be careful with the definition of  $SO(3)_X$  and of the Wilson operator  $W(p)$ . The boundary condition forces  $\vec{X}$  to be non-zero near the boundary, and is not invariant under naive rotation of  $\vec{X}$ . But it is invariant under a rotation of  $\vec{X}$  combined with a gauge transformation determined by  $\rho'$ . So the  $SO(3)_X$  symmetry of the boundary condition is the combination of an ordinary rotation of  $\vec{X}$  and a gauge transformation. As a result, the matrix elements of  $W(p)$  — once we define them — transform non-trivially under  $SO(3)_X$ , and some of them are negatively charged under  $SO(2)_X$ . This will lead to  $q_R < 0$ .

The Nahm pole leads to a subtlety in defining the operator  $W(p)$ . By definition, the Nahm pole means that  $\vec{X} \sim \vec{t}/y$ , near  $y = 0$ , where  $\vec{t}$  is the image under  $\rho'$  of a standard set of  $\mathfrak{su}(2)$  generators. So  $\mathcal{A} \sim it_3/y$ , with  $t_3 = \vec{n} \cdot \vec{t}$ . The pole in  $\mathcal{A}$  causes a problem in defining  $W(p)$ . We regularize the resulting divergences by letting  $S_\epsilon$  be the restriction of  $S$  to  $y \geq \epsilon$  and defining  $W_\epsilon(p) = P \exp \int_{S_\epsilon} \mathcal{A}$ . Taking  $\epsilon \rightarrow 0$ , we define the regularized Wilson operator<sup>13</sup>

$$\widehat{W} = \lim_{\epsilon \rightarrow 0} \epsilon^{-it_3} W_\epsilon(p). \tag{3.15}$$

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<sup>13</sup>Our gauge fields  $A$  and scalar fields  $\vec{X}$  are antihermitian, so  $t_3$  is antihermitian and  $it_3$  has real eigenvalues.

Gauge-invariant matrix elements of  $\widehat{W}$  are chiral operators of dimension and  $R$ -charge  $q_R = it_3$ .

Hence we predict that the  $S$ -dual of a boundary condition  $\mathcal{B}$  will break the dual gauge theory at the boundary if and only if monopole operators of non-positive charges can be found in the composite  $H$  gauge theory.

Even if the dual boundary condition breaks the dual gauge theory at the boundary, the D-brane realization of the system suggests that the composite  $H$  gauge theory still holds the information about the dual boundary condition  $\mathcal{B}^\vee$ . One may start with the “ ungauging ” configuration, and move away in the  $\vec{Y}$  direction the D5-branes which define the Nahm pole. The D3-segments attached to those D5-branes will need to move away as well, and so will those D5-branes at the Dirichlet boundary condition that are attached to them. What is left is a four-dimensional gauge theory on  $\mathbb{R}^3 \times I$  with a reduced gauge group  $H^\vee$ , with Dirichlet boundary conditions at one end, and coupled to some boundary theory  $\mathfrak{B}_{H^\vee}^\vee$  at the other end. The system will flow smoothly in the IR to  $\mathfrak{B}_{H^\vee}^\vee$ .

To carry out an  $S$ -dual of this process in the composite  $H$  gauge theory, and thereby construct the mirror of  $\mathfrak{B}_{H^\vee}^\vee$ , we will need to identify the FI parameters in the composite gauge theory which correspond to the motion of the D5-branes, and make them large. It is difficult to give a general prescription on how to do that without knowing anything about  $\mathcal{B}$ . The monopole operators with negative  $R$ -charges dual to the regularized Wilson line operators (3.15) will transform non-trivially under the Coulomb branch isometries that correspond to the appropriate FI parameters. One might be able to collect further information on  $\mathcal{B}^\vee$  by exploring modifications of the Dirichlet boundary condition: adding a Nahm pole, coupling some judiciously chosen SCFT, etc. We will not pursue this matter further.

### 3.5 Domain walls

In general, domain walls between gauge theories with one gauge group  $G_1$  on one side and another gauge group  $G_2$  on the other side are equivalent, after a folding trick, to boundary conditions in  $G_1 \times G_2$  gauge theory. (For more on this, see Section 2.6 of [1].) Hence all methods for studying boundary conditions can be adapted to domain walls.

If  $G_+ = U(n_+)$  and  $G_- = U(n_-)$  are unitary gauge groups realized by D3-branes for  $y > 0$  and  $y < 0$ , and the domain walls are constructed via fivebranes, we can study  $S$ -duality of a domain wall by the usual manipulations. It is also possible to apply an ungauging procedure similar to what we have just used in discussing boundary conditions.

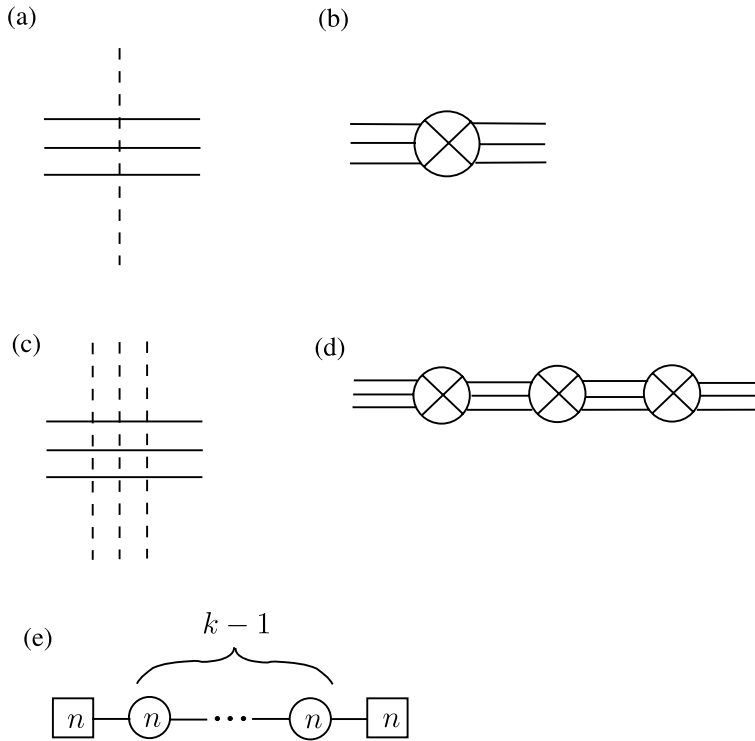


Figure 40: (a) A simple supersymmetric domain wall in  $U(n)$  gauge theory (drawn for  $n = 3$ ); intersection with a D5-brane creates a fundamental hypermultiplet supported at  $y = 0$ . (b) The  $S$ -dual domain wall is constructed from the intersection with a single NS5-brane. (c) Intersection with  $k$  D5-branes creates a defect supporting  $k$  fundamental hypermultiplets (sketched here for  $k = 3$ ). (d) The  $S$ -dual of (c). (e) For  $k > 1$ , the  $S$ -dual involves coupling to a non-trivial SCFT, which can be represented by a quiver. The quiver has been drawn for the general case of  $n$  D3-branes intersecting  $k$  D5-branes.

Consider a gauge theory with a domain wall  $\mathcal{D}$  at  $y = 0$ . The domain wall is constructed in general by a two-sided version of the construction used for boundary conditions.  $G_+$  and  $G_-$  are broken to subgroups  $H_{\pm}$  by two Nahm poles ( $\vec{X} \sim \vec{t}_{\pm}/y$  for  $y \rightarrow 0^{\pm}$ ). The product  $H_+ \times H_-$  is then further broken at  $y = 0$  to a subgroup  $H$ , which is then coupled to a three-dimensional defect theory  $\mathcal{D}$ .

As long as we consider only domain walls constructed from fivebranes, there is no problem determining the  $S$ -duals. A few examples are given in figure 40. In (a) we consider a single D5-brane crossing  $n$  D3-branes. There is then a fundamental hypermultiplet supported on the defect or domain wall

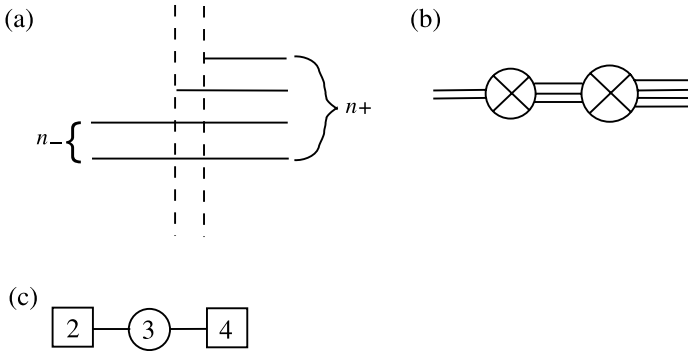


Figure 41: (a)  $U(n_+)$  gauge theory for  $y > 0$ , reduced to  $U(n_-)$  for  $y < 0$ , by letting D3-branes end one by one on  $n_+ - n_-$  D5-branes. Sketched here is the case  $n_+ = 4, n_- = 2$ . (b) The  $S$ -dual, which as in (c), can be represented by a balanced quiver gauge theory. The boxes represent global symmetries that can be coupled to four-dimensional gauge fields in the right or left half-spaces.

at  $y = 0$  and coupled to the bulk  $U(n)$  gauge fields. In the language of the last paragraph,  $G_+ = G_- = U(n)$ , there are no Nahm poles,  $H$  is a diagonal subgroup of  $G_+ \times G_-$ , and  $\mathfrak{D}$  describes a free fundamental hypermultiplet. The  $S$ -dual in (b) clearly corresponds to interaction with a single NS5-brane. In this case, there are distinct  $U(n)$  gauge theories on the two half-lines. There is no reduction of gauge symmetry at the interface and the full  $U(n) \times U(n)$  couples to a bifundamental hypermultiplet. We can generalize this example to have different numbers  $n_{\pm}$  of D3-branes on the two sides. In (a), there would then be a Nahm pole of rank  $|n_+ - n_-|$  on the side with more D3-branes, and in (b), there would be a bifundamental hypermultiplet coupled to  $U(n_+) \times U(n_-)$ .

Returning to the case  $n_+ = n_-$ , in (c), we generalize (a) to a defect made from  $k$  D5-branes and so supporting  $k$  fundamental hypermultiplets. The  $S$ -dual involves a chain of  $k$  NS5-branes leading to the balanced quiver gauge theory in (d) and (e). As usual, the chain of balanced nodes in the quiver leads to a global symmetry of the Coulomb branch, matching the global symmetry of the hypermultiplets of (c).

For a slightly different example, in figure 41(a), the gauge group  $U(n_+)$  is reduced to  $U(n_-)$  in crossing a domain wall by interaction with  $n_+ - n_-$  D5-branes; one D3-brane ends on each D5-brane. The  $S$ -dual is an analogous picture with NS5-branes (figure 41(b)), which can be represented by a balanced quiver gauge theory with  $U(n_+) \times U(n_-)$  symmetry, as in (c). The examples we have given are particularly simple because no brane

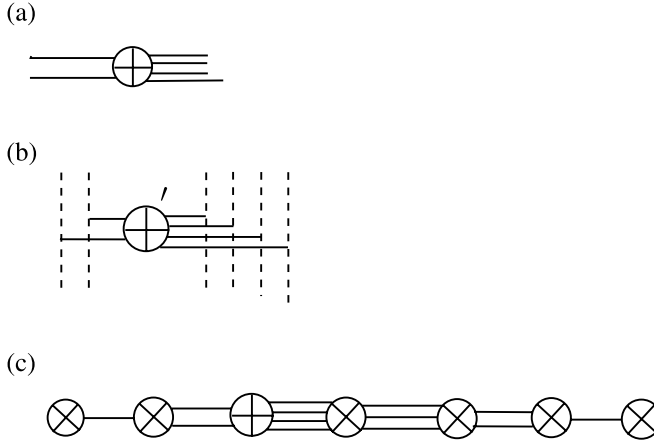


Figure 42: (a) The symbol  $\oplus$  represents a general domain wall between  $U(n_-)$  and  $U(n_+)$  gauge theory, shown here for  $n_- = 2$ ,  $n_+ = 4$ . (b) The symbol  $\oplus'$  represents the dual domain wall. If it arises by coupling to an SCFT  $\mathfrak{D}$  with  $U(n_-) \times U(n_+)$  symmetry, then this SCFT can be extracted by using Dirichlet boundary conditions on both sides to ungauged the gauge symmetry. As usual, Dirichlet boundary conditions are constructed using a chain of D5-branes. (c) Taking the  $S$ -dual, we get a quiver representation of the mirror  $\mathfrak{D}^\vee$  to  $\mathfrak{D}$ , in terms of the original domain wall coupled by  $U(n_+) \times U(n_-)$  gauge fields to  $T(SU(n_+)) \times T(SU(n_-))$ , which are here represented by quivers.

rearrangement is required, but in general, starting with any domain wall with fivebranes arranged to satisfy our rules, the  $S$ -dual can be rearranged in the usual way to also satisfy them, giving another domain wall with a simple gauge theory description.

It is also possible to develop a general recipe using  $T(SU(n))$ . Let  $\mathcal{D}$  be a domain wall and  $\mathcal{D}^\vee$  its  $S$ -dual. Suppose that  $\mathcal{D}^\vee$  arises by coupling to four-dimensional gauge fields a three-dimensional theory  $\mathfrak{D}^\vee$  with  $G_+ \times G_-$  symmetry. We can recover the theory  $\mathfrak{D}^\vee$  by terminating the four-dimensional gauge theory with Dirichlet boundary conditions at  $y = \pm L$ . The  $S$ -dual of this gives the mirror of the theory  $\mathfrak{D}^\vee$  as the infrared limit of a composite configuration described in figure 42. In this configuration, boundary conditions at the two ends are provided by coupling to  $T(SU(n_+))$  and  $T(SU(n_-))$ , and the original domain wall  $\mathcal{D}$  appears in the center. The low-energy limit, assuming that there are no Nahm poles in the definition of  $\mathcal{D}$ , is a three-dimensional gauge theory with gauge group  $H$  coupled to a product  $T(SU(n_+)) \times T(SU(n_-)) \times \mathfrak{D}$ . If there are Nahm poles, then just as in Section 3.4, the low-energy theory is a three-dimensional gauge

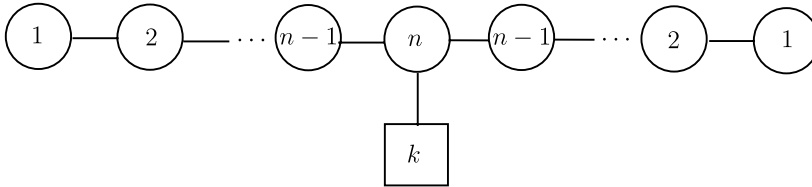


Figure 43: The result of applying the general recipe to the domain wall in  $U(n)$  gauge theory that consists of a coupling to  $k$  fundamental hypermultiplets (as sketched in figure 40(c)). This quiver is mirror to the one in figure 40(e).

theory with  $H$  coupled to  $T^{\rho+}(SU(n_+)) \times T^{\rho-}(SU(n_-)) \times \mathfrak{D}$ . If this three-dimensional gauge theory has a standard IR limit, that limit will be the mirror of the desired theory  $\mathfrak{D}^\vee$ .

A simple example is to use this procedure to study the configuration of figure 40(a) or (c) with a defect supporting  $k$  fundamental hypermultiplets. The composite gauge theory is a  $U(n)$  gauge theory coupled to two copies of  $T(SU(n))$  and to  $k$  fundamental hypermultiplets. This theory is described by the quiver of figure 43. For  $k > 1$ , this is a good quiver, whose mirror is the quiver that we found in figure 40(e) by direct brane manipulations.

For  $k = 1$ , the quiver of figure 43 is an ugly quiver with a single minimally unbalanced node in the center, and chains of  $n - 1$  balanced nodes on each side. As explained in Section 2.4.3, this will generate in the infrared an  $SU(n) \times SU(n) \times U(1)$  symmetry acting on a free bifundamental hypermultiplet. For this particular example, the dimension of the Coulomb branch is  $n^2$ , the same as the number of free hypermultiplets, so the infrared limit of the quiver theory describes the free hypermultiplets only, with no additional degrees of freedom. This is the expected answer of figure 40(b).

For  $k = 0$ , which describes an empty or trivial domain wall, we get a bad quiver. This is in accord with the fact that the  $S$ -dual, which is also the trivial domain wall, does not have the full  $U(n) \times U(n)$  gauge symmetry on the two sides, but only the diagonal  $U(n)$ . We will return to this example in Section 4.3, but for now we simply note that the hyper-Kähler dimension of the Coulomb branch is  $n^2$ , which is what one needs if the Coulomb branch is to break  $U(n) \times U(n)$  to the diagonal  $U(n)$ .

Similarly, we can apply this recipe to seek a dual of figure 40(b), where the domain wall, constructed from a single NS5-brane, has the full  $U(n) \times U(n)$  symmetry coupled to a single bifundamental hypermultiplet. The associated composite gauge theory is a  $U(n) \times U(n)$  gauge theory coupled to two copies



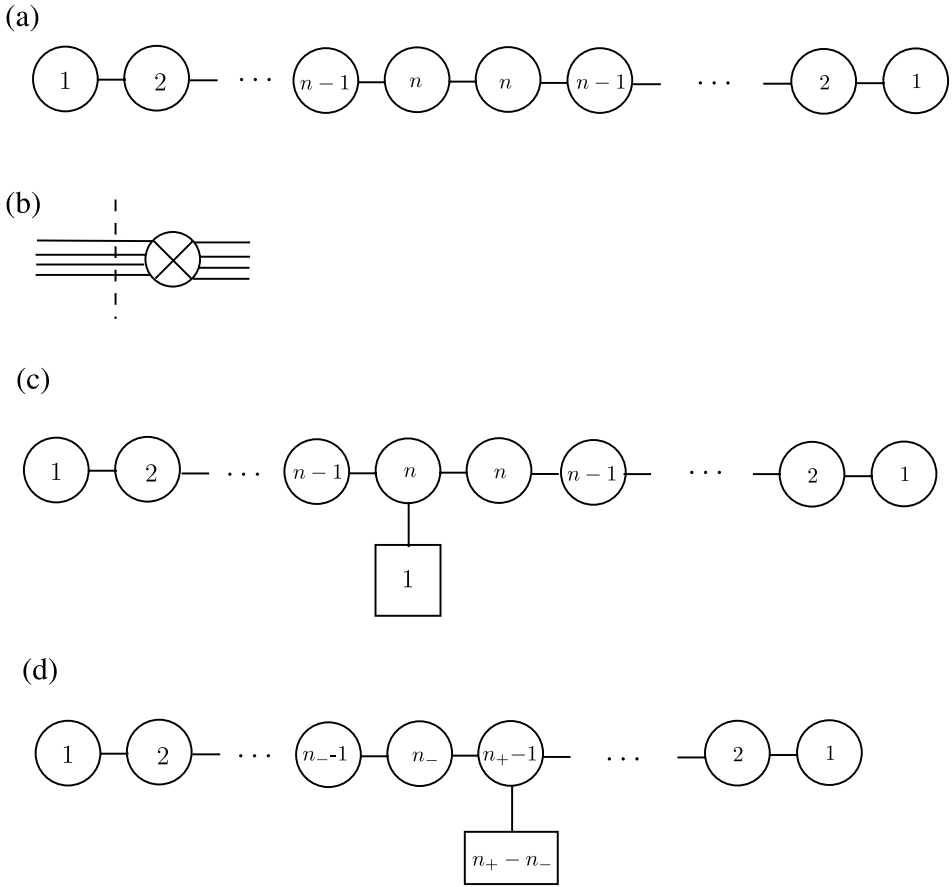


Figure 44: (a) The ungauging recipe, applied to the problem of generating an  $S$ -dual of the domain wall of figure 40(b), produces a composite  $U(n) \times U(n)$  gauge theory coupled to two copies of  $T(SU(n))$  and a bifundamental hypermultiplet. This is a bad quiver with two adjacent minimally unbalanced nodes, so the infrared analysis is not straightforward. (b) A domain wall constructed from a D5-brane and an NS5-brane (as opposed to the single NS5-brane of figure 40(b)).  $S$ -duality merely exchanges the two. (c) The associated quiver gauge theory, which differs from that in (a) by adding a fundamental hypermultiplet at one node. This is an ugly quiver with a single minimally unbalanced node, reflecting the fact that the mirror domain wall (which is that of (b) with the two fivebranes exchanged) has full  $U(n) \times U(n)$  gauge symmetry with free fundamental and bifundamental hypermultiplets. (d) The ungauging procedure applied to the domain wall of figure 41(a) leads to this quiver, after a rearrangement of branes.

of  $T(SU(n))$  and a bifundamental hypermultiplet, described by the quiver of figure 44(a). This is a bad quiver with two adjacent minimally unbalanced nodes, in keeping with the fact that the dual domain wall of figure 40(a) has reduced gauge symmetry.

There is an amusing modification of this problem for the domain wall in figure 44(b), which is constructed from a D5-brane to the left of an NS5-brane. The  $S$ -dual configuration is a mirror image, with a single D5-brane to the right of an NS5-brane. The composite gauge theory is described by the quiver in figure 44(c), which has a single minimally unbalanced node with  $n$  balanced nodes to the left,  $n - 1$  to the right. Again the dimension of the Coulomb branch,  $n(n + 1)$ , agrees with the number of free hypermultiplets realized as monopole operators. These are the expected bifundamental and fundamental free hypermultiplets at the dual boundary.

Finally we want to apply our construction to figure 41(a). The resulting composite gauge theory is described by the quiver depicted in figure 44(d), which is good as long as  $n_+ - n_- > 1$ . The quiver is mirror to the expected result in figure 41(b). If  $n_+ - n_- = 1$ , the quiver is ugly (and equivalent to that of figure 44(c)); its Coulomb branch flows in the infrared to the expected  $n \times (n + 1)$  bifundamental free twisted hypermultiplets.

To give an example of analyzing in detail the IR behavior of a bad quiver, we will consider the quiver in figure 43 at  $k = 0$ . This quiver gauge theory is supposed to be  $S$ -dual to a trivial domain wall. The trivial domain wall breaks the product of the  $U(n)$  gauge groups on the left and right to a diagonal  $U(n)$ . So the Coulomb branch of the bad quiver in question should break  $U(n) \times U(n)$  to the diagonal subgroup. The easiest way to find the Coulomb branch  $\mathcal{C}$  of the quiver is to use the  $S$ -dual description, which is simply a trivial domain wall cut off at both ends. In other words, the  $S$ -dual description is by four-dimensional  $\mathcal{N} = 4$  gauge theory on a slab  $\mathbb{R}^3 \times I$  with Dirichlet boundary conditions on both boundaries. The Coulomb branch  $\mathcal{C}$  is simply the moduli space of solutions of Nahm's equations on the interval  $I$ . Here Nahm's equations are the equations  $D\vec{X}/Dy + \vec{X} \times \vec{X} = 0$ , for fields  $\vec{X}$  and  $A = A_y$ ;  $\mathcal{C}$  is the moduli space of solutions of these equations, modulo gauge transformations that equal 1 at both ends of  $I$ . This is an important hyper-Kähler manifold described in [25] and reviewed in [1], Section 3.9.1. In the analysis of this manifold, the quantity  $W(p)$  defined in equation (3.14) plays an important role. It is holomorphic on  $\mathcal{C}$  in one of the complex structures; in physical terms, its expectation value  $\langle W(p) \rangle$  is a  $\mathrm{GL}(n, \mathbb{C})$ -valued function on the moduli space  $\mathcal{C}$  of vacua that because of the chiral nature of the operator  $W(p)$  is holomorphic in one complex structure. From a holomorphic point of view, this expectation value breaks the product of the left and right action of  $\mathrm{GL}(n, \mathbb{C})$  to a diagonal subgroup. In gauge

theory terms, the symmetry is  $U(n) \times U(n)$  broken to a diagonal  $U(n)$  (or a subgroup thereof, depending on the choice of a point in  $\mathcal{C}$ ).  $\mathcal{C}$  is smooth, so the quiver gauge theory certainly has no interesting critical point.

## 4 S-duality and Janus

An important ingredient in the last section was a self-mirror conformal field theory  $T(SU(n))$ , which we found most directly in studying the  $S$ -dual of Dirichlet boundary conditions. We used this, and its generalization  $T^\rho(SU(n))$ , to describe the  $S$ -dual of a very wide class of boundary conditions.

In this section, we will extend our results to any compact gauge group  $G$ . We will give an intrinsic definition of a three-dimensional conformal field theory  $T(G)$  analogous to  $T(SU(n))$  and use it, and a generalization  $T^\rho(G)$ , to formulate a recipe analogous to that of Section 3. As in Section 3, we will also consider a further generalization  $T_{\rho^\vee}^\rho(G)$ .

### 4.1 $T(G)$

One of the important properties of  $T(G)$  will be that it has global symmetry  $G \times G^\vee$ , where  $G^\vee$  is the dual group to  $G$ . (The groups acting faithfully are the adjoint forms of  $G$  and  $G^\vee$ , so the distinction between them is only important when they have different Lie algebras. That is why we did not encounter this distinction in Section 2.) The mirror of  $T(G)$  is  $T(G^\vee)$ .  $T(G)$  will appear as the dual of Dirichlet boundary conditions in  $G$  gauge theory.

$G$  acts on the Higgs branch of  $T(G)$ , and  $G^\vee$  acts on its Coulomb branch. The Higgs and Coulomb branches of  $T(G)$ , in any of their complex structures, are the nilpotent cones  $\mathcal{N}$  and  $\mathcal{N}^\vee$  of  $G$  and  $G^\vee$ , respectively. As reviewed more fully in Section 3 of [1],  $\mathcal{N}$  is the space of all nilpotent elements of the Lie algebra  $\mathfrak{g}_\mathbb{C}$  of the complexification  $G_\mathbb{C}$  of  $G$ . It is a union of finitely many nilpotent  $G_\mathbb{C}$  orbits. Each such orbit is the orbit of a nilpotent element  $\rho_+ \in \mathfrak{g}_\mathbb{C}$ , which is the image of the raising operator of  $\mathfrak{su}(2)$  under some homomorphism  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{g}_\mathbb{C}$ .  $\mathcal{N}$  is actually the closure of a single nilpotent orbit  $\mathcal{O}$  associated to a regular  $\mathfrak{su}(2)$  subalgebra. The other orbits are of positive codimension.

The moduli space  $\mathcal{M}_{T(G)}$  of vacua of  $T(G)$  does not just consist of Higgs and Coulomb branches, as there are also mixed Higgs–Coulomb branches.

The full structure of the moduli space is a union of components

$$\mathcal{M}_{T(G)} = \bigcup_{\alpha \in S} \mathcal{C}_\alpha \times \mathcal{H}_\alpha, \tag{4.1}$$

where  $S$  is the set of components and we call  $\mathcal{C}_\alpha$  and  $\mathcal{H}_\alpha$  the Coulomb and Higgs factors of the  $\alpha$ th component.  $G$  and  $SO(3)_X$  act non-trivially on  $\mathcal{H}_\alpha$  and trivially on  $\mathcal{C}_\alpha$ , and reciprocally  $G^\vee$  and  $SO(3)_Y$  act non-trivially on  $\mathcal{C}_\alpha$  and trivially on  $\mathcal{H}_\alpha$ . The Higgs branch is a component with  $\mathcal{C}_\alpha$  equal to a point and  $\mathcal{H}_\alpha$  equal to  $\mathcal{N}$ , and the Coulomb branch has  $\mathcal{H}_\alpha$  equal to a point and  $\mathcal{C}_\alpha = \mathcal{N}^\vee$ . We write simply  $\mathcal{H}$  and  $\mathcal{C}$  for the Higgs and Coulomb branches.

The reason that mixed branches exist is that, by adjusting parameters on  $\mathcal{C}$ , one can go to a locus at which a Higgs branch opens up. Let  $\mathcal{C}_\alpha$  be an irreducible component of the locus in  $\mathcal{C}$  at which this happens, and let  $\mathcal{H}_\alpha$  be the corresponding Higgs branch. Then the moduli space of vacua contains a component  $\mathcal{C}_\alpha \times \mathcal{H}_\alpha$ .  $\mathcal{C}_\alpha$  is a  $G^\vee$ -invariant hyper-Kähler subspace of  $\mathcal{C}$ . These properties imply that (as a complex manifold in one of its complex structures)  $\mathcal{C}_\alpha$  is a union of nilpotent orbits of  $G_{\mathbb{C}}^\vee$ . Since we have assumed  $\mathcal{C}_\alpha$  to be irreducible, it is actually the closure of a single such orbit, associated with some homomorphism<sup>14</sup>  $\rho_\alpha : \mathfrak{su}(2) \rightarrow \mathfrak{g}_{\mathbb{C}}^\vee$ . Applying the same argument starting on  $\mathcal{H}$ , we learn that each  $\mathcal{H}_\alpha$  is similarly the closure of a nilpotent orbit of  $G_{\mathbb{C}}$ .

Since each  $\mathcal{H}_\alpha$  or  $\mathcal{C}_\alpha$  is the closure of a nilpotent orbit  $\mathcal{O}_{\rho_\alpha}$  or  $\mathcal{O}_{\rho_\alpha^\vee}$ , the general form of the moduli space is

$$\mathcal{M}_{T(G)} = \bigcup_{\alpha \in S} \overline{\mathcal{O}_{\rho_\alpha}} \times \overline{\mathcal{O}_{\rho_\alpha^\vee}}. \tag{4.2}$$

The union in (4.2) is definitely not a disjoint union, as the various components meet on subspaces. All this can be made more explicit for  $G = SU(n)$ , using the representation of  $T(SU(n))$  as a quiver. In that case, every  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{su}(n)$  appears in the sum exactly once, and is paired with its dual  $\rho_D$ .

### 4.1.1 Janus and the $S$ -dual of Dirichlet

The strategy that we will follow to construct  $T(G)$  in general is as follows. We start with a half-BPS domain wall with  $G$  gauge theory for  $y < 0$  and  $G^\vee$  gauge theory for  $y > 0$ . There are many such domain walls, but there is a minimal one that we call the Janus domain wall.

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<sup>14</sup>The  $\rho_\alpha$  of distinct  $\alpha$  are not necessarily inequivalent.

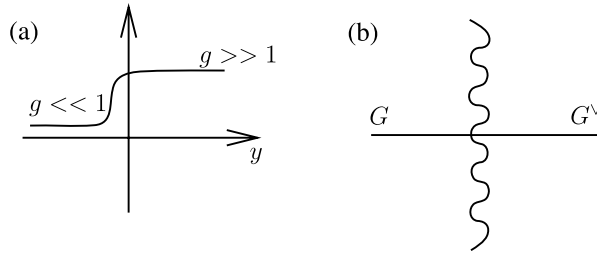


Figure 45: (a) A half-BPS configuration in  $G$  gauge theory in which the coupling  $g(y)$  depends on  $y$ ; it is small for  $y < 0$  and large for  $y > 0$ . (b) Applying  $S$ -duality for  $y > 0$ , we get an equivalent configuration described by weakly coupled  $G$  gauge theory on the left and  $G^\vee$  gauge theory on the right, coupled to a superconformal field theory  $T(G)$  with  $G \times G^\vee$  global symmetry. This SCFT is schematically denoted by the vertical wiggly line.

The general half-BPS Janus configuration [9–13] is a configuration in which the gauge coupling  $g$  is a general function  $g(y)$ . No additional degrees of freedom are added; one just considers  $\mathcal{N} = 4$  super Yang–Mills with  $y$ -dependent coupling. This configuration admits a smooth limit to a domain wall — a configuration in which  $g(y)$  is constant for  $y < 0$  and for  $y > 0$ , with a jump at  $y = 0$ . This limiting configuration is half-BPS and superconformally invariant, and we call it the Janus domain wall.

We consider a Janus domain wall with a coupling  $g(y)$  that is very small for  $y < 0$  and very large for  $y > 0$  (figure 45(a)). Making  $S$ -duality in the region  $y > 0$ , we reduce to a configuration with weak coupling on both sides. However, the gauge group is  $G$  to the left and  $G^\vee$  to the right. In the limit that the coupling is extremely small on both sides, we are left with some sort of superconformal field theory weakly coupled to  $G$  gauge fields in one half space and to  $G^\vee$  gauge fields in the other. We call this superconformal field theory  $T(G)$ .

We can investigate  $T(G)$  by ungauging the gauge fields on either or both sides. For instance, let us introduce Dirichlet boundary conditions for  $G^\vee$  at  $y = L$  (see figure 46). After making  $S$ -duality in the region  $y > 0$ , we have a Janus configuration (with  $y$ -dependent coupling  $g(y)$ ) that terminates at  $y = L$  with the  $S$ -dual of Dirichlet boundary conditions. If we regularize the Janus domain wall by choosing a smooth function  $g(y)$  and we flow to the IR, the non-trivial profile of  $g$  will just flow away, and we will be left with the dual of Dirichlet boundary conditions, and a constant, small  $g$  everywhere. This shows that  $T(G)$  is the boundary SCFT which defines the  $S$ -dual of a Dirichlet boundary condition for  $G^\vee$ .

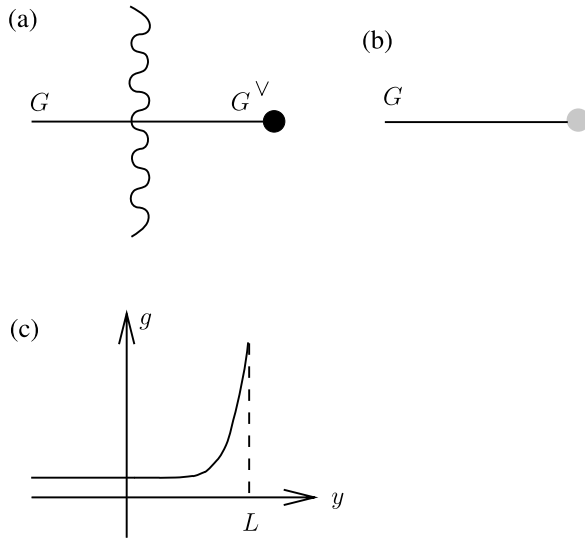


Figure 46: (a) The  $G^\vee$  symmetry of figure 45(b) can be ungauged — and converted to a global symmetry — by terminating the figure on the right with Dirichlet boundary conditions, which are schematically indicated by the black dot. (b) Applying  $S$ -duality on the right of the figure, we get  $G$  gauge theory coupled to the  $S$ -dual of Dirichlet boundary conditions — schematically indicated here with the shaded dot — and with a non-trivial coupling function  $g(y)$ . (c) The profile of  $g(y)$  after a deformation. The coupling is weak except very near the boundary at  $y = L$ . The IR limit is obtained by shrinking away the strongly coupled region, leaving weakly coupled  $G$  gauge theory on a half-space with boundary conditions that are the  $S$ -dual in  $G$  gauge theory of Dirichlet boundary conditions in  $G^\vee$  gauge theory. So the  $S$ -dual of Dirichlet is coupling to the SCFT  $T(G)$ .

Now consider  $G$  gauge theory on a half-space coupled to  $T(G)$  on the boundary. The gauging kills the Higgs branch of  $T(G)$ , but leaves the Coulomb branch, which we would like to identify. The easiest way to do this is to apply  $S$ -duality, which converts the gauge group into  $G^\vee$  and turns the boundary coupling to  $T(G)$  into Dirichlet boundary conditions. So we are simply left with  $G^\vee$  gauge theory on a half-space with Dirichlet boundary conditions. The moduli space of vacua of this theory (with fields vanishing at infinity) can be found by solving Nahm’s equations and is equal to the nilpotent cone  $\mathcal{N}^\vee$  of  $G^\vee$ , as explained in Section 3 of [1]. This therefore is also the Coulomb branch of  $T(G)$ .

If we introduce Dirichlet boundary conditions for both gauge groups, at  $y = L$  and  $y = -L$ , respectively, this will have the effect of ungauging both

Table 1: The Higgs and Coulomb branches  $\mathcal{H}$  and  $\mathcal{C}$  of the conformal field theory  $T(G)$  and its generalizations. Here  $\mathcal{S}_\rho$  denotes the Slodowy slice transverse to the raising operator  $\rho_+$  of  $\rho$ ,  $\mathcal{O}_\rho$  is the orbit of  $\rho_+$ , and  $\overline{\mathcal{O}}_\rho$  is the closure of this orbit. Finally,  $C_{\rho^\vee}$  is the set of all  $\alpha$  such that  $\rho_\alpha^\vee = \rho^\vee$ , and  $C_\rho$  is the set of  $\alpha$  such that  $\rho_\alpha = \rho$ .

	$\mathcal{H}$	$\mathcal{C}$
$T(G)$	$\mathcal{N}$	$\mathcal{N}^\vee$
$T_{\rho^\vee}(G)$	$\cup_{\alpha \in C_{\rho^\vee}} \overline{\mathcal{O}}_{\rho_\alpha}$	$\mathcal{S}_{\rho^\vee} \cap \mathcal{N}^\vee$
$T^\rho(G)$	$\mathcal{S}_\rho \cap \mathcal{N}$	$\cup_{\alpha \in C_\rho} \overline{\mathcal{O}}_{\rho_\alpha^\vee}$
$T_{\rho^\vee}^\rho(G)$	$\mathcal{S}_\rho \cap (\cup_{\alpha \in C_{\rho^\vee}} \overline{\mathcal{O}}_{\rho_\alpha})$	$\mathcal{S}_{\rho^\vee} \cap (\cup_{\alpha \in C_\rho} \overline{\mathcal{O}}_{\rho_\alpha^\vee})$

gauge groups. Moreover, we will be left with  $G$  global symmetry acting at  $y = -L$  and  $G^\vee$  at  $y = L$ . In the infrared, we will recover  $T(G)$ .

If we exchange the two ends (and  $\vec{X}$  and  $\vec{Y}$ , as well), we see the same configuration with  $G$  and  $G^\vee$  exchanged. So  $T(G)$  and  $T(G^\vee)$  are a pair of mirror SCFTs. In particular, the mirror symmetry implies that since  $\mathcal{N}^\vee$  is the Coulomb branch of  $T(G)$ , its Higgs branch is  $\mathcal{N}$ , the nilpotent cone of  $G$ .

### 4.2 Including the Nahm pole

The ungauging procedure can be generalized to include Nahm poles at the two ends  $y = -L$  and  $y = L$ , associated, respectively, with homomorphisms  $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{g}$  and  $\rho^\vee : \mathfrak{su}(2) \rightarrow \mathfrak{g}^\vee$ . If this configuration has a standard IR limit, we denote the resulting SCFT as  $T_{\rho^\vee}^\rho(G)$ . Its mirror, arrived at by exchanging the two ends of the picture, is  $T_{\rho^\vee}^\rho(G^\vee)$ .

We would like to determine the Higgs and Coulomb branches of these theories (the results are summarized in Table 1). In doing this, it is convenient to start with the case that  $\rho$  or  $\rho^\vee$  is trivial. In any event, this will be the most important case in the present paper.

Consider first  $T_{\rho^\vee}(G)$ . (The corresponding analysis of  $T^\rho(G)$  is made by simply exchanging  $G$  and  $G^\vee$  and using mirror symmetry.) By definition, it is obtained from a configuration with Dirichlet boundary conditions of  $G$  at the left of an interval, Janus in the center of the interval, and Dirichlet boundary conditions modified by  $\rho^\vee$  at the right of the interval (figure 47(a)). If we move Janus to the right (figure 47(b)), we get a configuration with gauge group  $G$ , the  $S$ -dual of  $\rho^\vee$  on the right, and Dirichlet boundary conditions on the left. As usual, Dirichlet boundary conditions

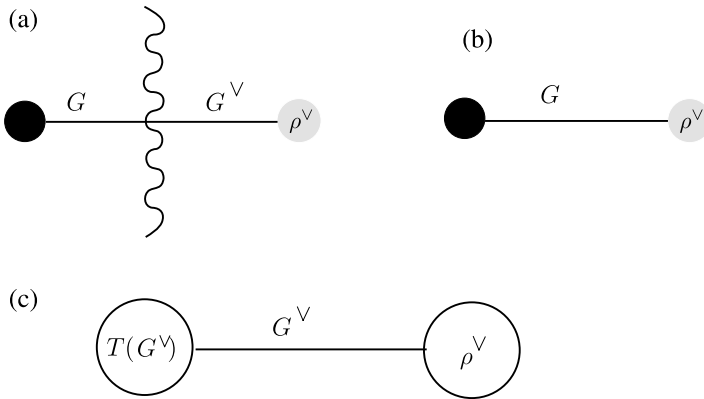


Figure 47: (a)  $G$  and  $G^\vee$  gauge theories joined by a Janus domain wall (wiggly line). On the left, we take Dirichlet boundary conditions for  $G$  (symbolized by the black dot), and on the right, Dirichlet boundary conditions for  $G^\vee$  modified by  $\rho^\vee$  (gray dot labeled  $\rho^\vee$ ). (b) Moving Janus to the right gives a  $G$  gauge theory with Dirichlet boundary conditions on the left and  $T_{\rho^\vee}(G)$ , the  $S$ -dual of  $\rho^\vee$  (that is, the  $S$ -dual of Dirichlet modified by  $\rho^\vee$ ) on the right. (c) Moving Janus to the left gives a  $G^\vee$  gauge theory, coupled on the left to  $T(G^\vee)$  (the  $S$ -dual of Dirichlet for  $G$ ) and to  $\rho^\vee$  on the right.

just ungauged  $G$ , leaving it as a global symmetry. The SCFT represented by figure 47(b) is therefore the  $S$ -dual of  $\rho^\vee$ , and this then is  $T_{\rho^\vee}(G)$ .

On the other hand, to describe the moduli space of vacua of  $T_{\rho^\vee}(G)$ , it is more convenient to move Janus to the left (figure 45(c)). Then we get a  $G^\vee$  gauge theory, coupled on the left to  $T(G^\vee)$ , and with boundary conditions set by  $\rho^\vee$  on the right. The Coulomb branch of vacua must be found by solving the  $G^\vee$  Nahm equations  $D\vec{X} + \vec{X} \times \vec{X} = 0$  with suitable conditions at the endpoints  $y = \pm L$ :

(1) At  $y = L$ ,  $\vec{X}$  must have a Nahm pole of type  $\rho^\vee$ , that is  $\vec{X} \sim \rho^\vee(\vec{t})/(y - L)$ .

(2) And at  $y = -L$ , we require that  $\vec{X}(-L) + \vec{\mu} = 0$ , where  $\vec{\mu}$  is the moment map for the action of  $G^\vee$  on the Higgs branch of  $T(G^\vee)$ , which is the nilpotent cone  $\mathcal{N}^\vee$  of  $G^\vee$ .

It is convenient to describe the result as a complex symplectic manifold in one of its complex structures. As is explained in Section 3.3 of [1], condition (1) gives the Slodowy slice transverse to the  $G^\vee_{\mathbb{C}}$  orbit of the nilpotent element  $\rho^\vee(t_+) \in \mathfrak{g}^\vee_{\mathbb{C}}$  (here  $t_+$  is the “raising” operator in  $\mathfrak{su}(2)$ ). Condition (2) gives



the intersection of this Slodowy slice with the nilpotent cone  $\mathcal{N}^\vee$  (since  $\vec{\mu}$  takes values in this cone). We write  $\mathcal{S}_{\rho^\vee}$  for the Slodowy slice transverse to  $\rho^\vee(t_+)$ . So the Coulomb branch of  $T_{\rho^\vee}(G)$  is the intersection  $\mathcal{S}_{\rho^\vee} \cap \mathcal{N}^\vee$ , accounting for one of the entries in Table 1. Mirror symmetry then also gives the Higgs branch of  $T^\rho(G)$ .

What we have determined so far is the component of the moduli space of vacua of  $T_{\rho^\vee}(G)$  on which  $SO(3)_X$  acts trivially. To describe fully the moduli space  $\mathcal{M}_{T_{\rho^\vee}(G)}$  of vacua of  $T_{\rho^\vee}(G)$ , it helps to be more systematic. One ingredient is the moduli space of vacua of  $T(G)$  (or its mirror  $T(G^\vee)$ ), whose general form was described in (4.2).

Once a particular vacuum is picked for  $T(G)$  at the end of the interval, to get a full description, we need to consider the behavior of  $\vec{X}^\vee$  and  $\vec{Y}^\vee$ .  $\vec{Y}^\vee$  will vanish, since the  $\rho^\vee$  boundary conditions set it to zero at  $y = L$ . However,  $\vec{X}^\vee$  can obey Nahm's equations. In solving Nahm's equations, the boundary condition (1) above is unchanged, but (2) is modified:

(2') At  $y = -L$ , we require that  $\vec{X}^\vee(-L) + \vec{\mu} = 0$ , where  $\vec{\mu}$  is the moment map for the action of  $G^\vee$  on  $\mathcal{M}_{T(G)}$ .

The full moduli space is therefore the intersection of  $\mathcal{M}_{T(G)}$  with the Slodowy slice  $\mathcal{S}_{\rho^\vee}$ . (We intersect  $\mathcal{S}_{\rho^\vee}$  with each Coulomb branch factor of  $\mathcal{M}_{T(G)}$  or Higgs branch factor of  $\mathcal{M}_{T(G^\vee)}$ .) The moduli space of vacua of  $T_{\rho^\vee}(G)$  is therefore

$$\mathcal{M}_{T_{\rho^\vee}(G)} = \bigcup_{\alpha \in S} \overline{\mathcal{O}}_{\rho_\alpha} \times (\mathcal{S}_{\rho^\vee} \cap \overline{\mathcal{O}}_{\rho_\alpha^\vee}). \tag{4.3}$$

For some  $\alpha$ , the intersection  $\mathcal{S}_{\rho^\vee} \cap \overline{\mathcal{O}}_{\rho_\alpha^\vee}$  may be empty, as the Slodowy slice transverse to  $\rho^\vee$  will not intersect an orbit that is too small. For other  $\alpha$ , this intersection has positive dimension, giving a branch of the moduli space that has a non-trivial Coulomb factor. To get a component of the Higgs branch of  $\mathcal{M}_{T_{\rho^\vee}(G)}$ , the intersection  $\mathcal{S}_{\rho^\vee} \cap \overline{\mathcal{O}}_{\rho_\alpha^\vee}$  should have dimension zero, which happens precisely if  $\rho_\alpha^\vee = \rho^\vee$ , in which case the intersection is a single point. Let  $C_{\rho^\vee}$  be the set of all  $\alpha$  such that  $\rho_\alpha^\vee = \rho^\vee$ . The Higgs branch of  $T_{\rho^\vee}(G)$  is then  $\bigcup_{\alpha \in C_{\rho^\vee}} \overline{\mathcal{O}}_{\rho_\alpha}$ . This accounts for another entry in the table, and of course mirror symmetry gives also the Coulomb branch of  $T^\rho(G)$ .

Finally, we would like to explain the last row in the Table 1, which describes the Higgs and Coulomb branches of  $T_{\rho^\vee}^\rho(G)$ . We start (figure 48)

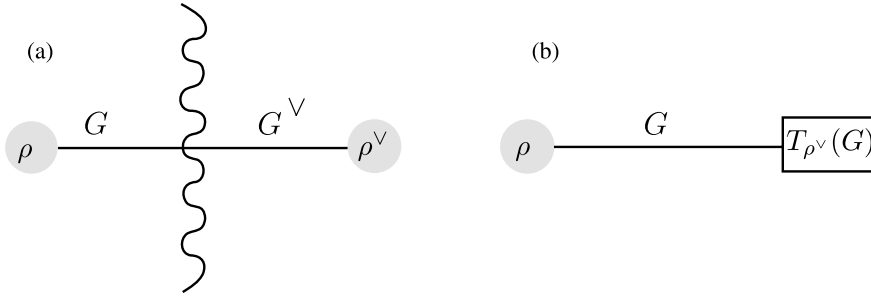


Figure 48: (a) The definition of the theory  $T_{\rho^\vee}^\rho(G)$ . On the left is  $G$  gauge theory with boundary conditions set by  $\rho$ . On the right is  $G^\vee$  with boundary conditions set by  $\rho^\vee$ . Between is the Janus domain wall. The infrared limit of this configuration gives  $T_{\rho^\vee}^\rho(G)$ . (b) Upon moving the Janus domain wall to the right, we get a  $G$  gauge theory with  $\rho$  on the left and  $T_{\rho^\vee}(G)$  on the right. The IR limit is the same.

with the definition of this theory in terms of gauge theory on a slab with boundary conditions set by  $\rho$  at one end and by  $\rho^\vee$  at the other end, with a Janus domain wall in the middle. Moving the Janus domain wall to the right, we get a configuration in  $G$  gauge theory with boundary conditions set by  $\rho$  on the left and by a coupling to  $T_{\rho^\vee}(G)$  on the right. To find a vacuum of  $T_{\rho^\vee}^\rho(G)$ , we start with a vacuum of  $T_{\rho^\vee}(G)$  and solve the  $G$  Nahm equations on an interval with the obvious modification of the above conditions:  $\vec{X}$  has a pole of type  $\rho$  at the left, and equals the moment map of  $T_{\rho^\vee}(G)$  on the right. As a complex manifold, the result is the intersection of the Slodowy slice  $\mathcal{S}_\rho$  transverse to  $\rho$  with the moduli space  $\mathcal{M}_{T_{\rho^\vee}}(G)$ . Using the description (4.3) of that moduli space, we arrive at a pleasantly symmetric description of the moduli space for  $T_{\rho^\vee}^\rho(G)$ :

$$\mathcal{M}_{T_{\rho^\vee}^\rho(G)} = \bigcup_{\alpha \in S} (\mathcal{S}_\rho \cap \overline{\mathcal{O}}_{\rho_\alpha}) \times (\mathcal{S}_{\rho^\vee} \cap \overline{\mathcal{O}}_{\rho_\alpha^\vee}). \tag{4.4}$$

Picking  $\alpha$  so that one factor or the other is a point, we arrive at the last row of the table.

For  $G = SU(n)$ , the theories  $T_{\rho^\vee}^\rho(G)$  are the general good linear quiver theories, made from a chain of unitary gauge groups, as we have explained in Section 3.3.3. We have therefore described the moduli space of vacua of the general such theory, which would have been very hard to get directly.

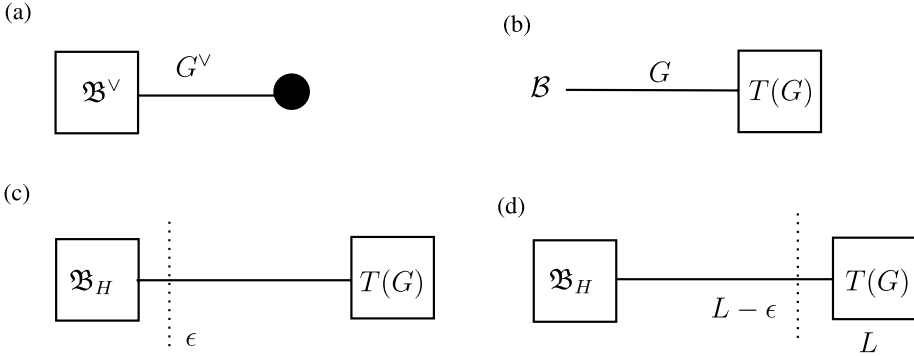


Figure 49: (a) The four-dimensional configuration which ungauges  $\mathfrak{B}^\vee$ .  $\mathfrak{B}^\vee$  is the  $S$ -dual of a general boundary  $\mathcal{B}$  given by a triple  $(\rho, H, \mathfrak{B}_H)$ , where  $H$  is a subgroup of the commutant of a Nahm pole  $\rho$ , and  $\mathfrak{B}_H$  is an SCFT with  $H$  symmetry. As usual the black dot represents Dirichlet boundary conditions. (b) The  $S$  dual of (a), involving a gauge theory with boundary condition  $\mathcal{B}$  on the left and coupling to  $T(G)$  on the right. (c) A split version of  $\mathcal{B}$ : the Nahm pole and reduction to  $H$  (indicated by the vertical dotted line) have been displaced to  $y = \epsilon > 0$ , leaving the SCFT  $\mathfrak{B}_H$  at the endpoint  $y = 0$ . (d) The Nahm pole is moved to the right. In the slab between  $y = L - \epsilon$  and  $y = L$ ,  $G$ -valued gauge fields are coupled to a Nahm pole at  $y = L - \epsilon$  and to  $T(G)$  at  $y = L$ . This generates  $T^\rho(G)$ . Between  $y = 0$  and  $y = L - \epsilon$  are  $H$ -valued gauge fields, coupled at  $y = 0$  to  $\mathfrak{B}_H$ , which was part of the original boundary condition, and at  $y = L - \epsilon$  to the global symmetry  $H$  of  $T^\rho(G)$ . Thus the mirror  $\mathfrak{B}^\vee$  is given by a composite gauge theory with  $H$  coupled to  $\mathfrak{B}_H \times T^\rho(G)$ .

### 4.3 A general duality prescription

Extending our arguments of Section 3 to an arbitrary gauge group  $G$ , we will now give a general recipe for understanding the  $S$ -dual of a boundary condition  $\mathcal{B}$  associated with a triple  $(\rho, H, \mathfrak{B}_H)$ , provided only that this  $S$ -dual has full  $G^\vee$  gauge symmetry, and therefore can be described by coupling to an SCFT  $\mathfrak{B}^\vee$  with  $G^\vee$  symmetry. The steps are depicted in figure 49.  $\mathfrak{B}^\vee$  can be extracted via the usual ungauging technique, introducing a Dirichlet boundary condition at  $y = L$  and flowing to the IR. The  $S$ -dual of that configuration is given as a  $G$  gauge theory on a slab  $\mathbb{R}^3 \times I$ , coupled to the original boundary condition  $\mathcal{B} = (\rho, H, \mathfrak{B}_H)$  at the left end, and to  $T(G)$  at the other end. When  $\rho$  is trivial and  $H = G$ , this gives at low energies a construction of the mirror  $\tilde{\mathfrak{B}}^\vee$  to  $\mathfrak{B}^\vee$  in terms of a three-dimensional gauge theory with gauge group  $G$  coupled to the product  $\mathfrak{B} \times T(G)$ .

In the general case (figure 49(c)), a boundary condition  $(\rho, H, \mathfrak{B}_H)$  can be “split” in space: as we approach the boundary from  $y > 0$  we may first encounter the Nahm pole  $\rho$ , followed by the reduction of the gauge symmetry to a subgroup  $H$  (which must commute with  $\rho$ ), and only then the coupling of  $H$  to a boundary theory  $\mathfrak{B}_H$ . This three stage nature of the boundary condition is illustrated in figure 1 (where a description of the first two stages by branes is assumed). The precise positions in  $y$  at which the Nahm pole and the reduction of the gauge group are located are not important when we flow to the infrared.

The usefulness of this splitting is that the domain wall that carries the Nahm pole and reduction of gauge symmetry can then be moved to the right as in figure 49(d), towards the  $T(G)$  boundary. Let us divide the configuration of figure 49(d) into two slabs. From the domain wall at  $y = L - \epsilon$  to the boundary at  $y = L$ , we have  $G$  gauge fields interacting with a Nahm pole at  $y = L - \epsilon$  and with  $T(G)$  at  $y = L$ . This (in the limit  $\epsilon \rightarrow 0$ ) is the definition of  $T^\rho(G)$ . Between  $y = 0$  and  $y = L - \epsilon$ , we have  $H$  gauge fields. These gauge fields interact at  $y = 0$  with the SCFT  $\mathfrak{B}_H$  that was part of the original boundary condition. At  $y = L - \epsilon$ , they couple to the fields in the other slab.

The slab in figure 49(d) between the domain wall at  $y = L - \epsilon$  and the boundary at  $y = L$  is the definition of  $T^\rho(G)$ : gauge fields of  $G$  with the Nahm pole at the left boundary and coupling to  $T(G)$  on the right. Gauge fields of  $H$  propagate to the left of  $y = L - \epsilon$ . Of course, the slabs to the left and right of the domain wall meet at their common boundary  $y = L - \epsilon$ . This means that the  $H$ -valued gauge fields on the left slab are coupled to a global  $H$ -symmetry of the matter system defined by the right slab. Indeed, since  $H$  commutes with  $\rho$ , it is a global symmetry of  $T^\rho(G)$ , acting on the Higgs branch.

Hence we can formulate a prescription for  $\tilde{\mathfrak{B}}^\vee$ , whenever it exists. It is the IR limit of a composite  $H$  gauge theory depicted in figure 49(d). This theory, which we call  $\mathfrak{B}_H \times_H T^\rho(G)$ , is a theory with gauge group  $H$  coupled to the product  $\mathfrak{B}_H \times T^\rho(G)$ . In flowing to the infrared, one can take zero modes of all vector multiplets in the  $y$ -direction, so that the composite gauge theory is purely three dimensional.

In using  $\mathfrak{B}_H \times_H T^\rho(G)$  to define a boundary condition in  $G^\vee$  gauge theory, the four-dimensional  $G^\vee$  gauge fields couple to symmetries of the Coulomb branch of  $T^\rho(G)$ . As usual, this construction is most useful if one can find a description in which the  $G^\vee$  symmetry of the Coulomb branch is visible.

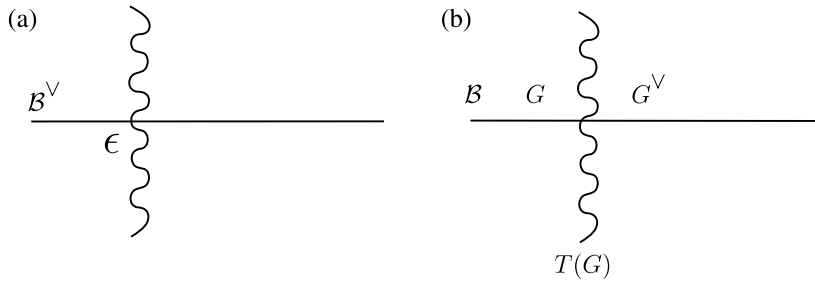


Figure 50: (a)  $G^\vee$  gauge theory on a half-space, with boundary condition  $\mathcal{B}^\vee$ . The gauge coupling is small except within a distance  $\epsilon$  from the boundary, to the left of the wiggly line. (b)  $S$ -duality to the left of  $y = \epsilon$  makes the gauge coupling small everywhere, but now the gauge group is  $G$  for  $y < \epsilon$  and there is a coupling to  $T(G)$  at  $y = \epsilon$ .

### 4.3.1 Symmetry breaking

If we are given a boundary condition  $\mathcal{B}$ , we can construct the composite gauge theory as above. If this theory has a standard infrared limit, with the usual  $R$ -symmetry and global  $G^\vee$  symmetry unbroken, then the dual boundary condition has full gauge symmetry and is obtained by coupling to the SCFT that emerges from the composite gauge theory.

What can we say when the composite gauge theory does not have a standard IR limit? To get some insight, we will describe a more conservative variant of the above procedure that is always valid.

Our basic procedure so far has been to ungauged the dual  $G^\vee$  gauge symmetry by imposing Dirichlet boundary conditions on the  $S$ -dual theory at  $y = L$ . This gives a quasi three-dimensional theory, since the fourth dimension is compact, to which we then apply  $S$ -duality, after which we flow to the infrared. Only after completing the infrared flow and constructing a boundary SCFT  $\tilde{\mathfrak{B}}^\vee$  do we “regauge” the theory, coupling to bulk  $G^\vee$  gauge fields so as to describe the dual of the original  $G$  theory on a half-space.

When the quasi three-dimensional theory does not have a standard IR limit, the reduction at an intermediate step to a quasi three-dimensional description is not helpful. Instead, it is better to formulate a completely four-dimensional procedure that is always valid (figure 50). For this, we introduce the theory  $T(G)$  in a slightly different way. Starting with the dual  $G^\vee$  configuration, we increase the gauge coupling in the region  $y < \epsilon$ , something that does not affect the infrared behavior. When the  $G^\vee$  gauge coupling is strong for  $y < \epsilon$  and weak for  $y > \epsilon$ , we make  $S$ -duality in the region  $y < \epsilon$ .

This gives a description in which the coupling is weak everywhere. For  $y < \epsilon$ , we have weakly coupled  $G$  gauge theory, say with coupling  $g_G$ , coupled to the original boundary condition  $\mathcal{B}$ . For  $y > \epsilon$ , we have weakly coupled  $G^\vee$  gauge theory. On the interface between the two theories at  $y = \epsilon$  lives the theory  $T(G)$ .

This description is always valid, and the question is what we can learn from it. There are two scales in the problem:  $\epsilon$  sets the scale of the Kaluza–Klein modes of the  $G$  vector multiplets, while  $\epsilon/g_G^2$  sets the scale of the three-dimensional gauge coupling. To reduce to a boundary condition, we want to take  $\epsilon \rightarrow 0$ . To flow to the infrared, we also want to take  $\epsilon/g_G^2$  to zero. When the composite gauge theory does not have a good IR limit, the second operation is not straightforward. However, there is no trouble in reducing to a three-dimensional boundary theory by taking  $\epsilon \rightarrow 0$  with  $1/g_3^2 = \epsilon/g_G^2$  fixed. The result is that the four-dimensional  $G^\vee$  gauge theory is coupled to a composite three-dimensional field theory at the boundary, given by the usual prescription of figure 49(d), but with finite gauge coupling for the three-dimensional gauge theory.

At this stage, we have  $G^\vee$  gauge theory in the half-space  $y \geq 0$  coupled to a boundary theory that has full supersymmetry and  $R$ -symmetry but is not superconformal. We still want to take the IR limit  $1/g_3^2 \rightarrow 0$ . By hypothesis, the boundary theory alone does not behave well in this limit. The final step of taking the infrared limit  $1/g_3^2 \rightarrow 0$  has to be taken for the combined theory on the half-space. This can produce a boundary condition for the  $G^\vee$  gauge fields that involves reduced gauge symmetry, possibly with a Nahm pole. (The case that it produces a boundary condition with full gauge symmetry at the boundary is precisely the case that the IR flow of the composite gauge theory could have been carried out in purely three-dimensional terms.)

If the IR flow of the composite gauge theory spontaneously breaks its  $G^\vee$  global symmetries, then the boundary condition will have reduced gauge symmetry. Moreover, if the moment map operators  $\vec{\mu}$  for the  $G^\vee$  isometries of the composite gauge theory receive expectation values in the IR, the boundary condition  $\vec{X} = \vec{\mu}$  forces the scalar fields  $\vec{X}$  to acquire expectation values at the boundary. Supersymmetry will then require that  $\vec{X}(y)$  should obey the Nahm equations. By dimensional analysis, the expectation value of  $\vec{X}(0)$  is proportional to the cutoff  $\epsilon^{-1}$ , so in this situation a Nahm pole will emerge when we remove the cutoff.

### 4.3.2 Examples

We will try to provide some simple examples of these phenomena, which involve the strong coupling dynamics of the three-dimensional gauge theory.

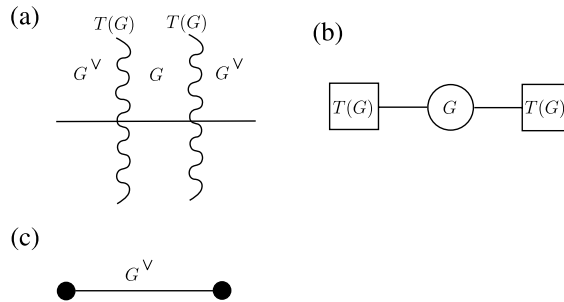


Figure 51: (a) The  $S$ -dual of a trivial domain wall in  $G$  gauge theory is a trivial domain wall in  $G^\vee$  gauge theory. Trying to show this using the ungauging procedure produces this configuration with two  $T(G)$  domain walls. The gauge group is everywhere  $G^\vee$  except in the central slab, where it is  $G$ . (b) The three-dimensional composite gauge theory has gauge group  $G$  coupled to two copies of  $T(G)$ . (c) A convenient way to compute its Coulomb branch is to describe it via  $G^\vee$  gauge theory on  $\mathbb{R}^3 \times I$  with Dirichlet boundary conditions at both ends. The Coulomb branch is the moduli space of solutions of Nahm’s equations.

*A Trivial domain wall*

The simplest example is a trivial domain wall for the gauge group  $G$ , which can be interpreted by the reflection trick as a boundary condition for a  $G \times G$  theory, broken to the diagonal  $G$  at the boundary. This example is the  $k = 0$  case of figure 40(c). Clearly the  $S$ -dual of a trivial  $G$  domain wall is a trivial  $G^\vee$  domain wall. We want to see how this result arises from the ungauging procedure. Naive application of this procedure leads to a bad quiver which is the  $k = 0$  case of figure 43.

We want to apply the general procedure of figure 50, but in the two-sided case of a domain wall, this procedure must be applied on both sides. The analog of figure 50 is therefore a two-sided configuration in  $G^\vee$  gauge theory with two Janus domain walls separating an interval in which the gauge group is  $G$  (figure 51). Clearly as we flow to the infrared, the two Janus domain walls will essentially meet and cancel each other, but we would like to understand how this happens when the Janus domain walls are represented by coupling to  $T(G)$ .

For this, we need to understand the Coulomb branch of  $G$  gauge theory on  $\mathbb{R}^3 \times I$  with coupling to  $T(G)$  at both ends. (Alternatively, we can consider a  $G$  gauge theory with finite coupling, coupled to two copies of  $T(G)$ .) The simplest way to do this is to use  $S$ -duality and the fact that  $T(G)$  is the  $S$ -dual of a Dirichlet boundary condition in  $G^\vee$  gauge theory. So we want

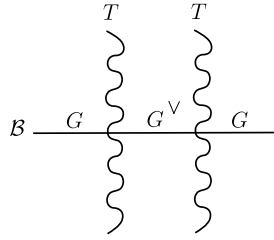


Figure 52: The square of an  $S$ -duality transformation of a boundary condition  $\mathcal{B}$ . At each step the original boundary condition is coupled to the Higgs branch of an appropriate theory  $T(G)$  or  $T(G^\vee)$ . Probing the IR dynamics of the  $G^\vee$  gauge theory in the central slab leads back to the original boundary condition. Alternatively, one can represent the two  $T$ 's by Janus domain walls and let the coupling constant profile flow away in the infrared.

the Coulomb branch of  $G^\vee$  gauge theory on the slab with Dirichlet boundary conditions at both ends. This is given by the moduli space of solutions of Nahm's equations  $D\vec{X}/Dy + \vec{X} \times \vec{X} = 0$  on  $I$ . Dirichlet boundary conditions simply mean that  $\vec{X}$  is arbitrary at both ends and that one divides only by gauge transformations that are trivial at both ends.

The hyper-Kähler manifold that arises from Nahm's equations in this situation was first studied by Kronheimer [25] and is described in [1], Section 3.9.1. (This manifold also entered at the end of Section 3.5 in relation to the same problem for  $SU(n)$ .) As a complex manifold in any of its complex structures, it is the cotangent bundle of  $G_{\mathbb{C}}^\vee$ . The  $G^\vee \times G^\vee$  symmetry acts by left and right multiplication on  $G_{\mathbb{C}}^\vee$ . A maximal unbroken subgroup is a diagonal  $G^\vee$  subgroup of  $G^\vee \times G^\vee$ .

In other words, on the Coulomb branch,  $G^\vee \times G^\vee$  is broken down to  $G^\vee$ , as one would expect for the trivial  $G^\vee$  domain wall, which is the  $S$ -dual of the trivial  $G$  domain wall with which we started.

*The square of S-duality*

We can use this construction to argue that our  $S$ -duality prescription defined using  $T(G)$  properly squares to the identity. Consider any  $G$  boundary condition  $\mathcal{B}$ , and apply our  $S$ -duality prescription twice, as in figure 52.

The result is essentially  $\mathcal{B}$  coupled through a  $G$  gauge theory to the Higgs branch of  $T(G)$ , whose Coulomb branch is in turn coupled through



a  $G^\vee$  gauge theory to the Higgs branch of  $T(G^\vee)$ . Finally, the bulk four-dimensional  $G$  gauge theory is coupled to the Coulomb branch of  $T(G^\vee)$ .

It is convenient to analyze the  $G^\vee$  dynamics first.  $G^\vee$  couples to two copies of  $T(G^\vee)$ . This is the  $S$ -dual of the system that we have just analyzed. We have learned that in the infrared the  $G \times G$  isometries of the Coulomb branch of this composite theory are broken to a diagonal  $G$ . As a result, we get a direct coupling of  $\mathcal{B}$  to the four-dimensional  $G$  gauge theory, as desired.

In effect, what we have just analyzed is very similar to the situation in figure 51, except that it is cut off on the left by the boundary condition  $\mathcal{B}$ .

There is a subtlety here, unrelated to the main ideas of this paper, which we mention only in the hope of avoiding some confusion. Our definition of  $S$  requires making a duality transformation in  $G$  gauge theory in a half-space and identifying the result, in that half-space, with  $G^\vee$  gauge theory. If  $G$  is a group such as  $SU(n)$  that has complex representations, then  $G$  and  $G^\vee$  gauge theory admit a non-trivial classical automorphism  $C$  of complex conjugation; in our definition of  $S$ , we did not pick an isomorphism with  $G^\vee$  gauge theory, so we did not distinguish  $S$  from the product  $SC$ . In defining the two  $S$  operations from  $G$  to  $G^\vee$  and from  $G^\vee$  to  $G$ , choices can be made such that  $S^2 = 1$ . However, taking account of the fact that in the relevant cases,  $G$  and  $G^\vee$  have the same Lie algebra, it is somewhat unnatural to define the two  $S$  operations independently, and a more natural set of choices actually leads to  $S^2 = C$ .  $C$  corresponds to the central element  $-1 \in SL(2, \mathbb{Z})$ .

### *Examples with symmetry breaking and Nahm pole*

The simplest example of a boundary condition  $\mathcal{B}$  whose dual has a Nahm pole is a boundary condition given by a coupling to  $T_{\rho^\vee}(G)$ . Almost by the definition of  $T_{\rho^\vee}(G)$ , the  $S$ -dual is the boundary condition  $\mathcal{B}^\vee$  in  $G^\vee$  gauge theory given by the  $\rho^\vee$  pole.

We want to understand how the general  $S$ -duality recipe can reproduce this fact. That prescription produces a composite gauge theory in which the three-dimensional gauge group  $G$  is coupled to the product  $T_{\rho^\vee}(G) \times T(G)$ . This theory does not have a good IR limit (as one can see from the quiver description if  $G = SU(n)$ ), so one should study it at finite gauge coupling. More conveniently, one can study  $G$  gauge theory on the slab  $\mathbb{R}^3 \times I$  with boundary conditions at the left and right set by coupling to  $T(G)$  and  $T_{\rho^\vee}(G)$ , respectively.

$S$ -duality converts this to  $G^\vee$  gauge theory on the slab with a  $\rho^\vee$  Nahm pole at one end and Dirichlet boundary conditions at the other end. The Coulomb branch is the moduli space of  $G^\vee$  Nahm equations on an interval, with  $\rho^\vee$  boundary conditions at the right end, and Dirichlet at the left end. This moduli space was described in [26] and reviewed in [1], Section 3.9.2. It has of course a  $G^\vee$  isometry. The moment map for this isometry is the initial value  $\vec{X}(0)$  of the scalar fields at the Dirichlet endpoint of  $I$ . The presence of the Nahm pole at the other endpoint forces this moment map to be non-zero. Indeed, any complex null combination such as  $\mu^1 + i\mu^2$  will be conjugate to the raising operator  $\rho^\vee(t_+)$ .

As a result, the coupling of the four-dimensional  $G^\vee$  gauge fields to the Coulomb branch of the composite gauge theory forces the bulk scalar fields  $\vec{X}$  to have non-zero expectation values at the boundary, in order to obey the boundary condition  $\vec{X} + \vec{\mu} = 0$ . By dimensional analysis,  $\vec{\mu}$  and therefore the boundary value of  $\vec{X}$  is proportional to  $1/L$ , the width of the slab. For  $L \rightarrow 0$ ,  $\vec{X}$  acquires a Nahm pole at the boundary, of type  $\rho^\vee$ .

#### 4.4 $T(U(1))$

Here we will describe  $T(G)$  in the abelian case  $G = U(1)$ .

For simple  $G$ ,  $T(G)$  is a non-trivial SCFT whose Higgs and Coulomb branches are copies of the nilpotent cones  $\mathcal{N}$  and  $\mathcal{N}^\vee$ . This suggests that  $T(U(1))$  might be trivial, since the nilpotent cone for the complexification  $GL(1)$  of  $U(1)$  consists of only one point (a non-zero element of the Lie algebra  $\mathfrak{gl}(1)$  is not nilpotent).

In a sense,  $T(U(1))$  is trivial as an SCFT but non-trivial as a recipe for coupling external vector multiplets. This means the following. Given the SCFT  $T(G)$ , we can couple it to  $G$ -valued and  $G^\vee$ -valued external gauge fields. The ability to do this is important in the way we use  $T(G)$  in constructing the  $S$ -dual of a given boundary condition.

For  $G = U(1)$ ,  $T(G)$  is trivial as an SCFT, and consists only of a prescription for coupling external vector fields. This turns out to be the following. Let  $V$  and  $W$  be two  $\mathcal{N} = 4$  vector multiplets that contain  $U(1)$  gauge fields  $B$  and  $C$ . Then the appropriate recipe is to couple them by the supersymmetric completion of the Chern–Simons coupling:

$$\frac{1}{2\pi} \int C \wedge dB. \tag{4.5}$$

An extension of this interaction with  $\mathcal{N} = 4$  supersymmetry does exist [4, 27]. Equation (4.5) is equivalent to saying that  $C$  is coupled to the current

$$J = \frac{\star F_B}{2\pi}, \tag{4.6}$$

where  $F_B$  is the curvature of  $B$ .

Assuming that this is  $T(U(1))$ , let us describe the appropriate recipe to find the dual of a general boundary condition  $\mathcal{B}$  in  $U(1)$  gauge theory. For brevity we suppose that  $\mathcal{B}$  is given by coupling to an SCFT  $\mathfrak{B}$  with  $U(1)$  symmetry. First, recall the case of simple  $G$ . In that case,  $T(G)$  has  $G \times G^\vee$  symmetry. We are supposed to introduce a  $\mathfrak{g}$ -valued vector multiplet  $V$  and couple it to the diagonal  $G$  symmetry of the product  $\mathfrak{B} \times T(G)$ . This then gives a theory  $\tilde{\mathfrak{B}}^\vee = \mathfrak{B} \times_G T(G)$  which is the mirror of the SCFT that defines the dual boundary condition.

As  $\tilde{\mathfrak{B}}^\vee$  has  $G^\vee$  symmetry (since  $T(G)$  does), we can couple it to a  $\mathfrak{g}^\vee$ -valued background vector multiplet  $W$ . If we do this, then at this stage, we have coupled gauge fields  $V$  and  $W$  of  $G \times G^\vee$  to the product  $\mathfrak{B} \times T(G)$ .  $V$  is dynamical and  $W$  is a background field.

For  $G = U(1)$ , we do exactly the same thing, but the meaning is a little different because  $T(U(1))$  is “trivial.” Coupling  $V$  and  $W$  to  $\mathfrak{B} \times T(G)$  means that we couple  $V$  to  $\mathfrak{B}$  in the usual sense, and we couple  $V$  and  $W$  to each other by means of the supersymmetric completion of (4.5).

In general, for any  $G$ , coupling  $W$  is just a way of formalizing what is the  $G^\vee$  symmetry of  $\tilde{\mathfrak{B}}^\vee$ . For  $G$  a simple group, this step is hardly necessary<sup>15</sup> because the only possible  $G^\vee$  symmetry is the one that comes from the  $G^\vee$  symmetry of the SCFT  $T(G)$ . For  $G = U(1)$ , since  $T(U(1))$  is trivial as an SCFT, explicitly spelling out the coupling of the two vector multiplets is the simplest way to describe the  $U(1)$  symmetry of  $\tilde{\mathfrak{B}}^\vee$ .

The duality procedure we have just described for  $U(1)$  boundary conditions is the same as the one that we arrived at by a different method in Section 3.1.4. Indeed, the coupling (4.5) appeared previously in equation (3.2). (The three-dimensional gauge field  $C$  can be understood as the boundary value of the gauge field  $A$  of (3.2), which is defined on a half-space.)

We can similarly compare the general  $T(G)$  procedure to the recipe of Section 3.4 for the case  $G = U(n)$ . The general recipe says to construct  $\tilde{\mathfrak{B}}^\vee$

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<sup>15</sup>If  $G$  and  $G^\vee$  have outer automorphisms, it is best to regard the choice of  $G$  and  $G^\vee$  action as part of the definition of  $T(G)$ .

by coupling  $U(n)$  gauge fields to  $\mathfrak{B} \times T(U(n))$ , while in Section 3.4, the prescription was to couple  $U(n)$  gauge fields to  $\mathfrak{B} \times T(SU(n))$ . Either way, we then want to find a  $U(n)^\vee = U(n)$  symmetry of the Coulomb branch of  $\widetilde{\mathfrak{B}}^\vee$ . This symmetry was constructed in Section 3.4 using the  $SU(n)^\vee$  Coulomb branch symmetry of  $T(SU(n))$  and a Chern–Simons coupling (between  $U(1)$  gauge fields that gauge the centers of  $U(n)$  and  $U(n)^\vee$ ) to exhibit the action of the center of  $U(n)^\vee$ . The local factorization  $U(n) \cong SU(n) \times U(1)$  means that  $T(U(n))$  is a product  $T(SU(n)) \times T(U(1))$ . The two procedures are equivalent by virtue of the above description of  $T(U(1))$ .

#### 4.4.1 Computation

So far we have shown that everything is consistent if  $T(U(1))$  is simply a recipe for a Chern–Simons coupling of two external  $U(1)$  vector multiplets. On the other hand, in Section 4.1.1, we gave a general recipe for defining  $T(G)$  by making a duality transformation in a half-space. Here we aim to show that the two approaches coincide for  $G = U(1)$ .

First we review electric–magnetic duality for  $U(1)$  in the absence of a boundary. Supersymmetry plays no essential role (since in the abelian case the additional fields required by supersymmetry are not coupled to the gauge field), so we focus on pure  $U(1)$  gauge theory. We follow [28] and Section 2.4 of [29] (see [30, 31] for the analog in two dimensions), but for brevity we take the  $\theta$ -angle to vanish. The action of the free  $U(1)$  gauge field  $B$ , whose curvature we call  $F_B$ , on a four-manifold  $M$  is

$$\mathbf{I} = \frac{1}{e^2} \int_M F_B \wedge \star F_B. \quad (4.7)$$

To establish electric–magnetic duality, we introduce a two-form field  $\mathbf{k}$  and require the extended gauge symmetry

$$\begin{aligned} B &\rightarrow B + b, \\ \mathbf{k} &\rightarrow \mathbf{k} + db, \end{aligned} \quad (4.8)$$

where  $b$  is any connection on a principal line bundle  $\mathcal{T}$  over spacetime, and  $db$  is its curvature.  $\mathbf{k}$  should really be regarded as a gerbe connection (the analog of the two-form field in string theory) with curvature  $\mathbf{h} = d\mathbf{k}$ ; the periods of  $\mathbf{h}$  are integer multiples of  $2\pi$ . The extended gauge symmetry reduces to ordinary Maxwell gauge symmetry for  $B$  if  $b = d\epsilon$  for some zero-form  $\epsilon$ . To get the extended gauge symmetry, it suffices to introduce  $\mathcal{F} = F_B - \mathbf{k}$  and replace  $F_B$  by  $\mathcal{F}$  in the action  $\mathbf{I}$ . The resulting theory, however, is trivial. To get something non-trivial, we introduce another abelian gauge

field  $C$ , and add to the action a coupling  $\frac{1}{2\pi} \int_M C \wedge d\mathbf{k} = \frac{1}{2\pi} \int_M F_C \wedge \mathcal{F}$ , with  $F_C = dC$  the curvature of  $C$ . At this stage, then, we have an extended action

$$\widehat{\mathbf{I}} = \frac{1}{2\pi} \int_M F_C \wedge \mathcal{F} + \frac{1}{e^2} \int_M \mathcal{F} \wedge \star \mathcal{F}. \tag{4.9}$$

Electric–magnetic duality is established by comparing two ways to study this theory. One approach is to first integrate over  $C$ , which leads to a delta function in the path integral by means of which one can set  $\mathbf{k} = 0$  modulo an extended gauge transformation of the form (4.8). (See the discussion of equation (2.20) in [28]. The argument depends on the precise coefficient  $1/2\pi$  in the first term of the extended action.) This shows that the extended theory is actually equivalent to the original theory with action (4.7). On the other hand, one can use the extended gauge symmetry to set  $B = 0$ , after which the integral over  $\mathbf{k}$  is Gaussian. Performing this integral, we arrive at an action for  $C$

$$\mathbf{I}_C = \frac{e^2}{16\pi^2} \int_M F_C \wedge \star F_C \tag{4.10}$$

that has the same form as the original action (4.7) except that  $\tau = 4\pi i/e^2$  has been replaced by  $-1/\tau$ . This is electric–magnetic duality.

Now let us consider the case that  $M$  has a boundary  $\partial M$ . Let us analyze the dual of Dirichlet boundary conditions. This means that we require that  $B$  and hence also its curvature  $F_B$  vanish when restricted to  $\partial M$ . This being so, in the definition of the extended gauge symmetry (4.8), we likewise have to require that  $b$  vanishes when restricted to  $\partial M$ . Now we have to ask what sort of boundary conditions on  $C$  and  $G$  will enable the above argument to work. The answer is that it works if we place Neumann boundary conditions on  $C$  and Dirichlet on  $G$ . (For example, we cannot place Neumann boundary conditions on both or Dirichlet on both as this does not give a well-posed boundary problem. If we place Dirichlet boundary conditions on  $C$  and Neumann on  $G$ , then the delta function from the  $C$  integral does not suffice to set  $G$  to zero modulo an extended gauge transformation.) For another explanation of the boundary condition on  $C$ , see equation (4.11) below.

So the dual gauge theory, in which the gauge field is  $C$ , obeys Neumann boundary conditions. We have shown that the  $S$ -dual of Dirichlet is Neumann.

Now we want to modify this by introducing a gauge field  $A$  on  $\partial M$ , with curvature  $F_A$ , and shifting the boundary condition on  $B$  from  $B|_{\partial M} = 0$  to  $B|_{\partial M} = A$ . We continue to require that  $b$  and  $\mathbf{k}$  vanish on  $\partial M$ . The extended theory with action  $\widehat{\mathbf{I}}$  still makes sense, and by doing first the integral over  $C$ , one can still show that this theory is equivalent to the original Maxwell theory (4.7) with the shifted boundary condition. What

happens if we proceed in the opposite order? We cannot gauge fix  $B$  to 0, since this does not obey the boundary conditions, but we can pick an arbitrary gauge field  $B_0$  on  $M$  that obeys the boundary conditions, and impose the gauge condition  $B| = B_0$ . Next we try to perform the integral over  $\mathbf{k}$ . The condition for the action to have a critical point as a function of  $\mathbf{k}$  that also obeys the boundary condition  $\mathbf{k}|_{\partial M} = 0$  is that

$$\frac{2}{e^2} \star F_C|_{\partial M} = \frac{1}{2\pi} F_A. \tag{4.11}$$

Once this boundary condition is imposed on  $C$ , one can perform the Gaussian integral over  $\mathbf{k}$ , leading back to the same bulk action (4.10) as before. The conclusion is that the shifted boundary condition on  $B$  is dual to the deformation (4.11) away from Neumann boundary conditions for  $C$ . The correction to the boundary condition is equivalent to the addition to the action of a boundary term. The total action for  $C$ , including the boundary term, is

$$I_C = \frac{e^2}{16\pi^2} \int_M F_C \wedge \star F_C + \frac{1}{2\pi} \int_{\partial M} C \wedge dA. \tag{4.12}$$

In other words, a shifted boundary condition  $B|_{\partial M} = A$  with specified  $A$  is dual to the boundary interaction just indicated. Another way to explain the result we have obtained is that the condition  $B|_{\partial M} = A$  implies that  $F_B|_{\partial M} = F_A$ . Under duality,  $F_B$  maps to  $\frac{4\pi}{e^2} \star F_C$ , so the boundary condition becomes  $\frac{4\pi}{e^2} \star F_C|_{\partial M} = F_A$ , as in (4.11).

Now it is straightforward to perform duality in a half-space and explain the claim we have made about  $T(U(1))$ . We start with a  $U(1)$  gauge field  $\widehat{A}$  on a four-manifold  $M$ . We select a three-dimensional submanifold  $N$  of  $M$  that divides  $M$  into two pieces  $M_1$  and  $M_2$ . We write  $A$  and  $B$  for the restrictions of  $\widehat{A}$  to  $M_1$  and  $M_2$ , respectively. Of course, they agree on the boundary:

$$B|_N = A|_N. \tag{4.13}$$

Now we carry out the above duality procedure on  $M_2$ . In the process,  $B$  is replaced by another  $U(1)$  gauge field  $C$  on  $M_2$ , and the boundary condition (4.13) is replaced by a boundary interaction

$$\frac{1}{2\pi} \int_N C \wedge dA. \tag{4.14}$$

We have justified the claim that after carrying out duality in half of space-time, the gauge fields  $A$  and  $C$  on the two sides are coupled in this fashion. This justifies our proposal for  $T(U(1))$ , and hopefully also makes more tangible the idea of defining  $T(G)$  by duality in a half-space.

#### 4.5 Massive deformations of $T(G)$

Here we will briefly discuss the massive deformations of  $T(G)$  and related theories. These are of interest physically, and also relevant to some aspects of recent mathematical work [32–34] on classical geometry related to three-dimensional mirror symmetry.

We begin by restating some standard facts. Let us start with a general superconformal  $\mathcal{N} = 4$  theory  $W$  in three dimensions. The moduli space of vacua has the general form

$$\mathcal{M} = \cup_{\alpha \in S} \mathcal{C}_\alpha \times \mathcal{H}_\alpha, \quad (4.15)$$

where  $S$  is the set of components, and each component is the product of Coulomb and Higgs factors  $\mathcal{C}_\alpha$  and  $\mathcal{H}_\alpha$ . Each factor is a conical hyper-Kähler manifold, with a scaling symmetry that leaves fixed only the apexes of the cones, and  $R$ -symmetry groups  $SO(3)_X$  and  $SO(3)_Y$  that rotate, respectively, the complex structures of the  $\mathcal{H}_\alpha$  and of the  $\mathcal{C}_\alpha$ . The union in (4.15) is not a disjoint union; the components meet on conical hyper-Kähler subvarieties, and in particular there is one point that they all share in common — the fixed point of the scaling symmetry. The Coulomb branch  $\mathcal{C}$  is the union of  $\mathcal{C}_\alpha$  for which  $\mathcal{H}_\alpha$  is a point, and the Higgs branch  $\mathcal{H}$  is the union of  $\mathcal{H}_\alpha$  for which  $\mathcal{C}_\alpha$  is a point.

It is not true that there is always a Higgs branch or a Coulomb branch, since there may be no component for which  $\mathcal{C}_\alpha$  or  $\mathcal{H}_\alpha$  is a point. In general, we define the maximal Higgs branch  $\mathcal{H}'$  as the union of  $\mathcal{H}_\alpha$  for which  $\mathcal{C}_\alpha$  is minimal, and the maximal Coulomb branch  $\mathcal{C}'$  as the union of  $\mathcal{C}_\alpha$  for which  $\mathcal{H}_\alpha$  is minimal. All  $\mathcal{H}_\alpha$  are subvarieties of  $\mathcal{H}'$  (on which Coulomb directions  $\mathcal{C}_\alpha$  appear) and all  $\mathcal{C}_\alpha$  are subvarieties of  $\mathcal{C}'$  (on which Higgs directions  $\mathcal{H}_\alpha$  appear).

Continuous symmetries act on either  $\mathcal{C}'$  or  $\mathcal{H}'$ , but not both. Let  $\tilde{H}$  and  $H$  be the groups that act on  $\mathcal{C}'$  and  $\mathcal{H}'$ , respectively. Let  $\tilde{T}$  and  $T$  be maximal tori in  $\tilde{H}$  and  $H$ , and write  $\tilde{\mathfrak{t}}$  and  $\mathfrak{t}$  for their Lie algebras. The actions of  $\tilde{T}$  and  $T$  preserve the conical hyper-Kähler structures of the  $\mathcal{C}'$  and  $\mathcal{H}'$ , and so always leave fixed the apexes of these cones. Massive deformations exist precisely if  $\tilde{T}$  and  $T$  have no other fixed points.

The massive deformations are constructed using Fayet–Iliopoulos (FI) parameters and mass parameters. These take values, respectively, in  $\tilde{\mathfrak{t}} \otimes \mathbb{R}^3$  and  $\mathfrak{t} \otimes \mathbb{R}^3$ , where one can think of  $\mathbb{R}^3$  as the space of imaginary quaternions. The effect of turning on FI parameters is to reduce the dimension of the  $\mathcal{C}_\alpha$ ,

possibly to zero, and to resolve/deform singularities of the  $\mathcal{H}_\alpha$ , possibly making them smooth. Indeed, each component of an FI parameter  $\vec{\zeta} \in \tilde{\mathfrak{t}} \otimes \mathbb{R}^3$  corresponds to a vector field on  $\mathcal{C}'$  and hence on each  $\mathcal{C}_\alpha$ . We refer to the common zeroes of these three vector fields as the zeroes or fixed points of  $\vec{\zeta}$ . The effect of perturbing the action by  $\vec{\zeta}$  is to reduce the moduli space  $\mathcal{M}$  to the zeroes of  $\vec{\zeta}$ . In the theory  $W$  perturbed by  $\vec{\zeta}$ , the  $\mathcal{C}_\alpha$  are replaced by the fixed point sets  $\mathcal{C}_\alpha^{\vec{\zeta}}$ , and the singularities of  $\mathcal{H}^\alpha$  are partially or completely deformed/resolved.

An important special case is that if  $\tilde{T}$  acting on  $\mathcal{C}'$  has no fixed point except for the apex, and we choose  $\vec{\zeta}$  generically, then the only zero of  $\vec{\zeta}$  is the apex. In this situation,  $\mathcal{H}'$  becomes smooth and becomes the moduli space of vacua. (Its singularities would be points of intersection with Coulomb branches, but such branches are absent, since the only zero of  $\vec{\zeta}$  is the apex.)

We say that  $\vec{\zeta}$  is regular if it leaves fixed only the apex of  $\mathcal{C}'$ . Otherwise we say that  $\vec{\zeta}$  is non-regular. If the set of regular  $\vec{\zeta}$  is non-empty, then the set of non-regular  $\vec{\zeta}$  has real codimension at least three. (To be non-regular,  $\vec{\zeta}$  must take values in  $\mathfrak{p} \otimes \mathbb{R}^3$  where  $\mathfrak{p}$  is a proper subspace of  $\tilde{\mathfrak{t}}$ ; the codimension of  $\mathfrak{p} \otimes \mathbb{R}^3$  in  $\tilde{\mathfrak{t}} \otimes \mathbb{R}^3$  is at least three.) This has the important consequence that the set of regular  $\vec{\zeta}$  is connected and simply connected.

Similarly, we call  $\vec{\mathfrak{m}}$  regular if it leaves fixed only the apex of  $\mathcal{H}'$ , and otherwise non-regular. Again, the set of regular  $\vec{\mathfrak{m}}$  is connected and simply connected (though possibly empty).

Conversely to what has been stated above, a perturbation by masses  $\vec{\mathfrak{m}} \in \mathfrak{t} \otimes \mathbb{R}^3$  reduces  $\mathcal{H}_\alpha$  to the fixed point set  $\mathcal{H}_\alpha^{\vec{\mathfrak{m}}}$  (defined again as the zero set of the vector fields corresponding to  $\vec{\mathfrak{m}}$ ), and deforms/resolves the singularities of  $\mathcal{C}_\alpha$ . If the only fixed point of  $T$  acting on  $\mathcal{H}'$  is at the apex, then after turning on a generic  $\vec{\mathfrak{m}}$ ,  $\mathcal{C}'$  becomes smooth and is the moduli space of vacua. We write  $\hat{\mathcal{H}}'$  and  $\hat{\mathcal{C}}'$  for generic smoothings of  $\mathcal{H}'$  and  $\mathcal{C}'$  resulting from  $\vec{\mathfrak{m}}$  or  $\vec{\zeta}$ , respectively.

Now let us focus on the situation that both  $T$  and  $\tilde{T}$  have only the apexes of the cones as fixed points. In this situation, if both  $\vec{\zeta}$  and  $\vec{\mathfrak{m}}$  are regular, the theory becomes massive. We can describe the massive vacua in two different limits,  $|\vec{\zeta}| \gg |\vec{\mathfrak{m}}|$  and  $|\vec{\mathfrak{m}}| \gg |\vec{\zeta}|$ .

If  $|\vec{\zeta}| \gg |\vec{\mathfrak{m}}|$ , then we first consider the effects of  $\vec{\zeta}$ . The moduli space of vacua reduces to the smooth space  $\hat{\mathcal{H}}'$ . Now turn on a regular  $\vec{\mathfrak{m}}$ . By hypothesis,  $\vec{\mathfrak{m}}$  has only a single fixed point on the conical space  $\mathcal{H}'$  — the



apex of the cone. After the smoothing to  $\widehat{\mathcal{H}}'$ , there are finitely many fixed points. (The apex of the cone is generically a singular point, and must be counted with an integer multiplicity, which becomes the number of fixed points once  $\mathcal{H}'$  is made smooth.) These are the massive vacua of the theory. So if  $|\vec{\zeta}| \gg |\vec{\mathfrak{m}}|$ , the set  $\mathfrak{M}$  of massive vacua has a natural correspondence<sup>16</sup> with the fixed points of  $T$  acting on  $\widehat{\mathcal{H}}'$ .

If  $|\vec{\mathfrak{m}}| \gg |\vec{\zeta}|$ , we carry out these steps in the opposite order, and establish a correspondence of  $\mathfrak{M}$  with the set of fixed points of  $\widetilde{T}$  acting on  $\widehat{\mathcal{C}}'$ . Since we can smoothly interpolate from one limit to the other with the theory remaining massive (as long as  $\vec{\zeta}$  and  $\vec{\mathfrak{m}}$  both remain regular) there must be a natural correspondence between  $\widetilde{T}$  fixed points on  $\widehat{\mathcal{C}}'$  and  $T$  fixed points on  $\widehat{\mathcal{H}}'$ . Such a correspondence has been found mathematically [32–34].

Now let  $\mathcal{W}$  and  $\widetilde{\mathcal{W}}$  be the Weyl groups of  $H$  and  $\widetilde{H}$ , respectively. There is always a natural action of the product  $\mathcal{W} \times \widetilde{\mathcal{W}}$  on  $\mathfrak{M}$ . The action of  $\mathcal{W}$  is clear<sup>17</sup> if we interpret  $\mathfrak{M}$  as the space of  $T$  fixed points on  $\widehat{\mathcal{H}}'$ , and the action of  $\mathcal{C}$  is clear if we interpret  $\mathfrak{M}$  as the space of  $\widetilde{T}$  fixed points on  $\widehat{\mathcal{C}}'$ .

To show that the actions of  $\mathcal{W}$  and  $\widetilde{\mathcal{W}}$  commute, we look at the problem more symmetrically. We write  $A_{\text{reg}}$  for the space of regular  $\vec{\zeta}$  and  $B_{\text{reg}}$  for the space of regular  $\vec{\mathfrak{m}}$ . Conjugation of  $\vec{\zeta}$  or  $\vec{\mathfrak{m}}$  by  $\widetilde{\mathcal{W}}$  or  $\mathcal{W}$  gives an equivalent theory. Hence the parameter space of massive deformations is  $P = A_{\text{reg}}/\widetilde{\mathcal{W}} \times B_{\text{reg}}/\mathcal{W}$ . Since  $A_{\text{reg}}$  and  $B_{\text{reg}}$  are connected and simply connected, the fundamental group of  $P$  is  $\pi_1(P) = \mathcal{W} \times \widetilde{\mathcal{W}}$ . The set  $\mathfrak{M}$  of massive vacua maps to  $P$ , and as  $\mathfrak{M}$  is a finite set, this map is a locally trivial fibration. The global monodromy action on the fiber gives the desired action of  $\mathcal{W} \times \widetilde{\mathcal{W}}$  on  $\mathfrak{M}$ .

### 4.5.1 Application to $T(G)$

The theory  $T(G)$  gives an excellent illustration of the ideas just summarized. The groups  $H$  and  $\widetilde{H}$  are  $G$  and the dual group  $G^\vee$ . The Higgs and Coulomb branches are the nilpotent cones  $\mathcal{N}$  and  $\mathcal{N}^\vee$ . The Weyl groups  $\mathcal{W}$  and  $\mathcal{W}^\vee$  of  $G$  and  $G^\vee$  naturally coincide.

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<sup>16</sup>It does not matter which smoothing  $\widehat{\mathcal{H}}'$  we use here. To compare fixed points for two different regular choices of  $\vec{\zeta}$ , we simply pick an interpolating path of regular  $\vec{\zeta}$ 's, and follow the fixed points along this path. The choice of path does not matter, since the space of regular  $\vec{\zeta}$ 's is simply connected.

<sup>17</sup>If  $h \in H$  normalizes  $T$ , then  $v \rightarrow hvh^{-1}$ ,  $v \in \mathfrak{t}$ , is a Weyl transformation  $w_h$ . Given  $w \in \mathcal{W}$ , we pick  $h$  such that  $w_h = w$ . Then the action of  $w_h$  on  $\widehat{\mathcal{H}}'$  permutes the fixed points of  $T$ , in a fashion that is independent of the choice of  $h$ ; moreover, this gives an action of  $\mathcal{W}$  on the set of fixed points.

The maximal tori  $T$  and  $T^\vee$  of  $G$  and  $G^\vee$  act on  $\mathcal{N}$  and  $\mathcal{N}^\vee$  with fixed points only at the origin. (In order for  $x \in \mathfrak{g}$  to be  $T$ -invariant, it must be an element of  $\mathfrak{t}$ ; but the intersection  $\mathfrak{t} \cap \mathcal{N}$  consists only of the point 0, since an element of  $\mathfrak{t}$  that is nilpotent must vanish.) The condition for  $\vec{\zeta}$  to be regular is that the subalgebra of  $\mathfrak{g}^\vee$  that commutes with all three components of  $\vec{\zeta}$  is precisely  $\mathfrak{t}^\vee$ ; similarly  $\vec{\mathfrak{m}}$  is regular if it commutes precisely with  $\mathfrak{t}$ .

A regular FI perturbation eliminates the Coulomb branch  $\mathcal{N}^\vee$  and deforms the nilpotent cone  $\mathcal{N}$  to the orbit of a semi-simple element, or a smooth resolution thereof. For a particular choice of FI perturbation,<sup>18</sup>  $\mathcal{N}$  is deformed to  $T^*(G/T)$ , the cotangent bundle of the compact flag manifold  $G/T$ . The fixed points of the  $T$  action on  $G/T$  correspond precisely to elements of the Weyl group  $\mathcal{W}$ . Indeed, let  $\pi : G \rightarrow G/T$  be the projection. For  $g \in G$ , the condition for  $\pi(g)$  to be a fixed point in the action of  $T$  on  $G/T$  is that for any  $t \in T$ , we should have  $tg = gt'$  for some  $t' \in T$ . In other words,  $g^{-1}tg = t'$ , so  $g$  normalizes  $T$  and generates a Weyl transformation. The action of  $\mathcal{W} \times \mathcal{W}$  on  $\mathfrak{M} = \mathcal{W}$  is simply the left and right action of  $\mathcal{W}$  on itself, as one can see by slightly extending this analysis.

We will give two additional ways to understand these statements. First, for  $G = SU(n)$ , we use the brane realization of  $T(G)$  that is sketched in figure 23(c) with  $n$  D3-branes stretched between  $n$  NS5-branes on the left and  $n$  D5-branes on the right. To realize  $T(SU(n))$ , the  $n$  NS5-branes should be coincident and likewise the  $n$  D5-branes should be coincident. The massive deformation is achieved by making generic displacements of the NS5-branes in  $\vec{X}$  and generic displacements of the D5-branes in  $\vec{Y}$ . (This is what is actually drawn in the figure, simply because it is easier to draw.) After this deformation, a supersymmetric vacuum corresponds to a situation, as drawn in the figure, in which each D3-brane ends at one end on an NS5-brane and at the other end on a D5-brane. The resulting vacuum is massive since a D3-brane with opposite endpoints of this kind supports no massless degrees of freedom. Because of the  $s$ -rule, no more than one D3-brane can end on any fivebrane, so as there are  $n$  branes of each kind, each fivebrane has precisely one D3-brane ending on it. Connecting the  $n$  NS5-branes with  $n$  D5-branes gives a map from one set of  $n$  objects to another. There are  $n!$  such maps, making up the set  $\mathfrak{M}$  of massive vacua.

The Weyl group of  $SU(n)$  is the group of permutations of  $n$  objects. The action of  $\mathcal{W} \times \mathcal{W}$  is visible in the brane picture: one factor acts by permuting the  $n$  NS5-branes, and one by permuting the  $n$  D5-branes. Clearly, we

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<sup>18</sup>One chooses  $\vec{\zeta} = (0, 0, b)$ , where  $b \in \mathfrak{t}$  is regular in the usual sense, so that in one complex structure  $\vec{\zeta}$  generates a resolution, not a deformation, of  $\mathcal{N}$ . The resolution in question is known as the Springer resolution.

can think of  $\mathfrak{M}$  as a copy of  $\mathcal{W}$ , on which  $\mathcal{W} \times \mathcal{W}$  acts by left and right multiplication.

An alternative approach is valid for any  $G$ . We can realize  $T(G)$  via gauge theory on  $\mathbb{R}^3 \times I$  (where  $I$  is the interval  $0 \leq y \leq L$ ), with  $G$  gauge theory with Dirichlet boundary conditions on the left,  $G^\vee$  gauge theory with Dirichlet boundary conditions on the right, and a Janus domain wall in between. (This is what we get by “ungauging” the configuration of figure 45 with Dirichlet boundary conditions at each end; it is also the special case of figure 47(a) with  $\rho^\vee$  trivial.) In that language, a massive deformation is made by shifting the Dirichlet boundary conditions at the two ends. On the left, instead of taking  $\vec{Y}(0) = 0$ , we take  $\vec{Y}(0) = \vec{\zeta}$ , and on the right, instead of taking  $\vec{Y}^\vee(L) = 0$ , we take  $\vec{Y}^\vee(L) = \vec{\mathfrak{m}}$ . (See Section 2.2.3 of [1] for this type of deformation.) Once we make a massive deformation, the Janus coupling profile  $g(y)$  in figure 45 is not important. We can deform to the case that  $g$  is constant, and small in the  $G$  description. It is then useful to use the  $G$  description everywhere. The duality transformation from  $G^\vee$  to  $G$  on the right part of the slab maps  $\vec{Y}^\vee$  to  $\vec{X}$ , and the boundary conditions become

$$\begin{aligned} \vec{Y}(0) &= \vec{\zeta}, \\ \vec{X}(L) &= \vec{\mathfrak{m}}. \end{aligned} \tag{4.16}$$

(Also, we impose Neumann boundary conditions on  $\vec{X}$  at  $y = 0$ , and on  $\vec{Y}$  at  $y = L$ . The gauge field  $A$  obeys Dirichlet boundary conditions at  $y = 0$  and Neumann at  $y = L$ .) Finally, we divide by gauge transformations that equal the identity at  $y = 0$ .

With these boundary conditions, the theory is massive. For example, having opposite boundary conditions on  $A$  at the two ends ensures that it has no massive modes, while the boundary conditions on  $\vec{X}$  and  $\vec{Y}$  ensure the same for them. To find a supersymmetric vacuum,  $\vec{X}$  and  $\vec{Y}$  must be covariantly constant, and all components of  $\vec{X}$  and  $\vec{Y}$  must commute with each other. In addition, the curvature  $F = dA + A \wedge A$  must vanish. Hence, by a gauge transformation that is trivial at  $y = 0$ , we can set  $A = 0$ . In this gauge,  $\vec{X}$  and  $\vec{Y}$  are simply constant. The boundary condition now tells us that  $\vec{Y}(y) = \vec{\zeta}$  for all  $y$ . As for  $\vec{X}$ , it is also constant, but we can no longer claim that it equals  $\vec{\mathfrak{m}}$ , since we have made a gauge transformation that may be non-trivial at  $y = L$ . However,  $\vec{X}$  must be a constant that commutes with  $\vec{Y} = \vec{\zeta}$ , so (as we assume  $\vec{\zeta}$  to be regular)  $\vec{X}$  must be  $\mathfrak{t}$ -valued. Moreover,  $\vec{X}$  must be gauge equivalent to  $\vec{\mathfrak{m}}$ . These conditions imply that in this gauge,  $\vec{X}$  is the conjugate of  $\vec{\mathfrak{m}}$  by some Weyl transformation  $w$ . Moreover, any  $w$

is possible. So again we see that the set  $\mathfrak{M}$  of massive vacua is a copy of  $\mathcal{W}$ . Using the definition of the  $\mathcal{W} \times \mathcal{W}$  action in terms of monodromy over the parameter space  $P$ , one can verify that this action is the left and right action of  $\mathcal{W}$  on itself.

## 5 Quivers with orthogonal and symplectic gauge groups

In the remainder of this paper, we will extend some of the explicit constructions to orthogonal and symplectic gauge groups. As a preliminary, in this section we extend to the orthogonal and symplectic case the analysis of good and bad gauge theories and quivers in Section 2.4. This will also enable us to describe quivers that are candidates for  $T(SO(n))$  and  $T(Sp(n))$ . We apply our results to  $S$ -duality of boundary conditions in Sections 6 and 7.

### 5.1 Orthogonal and symplectic gauge theory

The starting point is the general formula (2.15) for the  $R$ -charge of a monopole operator. The monopole operator is defined by specifying a  $U(1)$  embedding in the gauge group  $G$ . If  $h_i$  and  $v_j$  are the  $U(1)$  charges of hypermultiplets and vector multiplets, then the monopole operator has  $R$ -charge

$$q_R = \frac{1}{2} \left( \sum_i |h_i| - \sum_j |v_j| \right). \quad (5.1)$$

To implement this for  $SO(k)$ , we associate the  $U(1)$  embedding to a sequence of integer “eigenvalues”  $a_1, \dots, a_k$ ; the non-zero eigenvalues come in pairs of equal magnitude and opposite sign, since a generator of  $\mathfrak{so}(k)$  is conjugate to a sum of traceless  $2 \times 2$  blocks

$$\begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}. \quad (5.2)$$

Let us couple  $SO(k)$  gauge theory to fundamental hypermultiplets with flavor symmetry  $Sp(2n_f)$ . The evaluation of (2.15) gives

$$q_R = \frac{n_f}{2} \sum_i |a_i| - \frac{1}{2} \left( \sum_{1 \leq i < j \leq k} |a_i - a_j| - \sum_{1 \leq i \leq k} |a_i| \right). \quad (5.3)$$

With the  $a_i$  coming in equal and opposite pairs, this is equivalent to

$$q_R = \frac{n_f + 2 - k}{2} \sum_i |a_i| + \frac{1}{2} \sum_{i < j} (|a_i| + |a_j| - |a_i - a_j|). \tag{5.4}$$

This formula has an obvious similarity to (2.17), with  $k$  playing the role of  $2n_c$ . The condition for a good theory is

$$n_f \geq k - 1. \tag{5.5}$$

We call an  $SO(k)$  theory balanced if  $n_f = k - 1$ . In the balanced case there is a single monopole operator with  $q_R = 1$ , the sequence of charges being  $(1, -1, 0, 0, \dots, 0)$ ; this leads to an  $SO(2)$  symmetry of the Coulomb branch. An important point is that because the  $a_i$  come in pairs,  $q_R$  is always an integer and there are never free hypermultiplets. We define the excess  $e$  of an  $SO(k)$  gauge theory coupled to fundamentals by

$$e = n_f - k + 1. \tag{5.6}$$

We can make a similar analysis for gauge group  $Sp(2t)$  with fundamental hypermultiplets of flavor symmetry<sup>19</sup>  $SO(2n_f)$ . The analog of (5.3) is

$$q_R = \frac{n_f}{2} \sum_i |a_i| - \frac{1}{2} \left( \sum_{1 \leq i < j \leq 2t} |a_i - a_j| + \sum_{1 \leq i \leq 2t} |a_i| \right), \tag{5.7}$$

leading to

$$q_R = \frac{n_f - 2t}{2} \sum_i |a_i| + \frac{1}{2} \sum_{i < j} (|a_i| + |a_j| - |a_i - a_j|). \tag{5.8}$$

The condition for a good theory is now

$$n_f \geq 2t + 1, \tag{5.9}$$

and we call a theory balanced if  $n_f = 2t + 1$ . Again, a balanced theory has a single monopole operator with  $q_R = 1$ , with the same sequence of charges as before, leading to an  $SO(2)$  symmetry of the Coulomb branch. For the

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<sup>19</sup>The full flavor symmetry is actually  $O(2n_f)$ , not just  $SO(2n_f)$ , but we primarily consider the connected component except in Section 5.2.2. Classically, we could take flavor symmetry of the form  $O(2m + 1)$ . An anomaly would then force us to incorporate a half-integral Chern–Simons interaction for the gauge fields, modifying the properties of the monopole operators.

same reason as for orthogonal gauge groups, there is no value of  $n_f$  at which a free hypermultiplet appears. We define the excess  $e$  of an  $Sp(2t)$  gauge theory with fundamentals as

$$e = n_f - 2t - 1. \quad (5.10)$$

## 5.2 Orthosymplectic quivers

We now want to study linear quivers with unitary, orthogonal, and symplectic gauge groups. (For some examples, see figure 53.) To each link in such a quiver, we attach bifundamental hypermultiplets as usual; in the case of adjacent  $SO$  and  $Sp$  nodes, we place a reality condition on the hypermultiplets, halving the number of components.

We have already defined a notion of excess  $e_i$  for each kind of node. We also define  $\epsilon_i = 1$  for unitary nodes, and  $\epsilon_i = 1/2$  for orthogonal and symplectic nodes. Also, because of the reality condition on the bifundamentals, we define  $\epsilon_{i,i+1}$  to be  $1/2$  for a link connecting  $SO$  and  $Sp$  nodes, and  $1$  for any other link.

The general formula for the  $R$ -charge of a monopole operator is

$$q_R = \sum_i (\Delta_i + \epsilon_i A_i + \epsilon_{i,i+1} B_i), \quad (5.11)$$

where, as in Section 2.4,

$$\begin{aligned} \Delta_i &= \frac{e_i}{2} \sum_{k=1}^{n_i} |a_{i,k}|, \\ A_i &= \frac{1}{2} \sum_{k=1}^{n_i} \sum_{t=1}^{n_i} (|a_{i,k}| + |a_{i,t}| - |a_{i,k} - a_{i,t}|), \\ B_i &= -\frac{1}{2} \sum_{k=1}^{n_i} \sum_{t=1}^{n_{i+1}} (|a_{i,k}| + |a_{i+1,t}| - |a_{i,k} - a_{i+1,t}|). \end{aligned} \quad (5.12)$$

Because of the  $\epsilon$  factors, it is not so that a linear quiver with all nodes obeying  $e_i \geq 0$  is automatically good. A simple counterexample is the quiver in figure 53(a), which has two adjacent balanced symplectic nodes. The two  $e_i$  vanish, the two  $A_i$  equal 2, and the single  $B_i$  equals  $-2$ . Because of the  $\epsilon_i$  factors,  $q_R = 0$  if the charges at the two nodes are equal, and the quiver is bad.

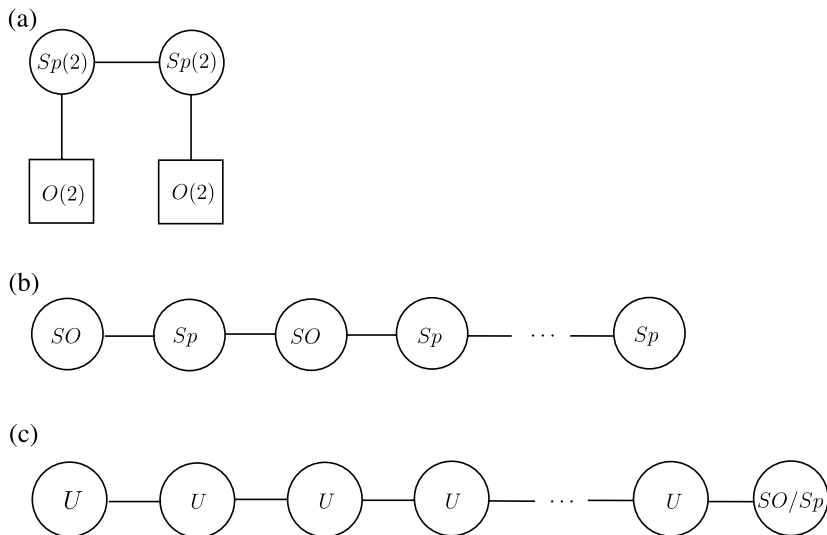


Figure 53: (a) A quiver with two adjacent balanced symplectic nodes is bad, as explained in the text. (b) An orthosymplectic quiver, with alternating orthogonal and symplectic nodes, each of which has  $e_i \geq 0$ , is good. (c) A chain of unitary nodes terminating at one end with an orthogonal or symplectic node is good, provided each node has  $e_i \geq 0$ .

Nevertheless, certain classes of quivers do have the property that if  $e_i \geq 0$  for each node, then  $q_R \geq 1$ . In the rest of this paper, two classes of such good quivers will play an important role. These classes, which are suggested by brane constructions with orientifolds, are as follows.

One is what we will call an “orthosymplectic” quiver, that is a linear quiver of alternating orthogonal and symplectic nodes (figure 53(b)). The second is the case of a linear quiver of unitary groups terminating at one end (or at each end) with an orthogonal or symplectic group (figure 53(c)).

First we consider orthosymplectic quivers. In this case, the  $\epsilon_i$  and  $\epsilon_{i,i+1}$  all equal  $1/2$ . This factor of  $1/2$  multiplies the sum  $\sum_i (A_i + B_i)$  considered in Section 2.4. So the key inequalities of that section, such as (2.28), have immediate analogs:

$$\begin{aligned}
 q_R \geq & \sum_{i=1}^{P-1} \frac{e_i}{2} \sum_{m=1}^{n_i} |a_{i,m}| + \frac{1}{2} \sum_{m=1}^{n_1} |a_{1,m}| (n_1 - m + 1) \\
 & + \frac{1}{2} \sum_{m=1}^{n_{P-1}} |a_{P-1,m}| (n_{P-1} - m).
 \end{aligned}
 \tag{5.13}$$

(There are  $P - 1$  nodes in the quiver. The group at the  $i$ th node is assumed to be  $SO(n_i)$  or  $Sp(n_i)$ ;  $a_{i,s}$  are the charges at the  $i$ th node.) There is also an analog of (2.30) in which one of the nodes, say the  $s$ th one, is singled out:

$$\begin{aligned}
 q_R \geq & \sum_{i=1}^{P-1} \frac{e_i}{2} \sum_{m=1}^{n_i} |a_{i,m}| + \frac{1}{2} \sum_{m=1}^{n_1} |a_{1,m}|(n_1 - m) + \frac{1}{2} \sum_{m=1}^{n_s} |a_{s,m}| \\
 & + \frac{1}{2} \sum_{m=1}^{n_{P-1}} |a_{P-1,m}|(n_{P-1} - m). \tag{5.14}
 \end{aligned}$$

The close relation of these formulas to the corresponding formulas for unitary quivers reflects the fact that an orthosymplectic quiver can be obtained as a  $\mathbb{Z}_2$  orbifold of a quiver of unitary groups. One divides by a  $\mathbb{Z}_2$  that reduces each  $U(n)$  gauge group to  $SO(n)$  or  $Sp(n)$ . The orthogonal and symplectic groups must alternate along the chain in order that the  $\mathbb{Z}_2$  action can be defined for the bifundamental hypermultiplets.

It immediately follows from these formulas that for a quiver of this kind with all  $e_i \geq 0$ , non-trivial monopole operators have  $q_R > 0$ . Allowing for the fact that the non-zero  $a_{i,s}$  are paired, the bound is  $q_R \geq 1$ , so there are no free hypermultiplets.

Furthermore, we can describe all of the operators of  $q_R = 1$  and therefore the symmetries of the Coulomb branch. From (5.14), it follows that the charges vanish at any node with  $e_i > 0$ . So it suffices to consider a chain of  $p$  balanced nodes. From (5.14), to get  $q_R \leq 1$ , at each node, the monopole charges either vanish or are  $(1, -1, 0, \dots, 0)$ . Just as in Section 2.4, if the subquiver of nodes with non-zero charges is disconnected, then  $q_R \geq 2$ . So to get  $q_R = 1$ , we must have a connected subquiver on which the monopole charges are  $(1, -1, 0, \dots, 0)$ , with all charges vanishing outside this subquiver. Conversely, all these operators have  $q_R = 1$ . So there are a total of  $p(p + 1)/2$  monopole operators of  $q_R = 1$ . (An exception to this counting is mentioned shortly.) This is the dimension of the Lie group  $SO(p + 1)$ , and we claim that is indeed the symmetry associated with the monopole operators.

If there is a single balanced node, the symmetry group is clearly  $SO(2)$ , as there is a single monopole operator. For any two consecutive balanced nodes, there are three monopole operators. A detailed computation of three point functions on the sphere would be needed to show directly that these three operators define an  $SO(3)$  algebra, as opposed to  $SO(2) \times SO(2) \times SO(2)$ .



In any event, this follows from the orthosymplectic mirror symmetry construction of [35].

Once one knows that the symmetry for two consecutive balanced nodes is  $SO(3)$ , one can reason by induction and build up an  $SO(p + 1)$  symmetry for a chain of  $p$  consecutive balanced nodes. Suppose that this is so for some value of  $p$ . Adding a  $(p + 1)$ th balanced node adds  $p + 1$  monopole operators, which must transform in a  $(p + 1)$ -dimensional representation of the group  $SO(p + 1)$  that is already present. This representation must be non-trivial, because of the hypothesis that two adjacent balanced nodes (one of which is the new one) generates  $SO(3)$  symmetry. Hence it must be the irreducible  $(p + 1)$ -dimensional representation, which combines with the  $SO(p + 1)$  from the first  $p - 1$  nodes to generate  $SO(p + 2)$ .

In one important case, the above counting needs modification. Suppose that the gauge symmetry of one of our balanced nodes is  $SO(2)$ . Such a node can only appear at the end of a quiver, as there is no way to divide its  $Sp(2)$  flavor symmetry. The group  $SO(2) \cong U(1)$  is abelian, so in a theory with  $SO(2)$  gauge symmetry, there is a classical symmetry of the Coulomb branch, the shift of the dual photon. Moreover, as  $SO(2)$  has no Weyl group, the elements

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.15}$$

of  $\mathfrak{so}(2)$  are not conjugate and correspond to distinct homomorphisms  $\mathfrak{u}(1) \rightarrow \mathfrak{so}(2)$ . Hence there are two different monopole operators of  $q_R = 1$ , roughly with positive or negative monopole charge, at a balanced  $SO(2)$  node. The group associated with the classical symmetry and the two monopole operators is  $SO(3)$ . Indeed,  $SO(2)$  with  $Sp(2)$  flavor symmetry is equivalent to  $U(1)$  with  $n_f = 2$ , and so was one of the basic examples in Section 2.4.1.

Now we claim that given a chain of  $p$  balanced orthogonal and symplectic nodes, of which the first has  $SO(2)$  gauge symmetry, the monopole operators and the classical symmetry of the Coulomb branch generate an  $SO(p + 2)$  symmetry. We have already established the case of  $p = 1$ . Proceeding by induction in  $p$ , adding a  $(p + 1)$ th balanced node adds now  $p + 2$  monopole operators, transforming non-trivially under  $SO(p + 2)$ , and so extending the symmetry to  $SO(p + 3)$ .

Similarly, an orthosymplectic chain of  $p$  balanced nodes, with  $SO(2)$  at each end, gives  $SO(p + 3)$  symmetry.

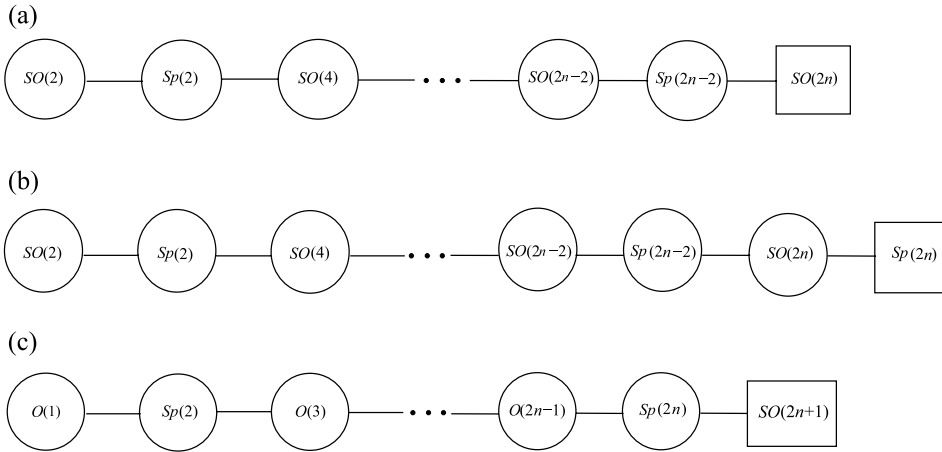


Figure 54: (a) A balanced chain ending with flavor symmetry  $SO(2n)$ . (b) A similar balanced chain ending with flavor symmetry  $Sp(2n)$ . (c) A bad quiver whose Higgs branch is the nilpotent cone of  $SO(2n + 1)$ .

### 5.2.1 Some significant examples

We will illustrate this idea with some significant examples. In Figure 54(a), we show a balanced orthosymplectic quiver with fundamental matter multiplets only at the last node. The sequence of groups is  $SO(2) - Sp(2) - SO(4) - Sp(4) - \dots - Sp(2n - 2)$ . The flavor symmetry at the last node is  $SO(2n)$ , and this is the classical symmetry of the Higgs branch. The symmetry of the Coulomb branch resulting from a chain of  $2n - 2$  balanced orthogonal and symplectic groups beginning with  $SO(2)$  is also  $SO(2n)$ . A short computation shows that the complex dimension of either the Higgs or the Coulomb branch is  $2n(n - 1)$ , which is the dimension of the nilpotent cone  $\mathcal{N}$  of the self-dual group  $SO(2n)$ . These facts suggest that the IR limit of this quiver describes  $T(SO(2n))$ . Indeed, it can be shown along the lines of [35] and our arguments in Section 3 that this quiver is self-mirror and describes the  $S$ -dual of Dirichlet boundary conditions for  $SO(2n)$ . We defer these matters to Section 7. Moreover, one can show directly by adapting<sup>20</sup> the arguments of [36] that the Higgs branch of this quiver is the nilpotent cone of  $SO(2n)$ .

<sup>20</sup>The Higgs branch of  $SO(2k)$  gauge theory coupled to fundamental hypermultiplets with flavor symmetry  $Sp(4k - 2)$  is the same as the Higgs branch of  $O(2k - 1)$  gauge symmetry with fundamental hypermultiplets of the same flavor symmetry. One can make this substitution for all of the orthogonal nodes in fig. 54(a) or (b). After doing so, the equivalence of the Higgs branches of these quivers to the nilpotent cones of  $SO(2n)$  and  $Sp(2n)$  is a special case of the results of [36] (see also a brief summary in [37]).

In figure 54(b), we show a similar balanced quiver but continued one step farther. Now, the symmetry of the Higgs branch is  $Sp(2n)$  rather than  $SO(2n)$ . The symmetry of the Coulomb branch — derived from  $2n - 1$  balanced nodes, the first group being  $SO(2)$  — is  $SO(2n + 1)$ . This is the dual group of  $Sp(2n)$ . The Higgs and Coulomb branches have the right dimensions to match the nilpotent cones  $\mathcal{N}$  and  $\mathcal{N}^\vee$  of  $T(Sp(2n))$ . Indeed, as we explain in Section 7, this theory describes the dual of Dirichlet boundary conditions for  $Sp(2n)$ . Again, the Higgs branch can be analyzed by adapting arguments in [36] and does coincide with the nilpotent cone of  $Sp(2n)$ .

As  $SO(2n + 1)$  is the dual group of  $Sp(2n)$ , the theory  $T(SO(2n + 1))$  is the mirror of  $T(Sp(2n))$ . It would be nice to have a direct construction of  $T(SO(2n + 1))$ , as opposed to its mirror. However, it seems that there is no good quiver that flows in the IR to  $T(SO(2n + 1))$ . The quiver of figure 54(c), based on the sequence  $O(1) - Sp(2) - O(3) - Sp(4) - \dots$ , is shown in [36] to have a Higgs branch that coincides<sup>21</sup> with the nilpotent cone of  $SO(2n + 1)$ . Moreover, its Coulomb branch has the same dimension as the nilpotent cone of  $Sp(2n)$ . However, this is not a good quiver, as the symplectic nodes are imbalanced. So it does not have a standard IR limit. We argue in Section 7 that this quiver theory flows to  $T(SO(2n + 1))$  in the infrared, but that the  $R$ -symmetry that is realized in the infrared is not the one that one sees in the underlying gauge theory.

### 5.2.2 Infrared flow of $T(SO(3))$

For  $SO(3)$ , we can argue this directly. First we ignore the  $O(1)$  gauge symmetry of the quiver and consider an  $Sp(2)$  (or  $SU(2)$ ) theory with flavor symmetry  $SO(4)$ . Classically, the Higgs branch  $\mathcal{H}$  consists of two copies<sup>22</sup> of the  $A_1$  singularity  $\mathbb{R}^4/\mathbb{Z}_2$ . The Coulomb branch of the model is  $\mathcal{C} = (\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$  [40]. Here  $\mathbb{Z}_2$  acts as a reflection of both  $\mathbb{R}^3$  and the circle  $S^1$ .  $\mathcal{C}$  has two singularities (coming from the fixed points of the reflection of  $S^1$ ), each of which is an  $A_1$  singularity. At each singularity of  $\mathcal{C}$ ,  $\mathcal{C}$  meets precisely one of the two components of  $\mathcal{H}$ . (This structure was found in [38] in a related four-dimensional model. It is true in three dimensions for similar reasons, as we explain below.)

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<sup>21</sup> For this result to hold classically, it is essential that the orthogonal gauge groups in the quiver are ordinary orthogonal groups  $O(k)$ , as indicated, rather than special orthogonal groups  $SO(k)$ . See Section 5.2.2.

<sup>22</sup> See the analysis of equation (3.4) in [38], where this Higgs branch arises in the study of a related four-dimensional model. From the point of view of [39], the Higgs branch is the one-instanton moduli space on  $\mathbb{R}^4$  with structure group the flavor group  $SO(4)$ , and has two components because the instanton can be in either factor of  $SO(4) \cong SU(2) \times SU(2)$ .

Let  $\mathcal{M}$  be the moduli space of vacua of the theory. Near either singularity of  $\mathcal{C}$ ,  $\mathcal{M}$  looks like two intersecting  $A_1$  singularities — one being  $\mathcal{C}$  and the other being the relevant component of  $\mathcal{H}$ . Either of these intersections gives the familiar picture of  $T(SU(2))$ , or equivalently  $T(SO(3))$ . However, the  $R$ -symmetry that is part of the superconformal structure of  $T(SO(3))$  is not the microscopic  $R$ -symmetry of the underlying  $Sp(2)$  gauge theory. The microscopic  $R$ -symmetry is of course an exact symmetry of  $\mathcal{C} = (\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$ , and comes from the rotation of  $\mathbb{R}^3$ . Near either of the  $A_1$  singularities of  $\mathcal{C}$ , the  $SO(3)$  symmetry of  $\mathcal{C}$  is enhanced to  $SO(4)/\mathbb{Z}_2 = SO(3)_1 \times SO(3)_2$ . Of these two factors, one of them, say  $SO(3)_1$ , is the superconformal  $R$ -symmetry, and the other, say  $SO(3)_2$ , is the expected  $SO(3)$  global symmetry that acts on the Coulomb branch of  $T(SO(3))$ . The microscopic  $R$ -symmetry is a diagonal subgroup of  $SO(3)_1 \times SO(3)_2$ . (The structure is the same as we described in footnote 4 for the free vector multiplet, and similar to what we will find in Section 7.1.1 for a certain splitting process involving branes.)

The  $Sp(2)$  gauge theory with  $SO(4)$  flavor symmetry can flow to  $T(SO(3))$  in two different ways, since we have to pick one of the singularities of the Coulomb branch  $\mathcal{C}$ . The bad quiver of figure 54(c) actually does not have this ambiguity, since the gauge group is not quite  $Sp(2)$  but  $O(1) \times Sp(2) = \mathbb{Z}_2 \times Sp(2)$ . Here the  $\mathbb{Z}_2$  factor exchanges the two components of  $\mathcal{H}$  and the two singularities of  $\mathcal{C}$ . Thus, after gauging this extra  $\mathbb{Z}_2$ , there is only one singularity at which the bad quiver flows to  $T(SO(3))$ .

To explain the claim about the action of  $O(1)$ , we note the following. BPS monopoles of the  $Sp(2)$  gauge symmetry appear in this theory as instantons. In an instanton field, each hypermultiplet flavor has a zero mode. The effective action of the instanton field is roughly  $\exp(i\phi)q_1q_2q_3q_4$ , where  $\phi$  is the dual photon, and  $q_i$  is a fermion of the  $i$ th real hypermultiplet. This effective action has  $SO(4)$  flavor symmetry, but it does not have  $O(4)$  flavor symmetry. However, the disconnected component of  $O(4)$  is a symmetry if combined with a  $\pi$  shift in  $\phi$ . Let  $\Theta$  be the product of a  $\pi$  shift of  $\phi$  and an element  $\text{diag}(-1, 1, 1, 1)$  of  $O(4)$ .

To determine the action of  $\Theta$ , we note that  $\phi$  parametrizes the  $S^1$  in  $\mathcal{C} = (\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$ . Hence a  $\pi$  shift of  $\phi$  exchanges the two singularities of  $\mathcal{C}$ . Also, classically, an  $O(4)$  transformation of determinant  $-1$  exchanges the two components of  $\mathcal{H}$ . (See again the discussion of equation (3.4) in [38]. Alternatively, in the instanton interpretation mentioned in footnote 22, the two components correspond to the two factors of  $SO(4) \cong SU(2) \times SU(2)$ , which are exchanged by a reflection in  $O(4)$ .) Thus,  $\Theta$  exchanges the two singularities of  $\mathcal{C}$  and the two components of  $\mathcal{H}$  that meet these two

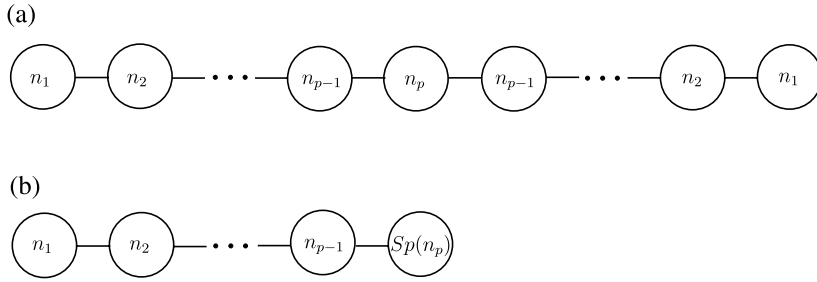


Figure 55: (a) A symmetric quiver of unitary groups. (A circle labeled by an integer  $k$  represents a  $U(k)$  gauge group; other groups will be indicated explicitly.) (b) Its  $\mathbb{Z}_2$  orbifold gives a quiver of unitary groups ending in an orthogonal or symplectic group. The symplectic case is depicted.

singularities. After dividing by  $\Theta$ , this bad quiver gauge theory has only one singular point with a flow to  $T(SO(3))$ .

To complete the story for this bad quiver, we simply note that the non-trivial element of the  $O(1)$  gauge symmetry of this quiver acts as  $-1$  on just one of the four real hypermultiplets, so it indeed corresponds to  $\Theta$ .

We will not make a similarly detailed analysis of the bad quivers of higher rank. We just note that for any  $k$ , the  $Sp(2k)$  theory with  $O(4k)$  flavor symmetry has many properties in common with the example just described: the Higgs branch has two components, exchanged by a flavor transformation of determinant  $-1$ , and meeting the Coulomb branch on different loci.

### 5.3 Unitary quivers with orthosymplectic groups at the end

Now we consider the situation of figure 53(c): a linear quiver of  $P - 1$  unitary gauge groups followed by a single orthogonal or symplectic node. Like orthosymplectic quivers, these can arise as  $\mathbb{Z}_2$  orbifolds of a quiver of unitary groups. In this case, the  $\mathbb{Z}_2$  symmetry must exchange the two ends of the quiver (figure 55).

The orbifold interpretation leads us to expect that there is a very simple inequality for the total  $R$ -charge. Just as in Section 2.4, the  $R$ -charge  $q_R$  of a monopole operator of charges  $a_{i,k}$  is a sum of separate contributions  $q_R^+$  and  $q_R^-$  from those  $a$ 's that are positive or negative, respectively:

$$q_R = q_R^+ + q_R^- \tag{5.16}$$

Each can be bounded as before. Indeed, the analog of equation (2.30) is the assertion that

$$q_R^+ \geq \sum_{i=1}^P \frac{e_i}{2} \sum_{k|a_{i,k} \geq 0} a_{i,k} + \sum_{k|a_{1,k} \geq 0} a_{1,k}(n_1 - k) + \sum_{k|a_{s,k} \geq 0}^{n_s} a_{s,k} - \frac{1}{2} \sum_{k|a_{P,k} \geq 0} a_{P,k} \tag{5.17}$$

for any choice of  $s$  with  $1 \leq s \leq P$ .  $q_R^-$  obeys a similar inequality with contributions from negative  $a_{i,k}$ .

Setting  $s = P$ , we see immediately that if all  $e_i$  are non-negative, then a monopole operator with non-zero charge at the orthogonal or symplectic node has  $q_R^\pm \geq 1/2$  and hence  $q_R \geq 1$ . This inequality also holds if the charges vanish at the last node, in view of our previous results for unitary quivers.

Now let us analyze the monopole operators of  $q_R = 1$  and hence the symmetry of the Coulomb branch. As usual, to get  $q_R = 1$ , monopole charges must vanish at any node with  $e > 0$ . So it suffices to consider a quiver of  $P$  nodes with  $e_i = 0$  at each node.

Monopole operators whose charge vanishes at the  $P$ th node will generate an  $SU(P)$  symmetry, the usual result for a chain of  $P - 1$  balanced unitary nodes. Now let us consider monopole operators that have non-zero charge at the  $P$ th node. The inequality (5.17) with  $s = P$  implies that to get  $q_R = 1$ , the charge at the  $P$ th node must be  $(1, -1, 0, \dots, 0)$ . Moreover  $q_R^\pm \geq 1/2$ , so to get  $q_R = 1$ , we need  $q_R^+ = q_R^- = 1/2$ . Considering the inequality with arbitrary  $s$ , we find that to get  $q_R^\pm = 1/2$ , the positive monopole charges at the  $s$ th node must be  $(1, 0, 0, \dots, 0)$ , up to permutation, and likewise the negative charges must be of the same form, up to permutation. Furthermore, by a familiar argument, the subquiver supporting the positive charges must be connected, and likewise the subquiver supporting the negative charges must be connected. Conversely, when all these conditions are imposed, we do get a monopole operator of  $q_R = 1$ .

Postponing for the moment some exceptions associated with  $U(1)$  and  $SO(2)$ , the conditions just described give  $P \times P = P^2$  monopole operators of  $q_R = 1$  with charge at the  $P$ th node. How these operators transform under  $SU(P)$  is completely determined by how they transform under the maximal torus of  $SU(P)$ , which is generated by the classical symmetries of the Coulomb branch. The non-zero weights that arise are differences between weights of the fundamental representation and weights of its conjugate (associated with positive and negative  $a$ 's, respectively), so the

monopole operators transform as the sum of the adjoint representation of  $SU(P)$  and a one-dimensional trivial representation.

This is enough to ensure that the symmetry of the Coulomb branch is (locally)  $SU(P) \times SU(P) \times U(1)$ .

Two exceptions should be pointed out. If a balanced  $U(1)$  node is present, there is no room for a monopole operator to have charges  $(1, -1, 0, \dots, 0)$  at that node. A balanced  $U(1)$  node must be at the left end of the quiver, so the only  $q_R = 1$  monopole operator that is removed by this limitation is the one that has both positive and negative charges at every node. The absence of this operator reduces the symmetry group to  $SU(P) \times SU(P)$ .

The other exception arises if the orthogonal or symplectic group at the  $P$ th node is  $SO(2)$ . The only balanced linear quiver with this property involves the sequence of groups  $U(1) - SO(2)$  for  $P = 2$ . The fact that  $SO(2)$  is abelian results in an enhancement of the symmetry of the Coulomb branch from  $SU(2) \times SU(2)$  (as suggested by the generic analysis) to  $Sp(4)$ . The Coulomb branch of this model has hyper-Kähler dimension 2, and we suspect that it is isomorphic to  $\mathbb{C}^4/\mathbb{Z}_2$ .

The hidden symmetries of the Coulomb branch that are associated with monopole operators can be seen using mirror symmetry if the special node is symplectic. They are classical symmetries of the mirror quivers, which we describe next. Mirror symmetry for these quivers was analyzed in [35], and will be considered in Section 6.6.

## 5.4 Bifurcated quivers

So far all our results, both in Section 2.4.3 and here, have involved linear quivers. But some of the results have close analogs for quivers of other types. These analogs will be important in the rest of the paper, when we include orbifolds and orientifolds.

We start with the basic question of understanding a general quiver of unitary gauge groups with only balanced nodes. Requiring that every node of a unitary quiver is balanced is actually quite restrictive, and there is a nice classification of such quivers. Consider an arbitrary graph  $\Gamma$  in which any two nodes are connected by at most one line.<sup>23</sup> Let  $N$  be the set of nodes, and let  $E$  be the set of edges, that is, the set of pairs of points in

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<sup>23</sup>The argument will show that this condition can be omitted, since without it the Cartan matrix cannot be positive definite.

$N$  that are connected by a line. A quiver and its associated gauge theory are defined as follows. To every node labeled by  $p_i \in N$  we attach a positive integer  $n_i$ , the rank of the group  $U(n_i)$  that we attach to that node. For each edge connecting points  $p_i$  and  $p_j$ , we attach a bifundamental hypermultiplet of  $U(n_i) \times U(n_j)$ . Finally, we assign  $m_i$  fundamental hypermultiplets to the  $i$ th node. Of course, the  $m_i$  must be non-negative.

The condition for every node to be balanced is that

$$2n_i - \sum_{j|(i,j) \in E} n_j = m_i \quad (5.18)$$

for all  $i$ . It is convenient to express this condition in terms of the Cartan matrix  $\mathfrak{C}$  of the graph  $\Gamma$ .  $\mathfrak{C}$  is a matrix that acts on a vector space  $V$  that has a basis element  $v_i$  for each node  $p_i \in N$ . We give  $V$  a metric  $(\cdot, \cdot)$  in which the  $v_i$  are orthonormal. In that basis, the Cartan matrix is  $\mathfrak{C}_{ij} = 2\delta_{ij} - e_{ij}$ , where  $e_{ij}$  is 1 if nodes  $p_i$  and  $p_j$  are connected by a link, and zero otherwise. The Cartan matrix is a discrete version of the one-dimensional Laplace operator  $\Delta = -d^2/dx^2$ , and like  $\Delta$ , it is real and symmetric.

In terms of the Cartan matrix, setting  $\mathbf{n} = \sum_i n_i v_i$  and  $\mathbf{m} = \sum_i m_i v_i$ , the condition for every node to be balanced reads

$$\mathfrak{C}\mathbf{n} = \mathbf{m}. \quad (5.19)$$

Like any real, symmetric matrix,  $\mathfrak{C}$  can be diagonalized with real eigenvalues.  $\mathfrak{C}$  shares with  $\Delta$  the property that its eigenvector  $\mathbf{q}$  with the lowest eigenvalue is unique up to a scalar multiple and can be chosen to have all entries positive,  $\mathbf{q} = \sum_i q_i v_i$  with all  $q_i > 0$ . Suppose that  $\mathfrak{C}\mathbf{q} = \lambda\mathbf{q}$ . We have  $\lambda(\mathbf{q}, \mathbf{n}) = (\mathfrak{C}\mathbf{q}, \mathbf{n}) = (\mathbf{q}, \mathfrak{C}\mathbf{n}) = (\mathbf{q}, \mathbf{m})$ . Here  $\mathbf{q}$  and  $\mathbf{n}$  have positive coefficients, and  $\mathbf{m}$  has non-negative coefficients, so the inner products involved are positive, except that  $(\mathbf{q}, \mathbf{m}) = 0$  if  $\mathbf{m} = 0$ . We deduce that  $\lambda \geq 0$  and if  $\lambda = 0$  then  $\mathbf{m} = 0$ .

Since  $\lambda$  was defined as the smallest eigenvalue of  $\mathfrak{C}$ , if  $\lambda > 0$  then  $\mathfrak{C}$  is positive definite. The graphs  $\Gamma$  with positive-definite Cartan matrix are nothing else than the ADE Dynkin diagrams.

If the smallest eigenvalue of  $\mathfrak{C}$  is zero, then  $\Gamma$  is the extended Dynkin diagram of a group of ADE type. Including this case does not add much, because the requirement that  $\mathbf{m} = 0$  means that one  $U(1)$  subgroup of the gauge group is decoupled. We may as well ungauged the extended node of the Dynkin diagram, reducing to the case that  $\Gamma$  is a Dynkin diagram rather



than an extended Dynkin diagram. This procedure gives a convenient example of a balanced quiver for every ADE diagram: start with the extended Dynkin diagram with  $\mathfrak{m} = 0$  and with  $\mathfrak{n}$  annihilated by  $\mathfrak{C}$ , and ungauged the extended node.

So we have shown that unitary quivers with all nodes balanced are associated with ADE Dynkin diagrams, and conversely that for every choice of an ADE diagram, there are balanced quivers.

If we let  $Q$  be a quiver based on a graph  $\Gamma$  of type  $G$ , where  $G$  is any ADE group, it is natural to suspect that the Coulomb branch of the gauge theory associated with  $Q$  (assuming that it has a standard IR limit) has  $G$  symmetry. This would generalize what we found in Section 2.4.3 for a quiver of type A. Actually, the result for any  $G$  essentially follows from the case of type A. For any  $\Gamma$ , the construction of Section 2.4.3 associates an  $SU(2)$  symmetry of the Coulomb branch to every balanced node. Moreover, for two balanced nodes that are not adjacent, the two  $SU(2)$ 's commute, and for any two adjacent nodes, the two  $SU(2)$ 's fit into an  $SU(3)$  symmetry. By the usual structure theory of Lie groups, it follows that the  $SU(2)$ 's associated with all of the nodes generate together a group of type  $G$ .

We define the excess of a quiver by  $\epsilon = -\mathfrak{C}\mathfrak{n} + \mathfrak{m}$ . A further conjecture along the lines of Section 2.4.3 would assert that if we are given a quiver of unitary groups in which each node is good, in the sense that the coefficients of  $\epsilon = \sum_i e_i v_i$  are all non-negative, then the whole quiver is good in the sense that all monopole operators satisfy  $q_R \geq 1$ . Given this, we would hope that gauge theories associated with good quivers flow to standard IR limits. The symmetry of the Coulomb branch would then presumably be the product of simply laced Lie groups corresponding to the various balanced subquivers, times a  $U(1)$  factor for each node with positive  $e$ . We are not in a position to prove this general conjecture.

In this subsection, we focus on quivers in the shape of a  $D_n$  diagram, or “bifurcated” quivers, as in figure 56. We will assume that all nodes are

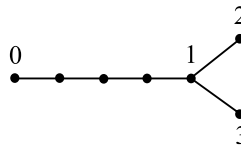


Figure 56: A “bifurcated” or  $D_n$  quiver. Some nodes that play an important role in the text are labeled. The bifurcation occurs at node 1, the short branches consist of nodes 2 and 3, and the outermost node of the long branch is labeled 0.

good and we aim to show that the whole quiver is good. The charge  $q_R$  of a monopole operator can be manipulated as usual: the contributions from positive and negative charges can be separated, and one can restrict to connected subquivers with non-zero charge at every node. The contribution from negative charges equals the contribution from an identical set of positive charges.

Let us consider the contribution to  $q_R$  from the positive monopole charges. It is the sum of three kinds of term. At each node we have contributions  $\Delta_i = \frac{e_i}{2} \sum_{k=1}^{n_i} a_{i,k}$  and  $A_i = \sum_{k=1}^{n_i} a_{i,k}(2n_i - 2k + 1)$  (the  $a_i$  are arranged here in non-decreasing order). At each link there is a contribution  $B_{ij} = -\frac{1}{2} \sum_{k=1}^{n_i} \sum_{t=1}^{n_j} 2 \min(a_{i,k}, a_{j,t})$ . Our study of linear quivers was based on inequalities such as

$$B_{ij} \geq - \sum_{k=1}^{n_i} a_{i,k}(n_i - k) - \sum_{t=1}^{n_j} a_{j,t}(n_j - t + 1). \tag{5.20}$$

The right-hand side is comparable to  $-\frac{1}{2}A_i - \frac{1}{2}A_j$ , enabling us to bound  $\sum_{ij} B_{ij}$  in terms of  $\sum_i A_i$ . This worked only because there were at most two  $B$  terms contributing at each node. As a result, we had a useful set of inequalities for  $q_R$  involving a sum of positive terms.

If we blindly use this method on a bifurcated quiver, the  $A$  at the node with three neighbors would not be sufficient to cancel negative contributions from the three corresponding  $B$  terms, leaving a net negative contribution to the inequality. We need a new trick, which fortunately is quite simple. As in the figure, let us label the node with three neighbors as  $i = 1$ , and the nodes at the ends of the short branches of the quiver as  $i = 2$  and  $i = 3$ . Without loss of generality, assume  $n_2 \geq n_3$ . Roughly, we want to bound  $B_{12} + B_{13}$  relative to  $-(\frac{1}{2}A_1 + A_2 + A_3)$ , after which there will still be a positive contribution  $\frac{1}{2}A_1$  to help in canceling the contribution from the  $B$ 's on the long branch of the quiver.

Let us introduce a fictitious node labeled  $c$  with  $n_c = n_2 + n_3$ , and define the sequence of charges  $a_{c,t}$ ,  $1 \leq t \leq n_c$  at this node by the following:  $a_{c,t} = a_{2,t}$  if  $t \leq n_2 - n_3$ , while otherwise  $a_{c,n_2-n_3+2t-1} = a_{3,t}$  and  $a_{c,n_2-n_3+2t} = a_{2,n_2-n_3+t}$ . This definition ensures that every  $a_{2,t}$  and  $a_{3,t}$  appears precisely once in the sequence  $a_{c,t}$ . Hence we can write

$$B_{1c} = B_{12} + B_{13} = -\frac{1}{2} \sum_{k=1}^{n_1} \sum_{t=1}^{n_2+n_3} 2 \min(a_{1,k}, a_{c,t}) \tag{5.21}$$

Even though the  $a_{c,t}$  are not ordered, the method used to derive the basic inequality (2.26) or (2.29) is perfectly valid, and gives

$$B_{1c} \geq - \sum_{k=1}^{n_1} a_{1,k}(n_1 - k + 1) - \sum_{t=1}^{n_2+n_3} a_{c,t}(n_2 + n_3 - t). \tag{5.22}$$

The first term is familiar, and will be canceled against  $\frac{1}{2}A_1$ . The second term receives contributions from both the 2 and 3 nodes, which need to be disentangled:

$$\begin{aligned} & - \sum_{t=1}^{n_2+n_3} a_{c,t}(n_2 + n_3 - t) \\ &= - \sum_{t=1}^{n_2-n_3} a_{2,t}(n_2 + n_3 - t) - \sum_{t=1}^{n_3} a_{3,t}(n_2 + n_3 - (n_2 - n_3 + 2t - 1)) \\ & \quad - \sum_{t=n_2-n_3+1}^{n_2} a_{2,t}(n_2 + n_3 - (2t - n_2 + n_3)). \end{aligned} \tag{5.23}$$

Finally, we have

$$B_{1c} \geq - \sum_{k=1}^{n_1} a_{1,k}(n_1 - k + 1) - \sum_{t=1}^{n_3} a_{3,t}(2n_3 - 2t + 1) - \sum_{t=1}^{n_2} a_{2,t}(2n_2 - 2t), \tag{5.24}$$

as desired. If we had exchanged  $2t$  and  $2t - 1$  in the definition of  $a_{c,t}$ , alternating the two kind of charges the opposite way, we would have

$$B_{1c} \geq - \sum_{k=1}^{n_1} a_{i,k}(n_i - k + 1) - \sum_{t=1}^{n_3} a_{3,t}(2n_3 - 2t) - \sum_{t=1}^{n_2} a_{2,t}(2n_2 - 2t + 1). \tag{5.25}$$

Starting instead with the mirror image of (5.22),

$$B_{1c} \geq - \sum_{k=1}^{n_1} a_{1,k}(n_1 - k) - \sum_{t=1}^{n_2+n_3} a_{c,t}(n_2 + n_3 - t + 1), \tag{5.26}$$

a similar argument gives

$$B_{1c} \geq - \sum_{k=1}^{n_1} a_{i,k}(n_i - k) - \sum_{t=1}^{n_3} a_{3,t}(2n_3 - 2t + 2) - \sum_{t=1}^{n_2} a_{2,t}(2n_2 - 2t + 1). \tag{5.27}$$

If  $j$  runs over the set  $N$  of all nodes and as in the figure  $i = 0$  labels the outermost node of the long branch, we have from (5.24) and (5.20), summed along the chain as in Section 2.4, the simple inequality

$$q_R \geq \sum_{j \in N} \Delta_j + \sum_{k=1}^{n_2} a_{2,k} + \sum_{k=1}^{n_0} a_{0,k}(n_0 - k). \tag{5.28}$$

For a quiver with all  $e_i > 0$ , so that  $\Delta_i \geq 0$ , this inequality implies that  $q_R \geq 1$ , unless the charges at node 2 vanish. But if those charges vanish, then we reduce to a quiver of type A, and again  $q_R \geq 1$  by virtue of our analysis in Section 2.4.3. We also learn from (5.28) and its analog for type A that to get  $q_R = 1$ , the  $\Delta_j$  must vanish, implying that  $e_j = 0$  at any node with non-zero charges. As usual then, in a good quiver, to analyze monopole operators of  $q_R = 1$ , we can omit nodes with  $e_j > 0$  and consider a connected balanced quiver — one in which all nodes are balanced.

A balanced quiver of type  $D_n$  has monopole operators that generate a  $D_n$  symmetry of the Coulomb branch, by virtue of an argument given earlier for all ADE quivers. It remains to show that there are no other monopole operators of  $q_R = 1$ . A useful fact is that, as in Section 2.4.3, to get  $q_R = 1$ , the set of nodes at which the charges are non-zero must be connected; otherwise each connected component contributes at least 1 to  $q_R$ . Inequality (5.28) implies that to get  $q_R = 1$ , the charge at the 2 node, if not zero, must be elementary:  $a_{2,k} = (0, 0, \dots, 0, 1)$ . It is then also true that  $a_{0,k} = (0, 0, \dots, 0, 1)$ . The modified inequality

$$q_R \geq \sum \Delta_i + \sum_{k=1}^{n_3} a_{3,k} + \sum_{k=1}^{n_0} a_{0,k}(n_0 - k) \tag{5.29}$$

derived from (5.25) shows that similarly  $a_{3,k} = (0, 0, \dots, 0, 1)$ . Then the third inequality (5.27) can similarly be used to show that

$$q_R \geq \sum \Delta_i - \sum_{k=1}^{n_3} a_{3,k} + \sum_{k=1}^{n_s} a_{s,k} + \sum_{k=1}^{n_0} a_{0,k}(n_0 - k), \tag{5.30}$$

for any  $s$  in the long chain, implying that  $\sum_k a_{s,k}$  must be no greater than 2. Hence  $a_{s,k} = (0, 0, \dots, 0, 2)$  or  $a_{s,k} = (0, 0, \dots, 0, 1, 1)$ . At this point it is straightforward, though tedious, to show that any charge  $(0, 0, \dots, 0, 2)$  leads to a monopole of  $q_R > 1$ , and that monopole operators with  $q_R = 1$  have at most one connected sequence of nodes with charges of the form  $a_{s,k} = (0, 0, \dots, 1, 1)$  starting at the node  $i = 1$ , with other non-zero charges being of the form  $a_{s,k} = (0, 0, \dots, 0, 1)$ . What we have enumerated here are

the positive roots of  $D_n$ , so this is the symmetry generated by the monopole operators.

#### 5.4.1 Orthosymplectic quivers of type D

We have learned in Section 5.2 that orthosymplectic quivers have the same expression for the  $R$ -charge as unitary quivers once the  $R$ -charge is expressed in terms of excesses, ranks, and monopole charges. Apart from some exceptions involving  $SO(2)$ , the number of monopole operators of  $q_R = 1$  in an orthosymplectic quiver is one-half of what it is for the corresponding unitary quiver.

So we can apply our results for a unitary quiver of type  $D_n$  to an orthosymplectic quiver of the same type. Let us start from the case with no  $SO(2)$  node involved. The total number of  $q_R = 1$  monopoles in the unitary case was  $\dim(SO(2n)) - \text{rk}(SO(2n)) = 2n(n - 1)$ . Hence we expect  $n(n - 1)$  monopoles in the orthosymplectic case. Monopole operators with charges supported on an  $A_{n-1}$  subquiver generate an  $SO(n)$  symmetry. It is possible to argue by induction that the full symmetry is actually  $SO(n) \times SO(n)$ .

If any endpoint of the quiver is an  $SO(2)$  group, the symmetry is enhanced. In particular an  $SO(2)$  node at the end of the long branch leads, for  $n > 4$ , to  $SO(n + 1) \times SO(n + 1)$  symmetry. An  $SO(2)$  node at the end of a short branch implies that the central node is  $Sp(2)$ , and then the quiver must be a  $D_4$  quiver, with three  $SO(2)$  gauge groups. (In particular, this is equivalent to the  $n = 4$  case of an  $SO(2)$  group at the end of the long branch.) It takes some patience to count all the 33 monopoles hiding in the quiver. The monopole operators and classical symmetries at each  $SO(2)$  node give rise to an  $SU(2)$  symmetry group. The remaining 27 monopole operators are in the representation  $(\mathbf{3}, \mathbf{3}, \mathbf{3})$  of this subgroup; hence the full group turns out to be  $Sp(8)$ . Notice that the Coulomb branch of the theory has hyper-Kähler dimension 4. This and the symmetry group suggest that the Coulomb branch is simply  $\mathbb{C}^8/\mathbb{Z}_2$ .

## 6 Orientifolds and orbifolds

Dirichlet and Neumann are the most obvious half-BPS boundary conditions in  $\mathcal{N} = 4$  super Yang–Mills theory. Intermediate between them are boundary conditions in which the gauge group  $G$  is reduced to a subgroup  $H$  along the boundary. Vector multiplets of  $H$  obey Neumann boundary conditions, while the rest of the  $G$  vector multiplets obey Dirichlet.

A particular case of this which is intuitively obvious and natural is the case that  $H$  is the subgroup of  $G$  that commutes with a symmetry  $\tau$  of  $G$  of order 2. A symmetry of order 2 is known as an involution, and may be either an inner automorphism or an outer automorphism. Boundary conditions associated with an involution can be obtained by a simple  $\mathbb{Z}_2$  orbifold of  $\mathcal{N} = 4$  super Yang–Mills on  $\mathbb{R}^4$ . One simply divides by the reflection  $y \rightarrow -y$  of space, accompanied by the gauge transformation  $\tau$ . Of course, this must be extended to the full  $\mathcal{N} = 4$  theory in a supersymmetric fashion. As explained in Section 2.2 of [1], if we decompose the Lie algebra of  $G$  as  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ , where  $\mathfrak{h}$  is the Lie algebra of  $H$  and  $\mathfrak{h}^\perp$  is its orthocomplement, and write  $\Phi^\pm$  for the projections of a field  $\Phi$  to  $\mathfrak{h}$  and  $\mathfrak{h}^\perp$ , then the necessary conditions for  $\vec{X}$  and  $\vec{Y}$  are

$$\vec{X}^+(0) = 0 = \vec{Y}^-(0). \quad (6.1)$$

Many instances of such boundary conditions can be implemented in string theory via orientifolds or orbifolds.

In this section, we will analyze, for  $G = U(n)$ , the  $S$ -duals of the boundary conditions associated with involutions. These examples illustrate in an interesting way some of the ideas of the present paper. They are quite different from examples that we have considered so far, but are rather tractable, partly because of their realizations in string theory. Another reason to study the  $S$ -duality of these examples is that one can compare to a mathematical theory developed by Nadler [41], though we will only go part way in that direction in the present paper.

### 6.1 Three types of involution

Let us first classify the possible involutions of  $G = U(n)$ .

An inner involution  $\tau$  is simply conjugation by an element  $h$  of  $U(n)$  that obeys  $h^2 = 1$ . Such an  $h$  has  $p$  eigenvalues 1 and  $q$  eigenvalues  $-1$ , for some  $p$  and  $q$  with  $p + q = n$ . We will loosely follow the notation of [41] and call this an involution of Class III. The subgroup  $H$  that commutes with a Class III involution is  $U(p) \times U(q)$ . If  $p = q$ , we say that the involution is symmetric.

An outer involution  $\tau$  is complex conjugation composed with an inner involution. In other words,  $\tau$  acts by  $g \rightarrow w\bar{g}w^{-1}$ , where  $g \rightarrow \bar{g}$  is complex conjugation and  $w$  is an element of  $G$ . There are two essentially different cases, depending on whether  $\tau^2$  equals 1 or  $-1$  when acting on the fundamental representation of  $U(n)$ .

If  $\tau^2 = 1$ , we can take  $\tau$  to be simply  $g \rightarrow \bar{g}$ . We will call this a Class I involution. It reduces the gauge symmetry from  $G = U(n)$  to  $H = O(n)$ .

It is only possible to have  $\tau^2 = -1$  if  $n$  is even. In that case, to realize  $\tau^2 = -1$ , we can take  $\tau$  to be  $g \rightarrow w\bar{g}w^{-1}$ , where  $w$  is the direct sum of  $n/2$  blocks of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.2}$$

We call this a Class II involution. It reduces the gauge symmetry from  $G = U(n)$  to  $H = Sp(n)$ .

Now let us begin to explore the  $S$ -duals of boundary conditions associated with  $\tau$  of Class I, II, or III. One of the most basic questions is whether the  $S$ -dual has full  $U(n)$  gauge symmetry and so is obtained by coupling to a  $U(n)$  invariant SCFT  $\mathfrak{B}^\vee$ . If so, the mirror of  $\mathfrak{B}^\vee$  is given by the familiar ungauging procedure. In the present case, this simply means (figure 57(a)) that we gauge a subgroup  $H$  of the symmetry group  $G$  of the Higgs branch of  $T(U(n))$ . Since  $T(U(n))$  can be represented by a quiver, the same is true for the candidate  $\mathfrak{B}^\vee$ , as indicated in figure 57(b).

However, of the three cases, only Class I leads to a good quiver. For example, for the rightmost two nodes of the quiver of Class III to be good, we need  $n - 1 \geq 2p$  and  $n - 1 \geq 2q$ , which is impossible since  $n = p + q$ . Concerning Class II, we note that according to (5.9), for an  $Sp(n)$  node to be good, the flavor symmetry of the fundamental hypermultiplets, when all other gauge couplings are turned off, must be at least  $SO(2n + 2)$ . However, the Class II quiver in figure 57(b) only leads to a flavor symmetry  $SO(2n - 2)$ .

The Class I quiver in figure 57(b) has an  $SO(n)$  node, and, according to (5.5), for this node to be good, the flavor symmetry of the fundamental hypermultiplets of  $SO(n)$ , when all other gauge couplings are turned off, should be at least  $Sp(2n - 2)$ . That is precisely what we get from the Class I quiver in the figure. So this quiver is good. In fact, it is a balanced quiver of a type considered in Section 5.3. The symmetry of the Coulomb branch is  $SU(n) \times SU(n)$  (or  $Sp(4)$  if  $n = 2$ ).

The Class I quiver has no Higgs branch. To show this, we use the fact that this quiver is the one considered in figure 24, which describes  $T(SU(n))$ , except that an  $SO(n)$  subgroup of the  $SU(n)$  global symmetry of that quiver has been gauged. As explained in Section 3.3.1, the hyper-Kähler quotient of the  $T(SU(n))$  quiver is parametrized by an  $n \times n$  matrix  $M$  that takes values in the nilpotent cone, or a deformation/resolution thereof if FI parameters are turned on. For our purposes, it is convenient to turn on complex FI

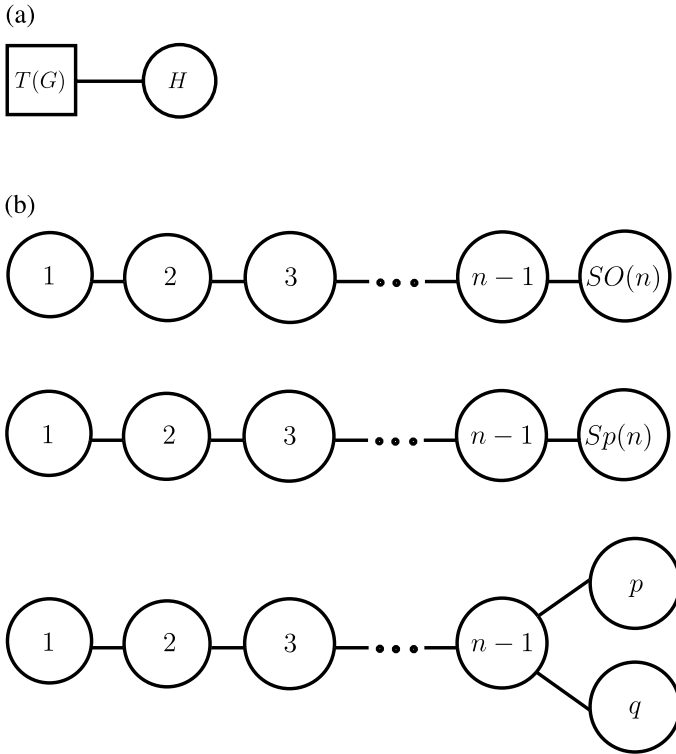


Figure 57: (a) Applying the ungauging procedure to find the  $S$ -dual of a boundary condition defined by reducing the gauge symmetry from  $G$  to  $H$  (with no Nahm pole or SCFT) means that we gauge an  $H$  subgroup of the  $G$  symmetry of  $T(G)$  and take the IR limit. (b) In the present case, for  $\tau$  of Class I, II, or III, this procedure leads to the quivers shown here; a subgroup of the symmetry of  $T(U(n))$  has been gauged. Only Class I leads to a good quiver. Here and later, a node of a quiver labeled simply by an integer  $k$  represents a  $U(k)$  gauge group; nodes that represent other types of gauge group are labeled in more detail.

parameters, so that the complex equation obeyed by  $M$  is deformed to (3.10), whose solutions parametrize a deformation  $\tilde{\mathcal{N}}$  of the nilpotent cone. We take the FI parameters to be generic so that the eigenvalues of  $M$  are distinct. To construct the Higgs branch of the Class I quiver in the presence of the FI parameters, we need to take the hyper-Kähler quotient of  $\tilde{\mathcal{N}}$  by  $SO(n)$ . The complex moment map is the antisymmetric part of  $M$ .  $M$  acts on a vector space  $V \cong \mathbb{C}^n$ ; the statement that it is symmetric means more invariantly that it preserves a complex bilinear form (not a hermitian form) on  $V$ . The group of linear transformations of  $V$  that preserve the bilinear



form is  $SO(n)_{\mathbb{C}}$ . To show that the Higgs branch is trivial, we must show that a symmetric matrix  $M$  that obeys (3.10) and in particular has distinct eigenvalues can be diagonalized by an  $SO(n)_{\mathbb{C}}$  transformation. Indeed, since  $M$  has distinct eigenvalues, its eigenvectors furnish a complex basis for  $V$ ; as  $M$  is symmetric, the eigenvectors are mutually orthogonal with respect to the quadratic form, and we can choose them to be an orthonormal basis. So there is an orthonormal basis in which  $M$  is diagonal and thus we can diagonalize  $M$  by an  $SO(n)_{\mathbb{C}}$  transformation.

The Coulomb branch  $\mathcal{C}$  of the Class I quiver has hyper-Kähler dimension  $n(n-1)/2 + [n/2]$ , where  $[ ]$  denotes the integer part. For  $n = 2$ , the hyper-Kähler dimension is 2, and in view of the  $Sp(4)$  symmetry and absence of free hypermultiplets, it is natural to suspect that  $\mathcal{C}$  may be  $\mathbb{C}^4/\mathbb{Z}_2$ . (Dividing  $\mathbb{C}^4$  by  $\mathbb{Z}_2$  preserves the  $Sp(4)$  symmetry and projects out the chiral operators of  $q_R = 1/2$ .) This together with the absence of a Higgs branch suggests that for  $n = 2$  the theory might be simply a  $\mathbb{Z}_2$  orbifold of a free theory. For  $n > 2$ , we have not been able to find a good candidate for  $\mathcal{C}$ , though because of its large symmetry this might be possible. It appears that for  $n > 2$  there is no good candidate for  $\mathcal{C}$  as an orbifold of some  $\mathbb{C}^{2s}$ , suggesting that the SCFT is non-Gaussian despite the absence of a Higgs branch. This is not a familiar state of affairs. Known examples of non-trivial  $\mathcal{N} = 4$  SCFTs in three dimensions generally have both Coulomb and Higgs branches.

For  $\tau$  of Class II or III, the badness of the quiver indicates that the dual of the orbifold boundary condition has reduced gauge symmetry. To understand the details, we will use a string theory construction.

## 6.2 String theory constructions

The gauge theory boundary conditions considered in this section arise in four-dimensional gauge theory from a reflection  $y \rightarrow -y$  together with a gauge transformation  $\tau$  and suitable transformations of other fields. We want to understand how to realize these boundary conditions in 10-dimensional string theory with D3-branes that generate the  $U(n)$  gauge symmetry interacting with orbifold or orientifold fiveplanes that will generate the boundary condition.

If  $\tau$  is an outer automorphism of  $U(n)$ , we have to use an orientifold fiveplane. The reason for this is that among the symmetries of D3-branes, it is the reversal of the worldsheet orientation of a string, often called  $\Omega$ , that acts on  $U(n)$  by an outer automorphism and maps the fundamental representation of  $U(n)$  to its complex conjugate. Thus, to relate Class I or

Class II boundary conditions to a string theory construction, we need to use some sort of orientifold.

For the same reason, if  $\tau$  is an inner automorphism, we have to use an orbifold operation — constructed using a symmetry that does not reverse the orientation of a string worldsheet.

Our basic constructions of boundary conditions use D5-branes that fill dimensions 012456 and NS5-branes that fill dimensions 012789. An orientifold fiveplane can preserve the same supersymmetry as a parallel D5-brane, and a properly chosen  $\mathbb{Z}_2$  orbifolding operation can preserve the same supersymmetry as an NS5-brane whose worldvolume coincides with its fixed point set.

Hence, we can represent Class I and Class II boundary conditions by an orbifolding operation that reverses  $y$  and  $\vec{Y}$  (that is, it acts as multiplication by  $-1$  on  $y = x^3$  and on  $x^7, x^8, x^9$ ) and leaves fixed the other coordinates — so its fixed point set coincides with the locus of a D5-brane in our constructions. And similarly, Class III boundary conditions come from a carefully chosen orbifolding operation that reverses  $y$  and  $\vec{X}$  (that is, one which acts as multiplication by  $-1$  on  $x^3, x^4, x^5, x^6$ ).  $S$ -duality exchanges  $\vec{X}$  and  $\vec{Y}$ , and as we will see, although it does not merely exchange Class II and Class III, it maps them to close cousins of one another. On the other hand,  $S$ -duality does not appear to map Class I to a perturbative configuration. This agrees with the fact that the dual of Class I involves a non-trivial SCFT, since the Class I quiver in figure 57(b) is good.

Now let us describe in more detail the relevant orbifolding and orientifolding operations, all of which have been well analyzed in the literature. First of all, there are several different kinds of orientifold fiveplane or O5-plane. The most basic distinction is according to whether the gauge symmetry of  $n$  D5-branes coincident with the O5-plane is  $SO(2n)$  or  $Sp(2n)$ . The former case is called an O5<sup>-</sup>-plane since it has a D5-brane charge of  $-1$ . The latter case has D5-brane charge  $+1$  and is called an O5<sup>+</sup>-plane.

In our problem, we consider  $n$  D3-branes with worldvolume spanning directions 0123, and we obtain a boundary condition by coupling to an O5-plane that spans directions 012456. The orbifolding operation gives a boundary condition in the four-dimensional  $U(n)$  gauge theory of the D3-branes, since it reverses the  $y = x^3$  direction.

In the case of a Class I boundary condition, which reduces  $U(n)$  to  $O(n)$  at the boundary, the flavor symmetry of boundary hypermultiplets in the fundamental representation is a symplectic group. Boundary hypermultiplets

come from coupling to D5-branes supported on the orientifold plane, and the flavor symmetry is the gauge symmetry of the D5-branes. Symplectic gauge symmetry of D5-branes corresponds to the case of an  $O5^+$ -plane, as asserted in the last paragraph. So this is the orientifold that leads to a Class I boundary condition.

Similarly, to get a Class II boundary condition, which reduces  $U(n)$  to  $Sp(n)$  at the boundary, the flavor symmetry of fundamental boundary hypermultiplets should be of orthogonal type. So the appropriate orientifold plane is of the type  $O5^-$ .

Finally, let us discuss the Class III boundary condition, which is supposed to involve some sort of orbifold reflection of directions 3456, leaving fixed directions 012789. Of course, we want to define this orbifold in Type IIB superstring theory since we want to do gauge theory on D3-branes. In Type IIA superstring theory, an orbifold that involves a simple reflection  $\mathcal{I}_4$  of four spatial coordinates preserves the same supersymmetry as an NS5-brane supported on the orbifold fixed set. But for Type IIB this is not so. (One way to see this is to observe that the  $\mathcal{I}_4$  orbifold is chiral for Type IIB — all unbroken supersymmetries have the same six-dimensional chirality — while the worldvolume theory of a Type IIB NS5-brane is not chiral.) To preserve the supersymmetry of an NS5-brane, one must divide not by  $\mathcal{I}_4$  but by  $\mathcal{I}_4(-1)^{FL}$ , where  $(-1)^{FL}$  is the operation that counts left-moving worldsheet fermions modulo 2.

The correspondence between involutions and string theory constructions is summarized in Table 2. In the last column of the table, we list the fivebrane charge of the appropriate orientifold or orbifold fiveplane. For orientifolds, this is the D5-brane charge, while for the  $\mathcal{I}_4(-1)^{FL}$  orbifold, it is the NS5-brane charge. This NS5-brane charge is zero; it can be computed as an integral at infinity of the  $H$ -field (the curvature of the string theory  $B$ -field) but the orbifold  $H$ -field vanishes.

Table 2: For each class of boundary condition derived from an involution, we indicate here what type of string theory orientifold or orbifold realizes it, and what is the corresponding O5-brane or NS5-brane charge.

Class of boundary condition	Orientifold or orbifold	Fivebrane charge
I	$O5^+$ orientifold	1
II	$O5^-$ orientifold	-1
III	$\mathcal{I}_4(-1)^{FL}$ orbifold	0

### 6.3 Non-perturbative duality

A non-perturbative duality relating some configurations of the type just described was discovered some years ago [42,43]. Since the duality symmetry  $S : \tau \rightarrow -1/\tau$  exchanges D5-branes, which are related to the first two rows in our table, with NS5-branes, which are related to the third, we look for a transformation that relates a Class III boundary condition to Class I or II.

The fivebrane charges make clear what the duality must be. The  $\mathcal{I}_4(-1)^{FL}$  orbifold has zero fivebrane charge, while the  $O5^\pm$ -planes have fivebrane charge  $\pm 1$ . We can make fivebrane charge 0 by combining an  $O5^-$ -plane with a single D5-brane. But there is no way to get to charge 0 by adding fivebranes to an  $O5^+$ -plane, which already has positive charge. (Adding anti-D5-branes would of course break supersymmetry.) So the only reasonable conjecture is that the  $\mathcal{I}_4(-1)^{FL}$  orbifold is  $S$ -dual to an  $O5^-$ -plane plus one D5-brane.

Apart from the fact that the fivebrane charges and the unbroken supersymmetries match, one of the original arguments for this assertion is that the gauge symmetries match. Quantization of the twisted sector of the  $\mathcal{I}_4(-1)^{FL}$  orbifold gives a single massless vector multiplet, with gauge group  $U(1)$ . On the other hand, a single D5-brane at an  $O5^-$ -plane has gauge symmetry  $SO(2)$ , or equivalently  $U(1)$ .

We want to apply this duality to the gauge theory on the D3-branes. In the field of the  $\mathcal{I}(-1)^{FL}$  orbifold, there are two kinds of fractional D3-brane, depending on whether the generator of the orbifold symmetry acts on the Chan–Paton bundle of the D3-brane as multiplication by  $+1$  or  $-1$ . If we pick  $p$  D3-branes of one type and  $q$  of the other type, with  $n = p + q$ , we get  $U(n)$  gauge symmetry broken at the boundary to  $U(p) \times U(q)$ . This is the general Class III boundary condition.

There is no such choice to be made for the Class II orientifold: we simply have in bulk  $n$  D3-branes, with gauge symmetry reduced from  $U(n)$  to  $Sp(n)$  at  $y = 0$ . Since there is no choice to be made, we have a puzzle to resolve. For what values of  $p$  and  $q$ , if any, is the orbifold dual to the orientifold?

This question actually has a very simple answer. In the orientifold, with  $n$  being even, it is possible to give expectation values to  $\vec{Y}$ , breaking  $Sp(n)$  to  $U(1)^{n/2}$ , and leaving no D3-branes at  $\vec{Y} = 0$  (that is, at  $x^7 = x^8 = x^9 = 0$ ). So in the  $S$ -dual orbifold, if there is one, it must similarly be possible to reduce the gauge group to  $U(1)^{n/2}$  by displacing all D3-branes in  $\vec{X}$  (recall that  $S$ -duality exchanges  $\vec{X}$  and  $\vec{Y}$ ) and leaving none at  $\vec{X} = 0$ . This is

possible precisely if  $p = q = n/2$ ; otherwise, there are “fractional D3-branes” that cannot be removed from  $\vec{X} = 0$ . So this must be the right case for the duality.

Let us look at this a little more closely, taking account of the boundary condition (6.1), which constrains  $\vec{X}(0)$  and  $\vec{Y}(0)$ . For the orientifold, (6.1) says that  $\vec{Y}(0)$  transforms in the adjoint representation of the unbroken  $Sp(n)$ . Its expectation value can indeed break  $Sp(n)$  to the maximal torus  $U(1)^{n/2}$ . For the orbifold, (6.1) implies that  $\vec{X}(0)$  transforms in the bifundamental representation of the unbroken  $U(p) \times U(q)$ . The case that  $\vec{X}(0)$  can break  $U(p) \times U(q)$  to  $U(1)^{n/2}$  is the symmetric case  $p = q = n/2$ . So again that is the right case for this duality between perturbative configurations. The unbroken groups  $U(1)^{n/2}$  that remain when the orbifold is perturbed in  $\vec{X}$  or the orientifold in  $\vec{Y}$  are equivalent as subgroups of  $U(n)$ , as one would expect from the  $S$ -duality between these two configurations.

So we have our first case of using orbifolds and orientifolds to answer the basic question from Section 6.1. The  $S$ -dual of the symmetric Class III boundary condition, with  $p = q = n/2$ , is given by a Class II boundary condition supplemented by boundary hypermultiplets.

### 6.3.1 The general case

We are obviously left with some questions. (1) What is the  $S$ -dual of a Class II boundary condition without boundary hypermultiplets? (2) And what is the  $S$ -dual of a Class III boundary condition with  $p \neq q$ ?

We will show in Sections 6.4 and 6.5 that the answers to these questions do not involve simple orbifold boundary conditions, but involve Nahm poles:

(1)' The  $S$ -dual of a Class II boundary condition that breaks  $U(n)$  to  $Sp(n)$  (without boundary hypermultiplets) is a boundary condition with a Nahm pole relative to the decomposition  $n = 2 + 2 + 2 + \dots + 2$ . This breaks  $U(n)$  to what we will call  $U(n/2)_2$ . Here  $U(n/2)_2$  is a diagonal subgroup of  $U(n/2) \times U(n/2) \subset U(n)$ .

(2)' The  $S$ -dual of a Class III boundary condition that breaks  $U(n)$  to  $U(p) \times U(q)$  with  $n = p + q$  and  $p \geq q + 2$  is a boundary condition built from a Nahm pole and a further reduction of gauge symmetry. The Nahm pole is associated with the decomposition  $n = (p - q) + 1 + 1 + \dots + 1$  and commutes with  $U(1) \times U(2q) \subset U(n)$ . The boundary condition further reduces  $U(1) \times U(2q)$  to  $H = Sp(2q)$ . If  $p - q = 1$ , there is no Nahm pole; the dual boundary condition simply reduces the symmetry from  $U(n)$  to  $Sp(n - 1)$ .

Table 3: The first column lists the unbroken subgroups  $H$  in boundary conditions in  $SU(n)$  gauge theory that are defined by an involution  $\tau$ . The second column lists the unbroken gauge symmetry  $\tilde{H}$  of the  $S$ -dual boundary condition. The third column describes the Nahm pole, if any, that is part of the reduction of the dual gauge group from  $SU(n)$  to  $\tilde{H}$ . The fourth column describes the matter system that is coupled to  $\tilde{H}$ . (The hypermultiplets indicated are in the fundamental representation of  $Sp(n)$ .)

$H$	$\tilde{H}$	Nahm Pole	Matter System
$SO(n)$	$SU(n)$	None	Non-trivial SCFT
$Sp(n)$	$SU(n/2)_2$	$n = 2 + 2 + \dots + 2$	None
$S(U(n/2) \times U(n/2))$	$Sp(n)$	None	Hypermultiplets
$S(U(p) \times U(q)),$ $p > q$	$Sp(2q)$	$n = (p - q) + 1$ $+1 + \dots + 1$	None

In each of these cases, and in contrast to the symmetric case  $p = q$ , there are no boundary hypermultiplets.

These results are summarized in Table 3. The table gives the group  $H$  that is left unbroken by an involution  $\tau$ , and the construction of the dual boundary condition in terms of a Nahm pole, a group  $\tilde{H}$  that commutes with the Nahm pole, and a matter system with  $\tilde{H}$  symmetry. The table has been written for  $G = SU(n)$  rather than  $U(n)$ . This is accomplished by merely dropping central  $U(1)$  factors (which in our constructions obey Dirichlet boundary conditions on one side, and Neumann on the other) from various entries.

Our table can be compared to the first three lines in Table 1 of [41], which refer to the group  $A_{n-1} = SU(n)$ . What is called  $\mathfrak{g}_{\mathbb{R}}$  in the first column of that table is the Lie algebra of a real group  $G_{\mathbb{R}}$  whose maximal compact subgroup we call  $H$ . (From our point of view, the data determining  $G_{\mathbb{R}}$  are the choice of compact gauge group  $G$  and involution  $\tau$ .) What is called  $\mathfrak{h}^{\vee}$  in the fourth column is the complexification of the Lie algebra of what we call  $\tilde{H}$ . With this translation, our table is in perfect agreement with that of [41].

To make clear why the two tables should match, we will briefly describe the problem treated in [41], but restated in gauge theory language. In effect,  $G$  gauge theory is studied on a four-manifold  $M$  with boundary; the boundary condition is determined by an involution  $\tau$  of  $G$ , which reduces  $G$  to a subgroup  $H$ . Then 't Hooft operators that are supported on the boundary are classified. It is shown that, although 't Hooft operators in the interior of  $M$  are classified by representations of the dual group  $G^{\vee}$ , 't Hooft operators supported at the boundary of  $M$  are related to representations of

a more mysterious group  $\tilde{H}$ , whose origin is not obvious. For example,  $\tilde{H}$  is not the dual group of either  $G$  or  $H$ . From our point of view,  $\tilde{H}$  is the subgroup of the dual gauge group  $G^\vee$  that is gauged in the dual boundary condition. So Wilson operators at the boundary are  $\tilde{H}$ -valued.

The last two columns in Table 3 involve matters that have apparently not been explored yet in the mathematical literature. The Nahm pole is plausibly related mathematically to Arthur's  $SL_2$ . What from our point of view is the matter system that is part of the dual boundary condition might show up mathematically in a precise study of the 't Hooft operators.

Some remarks about the C and D cases of the table in [41] are made at the end of Section 7.4.

## 6.4 Nahm poles

Let us return to the duality between (a) a boundary condition that breaks  $U(n)$  to  $Sp(n)$ , with coupling to a boundary hypermultiplet, and (b) a boundary condition that breaks  $U(n)$  to  $U(n/2) \times U(n/2)$ .

We want to understand the dual of (a) without the boundary hypermultiplet. Our strategy will be to use the fact that it is possible to preserve supersymmetry while giving a bare mass to the hypermultiplet. We will determine what parameter in (b) corresponds to the hypermultiplet bare mass, and then we will determine the limit of (b) when the bare mass becomes large. This will give us the dual of breaking  $U(n)$  to  $Sp(n)$ , without the hypermultiplet.

Before we introduce any perturbation, the boundary conditions obeyed by  $\vec{X}$  and  $\vec{Y}$  in the presence of boundary hypermultiplets are

$$\begin{aligned}\vec{X}^+(0) + \vec{\mu}_Z &= 0, \\ \vec{Y}^-(0) &= 0.\end{aligned}\tag{6.3}$$

where  $\vec{\mu}_Z$  is the moment map for the space  $Z$  that parametrizes the boundary hypermultiplets. (This condition coincides with equation (6.1), except that now we include the hypermultiplets.) As explained in Sections 2.2.3 and 2.3.5 of [1], it is possible to add central constants to these boundary conditions, which become

$$\begin{aligned}\vec{X}^+(0) + \vec{\mu}_Z &= \vec{v}, \\ \vec{Y}^-(0) &= \vec{w}.\end{aligned}\tag{6.4}$$

Here  $\vec{v}$  takes values in the center of  $\mathfrak{h}$ , and  $\vec{w}$  takes values in the subspace of  $\mathfrak{h}^\perp$  that commutes with  $\mathfrak{h}$ . (Moreover, the components of  $\vec{w}$  must commute with each other.)

The parameters  $\vec{v}$  are FI parameters in a generalized sense, transforming non-trivially under  $SO(3)_X$  and trivially under  $SO(3)_Y$ . The parameters  $\vec{w}$  are mass parameters, transforming non-trivially under  $SO(3)_Y$ . In enumerating parameters, we also must include the FI parameters and mass parameters of the boundary theory, if any. In the present discussion, the boundary theory consists of free hypermultiplets and has only mass parameters.

Now in (a),  $\mathfrak{h} = \mathfrak{sp}(n)$ , which is a simple Lie algebra with trivial center. So there are no FI parameters. There are two mass parameters. One arises because in (6.3), we can take  $\vec{Y}^-(0) = \vec{d} \cdot 1_n$ , where  $1_n$  is the identity  $n \times n$  matrix. The parameter  $\vec{d}$  would be absent if we took the underlying gauge group  $G$  to be  $SU(n)$  instead of  $U(n)$ . (We use  $U(n)$  because it arises more naturally from branes.) The second parameter is the hypermultiplet bare mass  $\mathbf{m}$ .

Dually in (b), there are no mass parameters since the condition that  $\vec{w}$  should commute with  $\mathfrak{h}$  forces  $\vec{w} = 0$ ; moreover, this boundary condition has no boundary hypermultiplets. However, in (b) there are two FI parameters, since the center of  $H = U(n/2) \times U(n/2)$  has rank two. Embedding  $U(n/2) \times U(n/2)$  in  $U(n)$  in terms of  $n/2 \times n/2$  blocks

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \tag{6.5}$$

we can take the boundary condition on  $\vec{X}$  to be

$$\vec{X}(0) = \begin{pmatrix} \vec{c}_1 \cdot 1_{n/2} & * \\ * & \vec{c}_2 \cdot 1_{n/2} \end{pmatrix}. \tag{6.6}$$

(There is no  $\vec{\mu}_Z$  term here as in this description there are no boundary hypermultiplets.)

So the two parameters  $\vec{c}_1$  and  $\vec{c}_2$  must match on the other side the parameters  $\vec{d}$  and  $\mathbf{m}$ . The matching is easy to do because one parameter on each side involves the center  $U(1)$  of  $G = U(n)$ . The boundary conditions that we are considering do not couple the two factors of  $G \sim U(1) \times SU(n)$ , so we can consistently remove the center of  $G$ , which means on one side setting  $\vec{d} = 0$  and on the other side  $\vec{c}_1 = -\vec{c}_2$ . So the dual of the hypermultiplet



bare mass is a boundary condition

$$\vec{X}(0) = \begin{pmatrix} \vec{\mathbf{m}} \cdot 1_{n/2} & * \\ * & -\vec{\mathbf{m}} \cdot 1_{n/2} \end{pmatrix}. \tag{6.7}$$

Now let us discuss how to preserve supersymmetry in the presence of this boundary condition. We take the vacuum at infinity to be given by  $\vec{X} = \vec{Y} = 0$ . Supersymmetry then requires that  $\vec{Y}$  vanishes everywhere, but  $\vec{X}$  cannot vanish identically in view of the boundary condition (6.7). Rather, we must look for a solution of Nahm's equations  $d\vec{X}/dy + \vec{X} \times \vec{X} = 0$  that obeys the boundary condition and has  $\vec{X}$  vanishing at infinity.

Let us first discuss how to do this for  $n = 2$ . The general solution of the  $SU(2)$  Nahm equations on the half-line  $y \geq 0$  with  $\vec{X} \rightarrow 0$  at infinity is

$$\vec{X} = f \frac{\vec{t}}{y + y_0} f^{-1}, \tag{6.8}$$

with  $y_0 > 0$  and  $f \in SU(2)/\mathbb{Z}_2$ ;  $\vec{t}$  are the usual  $\mathfrak{su}(2)$  generators. We must adjust the parameters  $f$  and  $y_0$  to obey the boundary condition. Without essential loss of generality, take  $\vec{\mathbf{m}} = (0, 0, \mathbf{m}_3)$  and work in the usual basis in which  $t_3$  is diagonal and  $t_1, t_2$  are purely off-diagonal. Then to obey (6.7), we need to take  $f = 1$  and  $y_0 = 1/|\mathbf{m}|$ .

The limit as  $\vec{\mathbf{m}} \rightarrow \infty$  is now easily described. In this limit,  $y_0 \rightarrow 0$  and  $\vec{X}(y)$  has an irreducible Nahm pole at  $y = 0$ .

Actually, for  $n = 2$  this is not really a new result. The groups  $SU(2)$  and  $Sp(2)$  coincide, so the boundary condition that reduces  $SU(n)$  to  $Sp(n)$  just coincides, when  $n = 2$ , with Neumann boundary conditions. We already know that Neumann boundary conditions are dual to an irreducible Nahm pole, and this is what we have just rediscovered from another point of view.

Now, however, we can immediately generalize to the case of any  $n$ . ( $n$  must be even for the question about reduction from  $U(n)$  to  $Sp(n)$  to make sense.) To solve the  $SU(n)$  Nahm equations with the boundary condition (6.7), we just take the tensor product of the  $SU(2)$  solution (6.8) with the rank  $n/2$  identity matrix  $1_{n/2}$ . Then, taking  $\vec{\mathbf{m}} \rightarrow \infty$  as before,  $\vec{X}$  acquires a Nahm pole which is obtained simply by taking the tensor product of the standard rank 2 Nahm pole with  $1_{n/2}$ .

This Nahm pole corresponds to a decomposition  $n = 2 + 2 + 2 + \dots + 2$ . It breaks  $U(n)$  to  $U(n/2)_2$ , a diagonal subgroup of  $U(n/2) \times U(n/2) \subset U(n)$ .

What we have learned is that the dual of the boundary condition defined by the involution that breaks  $U(n)$  to  $Sp(n)$  is a boundary condition defined by a Nahm pole that breaks  $U(n)$  to  $U(n/2)_2$ .

## 6.5 The last case

To complete our explanation of Table 3, we must justify the last line, which describes the  $S$ -dual of a boundary condition associated with an involution that breaks  $SU(p+q)$  to  $S(U(p) \times U(q))$  for  $p > q$ . One approach is to start with the third line of the table, which says what happens for  $p = q$ , and “flow” to  $q < p$  by giving suitable expectation values to the scalar fields  $\vec{X}$  at infinity.

We will follow another approach, which is simple and possibly illuminating. In this, we will use an elementary relation between Dirichlet and Neumann boundary conditions. Suppose that, in any gauge theory with gauge group  $H$ , we impose Dirichlet boundary conditions, meaning that the gauge field  $A$  and the generator  $\epsilon$  of a gauge transformation are both required to vanish at the boundary. Then we get a theory in which  $H$  acts as a global symmetry at the boundary. The global symmetry is just a gauge transformation with  $\epsilon$  constant, but not equal to 1, at the boundary. (See [1, Section 2.2.2].) This preserves the boundary condition  $A = 0$ . As with any global symmetry, we can seek to gauge this one. In the present case, this just means that we allow the boundary value of  $\epsilon$  to be non-constant and thus arbitrary. So we simply arrive at four-dimensional gauge theory with a gauge parameter that is unrestricted at the boundary. The gauge field is then also unconstrained at the boundary. In other words, gauging the global symmetry of Dirichlet boundary conditions produces Neumann boundary conditions. We will call this process the regauging trick.

The claim in the last row of the table is that if  $\mathcal{B}$  is a boundary condition defined by an involution  $\tau$  that breaks  $SU(p+q)$  to  $S(U(p) \times U(p))$ , then the dual is a boundary condition  $\mathcal{B}^\vee$  consisting of a Nahm pole of rank  $p - q$ , followed by a reduction of the structure group from  $U(2q)$  to  $Sp(2q)$ . It is convenient to start with  $\mathcal{B}^\vee$  and show that the dual is  $\mathcal{B}$ .  $\mathcal{B}^\vee$  can be conveniently represented by a configuration of  $p + q$  D3-branes intersecting a D5-brane and ending on an orientifold (figure 58(a)). As usual, to work with branes we extend the symmetry from  $SU(p+q)$  to  $U(p+q)$ . Later, we will factor out the central  $U(1)$  from the final statement.

The  $S$ -dual of the D5-brane in figure 58(a) is simply an NS5-brane. As for the orientifold in the figure, it has no simple string theory dual. If there

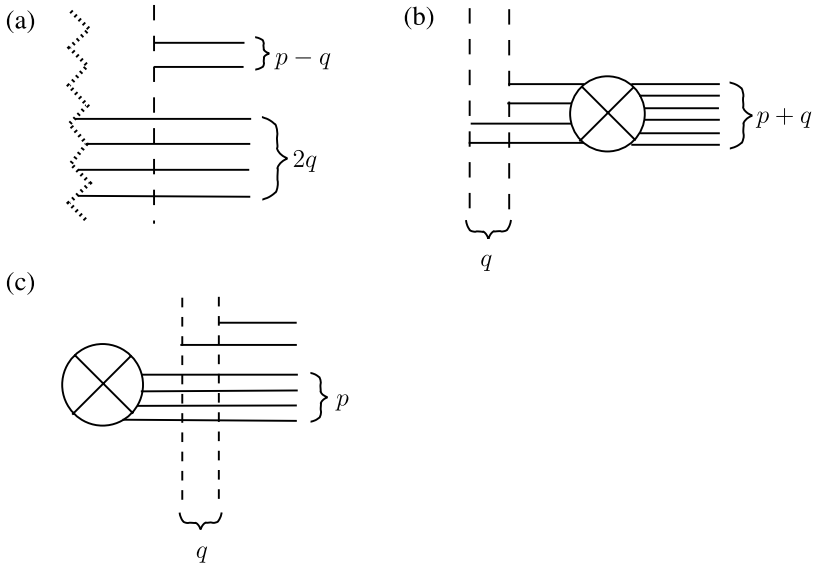


Figure 58: (a) A boundary condition in  $U(p+q)$  gauge theory (depicted here for  $p=4, q=2$ ) consisting of a Nahm pole of rank  $p-q$  followed by an orientifold (represented by the vertical jagged line) that reduces the gauge symmetry to  $Sp(2q)$ . (b) A brane configuration that is closely related to the  $S$ -dual of (a).  $q$  D5-branes create a Nahm pole that is associated to the decomposition  $2q = 2 + 2 + \dots + 2$ . In the limit that the separation between these fivebranes vanishes, there is a global  $U(q)$  symmetry; gauging this symmetry gives the  $S$ -dual of (a). (c) Instead, before gauging the symmetry, we can move the D5-branes across the NS5-brane, arriving at this configuration. Now if we take the D5-branes to coincide and gauge the global  $U(q)$  symmetry, we arrive at a gauge theory representation of the  $S$ -dual of the original configuration (a). This is a boundary condition in  $U(p+q)$  gauge theory in which a subgroup  $U(p) \times U(q)$  is gauged at the boundary.

were an additional D5-brane with no D3-branes ending on it, there would be a simple  $S$ -dual given by the  $\mathcal{I}_4 \cdot (-1)^{FL}$  orbifold. However, we learned in Section 6.4 how to describe the dual of the boundary condition due to the “bare” orientifold unaccompanied by an extra D5-brane: it is given by a Nahm pole for the decomposition  $2q = 2 + 2 + \dots + 2$ , with gauging of the resulting  $U(q)$  symmetry. We can easily represent the Nahm pole by incorporating  $q$  D5-branes with two D3-branes ending on each one, as in figure 58(b). In this figure, a  $U(q)$  global symmetry appears if we take the separations between the D5-branes to vanish. If we gauge this symmetry,

we arrive at the  $S$ -dual of the boundary condition set by the bare orientifold. There is no convenient way in a brane construction to gauge the  $U(q)$  symmetry. So we will simply remember to gauge it at the end of the construction.

With this understanding, figure 58(b) can be used to construct the  $S$ -dual of figure 58(a). On the other hand, we can make a standard brane manipulation. We simply move the D5-branes to the right of the NS5-brane, to arrive at figure 58(c). Now only a single D3-brane ends on each D5-brane, leaving  $p$  D3-branes to end on the NS5-brane. The collection of  $q$  D5-branes therefore reduces the gauge symmetry from  $U(p+q)$  to  $U(p)$ , which then obeys Neumann boundary conditions because of the NS5-brane. From a field theory point of view, this boundary condition admits a global  $U(q)$  symmetry (the commutant of the unbroken gauge group  $U(p)$ ). The  $U(q)$  symmetry appears as a symmetry of the brane configuration if the D5-branes are taken to be coincident. To construct the  $S$ -dual, we are now supposed to gauge this global  $U(q)$  symmetry. At this stage, we have a  $U(p+q)$  gauge symmetry in the half-space  $y \geq 0$ , with a subgroup  $U(p) \times U(q)$  gauged at the boundary.

Thus the dual of a boundary condition  $\mathcal{B}$  with  $U(p+q)$  reduced to  $U(p) \times U(q)$  at the boundary, for  $p > q$ , is a boundary condition  $\mathcal{B}^\vee$  with a Nahm pole reducing  $U(p+q)$  to  $U(1) \times U(2q)$ , which is then reduced to  $Sp(2q)$  at the boundary. Notice that in  $\mathcal{B}$ , the central  $U(1)$  obeys Neumann boundary conditions, while in  $\mathcal{B}^\vee$  it obeys Dirichlet boundary conditions. Factoring out this central  $U(1)$ , we arrive at the statement of the last line of Table 3 for gauge group  $SU(p+q)$ .

A key step in this argument — gauging the  $U(q)$  global symmetry at the very end of the process — was essentially the regauging trick in which Neumann boundary conditions (here for the subgroup  $H = U(q)$ ) can be obtained by gauging the global symmetry of Dirichlet boundary conditions.

## 6.6 More elaborate examples

Once one understands the  $S$ -duality between the  $O5^-$ -plane and the  $\mathcal{I}_4(-1)^{FL}$  orbifold, one can understand the  $S$ -duals of more general boundary conditions made by combining these with fivebranes. We will describe a few examples and compare the results we get to results of the standard  $T(SU(n))$  construction.

We can start by considering the configuration in figure 59(a), consisting of an the orbifold plane plus a D5-brane. This adds a fundamental flavor

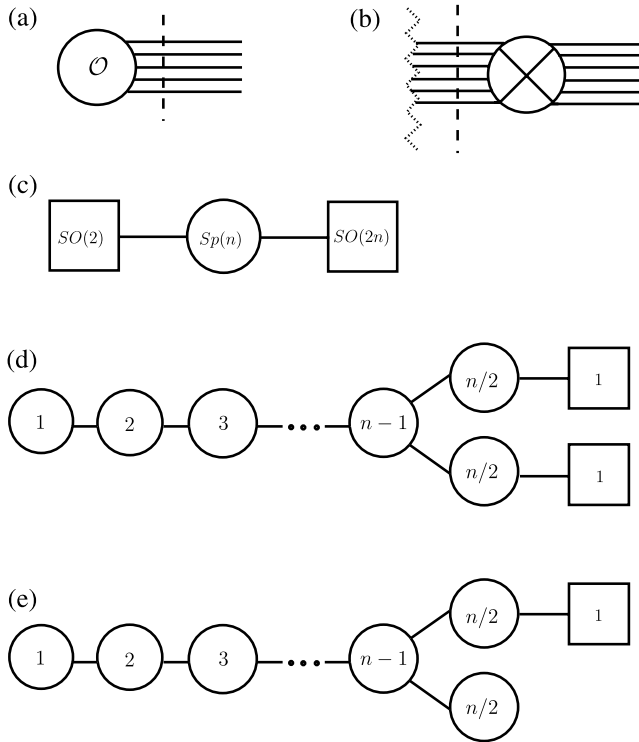


Figure 59: (a) A boundary condition in  $U(n)$  gauge theory created by a D5-brane together with an orbifold plane (which is depicted by a circle containing the symbol  $\mathcal{O}$ ). (b) The  $S$ -dual, constructed from an NS5-brane, D5-brane, and an orientifold plane, represented by the jagged line. (The D5-brane is generated from  $S$ -duality applied to the orbifold plane in (a), while the NS5-brane is dual to the D5-brane in (a).) (c) A quiver related to (b). It is balanced, so the Coulomb branch has  $U(1)$  symmetry. (d) The quiver description of the  $S$ -dual to (a) that comes by using  $T(SU(n))$ . This should be mirror to the quiver in (c). (e) A boundary condition in which only one of the two  $U(n/2)$  factors is coupled to a fundamental hypermultiplet leads to this quiver.

to both  $U(n/2)$  gauge groups at the boundary. The  $S$ -dual configuration in figure 59(b) involves  $n$  D3-branes crossing an NS5-brane followed by a D5-brane, and ending on an  $O5^-$ -plane. This configuration produces a non-trivial SCFT, which arises as the IR limit of a three-dimensional  $Sp(n)$  gauge theory coupled to fundamental hypermultiplets with flavor symmetry  $SO(2n+2)$ . This gauge theory is balanced; hence the Coulomb branch has a hidden  $SO(2)$  symmetry. The bulk  $U(n)$  gauge group is embedded in an  $SO(2n)$  subgroup of  $SO(2n+2)$ .

The  $T(SU(n))$  recipe corresponds to the quiver in figure 59(d). This is a balanced  $D_{n+1}$  type quiver, with a hidden  $SO(2n+2)$  symmetry in the Coulomb branch, and an obvious  $U(1) \cong SO(2)$  symmetry of the Higgs branch. The quivers in (c) and (d) must be mirror and indeed we see that their symmetries match. Such mirror quivers have been considered before in [44].

Alternatively, we can introduce a single flavor for one of the  $U(n/2)$  groups only. This makes sense as a half-BPS boundary condition, though we cannot realize it by a brane construction. It is entertaining to look at the  $T(SU(n))$  prescription. The quiver in figure 59(e) has a single minimally unbalanced node, and all other nodes are balanced. Though not a good quiver, this quiver can be readily analyzed with the inequalities of Section 5.4. The inequality (5.29), with the unbalanced node labeled as node 3, implies that a monopole operator with charge at that node has  $q_R \geq 1/2$ ; moreover, that value can indeed be achieved. Hence the quiver is ugly. Omitting the unbalanced node, the symmetry group is  $SU(n+1)$ . The unbalanced node has a classical  $U(1)$  symmetry. The monopole operators which previously extended  $SU(n+1) \times U(1)$  to  $SO(2n+2)$  now have  $q_R = 1/2$  instead of  $q_R = 1$ . They transform in the antisymmetric tensor of  $SU(n+1)$ , and have charge 1 under the classical  $U(1)$  at the unbalanced node. The dimension of the Coulomb branch is  $n(n+1)/2$ , which coincides with the number of free  $q_R = 1/2$  hypermultiplets, so the theory is completely free. The  $U(n)$  gauge symmetry is embedded in  $SU(n+1) \times U(1)$ . (How the center of  $U(n)$  is embedded is not quite clear.)

One can add more matter fields coupled to  $U(n/2) \times U(n/2)$  at the boundary, but this does not seem to add many new ideas. Instead we will do the opposite and add matter fields on the  $Sp(n)$  side of the duality. We begin with figure 60(a):  $n$  D3-branes cross two D5-branes and end on an  $O5^-$ -plane. This is a boundary at which the  $U(n)$  bulk gauge theory is broken to  $Sp(n)$ , and fundamental hypermultiplets with  $SO(4)$  flavor symmetry are coupled to the surviving gauge group.

The  $S$ -dual brane configuration in figure 60(b) involves an orbifold plane and a single NS5-brane. The  $4d$   $U(n)$  gauge theory in the slab between the orbifold and the fivebrane is broken to  $U(n/2) \times U(n/2)$  by the orbifold boundary condition. At the NS5-brane, there are bifundamental hypermultiplets coupling each of the  $U(n/2)$  groups to the bulk  $U(n)$  gauge theory on the half space. In the infrared we are led to a boundary condition in which the full  $U(n)$  gauge theory is preserved, and coupled diagonally to an SCFT which is the product of two copies of a  $U(n/2)$  three-dimensional gauge theory with  $n$  flavors, as in figure 60(c).

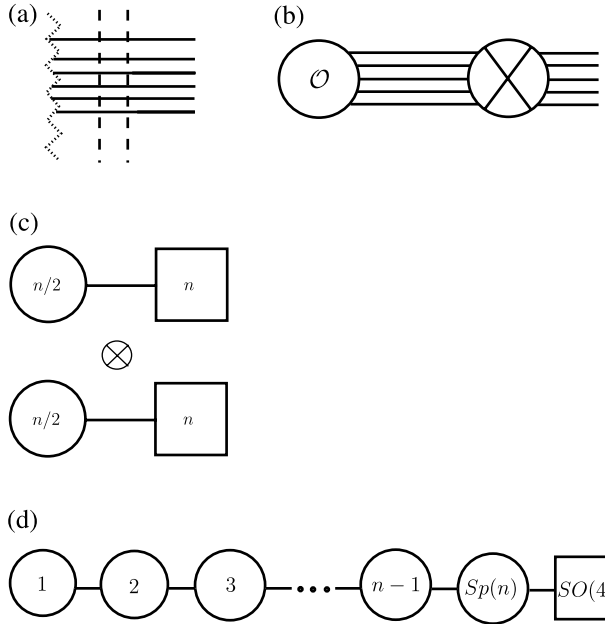


Figure 60: (a) The  $Sp(n)$  orientifold plus two half D5-branes. (b) The  $S$ -dual of (a), constructed from an orbifold plane and a half NS5-brane. (c) The corresponding quiver. (d) The result of applying the  $T(SU(n))$  recipe to (a).

This SCFT has a hidden  $SU(2) \times SU(2)$  symmetry on the Coulomb branch due to the two independent balanced nodes. This matches the  $SO(4)$  flavor symmetry of the original boundary condition.

It is also interesting to compare this to the  $T(SU(n))$  prescription for the  $S$ -dual boundary condition. The quiver is depicted in figure 60(d), and is a balanced quiver with a symplectic node at the end. We know from the monopole analysis that we should expect an  $SU(n) \times SU(n)$  hidden symmetry in the Coulomb branch, which matches well the symmetry of the Higgs branch of figure 60(c).

Finally, we may consider a similar situation with  $k > 2$  flavors, as in figure 61(a). The  $S$ -dual brane configuration in figure 61(b) leads to a boundary condition in which the full  $U(n)$  gauge group at the boundary is coupled to the SCFT depicted in figure 61(c). This SCFT is described by a balanced  $D_k$  quiver. The monopole analysis predicts an  $SO(2k)$  symmetry of the Coulomb branch, which matches the symmetry of the original boundary condition. The  $T(SU(n))$  prescription provides the quiver

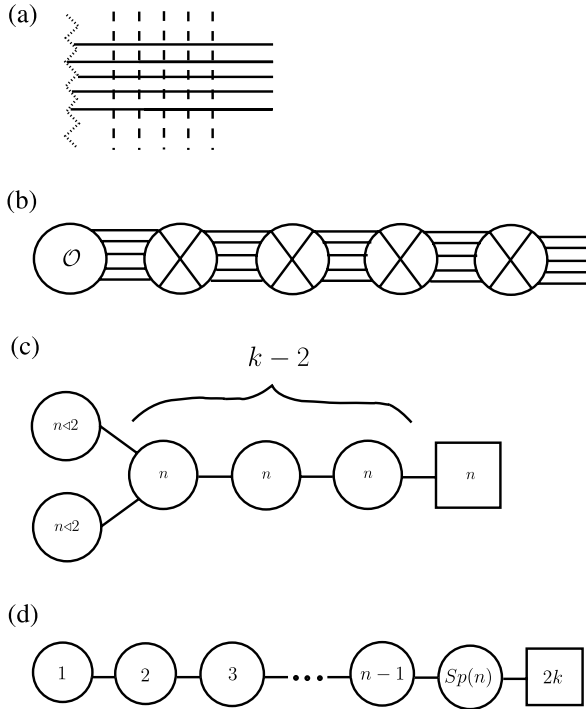


Figure 61: (a) The analog of figure 60(a) with  $k$  half D5-branes (sketched here for  $k = 5$ ) rather than 2. (b–d) The corresponding dual brane configurations and quivers. The main difference is that the quiver in (c) is now connected and is of type D. The quivers in (c) and (d) are still mirror.

in figure 61(d): only the unitary nodes are balanced, and indeed we only expected an  $SU(n)$  symmetry of the Coulomb branch, not  $SU(n) \times SU(n)$ .

### 7 Boundary conditions for orthogonal and symplectic gauge groups

We want to extend some of the explicit constructions of Section 3 to orthogonal and symplectic gauge groups. A well-known way to generate in string theory an  $\mathcal{N} = 4$  super Yang–Mills gauge theory with orthogonal or symplectic gauge groups is to consider D3-branes in the background of an O3-plane. Brane constructions involving D3-branes and fivebranes in the presence of an O3-plane have been introduced in [35] as a tool to construct mirror pairs of orthosymplectic linear quivers. We will review and clarify this construction in the next subsection, and then adapt it to describe  $S$ -duality of boundary conditions.



Table 4: The discrete RR and NS fluxes of an O3-plane, its name, its D3-brane charge, the type of gauge group is produces when combined with D3-branes, and its *S*-dual.

Fluxes	Name	D3-brane charge	Gauge group	<i>S</i> -dual
(0, 0)	O3 <sup>-</sup>	-1/4	SO(2 <i>n</i> )	O3 <sup>-</sup>
(0, 1)	$\tilde{\text{O}}3^-$	1/4	SO(2 <i>n</i> + 1)	O3 <sup>+</sup>
(1, 0)	O3 <sup>+</sup>	1/4	Sp(2 <i>n</i> )	$\tilde{\text{O}}3^-$
(1, 1)	$\tilde{\text{O}}3^+$	1/4	Sp'(2 <i>n</i> )	$\tilde{\text{O}}3^+$

### 7.1 Review of O3 planes

O3-planes in type IIB string theory come in four kinds, distinguished by  $\mathbb{Z}_2$  valued discrete fluxes of RR and NS B-fields. The *S*-duality  $SL(2, \mathbb{Z})$  symmetry acts on the discrete fluxes much like  $SL(2, \mathbb{Z})$  acts on the spin structures of a two-torus. If the fluxes are zero, the orientifold plane, indicated here as O3<sup>-</sup>, is invariant under *S*-duality and carries -1/4 unit of D3-brane charge. *n* D3-branes in the background of an O3<sup>-</sup>-plane carry an SO(2*n*) gauge theory.

Adding half a D3-brane changes the RR discrete flux to 1 and the D3-brane charge to +1/4. The resulting gauge group is SO(2*n* + 1). This  $\tilde{\text{O}}3^-$ -plane is invariant under  $T : \tau \rightarrow \tau + 1$  and transforms under  $S : \tau \rightarrow -1/\tau$  to an object O3<sup>+</sup> with NS discrete flux. *n* D3-branes in the background of an O3<sup>+</sup>-plane carry an Sp(2*n*) gauge group. Finally, a *T* transformation on O3<sup>+</sup> adds half a unit of theta-angle to the Sp(2*n*) gauge theory, leading to an object called  $\tilde{\text{O}}3^+$ . Whenever we label the O3-planes by the corresponding gauge group, we will label this as Sp'(2*n*). (The prime means the following: the theta-angle of the Sp'(2*n*) gauge theory differs by  $\pi$  from the theta-angle of the underlying Type IIB superstring theory. In the other cases, the two angles are equal.)

The most obvious question for us is what fivebrane configurations will lead to simple boundary conditions and domain walls in the presence of O3-planes. Roughly speaking one can introduce “half” D5 and NS5-branes, which lead to simple field theory constructions, similar to those for unitary groups. But there are important subtleties.

An NS5-brane is defined simply by a conformal field theory, although not one which is known explicitly. If we take the NS5-brane to have world-volume in the 012789 directions and to be localized at  $y = x^4 = x^5 = x^6 = 0$ , then the SCFT is invariant under reflection of directions 456789, so one can

construct an orientifold SCFT. This describes the O3-plane interacting with what is usually called a half NS5-brane. All fields and couplings in this orientifold spacetime are obtained by applying a  $\mathbb{Z}_2$  projection to whatever one has before orientifolding.

At large distances along the  $y$ -direction, the target spacetime of the orientifold must resemble one of the four flat space O3-planes. Because of the  $H$ -field of the half NS5-brane, the NS flux will jump across the NS5-brane and the type of O3-plane will be actually different at large positive or large negative  $y$ . This means that if the gauge group is of orthogonal type on one side of the NS5-brane, it will be of symplectic type on the other.

Before orientifolding, if D3-branes end on an NS5-brane from both sides, one gets bifundamental hypermultiplets of the appropriate  $U(n) \times U(m)$  group. In the present situation,  $U(n) \times U(m)$  is projected to  $SO(n) \times Sp(m)$  (or  $Sp(n) \times SO(m)$ ); the bifundamental representation of  $SO(n) \times Sp(m)$  is pseudoreal, making it possible to define the  $\mathbb{Z}_2$  projection for the hypermultiplets. It follows from what we have just explained that constructions based on half NS5-branes will generally produce the sort of orthosymplectic quivers considered in Section 5.2.

Given the existence of a half NS5-brane,  $S$ -duality implies the existence of a “half” D5-brane across which the RR flux of the O3-planes jumps. This will correspond to a domain wall between  $SO(2n)$  and  $SO(2n + 1)$  gauge theories, or between  $Sp(2n)$  and  $Sp'(2n)$ . The two cases are fundamentally different, however. In the  $Sp(2n)$  case, the half D5-brane can be constructed explicitly from free fields, and all its properties can be calculated. In the orthogonal case, there is no explicit construction, and this will lead to some unusual properties. The difference can be explained as follows.

In general, before introducing the O3-plane, we can consider for any positive integer  $k$  a system of  $k$  D5-branes of worldvolume 012456. Their gauge symmetry is  $U(k)$ . To make an orientifold projection of this object, we need to choose an outer involution  $\tilde{\tau}$  of the Chan–Paton bundle that squares to  $\pm 1$ . An outer involution is one which acts by complex conjugation times conjugation by an element of  $U(k)$ . Similarly to define the orientifold projection for D3-branes, one picks an outer involution  $\tau$  squaring to  $\pm 1$ . The D3-brane gauge symmetry is orthogonal or symplectic for  $\tau^2 = 1$  or  $-1$ . The flavor symmetry of hypermultiplets arising from 3–5 strings is the gauge symmetry of the D5-branes and is likewise orthogonal or symplectic for  $\tilde{\tau}^2 = 1$  or  $-1$ . As before, the D3–D5 strings admit an orientifold projection only if one group is orthogonal and one is symplectic, so we need  $\tau^2 = -\tilde{\tau}^2$ . (Of course, one can also give a conformal field theory explanation of this fact as in [45].)

To get a half D5-brane, we want  $k = 1$ , but this is possible only if  $\tilde{\tau}^2 = 1$  and hence  $\tau^2 = -1$ . So it is possible only for an O3-plane of  $Sp$  type. Making the orientifold projection of the usual D3–D5 system leads to a single fundamental hypermultiplet of  $Sp(2n)$ , with flavor symmetry  $O(1)$ . The jumping of the RR flux across the half D5-brane means that the theta-angle must jump by  $\pi$  in crossing it. This is consistent with the  $\mathbb{Z}_2$  anomaly of three-dimensional  $Sp(2n)$  gauge theory with an odd number of real hypermultiplets: the anomaly can be cancelled by a half-integral Chern–Simons term, or equivalently in our context by letting the four-dimensional theta-angle jump by  $\pi$ .

If  $\tilde{\tau}^2 = -1$ ,  $k$  must be even. After taking the  $\mathbb{Z}_2$  projection, the D5-brane charge  $k/2$  is an integer. The object with smallest D5-brane charge has  $k = 2$  and is called a full D5-brane.

We call the  $k = 2$  object a full D5-brane for either sign of  $\tilde{\tau}^2$ . However, for  $\tilde{\tau}^2 = 1$ , the full D5-brane is trivially a direct sum of two half D5-branes.

For  $\tilde{\tau}^2 = -1$ , the zero mode of the worldvolume scalar field  $\Phi$  which parametrizes the relative motion of the two original D5-branes along  $y$  is projected out by the orientifold. The relative motion of two D5-branes is described by fields valued in the adjoint representation of  $SU(2)$ , so let us omit the diagonal part of  $\Phi$  and consider only the adjoint-valued part. We also set  $\vec{x} = (x^4, x^5, x^6)$ . The orientifold projection on  $\Phi$  is

$$\Phi(-\vec{x}) = -\Phi(\vec{x}), \quad (7.1)$$

ensuring that  $\Phi$  has no zero mode.

There is no free field construction of a half D5-brane in the presence of an O3-plane of orthogonal type, but still, as already noted,  $S$ -duality implies its existence. In crossing such an object, since the RR flux jumps, the gauge group jumps between  $SO(2n)$  and  $SO(2m + 1)$  for some  $n, m$ . Actually it is possible to show that this object must exist without making use of  $S$ -duality. We use the fact that in Type IIB superstring theory, without any orientifolding, one can have a supersymmetric configuration consisting of a D5-brane with different numbers  $p$  and  $q$  of D3-branes ending on the two sides. The gauge group jumps from  $U(p)$  to  $U(q)$ , with a Nahm pole of rank  $|p - q|$ , but with no extra degrees of freedom supported at the intersection. (See Section 3.4.4 of [1] for more detail.) The orthogonal-type orientifold projection of this configuration has  $SO(p)$  gauge symmetry on one side, and  $SO(q)$  on the other. It still has no degrees of freedom supported at the intersection, since the projection of nothing is nothing. It is only consistent if  $p - q$  is odd, because that is the case that the Nahm pole is real. So this

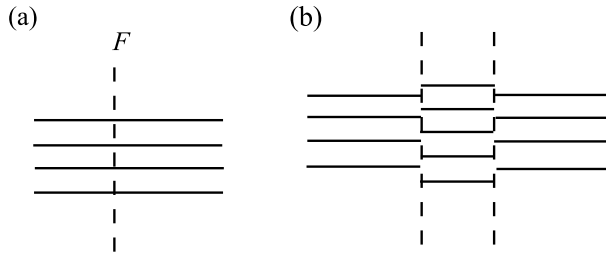


Figure 62: (a) D3-branes parallel to an O3-plane of orthogonal type (not drawn) intersecting a full D5-brane. Here and later, a vertical dotted line labeled by the letter  $F$  represents a full D5-brane, while an unlabeled vertical dotted line represents a half D5-brane. (b) The same O3-plane intersecting a pair of parallel half D5-branes.

type of D-brane configuration exists even though it cannot be constructed with free two-dimensional fields.

At this stage, for intersection with an O3-plane of orthogonal gauge symmetry, we have two superficially similar configurations with the same D5-brane charge, namely a full D5-brane or a pair of parallel half D5-branes (figure 62). If the number of half D3-branes on the left and right of the figure is  $p$  (corresponding to  $SO(p)$  gauge symmetry), then the number inside must differ from  $p$  by an odd number. For reasons that will become clear, we have chosen the number inside to be  $p + 1$ .

### 7.1.1 Splitting and $R$ -symmetry

Consider a full D5-brane that intersects an O3-plane of orthogonal type? Can we “split” the full D5-brane into a pair of half D5-branes by giving an expectation value to  $\Phi$ ? The orientifold projection (7.1) makes it impossible to give  $\Phi$  a constant expectation value, which we usually think of as the way to separate two branes. What saves the day is that the full D5-brane has an  $Sp(2) \cong SU(2)$  gauge symmetry. To preserve the supersymmetry of the orientifold plane, we need not take  $\Phi$  to be constant. It is enough to obey the Bogomolny equations of  $SU(2)$ :

$$F = *D\Phi. \tag{7.2}$$

The usual hedgehog solution of monopole charge 1 on  $\mathbb{R}^3$  is of the form

$$\Phi = f(r)\vec{x} \cdot \vec{t}, \tag{7.3}$$

where  $\vec{t}$  are the  $\mathfrak{su}(2)$  generators and  $r = |\vec{x}|$  (and the gauge field is given by an analogous ansatz). Moreover,  $f(r) \sim L/r$  for large  $r$  for some constant  $L$ , so that  $|\Phi|$  approaches  $L$  at infinity.

This solution obeys the orientifold condition (7.1). At large  $r$ , it describes two D5-branes separated by an amount  $L$ , just as in figure 62(b). (For small  $r$ , inside the core of the monopole, the solution is more complicated; the size of the core is of order  $1/L$  and so is negligible for large  $L$ . This is consistent with the semiclassical picture of figure 62(b) in which  $L$  is much larger than the string scale and the core is not seen.) Interpreting the solution on  $\mathbb{R}^3/\mathbb{Z}_2$  rather than  $\mathbb{R}^3$ , the monopole charge is  $1/2$ . This represents one extra half D3-brane stretched between the two D5-branes in the figure, in addition to those already present in figure 62(a). This explains the fact that in figure 62(b), there is precisely one extra half D3-brane between the two half D5-branes.

So the two configurations can be deformed into each other. But there is a very important point to be made about the symmetries. In figure 62(a), there is an  $SU(2)_X$  symmetry that rotates  $\vec{x}$ . (The group that acts faithfully on  $\vec{x}$  is of course  $SO(3)_X$ .) There is also an  $SU(2)$  gauge symmetry of the D5-branes, which is realized in the D3-brane theory as a flavor symmetry  $SU(2)_F$ .

On the other hand, in the separated configuration of figure 62(b), we see only one combination of these symmetries — the rotation symmetry  $SU(2)'_X$  that acts on  $\vec{x}$ . (Again, it is the quotient  $SO(3)'_X$  that acts faithfully on  $\vec{x}$ .) The second symmetry has been lost. The reason that we have given  $SU(2)'_X$  a different name from the  $SU(2)_X$  rotation symmetry of figure 62(a) is that actually they do not coincide even in the limit  $L \rightarrow 0$ . The hedgehog solution is not invariant under either the rotation symmetry  $SU(2)_X$  or the gauge symmetry  $SU(2)_F$ , but only under a diagonal subgroup. It therefore is this diagonal subgroup that is the symmetry of the separated configuration with two half D5-branes and so corresponds to  $SU(2)'_X$ .

Going back to the configuration with the full D5-brane, from the point of view of the  $SO(p)$  gauge theory on the D3-brane, the low-energy physics is described by the coupling of the bulk four-dimensional gauge fields to a three-dimensional SCFT. This SCFT is a free field theory that describes the bifundamental hypermultiplet  $H$ . The bosonic components of  $H$  transform as  $(1/2, 1/2)$  under  $SU(2)_X \times SU(2)_F$ . They also have conformal dimension  $1/2$ , telling us that they must transform with spin  $1/2$  under the  $R$  symmetry that is part of the superconformal algebra. Since  $SU(2)_F$  commutes with supersymmetry while  $SU(2)_X$  is an  $R$ -symmetry, the candidate  $R$  symmetries are  $SU(2)_X$  or a diagonal subgroup of  $SU(2)_X \times SU(2)_F$ .

The condition that  $H$  must transform with spin  $1/2$  tells us that the superconformal  $R$ -symmetry in the low-energy limit of figure 62(a) is actually  $SU(2)_X$ .

The split configuration has a mass scale  $L^{-1}$ . We can recover from it a superconformal field theory by going to a low-energy limit in which the separation of the half D5-branes is unimportant and we recover the physics of the full D5-brane. However, in view of what we have said, the  $R$ -symmetry  $SU(2)'_X$  that is visible in the split configuration is not the superconformal  $R$ -symmetry of the IR limit.

In a sense, this should come as no surprise. The split configuration does not obey the usual constraint that linking numbers should be non-decreasing from left to right. The linking numbers in the orthosymplectic case are defined for half branes in essentially the same way that they are defined in the unitary case for full branes: the linking number of a half fivebrane is the number of half fivebranes of the opposite kind to its left plus the jump in the half D3-brane charge across it, including the D3-brane charge of the O3-planes. From left to right of figure 62(b), the linking numbers are 1 and  $-1$ . So naturally, the ultraviolet and infrared  $R$ -symmetries are different.

Even though the configuration of figure 62(b) violates the linking number constraint, the above analysis implies that as the separation between the two half D5-branes is taken to zero, there is a smooth limit to the unsplit configuration of figure 62(b). This is rather special in that, as shown via Nahm's equations in Section 3.5.1 of [1], a generic D3–D5 configuration that violates the linking number constraint does not have a similar natural limit to an unsplit configuration when the D5 separations are taken to zero. This is shown in [1] for unitary groups (that is, without the O3-plane) and is also true for generic configurations that violate the linking number constraint in the presence of the O3-plane. But evidently (and as one can verify from Nahm's equations), it is not true when the violation of the linking number constraint comes only by splitting full D5-branes.

Violating the usual linking number constraint means that, under  $S$ -duality, we should expect to encounter bad quiver gauge theories. We will now give a simple example. We define a full NS5-brane to be the  $S$ -dual of a full D5-brane. We want to determine the IR dynamics at the intersection of an O3-plane with a full NS5-brane, by splitting the full NS5-brane to a pair of half NS5-branes and constructing a quiver.  $S$ -duality will enable us to determine exactly what is the correct quiver.

First we consider a full NS5-brane intersecting an  $\tilde{O}3^-$ -plane with gauge group  $SO(2r + 1)$  for some  $r$  (figure 63). The  $S$ -dual is a full D5-brane

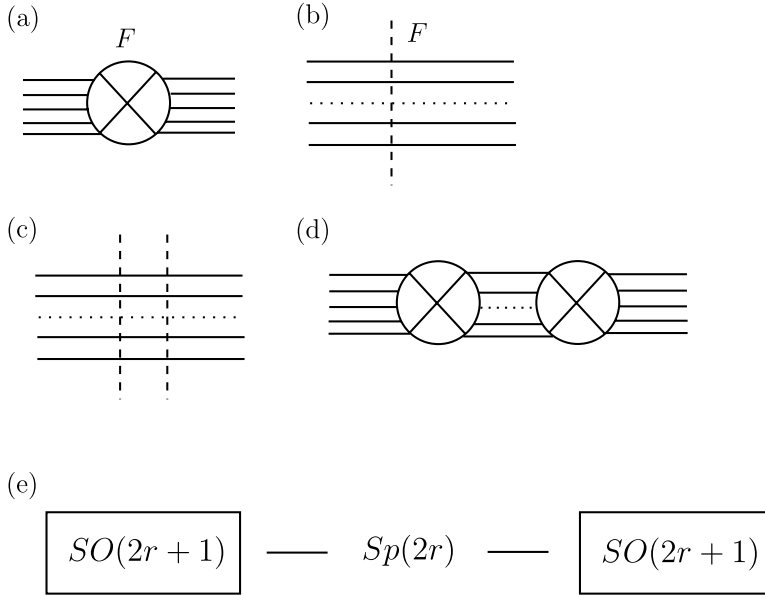


Figure 63: (a) A full NS5-brane intersecting an O3-plane of type  $SO(2r + 1)$  (sketched here for  $r = 2$ ).  $S$ -duality leads to (b), in which the O3-plane is now of symplectic type. Splitting the full D5-brane leads to (c), and finally  $S$ -duality brings us back to (d). The theory in the central slab is  $Sp'(2r)$ . The three-dimensional limit is described by the good quiver shown in (e). In (b) and (c), and below, the horizontal dotted line represents a half unit of D3-brane charge carried by an O3-plane of symplectic type. This dotted line contributes to linking numbers (though not to gauge symmetries) and drawing it lets us distinguish visually between O3-planes of orthogonal or symplectic type. By contrast, in these figures, an O3-plane of orthogonal type is not explicitly drawn.

intersecting an  $O3^+$ -plane with gauge group  $Sp(2r)$ . This can be straightforwardly split to two perturbative half D5-branes, with gauge group  $Sp'(2r)$  between them. Then applying  $S$ -duality again, we find that the original configuration with the full NS5-brane is dual to a system of two half NS5-branes with  $Sp'(2r)$  in the central slab. The difference between  $Sp(2r)$  and  $Sp'(2r)$  is unimportant in the low-energy three-dimensional limit, so that limit gives an  $Sp(2r)$  gauge theory with flavor symmetry  $SO(4r + 2)$ , coming from bifundamentals at the two ends. This is a good and in fact balanced theory, in the sense of Section 5.1, so the infrared and ultraviolet  $R$ -symmetries agree, consistent with the fact that in this case the splitting of the D5-brane was straightforward. Moreover, since the theory is balanced, the Coulomb

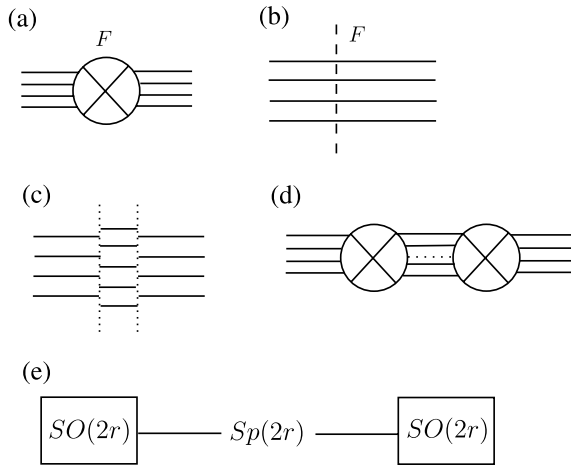


Figure 64: (a) A full NS5-brane intersecting an O3-plane of type  $SO(2r)$  (again sketched for  $r = 2$ ).  $S$ -duality leads again to (b), and splitting the full D5-brane and applying  $S$ -duality again leads to (c) and then (d). But now, as the O3-plane in (b) is of orthogonal type, the  $R$ -symmetry is modified in this process. Accordingly, the quiver in (e), which describes the three-dimensional limit of the configuration in (d), is now a bad one.

branch has an  $SO(2)$  symmetry in the infrared, which is dual to the flavor symmetry of two half D5-branes.

The splitting of a full NS5-brane intersecting an  $\tilde{O}3^+$ -plane is similar, since the dual D5-brane again intersects an O3-plane of symplectic type and can be split straightforwardly. The gauge theory describing the full NS5-brane turns out to be a balanced  $SO(2r + 1)$  gauge theory with flavor symmetry  $Sp(4r)$ . Again the ultraviolet and infrared  $R$ -symmetries match and the Coulomb branch has an  $SO(2)$  symmetry in the infrared.

The other two cases behave differently. A full NS5-brane intersecting an  $O3^-$ -plane is shown in figure 64(a). In carrying out the usual duality operations, we have to split a full D5-brane interacting with an O3-plane of orthogonal type, so we expect the  $R$ -symmetry to be modified. Indeed, the gauge theory description turns out to be an  $Sp(2r)$  theory with flavor symmetry  $SO(4r)$ . This theory is bad, by the criterion of Section 5.1, so it is indeed impossible to match the ultraviolet and infrared  $R$ -symmetries. In the infrared, the Coulomb branch is supposed to have an  $Sp(2)$  global symmetry, matching the flavor group of the full D5-brane. But as the gauge theory is bad, we cannot see this symmetry by studying monopole operators.



The last case of a full NS5-brane intersecting  $O3^+$ -plane is similar. It is dual to an  $SO(2r + 2)$  theory with flavor symmetry  $Sp(4r)$ . This is a bad theory according to Section 5, consistent with the fact that the  $R$ -symmetry is modified in splitting the full D5-brane.

### 7.1.2 Monopoles and orientifolds

Depending on the type of orientifold projection, only a net even or odd number of half D3-branes can end on a half D5-brane. From the viewpoint of the D3-brane theory, this reflects the reality property of the Nahm pole. We want to investigate the matter from the point of view of the D5-brane theory.

The endpoint of a half D3-brane looks like a singular monopole of charge 1 in the  $U(1)$  gauge theory of the D5-brane. So to use D5-brane field theory, we will omit the points where D3-branes end, and discuss the topology of the orientifold projection on a large two-sphere  $S$  given by  $r = \text{constant}$ .

Let  $\mathcal{L} \rightarrow S$  be any line bundle. Let  $\pi : S \rightarrow S$  be the antipodal map  $\vec{x} \rightarrow -\vec{x}$ . There exists an antiunitary isomorphism  $\phi$  between  $\mathcal{L}$  and  $\pi^*(\mathcal{L})$ , because  $\mathcal{L}$  is completely classified topologically by its first Chern class  $c_1(\mathcal{L})$ , which is odd under both  $\pi^*$  and complex conjugation. There is no natural choice of  $\phi$ , but  $\phi^2$  is independent of the choice and equals 1 or  $-1$ , depending only on the topology of  $\mathcal{L}$ . The formula is in fact that  $\phi^2 = (-1)^{c_1(\mathcal{L})}$ .

Now we can refine the criterion for when a half D5-brane exists. The definition of the orientifold projection requires a choice of antiunitary isomorphism  $\tilde{\tau}$  from the Chan–Paton gauge bundle of the D5-brane to itself. For a single half D5-brane, this bundle is a line bundle  $\mathcal{L}$ , and in view of what is said in the last paragraph, we have  $\tilde{\tau}^2 = (-1)^{c_1(\mathcal{L})}$ . If the D3-brane gauge theory is symplectic, we want  $\tilde{\tau}^2 = 1$ , so in this case  $c_1(\mathcal{L})$  must be even. Since a half D3-brane ending on a half D5-brane carries magnetic charge, this result means that only a net even number of half D3-branes can end on a single half D5-brane, as we already know.

If the D3-brane gauge theory is orthogonal, we want  $\tilde{\tau}^2 = -1$ , so  $c_1(\mathcal{L})$  must be odd. Hence only a net odd number of half D3-branes can end on the half D5-brane, as we also already know.

Now, taking the gauge theory of the D3-branes to be symplectic, we want to consider a problem of  $4s + 2$  half D3-branes ending on a full D5-brane (figure 65(a)). The D5-brane gauge bundle  $E \rightarrow S$  is now a rank two bundle with  $c_1(E) = 4s + 2$ . It is possible for such a configuration to have an  $SU(2)$  global symmetry which we call  $SU(2)_F$ . For this, we require  $E = \mathcal{L} \oplus \mathcal{L}$ , where  $\mathcal{L}$  is a line bundle with  $c_1(\mathcal{L}) = 2s + 1$ . The orientifold projection

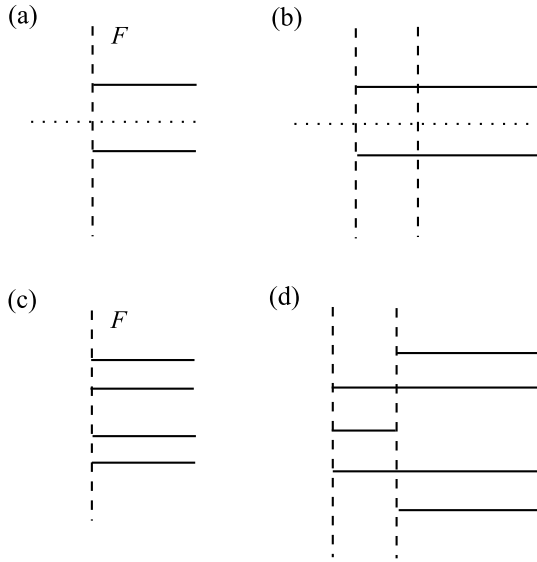


Figure 65: (a) An O3-plane of symplectic type intersecting a full D5-brane, on which  $4s + 2$  half D3-branes end, drawn here for  $s = 0$ . (b) The gauge bundle at infinity in the last figure is  $\mathcal{L} \oplus \mathcal{L}$ , so after splitting the full D5-branes, one might expect  $2s + 1$  half D3-branes to end on each half D5-brane. Instead, solving the Bogomolny equations leads to the situation drawn here, the numbers (from left to right) being  $2s + 2$  and  $2s$ . (c) An O3-plane of orthogonal type intersecting a full D5-brane on which  $4s$  half D3-branes end, drawn here for  $s = 1$ . (d) Upon splitting the D5-brane, an extra D3-brane fragment is created via the Bogomolny equations, with the result that a net odd number of half D3-branes ends on each half D5-brane.

now has an unusual property. The antiunitary isomorphism  $\tilde{\tau} : E \rightarrow \pi^*E$  must square to 1, even though any antiunitary isomorphism  $\phi : \mathcal{L} \rightarrow \pi^*\mathcal{L}$  obeys  $\phi^2 = -1$ . So we pick

$$\tilde{\tau} = \begin{pmatrix} 0 & \phi \\ -\phi & 0 \end{pmatrix}, \tag{7.4}$$

still leaving  $SU(2)_F$  symmetry.

Now we would like to split the D5-brane to a pair of half D5-branes. Because of the form of  $\tilde{\tau}$ , we are in a situation similar to what happened when we tried to split a full D5-brane in the presence of an orthogonal O3-plane. The zero mode of the relevant adjoint-valued field  $\Phi$  is projected out, but the splitting is possible anyway with the help of an 't Hooft-Polyakov monopole. The result is that the split configuration exists, but it is not true, as one

might expect from the choice  $E = \mathcal{L} \oplus \mathcal{L}$  with  $c_1(\mathcal{L}) = 2s + 1$ , that  $2s + 1$  half D3-branes end on each half D5-brane. Rather, the 't Hooft-Polyakov monopole gives an extra half D3-brane between the two half D5-branes, so from left to right the net numbers are  $2s + 2$  and  $2s$ , as in figure 65(b). As before,  $SU(2)_X \times SU(2)_F$  is broken to a diagonal subgroup.

This has a variant (figure 65(c)) for a full D5-brane interacting with an O3-plane of orthogonal type. We recall that this is obtained by orientifolding a configuration with a pair of D5-branes intersecting the O3-plane. Suppose that, before the orientifolding, a net of  $4s$  D3-branes end on the pair of D5-branes. We can take the D5-brane gauge bundle  $E \rightarrow S$  to be  $E = \mathcal{L} \oplus \mathcal{L}$ ,  $c_1(\mathcal{L}) = 2s$ , giving a configuration again with flavor symmetry  $SU(2)_F$ . The orientifold projection must obey  $\tilde{\tau}^2 = -1$ , and since  $c_1(\mathcal{L})$  is even, ensuring that  $\phi^2 = 1$ , we can accomplish this with  $\tilde{\tau}$  of the same form as in (7.4). With the help of the Bogomolny equations, we can again split the full D5-brane, once again breaking  $SU(2)_X \times SU(2)_F$  to a diagonal subgroup and arriving at figure 65(d).

## 7.2 Simple boundary conditions from branes

The basic idea of Section 3 was to realize boundary conditions in  $U(n)$  gauge theory in terms of D3-branes ending on a system of fivebranes, and use the properties of the branes to determine the action of  $S$ -duality. We have developed the tools we need to do the same for orthogonal and symplectic groups, by adding an O3-plane to the previous constructions.

$n$  D3-branes ending on a single NS5-brane gives Neumann boundary conditions in  $U(n)$  gauge theory. So, applying an orientifold projection,  $n$  half D3-branes ending on a single half NS5-brane gives Neumann boundary conditions in  $SO(n)$  or  $Sp(n)$  gauge theory.<sup>24</sup>

The dual of Neumann boundary condition then corresponds to a configuration in which all D3-branes end on a single half D5-brane. This must correspond to a regular Nahm pole, as otherwise upon solving Nahm's equations, it would support a moduli space of vacua. If the bulk gauge group is  $Sp(2n)$ , the regular  $\mathfrak{su}(2)$  embedding corresponds to the  $2n$ -dimensional irreducible representation of  $SU(2)$ , and similarly for  $SO(2n + 1)$  it corresponds to the  $2n + 1$ -dimensional irreducible representation. On the other hand, the regular embedding of  $SO(2n)$  corresponds to the decomposition

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<sup>24</sup>A variant is the case that an  $Sp'$  theory ends on a half NS5-brane, with a single half D3-brane on the other side. There is then a real fundamental hypermultiplet of  $Sp(n)$  at the boundary.

$2n = (2n - 1) + 1$ . This is consistent with the fact that on the other side of the half D5-brane the orientifold is of the  $\tilde{O}3^-$  type, which supports odd orthogonal gauge symmetry. Only  $2n - 1$  of the  $2n$  half D3-branes stop at the half D5-brane, leaving  $O(1)$  on the other side.

More general Nahm poles can be produced by combining several half D5-branes. Consider an  $SO(n)$  bulk gauge theory. We have learned that at a single half D5-brane only an odd number of half D3-branes can end. This produces naturally any Nahm pole of odd rank. We can split any Nahm pole in which every summand is odd dimensional into a sequence of elementary Nahm poles, ordered by increasing dimension, in complete parallel with the unitary construction.  $S$ -duals of such boundary conditions are trivially found. The corresponding set of half NS5-branes gives rise to a good linear orthosymplectic quiver. The quiver has balanced nodes whenever successive summands of the Nahm pole have the same dimension, and the enhanced orthogonal symmetries of the Coulomb branch match the subgroup of  $SO(n)$  which commutes with the Nahm pole.

From a field theory point of view, this is not the end of the story. In the most general  $\mathfrak{su}(2)$  embedding in  $\mathfrak{so}(n)$ , it is possible to have an even number of summands of the same even dimension. If there are  $2k$  such summands, the commutant of this embedding contains a factor of  $Sp(2k)$ . This is the symmetry group associated to  $k$  full D5-branes; hence it is natural to suspect that full D5-branes will be needed to describe these even rank Nahm poles. The Nahm pole can indeed be realized by orientifold projection of  $2k$  poles of rank  $2d$  in the unitary gauge theory.

To describe the  $S$ -dual configuration via gauge theory, we need to split the D5-branes to half D5-branes (whose duals are half NS5-branes that have a simple gauge theory interpretation). This splitting, however, involves the process of figure 65(c),(d) in which a half D3-brane is created and the half D5-branes violate the linking number constraint. After the splitting, an odd number of half D3-branes end on each half D5-brane, but the  $R$ -symmetry no longer coincides with the  $R$ -symmetry of the unsplit configuration. Once the configuration has been split in this way, it is straightforward to identify its  $S$ -dual as an orthosymplectic quiver. The only drawback is that this quiver will be a bad quiver, with a bad node corresponding to each unusually ordered pair of half D5-branes.

We can proceed in much the same fashion for symplectic gauge theories. Now the simplest Nahm poles to describe are the ones of even rank. Those are easily mapped to half D5-branes on which an even numbers of half D3-branes end, with the gauge group changing from  $Sp(2n)$  to  $Sp'(2m)$  or vice versa. As usual a Nahm pole with several even summands can be

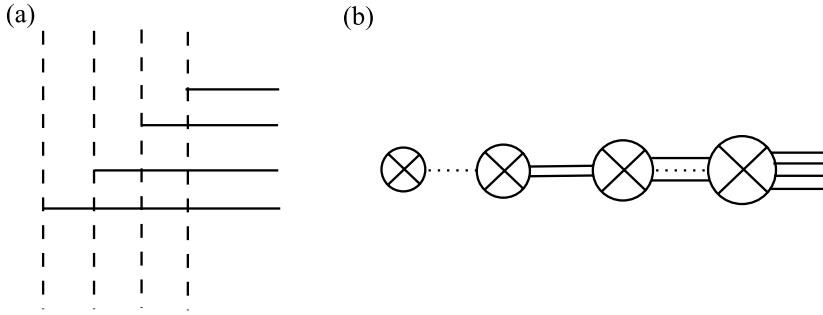


Figure 66: (a) Dirichlet boundary conditions in  $SO(2n)$  gauge theory can be achieved by letting  $2n$  half D3-branes end on  $2n$  half D5-branes (sketched here for  $n = 2$ ). (b) The  $S$ -dual configuration, which leads to the self-mirror quiver of fig. 54(a).

decomposed into a sequence of elementary Nahm poles of increasing rank, and then  $S$ -duality is straightforward. The result is a good orthosymplectic quiver. Again the commutant of the  $\mathfrak{su}(2)$  embedding has orthogonal factors for every set of summands of the same dimension. This will map to sequences of consecutive balanced nodes in the  $S$ -dual quiver gauge theory.

Just as in the case of an orthogonal gauge group, the full story is more complicated. In the most general  $\mathfrak{su}(2)$  embedding in  $\mathfrak{sp}(2n)$ , it is possible to have a pair of summands of the same odd dimension  $d = 2s + 1$ . The simplest example of this is Dirichlet boundary conditions, where all summands are of dimension 1. We have learned in Section 7.1.2 that we can realize two Nahm poles of the same odd dimension with a full D5-brane using the orientifold projection (7.4). We have also learned how to split such a full D5-brane, with the help of the Bogomolny equations. This leads to a configuration like that of figure 65(b) in which, from left to right,  $d + 1$  half D3-branes end on the first half D5-brane and  $d - 1$  on the second. As before, this splitting modifies the  $R$ -symmetry.

$S$ -duality is straightforward and leads to a linear orthosymplectic quiver. The quiver, however, will have a bad node for each pair of unusually ordered Nahm poles in the split configuration.

**7.2.1 Quiver representations for  $T(G)$**

We are finally prepared to give a quiver description of  $T(G)$  for orthogonal and symplectic groups. The simplest example is  $T(SO(2n))$ . The Dirichlet boundary condition for  $SO(2n)$  is realized by  $2n$  half D3-branes, each ending on a separate half-D5 brane, as in figure 66(a). The  $S$ -dual of this brane configuration is depicted in figure 66(b), and corresponds to the

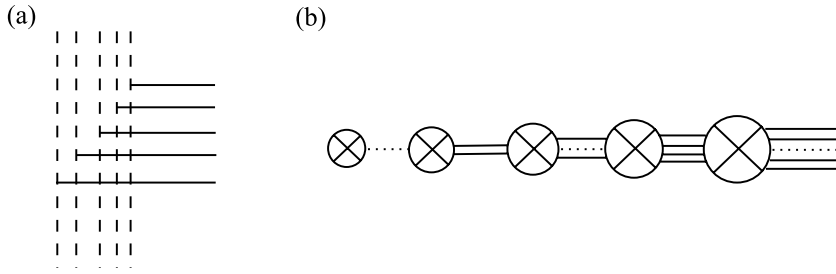


Figure 67: (a) A brane realization of Dirichlet boundary conditions for  $SO(2n + 1)$ , sketched here for  $n = 2$ . (b) The  $S$ -dual, which leads to the quiver of figure 54(b).

good and perhaps even beautiful self-mirror quiver in figure 54(a), which describes  $T(SO(2n))$ .

One more half D3 and half D5-brane leads (figure 67) to a Dirichlet boundary condition for  $SO(2n + 1)$ , and to a slightly longer quiver in figure 54(b). This is a good quiver description of  $T(Sp(2n))$ .

The theory  $T(SO(2n + 1))$  (which is also the mirror of  $T(Sp(2n))$ ) is the dual of Dirichlet boundary conditions for an  $Sp(2n)$  gauge group. Dirichlet boundary conditions are a very special case of an even number of odd dimensional summands in the decomposition of  $2n$ . As in the examples just treated, we can realize Dirichlet boundary conditions for  $Sp(2n)$  by letting  $2n$  half D3-branes end on  $2n$  half D5-branes. But now, when we separate the half D5-branes in  $y$  in preparation for  $S$ -duality, the  $R$ -symmetry is modified and several half D3-branes are added, to lead to the configuration depicted in figure 68(a). The dual NS5-brane configuration is depicted in figure 68(b), and the resulting bad quiver in figure 54(c). This gives a description of  $T(SO(2n + 1))$  which completely obfuscates the symmetries of the Coulomb branch. For  $T(SO(3))$ , this description was analyzed in Section 5.2.2.

Finally, we can ask for the  $S$ -dual of a Dirichlet boundary condition for  $Sp'(2n)$ . The half D5-brane configuration is essentially identical to the one in figure 68(a), but with the labelling  $Sp$  and  $Sp'$  permuted. The  $S$ -dual boundary condition involves the bad quiver in figure 68(c). The resulting theory, which we could still call  $T(Sp'(2n))$ , is apparently self-mirror, and would be useful to find the  $S$ -dual of boundary conditions for an  $Sp'$  gauge theory.

We can similarly give quiver descriptions of the various  $T_{\rho_v}^\rho(G)$  for orthogonal and symplectic gauge groups. The simplest case, which does not require

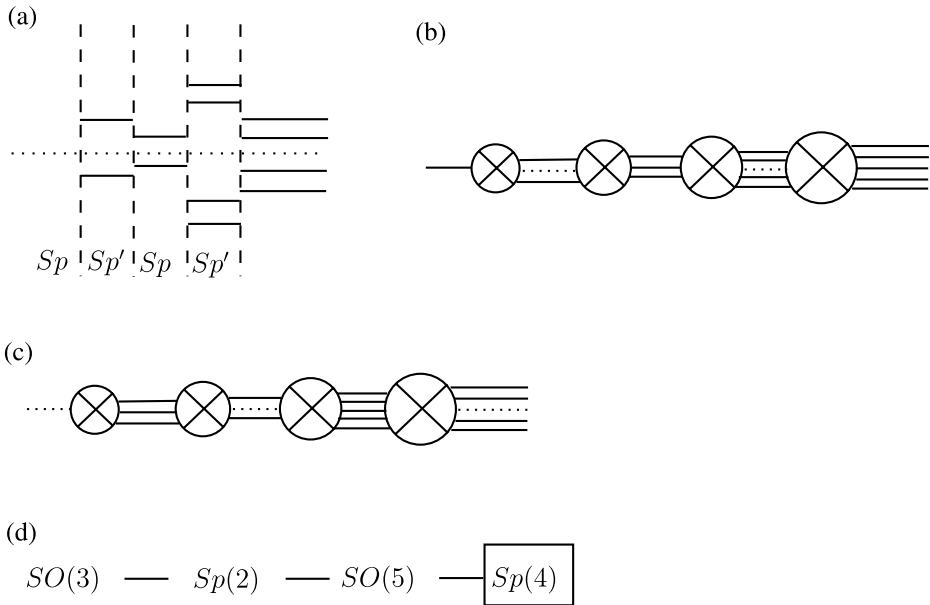


Figure 68: (a) A brane realization of the dual of Dirichlet boundary conditions for  $Sp(2n)$ , sketched here for  $n = 2$ . Between the D5-branes, the gauge theories are alternately  $Sp$  and  $Sp'$  theories; that is, the  $\theta$ -angle jumps by  $\pi$  in crossing each half D5-brane. (b) The  $S$ -dual of (a), leading to the bad quiver of figure 54(c) which describes  $T(SO(2n + 1))$ . (c) If we exchange all  $Sp$  and  $Sp'$  labels in (a), the dual looks like this, and the gauge theory limit is the bad quiver in (d). In (a), (b), and (c), linking numbers are alternately 2 and 0.

brane manipulations, is  $T_{\rho^\vee}(G)$ , which is the  $S$ -dual of a  $\rho^\vee$  Nahm pole for  $G^\vee$ . The Nahm pole is built out of half D5-branes by following the rules already formulated, and the  $S$ -dual orthosymplectic quiver has the same structure as the  $T(G)$  quiver, but with missing nodes.

It is almost as easy to describe a general  $T_{\rho^\vee}^\rho(G)$ :  $\rho^\vee$  can be realized as a configuration of D5-branes in the  $G^\vee$  duality frame, and converted to an identical configuration of NS5-branes in the  $G$  duality frame. In some cases, splitting of these NS5-branes will modify the  $R$ -symmetry, but we do not need to split fully the D5-brane configuration that generates  $\rho$ . It is convenient to represent summands in  $\rho$  of the “correct” dimension (odd for orthogonal  $G$ , even for symplectic  $G$ ) by a half D5-brane, and pairs of summands of the “wrong” dimension by full D5-branes. To order the branes properly so as to get a gauge theory description, a half D5-brane with a Nahm pole of odd dimension  $d$  needs to be moved across  $d$  half NS5-branes, and ends up representing a single real flavor at the  $d$ th node

of the quiver (which is symplectic). A full D5-brane with two poles each of even dimension  $d$  also is moved across  $d$  half NS5-branes and ends up as a full flavor at the  $d$ th node, which now is orthogonal. The ranks of the gauge groups at the nodes are then determined by following the brane manipulations or more simply from the linking numbers of the original half NS5-brane configuration.

The set of  $T_{\rho\nu}^{\rho}(G)$  includes all mirror pairs built from linear orthosymplectic quivers.

### 7.3 Examples of interesting boundary conditions

As in Section 3, we can use these methods to understand  $S$ -duality for a much wider class of examples. We consider some illustrative cases involving Neumann boundary conditions with fundamental matter at the boundary.

We begin with the brane configuration in figure 69(a). It produces a Neumann boundary condition in  $SO(2n)$  gauge theory coupled to a fundamental hypermultiplet with an  $Sp(2)$  flavor symmetry.  $2n$  half D3-branes end on the NS5-brane, but the half D3-brane charge of the O3-plane also jumps by  $-1$  across the NS5-brane, so its linking number is  $2n - 1$ . To do  $S$ -duality, the full D5-brane has to be split as in figure 69(b). The resulting configuration has two half D5-branes of linking numbers 2 and 0. The reordered  $S$ -dual configuration is shown in figure 69(c). The  $SO(2n)$  gauge group is broken to  $SO(3)$  by a Nahm pole of dimension  $2n - 3$ , and it is coupled to the CFT associated to the quiver in figure 69(d), which describes an  $Sp(2)$  gauge theory with  $SO(4)$  flavor symmetry group. Notice that one of the four flavors arises from the bifundamental hypermultiplet at the leftmost NS5-brane, where the gauge theory jumps from  $Sp'(2)$  to  $O(1)$ . This bad quiver was analyzed in Section 5.2.2. Its low-energy limit is  $T(SO(3)) = T(SU(2))$ , though in this flow the microscopic  $R$ -symmetry is not the one that is relevant in the infrared.

We can check that our general prescription based on  $T(SO(2n))$  agrees with this answer. We want to reproduce the original boundary condition as the  $S$ -dual of the boundary condition in figure 69(c). As a Nahm pole  $\rho$  (related to the decomposition  $2n = (2n - 3) + 1 + 1 + 1$ ) is present, we need to use  $T^{\rho}(SO(2n))$ , which is the infrared limit of the good quiver in figure 69(e). We could couple this to the quiver in figure 69(d) to produce a dual boundary condition, but as that quiver is bad we would learn little from it. Alternatively, we can couple it to a more useful description of the theory: the usual  $T(SU(2))$  realization as  $U(1)$  with two flavors. The price



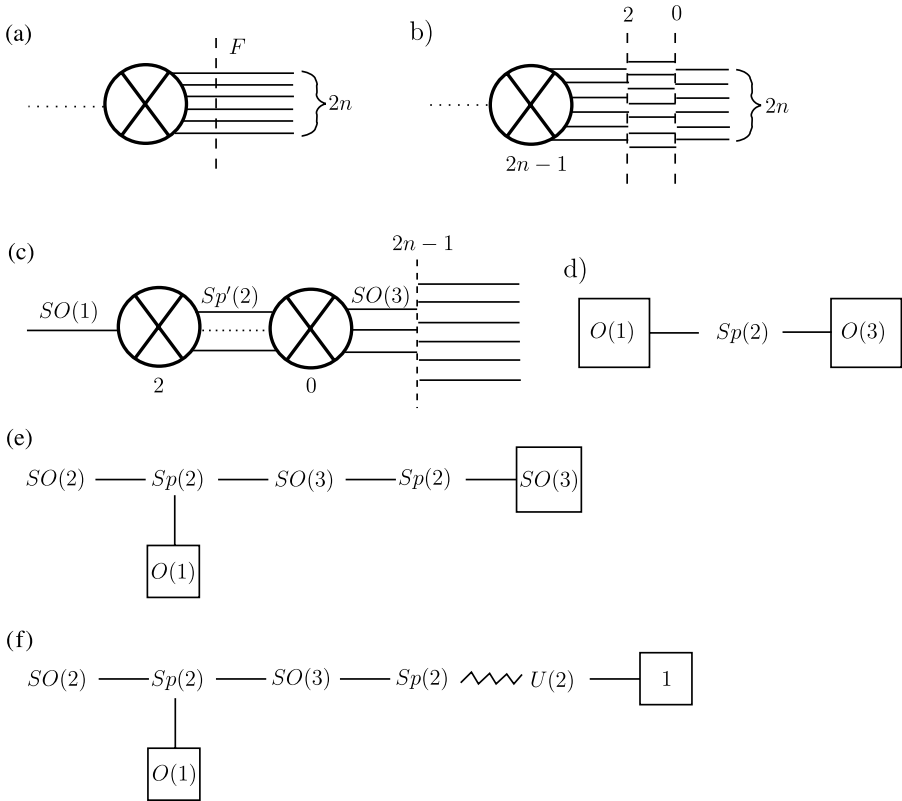


Figure 69: (a) D3-branes intersecting a full D5-brane, leading to  $SO(2n)$  gauge theory coupled to a fundamental hypermultiplet of flavor symmetry  $Sp(2)$ . (b) Splitting the D5-brane. Linking numbers are indicated. (c) The  $S$ -dual of (b). The gauge symmetry is reduced to  $SO(3)$  by a Nahm pole. (d) The  $SO(3)$  gauge symmetry near the boundary is coupled to the SCFT that is generated by this quiver. This is a bad quiver representing  $T(SU(2))$ . (e) The quiver representing  $T^\rho(SO(2n))$  for the decomposition  $2n = (2n - 3) + 1 + 1 + 1$ , sketched here for  $n = 3$ . (f) The result of diagonally coupling  $SO(3)$  gauge fields to  $T^\rho(SO(2n)) \times T(SU(2))$  is this “quiver,” in which the jagged line represents the tensor product of the fundamental representation of  $Sp(2)$  and the fundamental of  $SO(3)$ , viewed as a  $U(2)$  representation.

to pay is that the resulting theory is not a quiver in the strict sense, as the  $SO(3) \sim SU(2)$  node is coupled to hypermultiplets in both the triplet and the doublet of  $SU(2)$ . One can verify that monopoles with  $q_R = 1/2$  exist in this quiver, with weights which match the ones of an  $SO(2n) \times Sp(2)$  bifundamental free hypermultiplet. (To get all the monopoles, one needs to know that in coupling  $SO(3)$  to the  $T(SU(2))$  quiver the gauge group is really  $U(2)$ , not  $SO(3) \times U(1)$ .)

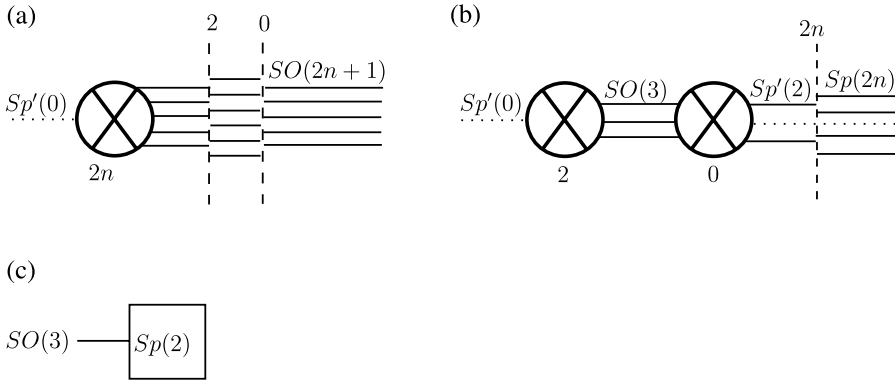


Figure 70: (a) Neumann boundary condition for  $SO(2n + 1)$  (in the figure  $n = 2$ ) with addition of a full D5-brane, which has been split to arrive at this picture. Linking numbers are indicated. The  $R$ -symmetry has been modified by splitting D5-branes. (b) The  $S$ -dual configuration after standard reordering. (c) The bad quiver representing the boundary SCFT.

A similar brane configuration in figure 70(a) gives the same boundary condition for  $SO(2n + 1)$ . The construction of the  $S$ -dual boundary condition is rather similar to the previous example; in particular the linking number of the NS5-brane is  $2n$  and those of the two half D5-branes are again  $2$  and  $0$ . The final result is shown in figure 70(b) and corresponds to a boundary condition for an  $Sp(2n)$  gauge theory reduced to  $Sp(2)$  by a Nahm pole of dimension  $2n - 2$ . The SCFT living at the boundary is depicted in figure 70(c): it is the IR limit of an  $SO(3)$  gauge theory with  $Sp(2)$  flavor symmetry. This is again a bad quiver theory, but it is also a very well-known theory in disguise:  $\mathcal{N} = 8$   $SO(3)$  gauge theory. In the infrared, the moduli space of this theory is a mixed branch:  $\mathbb{R}^8/\mathbb{Z}_2$ , with an  $SO(8)$   $R$ -symmetry group. Since the  $Sp(2)$  flavor symmetry is coupled to the bulk gauge fields, only the Coulomb factor of the moduli space is really visible, and corresponds to the moduli space of Nahm equations in figure 70(a).

The third example, a Neumann boundary condition for  $Sp(2n)$  together with a coupling to a fundamental hypermultiplet of flavor symmetry  $SO(2)$ , is depicted in figure 71(a). The brane manipulations are more elementary, as the full D5-brane is equivalent to two half D5-branes, as in figure 71(b). The  $S$ -dual configuration in figure 71(c) shows clearly that the dual  $SO(2n + 1)$  gauge theory is broken to an  $SO(2)$  subgroup by a Nahm pole of dimension  $2n - 1$ , and the  $SO(2)$  subgroup is gauged at the boundary.

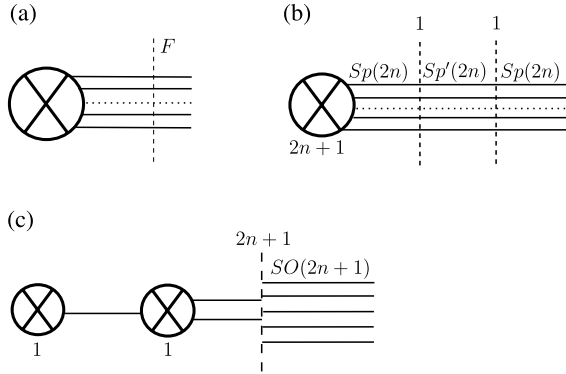


Figure 71: (a)  $Sp(2n)$  gauge theory interacting with a full D5-brane at the boundary. (b) The split configuration, with linking numbers indicated. (c) The  $S$ -dual.

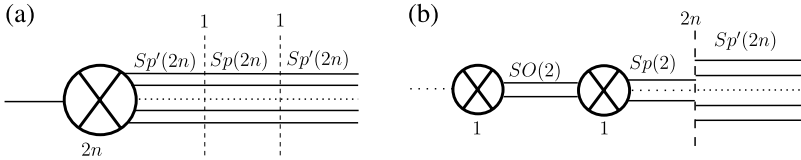


Figure 72: (a)  $Sp'(2n)$  gauge theory coupled to fundamental hypermultiplets with  $SO(3)$  flavor symmetry. In the split version, the hypermultiplets come from both the two half D5-branes and the boundary. (b) The  $S$ -dual configuration.

Finally, the same brane configuration for an  $Sp'(2n)$  gauge group represents an  $Sp'(2n)$  gauge theory coupled to fundamental hypermultiplets with an  $SO(3)$  flavor symmetry at the boundary, because of the extra  $O(1) \times Sp'(2)$  bifundamental hypermultiplet at the location of the NS5-brane. The split brane configuration and the  $S$ -dual brane configuration are shown in figure 72. The dual  $Sp'(2n)$  gauge theory is broken to  $Sp(2)$  by a Nahm pole of dimension  $2n - 2$  and coupled to an  $SO(2)$  gauge theory with  $Sp(2)$  flavor symmetry group, which is nothing else but  $T(SU(2))$ .

**7.4 O3-planes with O5-planes**

Now we will study orthosymplectic gauge groups realized by the combination of an O3-plane and an O5-plane. O5-planes were described in Section 6, where we also studied an  $\mathcal{I}_4(-1)^{FL}$  orbifold fiveplane. So in toto we have three objects: the O3-plane with reflection of the coordinates 456789, the O5-plane with reflection of coordinates 3789, and the orbifold fiveplane with

reflection of 3456. We will now study models in which all three of these objects are present. In fact, as soon as one has any two of them, the third arrives for free, since the product of any two of these reflection symmetries is the third.

To some extent, we can  $S$ -dualize the product of these objects if we know how to  $S$ -dualize them separately, since we can go to a region in spacetime in which only one of the reflection symmetries has a fixed point. Though  $O5^+$  does not have a useful  $S$ -dual, there is a useful duality [42, 43] involving  $O5^-$ . The following two objects are  $S$ -dual: (i) the combination of  $O5^-$  with a D5-brane and (ii) the orbifold fiveplane.

Now suppose that objects (i) and (ii) are both present. This gives a configuration that is invariant under  $S$ -duality at least away from the locus where the two objects meet. That locus, which is where  $\vec{x}$  and  $\vec{y}$  vanish, will be the locus of an  $O3$ -plane. The  $O5^-$  creates orthogonal gauge symmetry for D5-branes, so D3-branes intersecting it should have symplectic gauge symmetry. Hence the  $O3$ -plane is of symplectic type. For the overall configuration to be  $S$ -dual, the gauge group must be  $Sp'(2n)$  and so the  $O3$ -plane is of type  $\tilde{O}3^+$ . We therefore propose that the combination of the following objects is  $S$ -dual: an  $O5^-$ -plane together with a D5-brane; an orbifold fiveplane; and an  $\tilde{O}3^+$ -plane.

This statement implies the  $S$ -duality of a certain field theory boundary condition. This is a boundary condition in  $Sp'(2n)$  gauge theory in which  $Sp'(2n)$  couples to a fundamental hypermultiplet and is broken at the boundary to  $Sp(n) \times Sp(n)$ . Of course,  $n$  must be even.

Once we understand  $S$ -duality for this example, we can deduce how  $S$ -duality must act for a very large class of additional examples. We will give an interesting and illustrative example, using the knowledge gained in Section 5.4. We add one more half D5-brane to the system, so that it represents a boundary condition for an  $Sp(2n)$  gauge theory, broken to  $Sp(n) \times Sp(n)$ , with each factor coupled to a hypermultiplet with  $SO(2)$  flavor symmetry. The global symmetry of the Higgs branch is therefore  $SO(2) \times SO(2)$ .

The brane construction and its  $S$ -dual are depicted in figures 73(a),(b). The  $S$ -dual boundary condition for the  $SO(2n+1)$  gauge theory involves coupling with the product of two copies of a certain boundary theory. That theory is the IR limit of  $Sp(n)$  gauge theory coupled to hypermultiplets with  $SO(2n+2)$  flavor symmetry, as depicted in figure 73(c). These balanced theories each have an  $SO(2)$  symmetry of the Coulomb branch, matching the  $SO(2) \times SO(2)$  flavor symmetry of the original boundary condition. We can reproduce the mirror of this theory through  $T(Sp(2n))$ , as usual. The

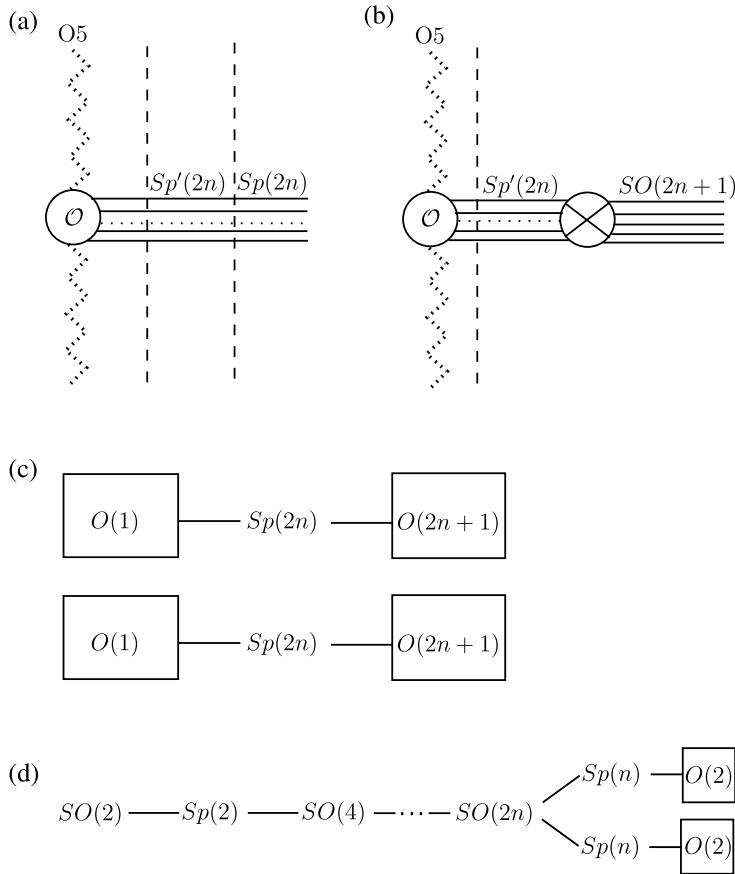


Figure 73: (a) A configuration involving a combination of O3 and O5 orientifold planes and an orbifold plane, together with two half D5-branes. (The circle containing the symbol  $\mathcal{O}$  represents the orbifold plane.) (b)  $S$ -duality converts the rightmost half D5-brane to a half NS5-brane, while, as explained in the text, the rest of the configuration is self-dual. (c) The corresponding boundary CFT. (d) The mirror of the same CFT, built through the use of  $T(Sp(2n))$ .

resulting composite gauge theory is depicted in figure 73(d): it is a balanced orthosymplectic quiver with the shape of a  $D_n$  Dynkin diagram, and its Coulomb branch has  $SO(2n+2) \times SO(2n+2)$  symmetry, as desired.

Our starting point has been a self-dual configuration that describes  $Sp'(2n)$  broken at the boundary to  $Sp(n) \times Sp(n)$  with a boundary hypermultiplet. Can we find the  $S$ -dual with the hypermultiplet removed? This question is superficially similar to the one treated in Section 6.4. One could try to imitate the approach used there by adding an extra half D5-brane, as we

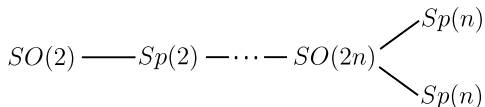


Figure 74: A bad quiver resulting from application of the  $T(Sp(2n))$  recipe.

have done in, figure 72(a), and then turning on a mass parameter to remove the boundary hypermultiplets (which means, in terms of branes, that the two half D5-branes recombine into a full D5-brane which is then displaced to large  $\vec{y}$ ). Unfortunately, that mass parameter is dual to the “hidden” FI parameter in figure 73(c), which corresponds to the  $SO(2)$  symmetry of the Coulomb branch. That deformation is difficult to analyze, so we will not follow this path.

Alternatively, we can use the standard  $S$ -duality prescription involving  $T(Sp(2n))$ . This gives rise to the bad quiver in figure 74, indicating that in the desired dual boundary condition, the  $SO(2n + 1)$  dual gauge symmetry is broken.

We will just conjecture an answer based on the analogy with the similar boundary conditions for the unitary case: a Nahm pole related to the decomposition  $2n + 1 = 2 + 2 + \dots + 2 + 1$  together with a gauging of the “level two” subgroup  $Sp(n)_2$  which commutes with the Nahm pole. We can extend this conjecture to a boundary condition breaking  $Sp(p + q)$  to  $Sp(p) \times Sp(q)$  at the boundary ( $p > q$ ): the dual has a Nahm pole related to the decomposition  $2n + 1 = 2 + 2 + \dots + 2 + (p - q)$  together with a gauging of the “level two” subgroup  $Sp(q)_2$  which commutes with the Nahm pole. This statement, which we suspect can be justified by adapting the arguments of Section 6.5, is related to the case CII in Table 1 of [41]. The case CI in that table is a case in which the  $T(G)$  prescription leads to a good quiver, so there is full gauge symmetry at the boundary; the split case of DI/DII is similar.

### 8 S-duality and theta-angle

It is natural to wonder if our construction of  $S$ -dual pairs of boundary conditions extends to an action of<sup>25</sup>  $SL(2, \mathbb{Z})$  on half-BPS boundary conditions corresponding to the full duality group of  $\mathcal{N} = 4$  SYM. In this section, we will argue that the answer is positive, and define a pair of abstract transformations  $S$  and  $T$  which generate the duality group.

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<sup>25</sup>When the gauge group is  $G_2$  or  $F_4$ , the appropriate duality group is not actually  $SL(2, \mathbb{Z})$ , but a group generated by  $T : \tau \rightarrow \tau + 1, S : \tau \rightarrow -1/n_{\mathfrak{g}}\tau$ , where  $n_{\mathfrak{g}}$  is 2 for  $G_2$  and 3 for  $F_4$ , as shown in [46]. For simplicity, we will refer to the duality group as  $SL(2, \mathbb{Z})$ .

Before trying to do this, we should explain what exactly the statement is supposed to mean. Let us first review the case that we have focused on so far:  $\tau$  is imaginary and we want to know how a boundary condition transforms under  $S : \tau \rightarrow -1/\tau$ . We start with a boundary condition in weakly coupled  $G$  gauge theory, and  $S$  maps us to an equivalent configuration in strongly coupled  $G^\vee$  gauge theory. This is not really illuminating, however. What we really mean by the dual boundary condition is a boundary condition in *weakly* coupled  $G^\vee$  gauge theory. To find it, after acting with  $S$ , we have to continue the  $G^\vee$  boundary condition from strong coupling back to weak coupling. This makes sense because all boundary conditions that we have studied in this paper can be defined (at  $\theta = 0$ ) for any value of the gauge coupling  $g$ . The ability to continue a boundary condition in  $g$  is built into what we mean in studying the action of  $S$  on boundary conditions.

This has an analog when we include  $\theta$  and consider a more general duality transformation  $\gamma : \tau \rightarrow (a\tau + b)/(c\tau + d)$ . We start at  $\text{Im } \tau \gg 1$  and, say,  $\text{Re } \tau = 0$ . After acting with  $\gamma$ , we land in the strongly coupled region, generically with a different value of  $\theta = 2\pi \text{Re } \tau$ . Then we have to continue back to the starting point. For this, we have to restrict ourselves to boundary conditions that can be varied with both  $\text{Im } \tau$  and  $\text{Re } \tau$  in a natural way.

Another way to describe the situation is this. If one is given a boundary condition that can be naturally continued as  $\tau$  varies in the upper half-plane, then one can approach any cusp on the real  $\tau$  axis and ask what the boundary condition looks like in that duality frame. So under these conditions it makes sense to ask how  $SL(2, \mathbb{Z})$  acts on a boundary condition. If instead one is given a boundary condition that is only defined for imaginary  $\tau$ , then it only makes sense to ask what happens under  $\tau \rightarrow -1/\tau$ .

While very general half-BPS boundary conditions allow a natural variation of the gauge coupling  $g^2 = 4\pi/\text{Im } \tau$ , only very special ones admit a similar variation of  $\theta$ . For example, consider a boundary condition with full  $G$  gauge symmetry coupled to a boundary SCFT  $\mathfrak{B}$  with  $\mathcal{N} = 4$  supersymmetry. We suppose that  $G$  acts on the Higgs branch of  $\mathfrak{B}$ . A generalization of such a boundary condition, preserving its full supersymmetry, is possible [1] precisely if the moment map  $\vec{\mu}$  satisfies the so-called fundamental identity: the complex moment map  $\mu_{\mathbb{C}}$  associated with any choice of an  $\mathcal{N} = 2$  subalgebra must obey<sup>26</sup>

$$\text{Tr } \mu_{\mathbb{C}}^2 = \text{constant}. \tag{8.1}$$

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<sup>26</sup>See equation (3.57) of [2] for the fact that there can be a constant on the right-hand side of the fundamental identity.

The same condition allows coupling of  $\mathfrak{B}$  to a three-dimensional gauge theory with Chern–Simons action.

Similarly the Coulomb branch of an  $\mathcal{N} = 4$  theory can be coupled to bulk gauge fields if the twisted moment map obeys the fundamental identity. A perhaps surprising generalization [47] is that if we are given a pair of theories, one with a  $G$ -action on the Higgs branch and one with a  $G$ -action on the Coulomb branch, both satisfying the fundamental identity, their product can be coupled to a  $G$  Chern–Simons gauge theory preserving the full  $\mathcal{N} = 4$  supersymmetry. By contrast, if both theories have  $G$  action on the same branch, such a coupling is generally not possible, as the fundamental identity is not additive in  $\mu$ .

A wide class of  $G$ -invariant  $\mathcal{N} = 4$  SCFTs that obey the fundamental identity has appeared in this paper. These are the theories  $T_{\rho^\vee}(G)$  for any  $G$  and  $\rho^\vee$ . The Higgs branch of such a theory is always a union of nilpotent orbits. The complex moment map  $\mu_{\mathbb{C}}$  takes values in those orbits; hence its quadratic Casimir vanishes, and the fundamental identity is obeyed, with the constant being zero. The examples given in [2] are special cases of these. If one makes an FI deformation of  $T_{\rho^\vee}(G)$ , smoothing the singularities and preserving the  $G$  symmetry, the nilpotent orbits are deformed to semisimple ones, and the (8.1) remains valid, now with a possibly non-zero constant.

A more obvious example of a boundary condition that can be naturally continued for all  $\tau$  is Dirichlet modified by a Nahm pole. With such a boundary condition, the topological term  $(\theta/8\pi^2) \int \text{Tr } F \wedge F$  can be added to the action, preserving all supersymmetry. This gives a natural way to vary  $\theta$ . So such boundary conditions should lie on an  $SL(2, \mathbb{Z})$  orbit. Not coincidentally, the duality transformation  $S : \tau \rightarrow -1/\tau$  converts Dirichlet with a pole of type  $\rho^\vee$  to a boundary condition  $\mathcal{B}$  associated with the theory  $T_{\rho^\vee}(G)$ . Since this theory obeys the fundamental identity, the boundary condition  $\mathcal{B}$  can again be contained in an  $SL(2, \mathbb{Z})$  orbit. Orbits of this type are the only  $SL(2, \mathbb{Z})$  orbits of half-BPS boundary conditions that we know about.

## 8.1 Definition of $S$ and $T$

For simplicity, we will describe the action of  $SL(2, \mathbb{Z})$  in terms of a transformation on the three-dimensional SCFT  $\mathfrak{B}$  that lives at the boundary. When the dual does not have full gauge symmetry but contains a Nahm pole or a gauge group reduction, one must adapt the following procedure along the lines of Section 4.3.1.



When we say that  $\mathfrak{B}$  is a theory with  $G$  symmetry, what we mean, to be more exact, is that we are given a precise recipe to couple  $\mathfrak{B}$  to a background  $\mathcal{N} = 4$  supermultiplet with gauge group  $G$ . Assuming the fundamental identity is obeyed, such a coupling can include a Chern–Simons coupling. We define the  $T$  operation as a unit shift of the Chern–Simons coefficient.

To define an  $S$  operation, we need to refine what we mean by saying that  $T(G)$  has  $G \times G^\vee$  symmetry. Again, we need to specify a standard coupling of  $T(G)$  to background  $G \times G^\vee$  vector multiplets. We specify this coupling by asking that it should be parity-symmetric. (This excludes the possibility of adding  $\mathcal{N} = 4$  Chern–Simons couplings, which otherwise are possible since  $T(G)$  obeys the fundamental identity.)

We call the  $G$  symmetry of  $T(G)$  a direct action, and the  $G^\vee$  symmetry a twisted action. The terminology is motivated by the idea that  $G$  acts on the Higgs branch, which is parameterized by hypermultiplets, while  $G^\vee$  acts on the Coulomb branch, which is parameterized by twisted hypermultiplets.

Now we are in position to define the  $S$  action: a theory  $\mathfrak{B}$  with a direct action of  $G$  is mapped by  $S$  to a composite theory  $T(G^\vee) \times_G \mathfrak{B}$  with a direct  $G^\vee$  action. Here we define  $T(G^\vee) \times_G \mathfrak{B}$  as the result of gauging the product of the twisted  $G$  action on  $T(G^\vee)$  and the direct  $G$  action on  $\mathfrak{B}$ . In general, there will be a Chern–Simons action for  $G$  implicit in the coupling to  $\mathfrak{B}$ ; otherwise we add a supersymmetric Yang–Mills coupling and then flow to the infrared. This  $S$  operation was essentially defined in Section 4.3; in the analysis of figure 52, we argued that it satisfies  $S^2 = 1$ . We now plan to show<sup>27</sup> that  $(ST)^3$  is also 1, so that the two transformations generate an  $SL(2, \mathbb{Z})$  duality group. (For  $G = U(1)$ , where everything is much more elementary, the fact that  $(ST)^3 = 1$  can be shown by a direct path integral computation [22].)

We would like to mimic the  $S^2 = 1$  proof, which used the Janus interpretation of  $S$ . Unfortunately the  $\mathcal{N} = 4$  Janus configurations are relatively “rigid” (only certain paths in the upper half plane are allowed), so that different  $\mathcal{N} = 4$  Janus walls cannot be concatenated in a fashion which preserves  $\mathcal{N} = 4$  supersymmetry. On the other hand, it is possible to relax this constraint at the price of reducing supersymmetry from  $\mathcal{N} = 4$  to  $\mathcal{N} = 3$ . Indeed, there is a relatively straightforward description of an  $\mathcal{N} = 3$  Janus configuration which allows for a generic  $y$ -dependence of  $\tau$ . We start with

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<sup>27</sup>As mentioned in footnote 25, for gauge group  $G_2$  or  $F_4$ ,  $S$  acts differently on the upper half plane. Consequently, the appropriate relation is not  $(ST)^3 = 1$ . The argument that follows really shows that any word in  $S$  and  $T$  that acts trivially on the upper half plane acts trivially on the theory; in this form, it applies also to  $G_2$  and  $F_4$ .

the fact [4, 48–50] that for any group  $\widehat{G}$  and hypermultiplet representation  $R$  there is an  $\mathcal{N} = 3$  action with a gauge coupling and a Chern–Simons coupling. Both of these couplings depend on choices of invariant quadratic forms on the Lie algebra, of which the first should be positive definite. We take  $\widehat{G}$  to be the infinite dimensional group of  $G$ -valued functions  $g(y)$  of the real variable  $y$ , and we take  $R$  to be the twisted version of the adjoint representation described in Section 2.3.1 of [1]. We pick quadratic forms on the Lie algebra that depend on arbitrary functions of  $y$ , as at the end of section 2.3.1 in that reference. The result is an  $\mathcal{N} = 3$  Janus configuration with an arbitrary  $\tau(y)$ .

We can represent the  $S$  operation by a Janus domain wall which interpolates from  $\tau$  to  $-1/\tau$ . Similarly  $T$  is a Janus wall which interpolates between  $\tau$  and  $\tau + 1$ . So  $(ST)^3$  comes from a succession of six Janus domain walls, at the end of which we return to the initial value of the coupling parameter  $\tau$ . Each of the six domain walls preserves  $\mathcal{N} = 4$  supersymmetry, but the combination has only  $\mathcal{N} = 3$ . As we flow to the infrared, the details of the path are forgotten, and we only remember the initial and final points of the path. Since these coincide, the infrared limit is a trivial domain wall, confirming that  $(ST)^3 = 1$ . Notice that a similar temporary  $\mathcal{N} = 3$  deformation which flows to a fixed point with enhanced supersymmetry has been used in [51].

## 8.2 Effective action for interaction with a $(p, q)$ fivebrane

A long standing puzzle in string theory has been to describe the intersection between  $n$  D3-branes and a  $(p, q)$  fivebrane. As an illustration of our construction, we will use it to resolve this puzzle. We start from a configuration that is already understood,  $n$  D3-branes intersecting a single NS5-brane, and apply a general  $SL(2, \mathbb{Z})$  transformation. To put this in our framework, we use the folding trick to describe this intersection as a boundary condition for  $U(n) \times U(n)$  gauge theory.

An important fact is that the folding trick reverses orientation, so it maps  $T$  to  $T^{-1}$  while preserving  $S$ . The initial boundary condition consists of a parity-invariant coupling to a bifundamental hypermultiplet of  $U(n) \times U(n)$ . The action of  $T^k$  gives Chern–Simons coefficients  $(k, -k)$  for  $U(n) \times U(n)$ . A single bifundamental hypermultiplet with these Chern–Simons coefficients gives a basic solution of the fundamental identity [2]. This theory, which was described in detail in the reference, describes the intersection of  $n$  D3-branes with a  $(1, k)$  fivebrane.

An application of  $S$  maps this theory to the “quiver” in figure 75(a), which therefore describes the intersection of  $n$  D3-branes and one  $(k, 1)$  fivebrane.

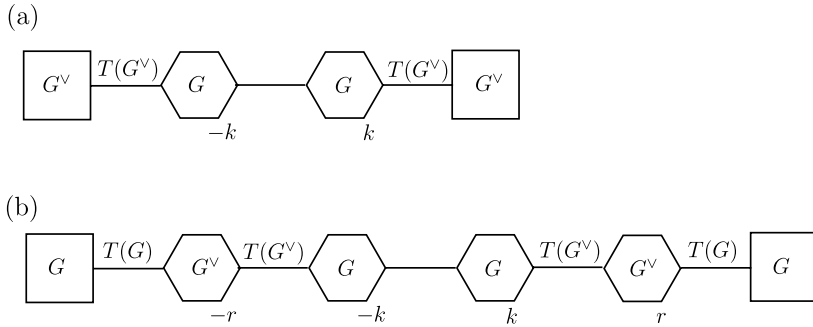


Figure 75: Schematic representations of “quivers” in which the hexagons represent Chern–Simons gauge theories, with levels indicated by subscripts; the central edges represent ordinary bifundamental hypermultiplets, while other edges represent couplings to  $T(G)$  or  $T(G^\vee)$ , as labeled; with levels indicated by subscripts; and the squares indicate the remaining flavor symmetries.

A hexagon in the figure represents a  $U(n)$  gauge group with the Chern–Simons coefficient indicated by the subscript. Repeated action of  $T$  and  $S$  generates longer “quivers” of this type. For example, the result of  $ST^r ST^k$  is represented in figure 75(b).

Continuing in this way, we get a “quiver” describing the interaction of  $n$  D3-branes with a  $(p, q)$  five-brane whenever  $p$  and  $q$  are relatively prime. Even if  $p$  and  $q$  are not relatively prime, but have greatest common divisor  $k > 1$ , we can make the same type of construction starting with  $k$  NS5-branes. We simply begin with the conventional linear quiver of figure 40(e), which describes the intersection of  $n$  D3-branes with a chain of  $k$  NS five-branes, and then apply the above operations.

We can generalize this slightly to the case that the numbers of D3-branes on the two sides of the  $(p, q)$  fivebranes are different. A special case is that there are D3-branes on only one side, leading to a boundary condition in  $U(n)$  gauge theory. A boundary condition involving a chain of NS5-branes with varying linking numbers leads to the  $SL(2, \mathbb{Z})$  orbit containing the theories  $T_{\rho^\vee}(SU(n))$ .

### 8.3 $(p, q)$ Fivebranes and fractional Chern–Simons couplings

We will conclude with an analysis of a single D3-brane ending on a  $(p, q)$  fivebrane. (Supersymmetry will play no important role and is omitted.) We begin with the case of a  $(1, 0)$  fivebrane. The boundary theory is trivial;

that is, the  $U(1)$  gauge theory of the D3-brane obeys Neumann boundary conditions, coupled to nothing else. We regard this trivial boundary theory as a theory with  $U(1)$  symmetry by introducing a background  $U(1)$  gauge field  $B$  whose couplings are zero. Then we act with  $T^k$ , after which the action for  $B$  is a level  $k$  Chern–Simons action  $k \int B \wedge dB/4\pi$ . In view of the description of  $T(U(1))$  in Section 4.4, acting with  $S$  means that we add a second  $U(1)$  gauge field  $A$  with coupling  $\int A \wedge dB/2\pi$ . At this point, then, the boundary action is

$$\frac{1}{2\pi} \int_{\partial M} A \wedge dB + \frac{k}{4\pi} \int_{\partial M} B \wedge dB. \quad (8.2)$$

This is the boundary action for a single D3-brane ending on a  $(k, 1)$  fivebrane. In that application,  $B$  is defined only on the boundary, but  $A$  is defined in bulk (and has a conventional bulk kinetic energy).

A somewhat inaccurate procedure that is frequently followed at this stage is to treat  $B$  as a linear field, ignoring the fact that it may have quantized Dirac fluxes. In this approximation, one can perform a Gaussian integral over  $B$ , leading to a boundary Chern–Simons coupling for  $A$  that is not properly quantized:

$$-\frac{1}{4\pi k} \int_{\partial M} A \wedge dA. \quad (8.3)$$

This is not really the right answer, because in deriving it one has omitted the sum over fluxes of  $B$ . Still, this computation sheds light on the sense in which one might claim [52] that ending a D3-brane on a  $(k, 1)$  fivebrane induces a Chern–Simons coupling  $-1/k$ .

An action much like (8.2) is often studied in relation to the fractional quantum Hall effect. (For example, see equation (2.11) in [53].) In that context,  $A$  is the ordinary electromagnetic vector potential, and  $B$  is an effective  $U(1)$  gauge field induced by strong coupling effects in a two-dimensional material. The couplings (8.2) in that context are supported on a defect in spacetime — the world-volume of the material — rather than on a boundary. In that context, the effective Chern–Simons coefficient for  $A$  is the quantum Hall conductivity. This conductivity is unaffected by the sum over fluxes of  $B$ , so the computation leading to (8.3) is a valid way to explain the fractional quantum Hall effect.

On the other hand, we will get into trouble if we take (8.3) literally. For example, consider a D3-brane suspended between a  $(1, 0)$  fivebrane, which generates Neumann boundary conditions, and a  $(1, k)$  fivebrane. (The configuration is not supersymmetric, but that does not affect the point we

are about to make.) The effective three-dimensional physics is given by the action (8.2), now understood in purely three-dimensional terms. This theory is completely consistent, but if we naively treat  $B$  as a Gaussian field and integrate it out, we will arrive at the theory (8.3) which is inconsistent, because the Chern–Simons coefficient is not properly quantized.

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