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(σ,λ)- ASYMPTOTICALLY STATISTICAL EQUIVALENT SEQUENCES

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Abstract. This paper presents the following definition which is a natural combination of the definition for asymptotically equivalent, λ- statistical convergence and σ- convergence. Two nonnegative sequences $[x]$ and $[y]$ are said to be $S_{\sigma,\lambda}$- asymptotically equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_{n} \frac{1}{\lambda_n} \quad k \in I_n : \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \geq \epsilon \quad = 0$$

uniformly in $m = 1, 2, 3, ...$ (denoted by $x \overset{S_{\sigma,\lambda}}{\sim} y$) and simply $S_{\sigma,\lambda}$- asymptotically equivalent, if $L = 1$. Using this definition we shall prove $S_{\sigma,\lambda}$- asymptotically equivalent analogues of Mursaleen's theorems in [8].

1. Introduction

Let l_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\| x \| = \sup_n | x_n |$. A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent of all of its Banach limits coincide. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [4] proved that

$$\hat{c} = \{x \in l_\infty \ : \ \lim_m t_{mn}(x) \text{ exists uniformly in n}\}$$

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where

$$t_{mn}(x) = \frac{x_n + x_{n+1} + \ldots + x_{n+m}}{m+1}$$

The space $[\hat{c}]$ is of strongly almost convergent sequence was introduced by Maddox [5] as follows:

$$[\hat{c}] = \{x \in l_\infty : \lim_m t_{mn}\left(|x - \ell e|\right) \text{ exists uniformly in } n \text{ for some } \ell \in C\},$$

where $e = (1, 1, \ldots)$.

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \left(\sigma^{m-1}(n)\right), m = 1, 2, 3, \ldots$. A continuous linear functional φ on l_∞ is said to be an invariant mean or a σ- mean if and only if

(1) $\varphi \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n.
(2) $\varphi(e) = 1$, where $e = (1, 1, \ldots)$ and
(3) $\varphi\left(x_{\sigma(n)}\right) = \varphi(x)$ for all $x \in l_\infty$.

For a certain kinds of mapping σ every invariant mean φ extends the limit functional on space c, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$ where V_σ is the bounded sequences all of whose σ-means are equal.

If $x = (x_k)$, set $Tx = (Tx_k) = \left(x_{\sigma(k)}\right)$ it can be shown that (see, Schaefer [13]) that

(1.1)

$$V_\sigma = \left\{x \in l_\infty \ : \ \lim_k t_{km}(x) = Le \text{ uniformly in m for some } L = \sigma - \lim x\right\}$$

where

$$t_{km}(x) = \frac{x_m + Tx_m + \ldots + T^k x_m}{k+1} \quad \text{and} \quad t_{-1,m} = 0$$

We say that a bounded sequence $x = (x_k)$ is σ-convergent if and only if $x \in V_\sigma$ such that $\sigma^k(n) \neq n$ for all $n \geq 0$, $k \geq 1$.

Just as the concept of almost convergence lead naturally to the concept of strong almost convergence, σ- convergence leads naturally to the concept of strong σ-convergence. A sequence $x = (x_k)$ is said to be strongly σ-convergent (see, Mursaleen [7]) if there exists a number ℓ such that

$$\frac{1}{k}\sum_{i=1}^{k}\left|x_{\sigma^i(m)} - l\right| \to 0 \tag{1.2}$$

as $k \to \infty$ uniformly in m. We write $[V_\sigma]$ as the set of all strong σ- convergent sequences. When (2) holds we write $[V_\sigma] - \lim x = \ell$. Taking $\sigma(m) = m + 1$, we obtain $[V_\sigma] = [\hat{c}]$ so strong σ- convergence generalizes the concept of strong almost convergence. Note that

$$[V_\sigma] \subset V_\sigma \subset l_\infty.$$

Let $\Lambda = (\lambda_n)$ be a nondecreasing sequence of positive reals tending to infinity

and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$.
The generalized de la Vallée - Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, ...$
A sequence $x = (x_k)$ is said to be (V, λ)- summable to a number L (see [3]) if $t_n(x) \to L$ as $n \to \infty$.
We write

$$[V, \lambda] = \left\{ x : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0, \text{ for } \text{ some } \ell \in \mathbb{R} \right\}$$

for the set of sequences that are strongly summable by the de la Valée-poussin method. In the special case where $\lambda_n = n$, for $n = 1, 2, 3, \ldots$, the sets $[V, \lambda]$ reduces to the set $[C, 1]$- summability defined as follows:

$$[C, 1] = \left\{ x : \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0, \text{ for some } \ \ell \in \mathbb{R} \right\}.$$

In 1993 Marouf presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003 Patterson extend these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Savas [11] presented the definition which is a natural combination of the definitions for asymptotically equivalent and λ- statistical convergence. In this paper we define and study $S_{\sigma,\lambda}$- asymptotically equivalent of multiple L. In addition to these definition, natural inclusion theorems shall also be presented.

2. Definitions and Notations

Definition 2.1 (Marouf, [6])**.** *Two nonnegative sequences* $[x]$*, and* $[y]$ *are said to be* **asymptotically equivalent** *if*

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$*).*

Definition 2.2 (Fridy, [2])**.** *The sequence* $[x]$ *has* **statistic limit** L*, denoted by* $st - \lim s = L$ *provided that for every* $\epsilon > 0$*,*

$$\lim_n \frac{1}{n} \{\text{the number of } k \leq n : |x_k - L| \geq \epsilon\} = 0.$$

Quite recently, following this definition Mursaleen [8] introduced the concept of λ-statistical convergence as follows:

Definition 2.3. *A sequence $x = (x_n)$ is said to be λ-statistically convergent or S_λ-convergent to L if for every $\epsilon > 0$,*

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case we write $S_\lambda - limitx = L$ or $x_k \longrightarrow L(S_\lambda)$, and

$$S_\lambda = \{x : \exists L \in \mathbb{R}, S_\lambda - limitx = L\}$$

The next definition is natural combination of definition (2.1) and(2.2).

Definition 2.4 (Patterson, [9])**.** *Two nonnegative sequence $[x]$ and $[y]$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\epsilon > 0$,*

$$\lim_n \frac{1}{n} \{\textit{the number of } k < n : |\frac{x_k}{y_k} - L| \geq \epsilon\} = 0,$$

(denoted by $x \overset{S_L}{\sim} y$), and simply asymptotically statistical equivalent, if $L = 1$.

It is quite natural to expect that the concept of λ-statistical convergence can be generalize by using σ - means.

Definition 2.5. *A sequence $x = (x_n)$ is said to be $S_{\sigma,\lambda}$- convergent to L if for every $\epsilon > 0$,*

$$\lim_n \frac{1}{\lambda_n} \left|\left\{k \in I_n : |x_{\sigma^k(m)} - L| \geq \epsilon\right\}\right| = 0,$$

uniformly in m =1, 2, 3, ... In this case we write $S_{\sigma,\lambda} - limitx = L$ or $x_k \longrightarrow L(S_{\sigma,\lambda})$, and

$$S_{\sigma,\lambda} = \{x : \exists L \in \mathbb{R}, S_{\sigma,\lambda} - limitx = L\}$$

We shall now introduce three new notions S_σ- asymptotically equivalent of multiple L, $S_{\sigma,\lambda}$- asymptotically equivalent of multiple L, and strong σ-asymptotically equivalent of multiple L. We also prove S_σ- asymptotically equivalent analogues of Mursaleen's theorems in [8].

Definition 2.6. *Two nonnegative sequences $[x]$ and $[y]$ are S_σ- asymptotically equivalent of multiple L provided that for every $\epsilon > 0$*

$$\lim_n \frac{1}{n} \left\{ \textit{ the number of } k \leq n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \epsilon \right\} = 0,$$

uniformly in m=1, 2, 3, ... , (denoted by $x \overset{S_\sigma}{\sim} y$), and simply σ- asymptotically statistical equivalent, if $L = 1$.

Definition 2.7. *Two nonnegative sequences $[x]$ and $[y]$ are $S_{\sigma,\lambda}$- asymptotically equivalent of multiple L provided that for every $\epsilon > 0$*

$$\lim_n \frac{1}{\lambda_n} \left|\left\{k \in I_n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \epsilon\right\}\right| = 0,$$

uniformly in m=1, 2, 3, ... , (denoted by $x \overset{S_{\sigma,\lambda}}{\sim} y$*), and simply* $S_{\sigma,\lambda}$*- asymptotically equivalent, if* $L = 1$.

Definition 2.8. *Two nonnegative sequences* $[x]$ *and* $[y]$ *are strong* (σ,λ)*-asymptotically equivalent of multiple L provided that for every* $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| = 0,$$

uniformly in m=1, 2, 3, ... , (denoted by $x \overset{[V_{\sigma,\lambda}]}{\sim} y$*), and simply strong* (σ,λ)*-asymptotically equivalent, if* $L = 1$.

If we take $\sigma(n) = n + 1$ the above definitions reduce the following definitions:

Definition 2.9. *Two nonnegative sequences* $[x]$ *and* $[y]$ *are* $\hat{S}$*- asymptotically equivalent of multiple L provided that for every* $\epsilon > 0$

$$\lim_n \frac{1}{n} \left\{ \text{ the number of } k \leq n : \left| \frac{x_{k+m}}{y_{k+m}} - L \right| \geq \epsilon \right\} = 0,$$

uniformly in m=1, 2, 3, ... , (denoted by $x \overset{\hat{S}}{\sim} y$*), and simply almost asymptotically statistical equivalent, if* $L = 1$.

Definition 2.10. *Two nonnegative sequences* $[x]$ *and* $[y]$ *are* $\hat{S}_\lambda$*- asymptotically equivalent of multiple L provided that for every* $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{x_{k+m}}{y_{k+m}} - L \right| \geq \epsilon \right\} \right| = 0,$$

uniformly in m=1, 2, 3, ... , (denoted by $x \overset{\hat{S}_\lambda}{\sim} y$*), and simply almost* λ*-asymptotically equivalent, if* $L = 1$.

Definition 2.11. *Two nonegative sequences* $[x]$ *and* $[y]$ *are strong almost* λ*-asymptotically equivalent of multiple L provided that for every* $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left| \frac{x_{k+m}}{y_{k+m}} - L \right| = 0,$$

uniformly in m=1, 2, 3, ... , (denoted by $x \overset{[\hat{V}_\lambda]}{\sim} y$*), and simply strong almost* λ*- asymptotically equivalent, if* $L = 1$.

Let Λ denote the set of all non-decreasing sequences $\Lambda = (\lambda_n)$ of positive numbers tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. The following theorem analogues of ([8]; theorem 2.1.)

3. Main Result

Theorem 3.1. *Let $\lambda \in \Lambda$, then*

(1) *If $x \overset{[V_{\sigma,\lambda}]}{\sim} y$ then $x \overset{S_{\sigma,\lambda}}{\sim} y$,*
(2) *If $x \in l_\infty$ and $x \overset{S_{\sigma,\lambda}}{\sim} y$ then $x \overset{[V_{\sigma,\lambda}]}{\sim} y$ and hence $x \overset{(C,1)}{\sim} y$, and*
(3) *$S_{\sigma,\lambda} \cap l_\infty = [V_{\sigma,\lambda}] \cap l_\infty$.*

Proof. Part (1): If $\epsilon > 0$ and $x \overset{[V_{\sigma,\lambda}]}{\sim} y$, then

$$\begin{aligned}\sum_{k\in I_n}\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right| &\geq \sum_{k\in I_n \,\&\, \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\ \geq\epsilon}\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right| \\ &\geq \epsilon\left|\left\{k\in I_n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right|\geq\epsilon\right\}\right|.\end{aligned}$$

Therefore $x \overset{[V_{\sigma,\lambda}]}{\sim} y$. Part (2): Suppose $[x]$ and $[y]$ are in l_∞ and $x \overset{S_{\sigma,\lambda}}{\sim} y$. Then we can assume that

$$\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right| \leq M \text{ for all } k \text{ and } m.$$

Given $\epsilon > 0$

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right| &= \frac{1}{\lambda_n}\sum_{k\in I_n \,\&\, \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\ \geq\epsilon}\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right| \\ &+ \frac{1}{\lambda_n}\sum_{k\in I_n \,\&\, \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\ <\epsilon}\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right| \\ &\leq \frac{M}{\lambda_n}\left|\left\{k\in I_n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right|\geq\epsilon\right\}\right|+\epsilon.\end{aligned}$$

Therefore $x \overset{[V_{\sigma,\lambda}]}{\sim} y$. Further, we have

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right) &= \frac{1}{n}\sum_{k=1}^{n-\lambda_n}\left(\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right)+\frac{1}{n}\sum_{k\in I_n}\left(\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right) \\ &\leq \frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n}\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right|+\frac{1}{\lambda_n}\sum_{k\in I_n}\left(\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right) \\ &\leq \frac{2}{\lambda_n}\sum_{k\in I_n}\left(\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}}-L\right).\end{aligned}$$

Hence $x \overset{(C,1)}{\sim} y$, since $x \overset{[V_{\sigma,\lambda}]}{\sim} y$.
Part (3): This immediately follows from (1) and (2). □

In the next theorem we prove the following relation.

Theorem 3.2.

$$x \overset{S_\sigma}{\sim} y \textit{ implies } x \overset{S_{\sigma,\lambda}}{\sim} y$$

if

$$liminf \frac{1}{\lambda_n} > 0 \tag{3.1}$$

Proof. For given $\varepsilon > 0$ we have

$$\frac{1}{n}\left\{k \le n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \ge \epsilon\right\} \supset \left\{k \in I_n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \ge \epsilon\right\}.$$

Therefore

$$\begin{aligned}\frac{1}{n}\left|\left\{k \le n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \ge \epsilon\right\}\right| &\ge \frac{1}{n}\left|\left\{k \in I_n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \ge \epsilon\right\}\right| \\ &\ge \frac{\lambda_n}{n}\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \ge \epsilon\right\}\right|.\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ and using (3.1), we get desired result. This completes the proof. □

Remark 1. *In case $\sigma(n) = n+1$, the above results reduce to the results for almost convergence.*

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