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Research Article

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Σ-Shaped Bifurcation Curves

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Abstract: We study positive solutions to the steady state reaction diffusion equation of the form:

$$\begin{cases} -\Delta u = \lambda f(u);\ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0;\ \partial\Omega \end{cases}$$

where $\lambda > 0$ is a positive parameter, Ω is a bounded domain in $\mathbb{R}^N$ when $N > 1$ (with smooth boundary $\partial\Omega$) or $\Omega = (0, 1)$, and $\frac{\partial u}{\partial \eta}$ is the outward normal derivative of u. Here $f(s) = ms + g(s)$ where $m \geq 0$ (constant) and $g \in C^2[0, r) \cap C[0, \infty)$ for some $r > 0$. Further, we assume that g is increasing, sublinear at infinity, $g(0) = 0$, $g'(0) = 1$ and $g''(0) > 0$. In particular, we discuss the existence of multiple positive solutions for certain ranges of λ leading to the occurrence of Σ-shaped bifurcation diagrams. We establish our multiplicity results via the method of sub-supersolutions.

Keywords: Σ-Shaped Bifurcaion Curves, Positive Solutions, Sub-Super Solutions

MSC: 35J15, 35J25, 35J60

1 Introduction

In the recent literature there has been considerable interest in reaction diffusion models where a parameter influences the equation as well as the boundary conditions. See [1, 2, 3] for recent studies in this direction. In this paper, we enhance this study to show that for certain classes of such models the bifurcation diagram $(\lambda, \|u\|_\infty)$ for positive solutions is at least Σ-shaped. Namely, we study boundary value problems of the form:

$$\begin{cases} -\Delta u = \lambda f(u);\ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0;\ \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a positive parameter, Ω is a bounded domain in $\mathbb{R}^N$ when $N > 1$ (with smooth boundary $\partial\Omega$) or $\Omega = (0, 1)$, and $\frac{\partial u}{\partial \eta}$ is the outward normal derivative of u. Here $f(s) = ms + g(s)$ where $m \geq 0$ (constant) and $g \in C^2[0, r) \cap C[0, \infty)$ for some $r > 0$. Further, we assume that g is increasing and satisfies:
(H_1) $g(0) = 0$, $g'(0) = 1$, $g''(0) > 0$, and $\lim_{s\to\infty} \frac{g(s)}{s} = 0$.
First, we recall some results from [3]. Namely, for $k > 0$, let A_k be the principal eigenvalue of the problem:

$$\begin{cases} -\Delta \phi = Ak\phi;\ \Omega \\ \frac{\partial \phi}{\partial \eta} + \sqrt{A}\phi = 0;\ \partial\Omega. \end{cases} \tag{1.2}$$

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Then A_k is a strictly decreasing function of k with

$$\lim_{k\to 0} A_k = \infty. \tag{1.3}$$

Further, for a fixed $\lambda > 0$, let $\sigma_{\lambda,k}$ be the principal eigenvalue and $\theta_{\lambda,k} > 0$ on $\overline{\Omega}$ be the corresponding normalized eigenfunction of:

$$\begin{cases} -\Delta\theta = (\sigma+\lambda)k\theta; \ \Omega \\ \frac{\partial\theta}{\partial\eta} + \sqrt{\lambda}\theta = 0; \ \partial\Omega. \end{cases} \tag{1.4}$$

We note that $\sigma_{\lambda,k} > 0$ when $\lambda < A_k$, $\sigma_{\lambda,k} < 0$ when $\lambda > A_k$, and $\sigma_{\lambda,k} \to 0$ as $\lambda \to A_k$. Next, let $C_N = \frac{(N+1)^{N+1}}{2N^N}$, R be the radius of the largest inscribed ball in Ω, v be the unique solution of

$$\begin{cases} -\Delta v = 1; \ \Omega \\ \frac{\partial v}{\partial\eta} + v = 0; \ \partial\Omega, \end{cases} \tag{1.5}$$

and let w be the unique solution of

$$\begin{cases} -\Delta w = 1; \ \Omega \\ \frac{\partial w}{\partial\eta} + \sqrt{\frac{A_1}{2}} w = 0; \ \partial\Omega. \end{cases} \tag{1.6}$$

Now, we introduce hypotheses (H_2) and (H_3).
(H_2) There exist $a_1 > 0$, $b_1 > 0$ such that $a_1 < b_1$ and
$\min\{A_m, \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}\} > \max\{\frac{b_1}{f(b_1)}\frac{2NC_N}{R^2}, A_{m+1}, 1\}$.
(H_3) There exist $a_2 > 0$, $b_2 > 0$ such that $a_2 < b_2$ and
$\frac{a_2}{f(a_2)}\frac{1}{\|w\|_\infty} \geq A_{m+1} > \max\{\frac{b_2}{f(b_2)}\frac{2NC_N}{R^2}, \frac{A_1}{2}\}$.
We note that functions satisfying (H_1) – (H_3) are such that $\frac{s}{f(s)}$ has the shape as in Figure 1 (with $\frac{l_1}{l_2} \gg 1$).

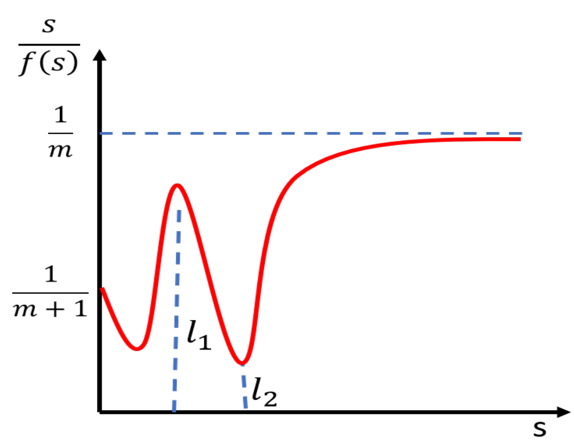
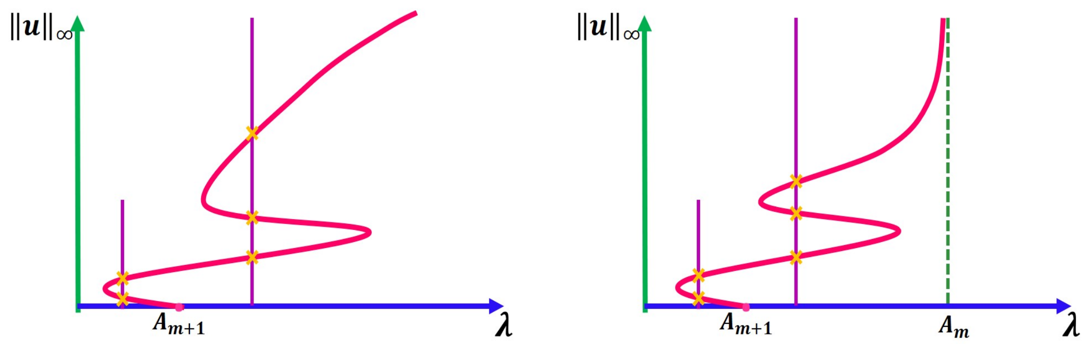


Fig. 1: Shape of $\frac{s}{f(s)}$ when our hypotheses are satisfied.

We now state our main results:

Theorem 1.1.
a) Let (H_1) hold. Then (1.1) has a positive solution for $\lambda \in [A_{m+1}, A_m)$. Also, a positive solution u_λ for $\lambda < A_m$ and $\lambda \approx A_m$ such that $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to A_m^-$. Further, there exists $\overline{\lambda} < A_{m+1}$ such that (1.1) has at least two positive solutions for $\lambda \in [\overline{\lambda}, A_{m+1})$. (Here, by $\lambda \approx A_m$, we mean λ is close to A_m.)

b) Let (H_1) and (H_2) hold. Then (1.1) has at least three positive solutions for
$\lambda \in \left(\max\{\frac{b_1}{f(b_1)}\frac{2NC_N}{R^2}, A_{m+1}, 1\}, \min\{A_m, \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}\}\right)$.

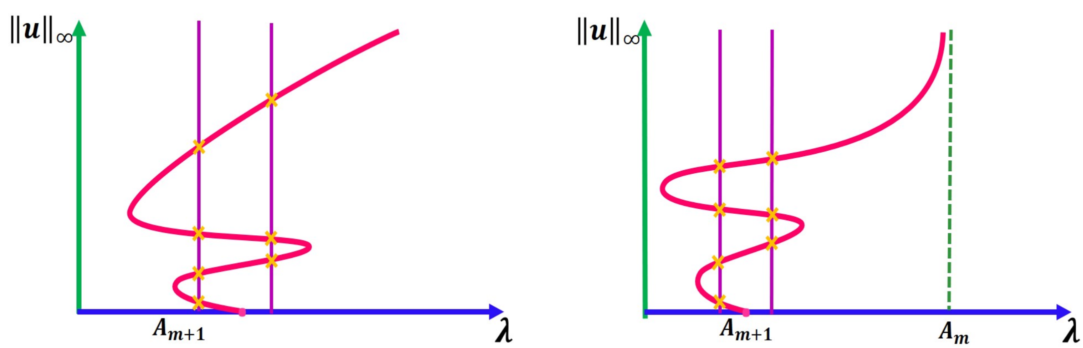


(a) When $m = 0$ **(b)** When $m > 0$

Fig. 2: An expected bifurcation diagram for (1.1) when hypotheses of Theorem 1.1(b) are satisfied.

Theorem 1.2. *Let* (H_1) *and* (H_3) *hold. Then there exists* $\lambda^* \in \left(\max\{\frac{b_2}{f(b_2)}\frac{2NC_N}{R^2}, \frac{A_1}{2}\}, A_{m+1}\right)$ *such that* (1.1) *has at least four positive solutions for* $\lambda \in [\lambda^*, A_{m+1})$.

Corollary 1.3. *Let* (H_1) - (H_3) *hold. Then there exists* λ^* *such that* (1.1) *has a positive solution for* $\lambda \in [\lambda^*, A_m)$, *a positive solution* u_λ *for* $\lambda < A_m$ *and* $\lambda \approx A_m$ *such that* $||u_\lambda||_\infty \to \infty$ *as* $\lambda \to A_m^-$, *at least four positive solutions for* $\lambda \in [\lambda^*, A_{m+1})$ *and at least three positive solutions for*
$\lambda \in \left(\max\{\frac{b_1}{f(b_1)}\frac{2NC_N}{R^2}, A_{m+1}, 1\}, \min\{A_m, \frac{a_1}{f(a_1)}\frac{1}{||v||_\infty}\}\right)$.



(a) When $m = 0$ **(b)** When $m > 0$

Fig. 3: An expected bifurcation diagram for (1.1) when hypotheses of Corollary 1.3 are satisfied.

Remark 1.1. *It is easy to show that* (1.1) *has no positive solutions for* $\lambda \approx 0$, *and when* $m > 0$ *for* $\lambda > A_m$ *(see Appendix).*

Remark 1.2. *A typical* f *which is likely to produce such a* Σ*–shaped bifurcation curve is as follows: Convex on* $(0, \alpha)$ *for some* $\alpha > 0$ *driving the bifurcation curve initially to the left, a strong concavity on* (α, β) *with* $\beta > \alpha$ *making the bifurcation curve go back to the right, a strong convexity on* (β, γ) *with* $\gamma > \beta$ *driving the bifurcation curve back again to the left, and then a strong concavity on* (γ, ∞) *bringing the curve eventually to the right (see Figure 4).*

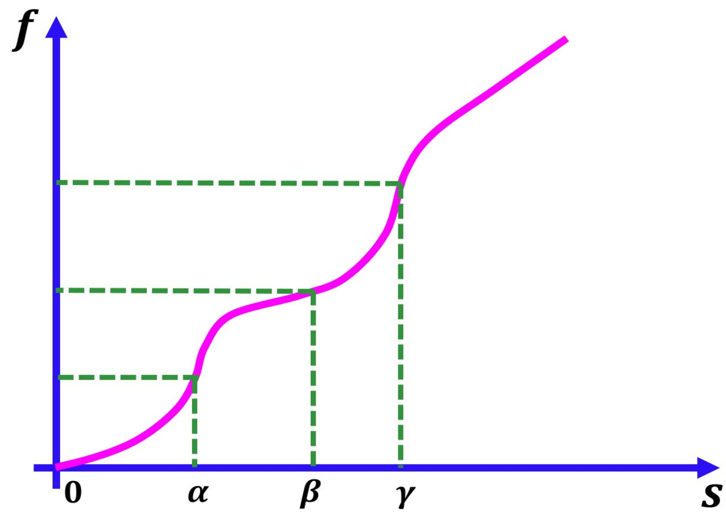


Fig. 4: Shape of f producing multiplicity.

For related study of models in biology see also [4, 5].

Finally, for an example for which Theorem 1.1, Theorem 1.2, and Corollary 1.3 hold, consider

$$\begin{cases} -\Delta u = \lambda f(u) = \lambda[mu + g(u)]; \ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0; \ \partial\Omega, \end{cases}$$

with

$$g(s) = g_{\alpha,k}(s) = \begin{cases} e^{\frac{cs}{c+s}} - 1; s \le k \\ [e^{\frac{\alpha s}{\alpha+s}} - e^{\frac{\alpha k}{\alpha+k}}] + [e^{\frac{ck}{c+k}} - 1]; s > k, \end{cases}$$

where $c > 2$ is a fixed number, $m \ge 0$, $\alpha > 0$ and $k > 0$ are parameters. We will discuss this example in detail in Section 4.

We present some preliminaries in Section 2. We provide proofs of Theorems 1.1 - 1.2 and Corollary 1.3 in Section 3. In Section 4, we discuss in detail the example f we introduced above and show that Theorems 1.1 - 1.2 and Corollary 1.3 hold for certain parameter values. In Section 5, when $\Omega = (0, 1)$, via the quadrature method discussed in [3], we provide approximations to the exact bifurcation diagrams via Mathematica computations for the example discussed in Section 4. Our existence and multiplicity results are established via a method of sub-supersolutions.

2 Preliminaries

In this section, we introduce definitions of a (strict) subsolution and a (strict) supersolution of (1.1), and state a sub-supersolution theorem and a three solution theorem that we will use.

By a subsolution of (1.1) we mean $\psi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that satisfies

$$\begin{cases} -\Delta \psi \le \lambda f(\psi); \ \Omega \\ \frac{\partial \psi}{\partial \eta} + \sqrt{\lambda} \psi \le 0; \ \partial\Omega. \end{cases}$$

By a supersolution of (1.1) we mean $Z \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that satisfies

$$\begin{cases} -\Delta Z \ge \lambda f(Z); \ \Omega \\ \frac{\partial Z}{\partial \eta} + \sqrt{\lambda} Z \ge 0; \ \partial\Omega. \end{cases}$$

By a strict subsolution of (1.1) we mean a subsolution which is not a solution. By a strict supersolution of (1.1) we mean a supersolution which is not a solution.

Then the following results hold (see [6, 7]):

Lemma 2.1. *Let ψ and Z be a subsolution and a supersolution of* (1.1) *respectively such that $\psi \leq Z$. Then* (1.1) *has a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $u \in [\psi, Z]$.*

Lemma 2.2. *Let $\underline{u}_1$ and $\overline{u}_2$ be a subsolution and a supersolution of* (1.1) *respectively such that $\underline{u}_1 \leq \overline{u}_2$ in Ω. Let $\underline{u}_2$ and $\overline{u}_1$ be a strict subsolution and a strict supersolution of* (1.1) *respectively such that $\underline{u}_2, \overline{u}_1 \in [\underline{u}_1, \overline{u}_2]$ and $\underline{u}_2 \not\leq \overline{u}_1$. Then* (1.1) *has at least three solutions u_1, u_2 and u_3 where $u_i \in [\underline{u}_i, \overline{u}_i]$ for $i = 1, 2$ and $u_3 \in [\underline{u}_1, \overline{u}_2] \backslash ([\underline{u}_1, \overline{u}_1] \cup [\underline{u}_2, \overline{u}_2])$.*

3 Proofs of Theorems 1.1-1.2 and Corollary 1.3

First we construct sub-super solutions for certain λ ranges. Recall $\theta_{\lambda,k}$ and $\sigma_{\lambda,k}$ (see (1.4)).

Construction of a small strict subsolution ψ_1 for $\lambda < A_{m+1}$ and $\lambda \approx A_{m+1}$ when (H_1) is satisfied

We first note that $f''(s) > 0$ for $s \approx 0$ since $g''(0) > 0$. Hence there exists $A^* > 0$ and $s_1 > 0$ such that $f''(s) > A^*$ for $s < s_1$. Let $\psi_1 = \delta_\lambda \theta_{\lambda,m+1}$ where $\delta_\lambda = \frac{2(m+1)\sigma_{\lambda,m+1}}{\lambda A^* \min_{\overline{\Omega}} \theta_{\lambda,m+1}}$. We note that $\sigma_{\lambda,m+1} > 0$, $\sigma_{\lambda,m+1} \to 0$ as $\lambda \to A_{m+1}^-$, and $\min_{\overline{\Omega}} \theta_{\lambda,m+1} \not\to 0$ as $\lambda \to A_{m+1}^-$. Thus $\delta_\lambda \to 0^+$ as $\lambda \to A_{m+1}^-$. Now by Taylor's Theorem, we have $f(\psi_1) = f(0) + f'(0)\psi_1 + \frac{f''(\zeta)}{2}{\psi_1}^2 = (m+1)\psi_1 + \frac{f''(\zeta)}{2}{\psi_1}^2$ for some $\zeta \in [0, \psi_1]$. Then we have

$$\begin{aligned} -\Delta\psi_1 - \lambda f(\psi_1) &= \delta_\lambda(\sigma_{\lambda,m+1} + \lambda)(m+1)\theta_{\lambda,m+1} - \lambda\left[(m+1)\delta_\lambda\theta_{\lambda,m+1} + \frac{f''(\zeta)}{2}(\delta_\lambda\theta_{\lambda,m+1})^2\right] \\ &< \delta_\lambda\theta_{\lambda,m+1}\left[(m+1)\sigma_{\lambda,m+1} - \frac{\lambda A^*}{2}\delta_\lambda \min_{\overline{\Omega}} \theta_{\lambda,m+1}\right] = 0; \ \Omega \end{aligned}$$

by our choice of δ_λ, for $\lambda < A_{m+1}$ and $\lambda \approx A_{m+1}$ such that $\psi_1 < s_1$. Also, $\frac{\partial\psi_1}{\partial\eta} + \sqrt{\lambda}\psi_1 = 0$ on $\partial\Omega$ since $\theta_{\lambda,m+1}$ satisfies this boundary condition. Thus, there exists $\overline{\lambda} < A_{m+1}$ such that ψ_1 is a strict subsolution of (1.1) for $\lambda \in [\overline{\lambda}, A_{m+1})$.

Construction of a small subsolution ψ_2 for $\lambda \in [A_{m+1}, A_m)$ when (H_1) is satisfied

We note that $f'(0) = m+1$, $\sigma_{\lambda,m+1} \leq 0$ for $\lambda \in [A_{m+1}, A_m)$ and $\sigma_{\lambda,m+1} \to 0$ as $\lambda \to A_{m+1}$. Let $\psi_2 = n_\lambda\theta_{\lambda,m+1}$ with $n_\lambda > 0$. Now, consider $H(s) = (\sigma_{\lambda,m+1} + \lambda)(m+1)s - \lambda f(s)$. Then we have $H(0) = 0$, $H'(0) = \sigma_{\lambda,m+1}(m+1) \leq 0$ and $H''(0) = -\lambda f''(0) < 0$ since $f''(0) > 0$. This implies that $-\Delta\psi_2 = n_\lambda(\sigma_{\lambda,m+1} + \lambda)(m+1)\theta_{\lambda,m+1} < \lambda f(n_\lambda\theta_{\lambda,m+1}) = \lambda f(\psi_2)$ in Ω for $n_\lambda \approx 0$. We also have $\frac{\partial\psi_2}{\partial\eta} + \sqrt{\lambda}\psi_2 = 0$ on $\partial\Omega$ since $\theta_{\lambda,m+1}$ satisfies this boundary condition. Thus ψ_2 is a subsolution of (1.1) for $n_\lambda \approx 0$ when $\lambda \in [A_{m+1}, A_m)$.

Construction of a subsolution ψ_3 for $\lambda < A_m$ and $\lambda \approx A_m$ such that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to A_m^-$ when (H_1) is satisfied

Let $m > 0$ and $\psi_3 = \epsilon_\lambda\theta_{\lambda,m}$ where $\epsilon_\lambda = \frac{\lambda g\left(\min_{\overline{\Omega}} \theta_{\lambda,m}\right)}{m\sigma_{\lambda,m}\|\theta_{\lambda,m}\|_\infty}$. We note that $\epsilon_\lambda > 0$ since $\sigma_{\lambda,m} > 0$ for $\lambda < A_m$. Further, $\epsilon_\lambda \to \infty$ as $\lambda \to A_m^-$ since $\sigma_{\lambda,m} \to 0^+$ as $\lambda \to A_m^-$ and $\min_{\overline{\Omega}} \theta_{\lambda,m} \not\to 0$. This implies that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to A_m^-$. Now we have

$$\begin{aligned} -\Delta\psi_3 - \lambda f(\psi_3) &= \epsilon_\lambda[(\lambda + \sigma_{\lambda,m})m\theta_{\lambda,m}] - \lambda[m\epsilon_\lambda\theta_{\lambda,m} + g(\epsilon_\lambda\theta_{\lambda,m})] \\ &= \epsilon_\lambda m\sigma_{\lambda,m}\theta_{\lambda,m} - \lambda g(\epsilon_\lambda\theta_{\lambda,m}) \\ &\leq \epsilon_\lambda m\sigma_{\lambda,m}\|\theta_{\lambda,m}\|_\infty - \lambda g(\epsilon_\lambda\theta_{\lambda,m}) \\ &= \lambda[g\left(\min_{\overline{\Omega}} \theta_{\lambda,m}\right) - g(\epsilon_\lambda\theta_{\lambda,m})] \\ &\leq 0; \ \Omega \end{aligned}$$

for $\lambda \approx A_m$, since $\epsilon_\lambda > 1$ for $\lambda \approx A_m$ and g is increasing. Hence, we have $-\Delta\psi_3 \le \lambda f(\psi_3)$ in Ω. Also, on the boundary we have $\frac{\partial\psi_3}{\partial\eta} + \sqrt{\lambda}\psi_3 = 0$ since $\theta_{\lambda,m}$ satisfies this boundary condition. Consequently ψ_3 is a subsolution of (1.1) such that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to A_m^-$.

Next, let $m = 0$. Here we can show (1.1) has a subsolution ψ_3 such that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to \infty$ by using a well known result in [8] for semipositone problems. Namely, define $h \in C^2([0,\infty))$ such that $h(0) < 0$, $h(s) \le f(s)$ for $s \in (0,\infty)$ and $\lim_{s\to\infty} h(s) > 0$. Then the boundary value problem

$$\begin{cases} -\Delta w = \lambda h(w); & \Omega, \\ w = 0; & \partial\Omega, \end{cases}$$

has a solution $\overline{w}_\lambda > 0$ for $\lambda \gg 1$ such that $\|\overline{w}_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$. Since by the Hopf maximum principle $\frac{\partial\overline{w}_\lambda}{\partial\eta} < 0$ on $\partial\Omega$, it is easy to show that $\psi_3 = \overline{w}_\lambda$ is a subsolution of (1.1) for $\lambda \gg 1$ such that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to \infty$.

Construction of a strict subsolution ψ_4 for $\lambda > \frac{b}{f(b)}\frac{2NC_N}{R^2}$ where $b = b_1$ when (H_2) is satisfied and $b = b_2$ when (H_3) is satisfied

Here we construct a strict subsolution ψ_4 for $\lambda > \frac{b}{f(b)}\frac{2NC_N}{R^2}$ using the iteration of a subsolution $\tilde{\psi}$ created originally in [9] and later also used in [10]. Namely, the authors in [10] take ψ to be the solution of:

$$\begin{cases} -\psi''(r) - \frac{N-1}{r}\psi'(r) = \lambda f(w(r)); & r \in (0,R) \\ \psi'(0) = 0 = \psi(R), \end{cases} \tag{3.1}$$

where R is the radius of the largest inscribed ball, B_R, in Ω (see Figure 5) and $w(r) = b\rho(r)$ with

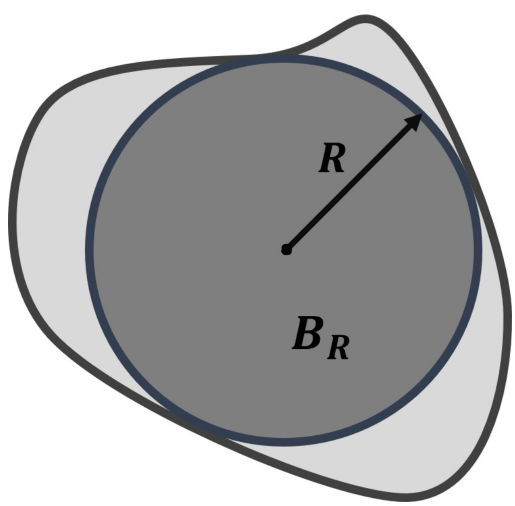


Fig. 5: Largest inscribed ball in Ω.

$$\rho(r) = \begin{cases} 1; & r \in [0,\epsilon] \\ 1 - \left[1 - \left(\frac{R-r}{R-\epsilon}\right)^\beta\right]^\alpha; & r \in (\epsilon, R], \alpha, \beta > 1. \end{cases}$$

When $\lambda > \frac{b}{f(b)}\frac{2NC_N}{R^2}$ for certain choices of $\alpha > 1$, $\beta > 1$, and $\epsilon \in (0,1)$ it was proven that (see [9] for details) $\psi \ge w$ on $[0,R]$ and hence a subsolution of (3.1) since f is increasing. Now since $f(0) = 0$ it follows that

$$\tilde{\psi} = \begin{cases} \psi; & B_R \\ 0; & \Omega \backslash B_R, \end{cases}$$

is a strict subsolution of:

$$\begin{cases} -\Delta u = \lambda f(u);\ \Omega \\ u = 0;\ \partial\Omega, \end{cases}$$

for $\lambda > \frac{b}{f(b)} \frac{2NC_N}{R^2}$ such that $\|\tilde{\psi}\|_\infty \geq b$.

Now let ψ_4 be the first iteration of $\tilde{\psi}$, namely, ψ_4 be the solution to the problem:

$$\begin{cases} -\Delta\psi_4 = \lambda f(\tilde{\psi});\ \Omega \\ \frac{\partial\psi_4}{\partial\eta} + \sqrt{\lambda}\psi_4 = 0;\ \partial\Omega. \end{cases}$$

Then we have $-\Delta(\psi_4 - \tilde{\psi}) \geq 0$ and $\frac{\partial(\psi_4-\tilde{\psi})}{\partial\eta} + \sqrt{\lambda}(\psi_4 - \tilde{\psi}) = -\frac{\partial\tilde{\psi}}{\partial\eta} > 0$ by the Hopf maximum principle. This implies that $\psi_4 > \tilde{\psi}$ in Ω. Hence, ψ_4 is a strict subsolution of (1.1) for $\lambda > \frac{b}{f(b)} \frac{2NC_N}{R^2}$.

Construction of a large supersolution Z_1 for $\lambda < A_m$ when (H_1) is satisfied

Let $m > 0$. Choose $Z_1 = M\theta_{\lambda,m}$ for $M > 0$. Then $-\Delta Z_1 - \lambda f(Z_1) = M(\sigma_{\lambda,m}+\lambda)m\theta_{\lambda,m} - \lambda[mM\theta_{\lambda,m}+g(M\theta_{\lambda,m})] = mM\theta_{\lambda,m}\left[\sigma_{\lambda,m} - \frac{\lambda g(M\theta_{\lambda,m})}{mM\theta_{\lambda,m}}\right] > 0$ in Ω for $M \gg 1$ since $\sigma_{\lambda,m} > 0$ for $\lambda < A_m$ and $\frac{g(s)}{s} \to 0$ as $s \to \infty$. Further, $\frac{\partial Z_1}{\partial\eta} + \sqrt{\lambda}Z_1 = 0$ on $\partial\Omega$ since $\theta_{\lambda,m}$ satisfies this boundary condition. Hence, Z_1 is a supersolution of (1.1) for $M \gg 1$.

Next, let $m = 0$. Here we choose $Z_1 = Me_\lambda$, where e_λ is the unique solution of $-\Delta e = 1$ in Ω and $\frac{\partial e}{\partial\eta} + \sqrt{\lambda}e = 0$ on $\partial\Omega$. Note $e_\lambda > 0$ on $\overline{\Omega}$. Then $-\Delta Z_1 - \lambda f(Z_1) = M - \lambda g(Me_\lambda) \geq M\left[1 - \lambda\frac{g(M\|e_\lambda\|_\infty)}{M\|e_\lambda\|_\infty}\|e_\lambda\|_\infty\right] > 0$ for $M \gg 1$ since g is increasing and $\frac{g(s)}{s} \to 0$ as $s \to \infty$. Also, $\frac{\partial Z_1}{\partial\eta} + \sqrt{\lambda}Z_1 = 0$ on $\partial\Omega$ since e_λ satisfies this boundary condition. Hence, Z_1 is a supersolution of (1.1) for $M \gg 1$.

Construction of a strict supersolution Z_2 for $\lambda < A_{m+1}$ when (H_1) is satisfied

Let $Z_2 = m_\lambda\theta_{\lambda,m+1}$ and $l(s) = (\sigma_{\lambda,m+1} + \lambda)(m + 1)s - \lambda f(s)$. We note that $\sigma_{\lambda,m+1} > 0$ for $\lambda < A_{m+1}$. Then we have $l(0) = 0$ and $l'(0) = (\sigma_{\lambda,m+1} + \lambda)(m + 1) - \lambda f'(0) = \sigma_{\lambda,m+1}(m + 1) > 0$ since $f'(0) = m + 1$. This implies that $-\Delta Z_2 = m_\lambda(\sigma_{\lambda,m+1} + \lambda)(m + 1)\theta_{\lambda,m+1} > \lambda f(m_\lambda\theta_{\lambda,m+1}) = \lambda f(Z_2)$ in Ω for $m_\lambda \approx 0$. On the boundary, we have $\frac{\partial Z_2}{\partial\eta} + \sqrt{\lambda}Z_2 = 0$ since $\theta_{\lambda,m+1}$ satisfies this boundary condition. Thus Z_2 with $m_\lambda \approx 0$ is a strict supersolution of (1.1) for $\lambda < A_{m+1}$.

Construction of a strict supersolution Z_3 for $\lambda \in \left(1, \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}\right)$ when (H_2) is satisfied

Let $Z_3 = \frac{a_1 v}{\|v\|_\infty}$ where v is as in (1.5). Then $-\Delta Z_3 = \frac{a_1}{\|v\|_\infty} > \lambda f(a_1) \geq \lambda f(Z_3)$ since $\lambda < \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}$ and f is increasing. Further, Z_3 satisfies $\frac{\partial Z_3}{\partial\eta} + \sqrt{\lambda}Z_3 = \frac{a_1}{\|v\|_\infty}\frac{\partial v}{\partial\eta} + \sqrt{\lambda}\frac{a_1 v}{\|v\|_\infty} > \frac{a_1}{\|v\|_\infty}[\frac{\partial v}{\partial\eta} + v] = 0$ on $\partial\Omega$ since $\lambda > 1$. Thus Z_3 is a strict supersolution of (1.1) for $\lambda \in \left(1, \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}\right)$.

Construction of a strict supersolution Z_4 for $\lambda \in \left(\frac{A_1}{2}, \frac{a_2}{f(a_2)}\frac{1}{\|w\|_\infty}\right)$ when (H_3) is satisfied

Let $Z_4 = \frac{a_2 w}{\|w\|_\infty}$ where w is as in (1.6). Then $-\Delta Z_4 = \frac{a_2}{\|w\|_\infty} > \lambda f(a_2) \geq \lambda f(Z_4)$ since $\lambda < \frac{a_2}{f(a_2)}\frac{1}{\|w\|_\infty}$ and f is increasing. Further, Z_4 satisfies $\frac{\partial Z_4}{\partial\eta} + \sqrt{\lambda}Z_4 = \frac{a_2}{\|w\|_\infty}\frac{\partial w}{\partial\eta} + \sqrt{\lambda}\frac{a_2 w}{\|w\|_\infty} > \frac{a_2}{\|w\|_\infty}[\frac{\partial w}{\partial\eta} + \sqrt{\frac{A_1}{2}}w] = 0$ on $\partial\Omega$ since $\lambda > \frac{A_1}{2}$. Thus Z_4 is a strict supersolution of (1.1) for $\lambda \in \left(\frac{A_1}{2}, \frac{a_2}{f(a_2)}\frac{1}{\|w\|_\infty}\right)$.

Now we prove Theorems 1.1-1.2 and Corollary 1.3.

Proof of Theorem 1.1: a) Let M be as in the construction of the supersolution Z_1 and n_λ be as in the construction of the subsolution ψ_2. We choose $M \gg 1$ and $n_\lambda \approx 0$ such that $Z_1 \geq \psi_2$. By Lemma 2.1, (1.1) has a positive solution $u_\lambda \in [\psi_2, Z_1]$ for $\lambda \in [A_{m+1}, A_m)$.

Recall the subsolution ψ_3 of (1.1). Now we choose $M \gg 1$ such that $\psi_3 \leq Z_1$. Hence, recalling that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to A_m^-$, by Lemma 2.1, (1.1) has a positive solution $u_\lambda \in [\psi_3, Z_1]$ such that $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to A_m^-$.

Next, let $\lambda \in [\overline{\lambda}, A_{m+1})$ where $\overline{\lambda}$ be as in the construction of the strict subsolution ψ_1. We note that $\psi_0 = 0$ is a solution and hence a subsolution of (1.1). Recall the strict supersolution Z_2 of (1.1). Now we choose m_λ

small enough such that $\|Z_2\|_\infty < \|\psi_1\|_\infty$. Next, we choose $M \gg 1$ such that $\psi_1 \le Z_1$ and $Z_2 \le Z_1$ (see Figure 6). By Lemma 2.2, (1.1) has at least two positive solutions $u_1 \in [\psi_1, Z_1]$ and $u_2 \in [\psi_0, Z_1]\backslash([\psi_0, Z_2] \cup [\psi_1, Z_1])$

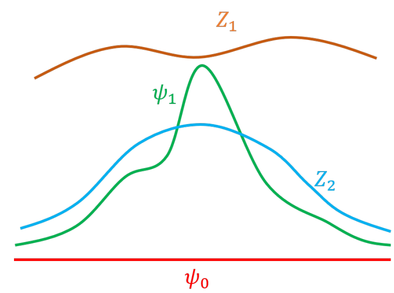


Fig. 6: Subsolutions ψ_0, ψ_1 and supersolutions Z_1, Z_2.

for $\lambda \in [\bar{\lambda}, A_{m+1})$.

b) Recall the strict subsolution ψ_4 when $b = b_1$ and the strict supersolution Z_3 of (1.1). Now we choose n_λ small enough such that $\psi_2 \le \psi_4$ and $\psi_2 \le Z_3$. Next we choose $M \gg 1$ such that $\psi_4 \le Z_1$ and $Z_3 \le Z_1$ (see Figure 7). We note that $\|\psi_4\|_\infty \ge b_1 > a_1 = \|Z_3\|_\infty$. By Lemma 2.2, (1.1) has at least three positive solutions for $\lambda \in \left(\max\{\frac{b_1}{f(b_1)}\frac{2NC_N}{R^2}, A_{m+1}, 1\}, \min\{A_m, \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}\}\right)$. We note that in the construction of ψ_2, ψ_4, Z_1, and Z_3, the intersection of intervals of λ is $\left(\max\{\frac{b_1}{f(b_1)}\frac{2NC_N}{R^2}, A_{m+1}, 1\}, \min\{A_m, \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}\}\right)$. This completes the proof.

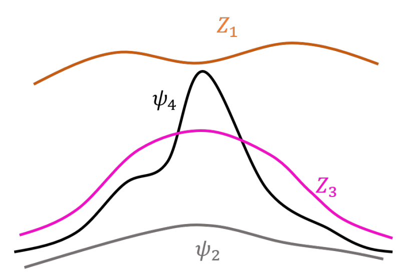


Fig. 7: Subsolutions ψ_2, ψ_4 and supersolutions Z_1, Z_3.

Proof of Theorem 1.2: Let $\lambda^* = \bar{\lambda}$ and ψ_0 be as in the proof of Theorem 1.1. Recall the strict supersolution Z_4 and the strict subsolution ψ_4 when $b = b_2$. First we choose $\lambda^* > \max\{\frac{b_2}{f(b_2)}\frac{2NC_N}{R^2}, \frac{A_1}{2}\}$, $\lambda^* < A_{m+1}$, and $\lambda^* \approx A_{m+1}$ (making $\delta_\lambda \approx 0$) such that $\psi_1 < \psi_4$ and $\psi_1 < Z_4$ for $\lambda \in [\lambda^*, A_{m+1})$. Next, we choose m_λ small enough such that $\|Z_2\|_\infty < \|\psi_1\|_\infty$. Further, we can choose $M \gg 1$ such that $\psi_1 \le Z_1$ and $Z_2 \le Z_1$ (see Figure (8)). By Lemma 2.2, (1.1) has a positive solution $u_1 \in [\psi_0, Z_1]\backslash([\psi_0, Z_2] \cup [\psi_1, Z_1])$ for $\lambda \in [\lambda^*, A_{m+1})$. We also have $\psi_4 \le Z_1$, $Z_4 \le Z_1$ for $M \gg 1$ and $\|\psi_4\|_\infty \ge b_2 > a_2 = \|Z_4\|_\infty$ (see Figure 8). Again, by Lemma 2.2, (1.1) has at least three positive solutions $u_2 \in [\psi_1, Z_4]$, $u_3 \in [\psi_4, Z_1]$, and $u_4 \in [\psi_1, Z_1]\backslash([\psi_1, Z_4] \cup [\psi_4, Z_1])$ for $\lambda \in [\lambda^*, A_{m+1})$. Hence (1.1) has at least four positive solutions for $\lambda \in [\lambda^*, A_{m+1})$. This completes the proof.

Proof of Corollary 1.3: We note that the proof of Corollary 1.3 is an immediate consequence of the proof of Theorem 1.1 and Theorem 1.2.

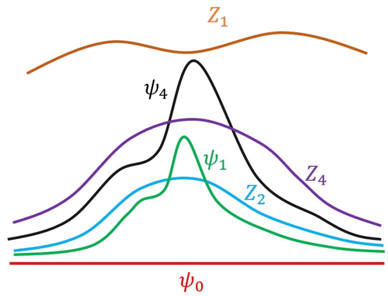


Fig. 8: Subsolutions ψ_0, ψ_1, ψ_4 and supersolutions Z_1, Z_2, Z_4.

4 Example

In this section, we provide an example for which Theorems 1.1 - 1.2 and Corollary 1.3 hold. Consider

$$\begin{cases} -\Delta u = \lambda f(u) = \lambda[mu + g(u)]; \ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0; \ \partial\Omega, \end{cases} \tag{4.1}$$

where

$$g(s) = g_{\alpha,k}(s) = \begin{cases} e^{\frac{cs}{c+s}} - 1; s \leq k \\ [e^{\frac{\alpha s}{\alpha+s}} - e^{\frac{\alpha k}{\alpha+k}}] + [e^{\frac{ck}{c+k}} - 1]; s > k. \end{cases}$$

Here $c > 2$ is a fixed number, $m \geq 0$, $\alpha > 0$ and $k > 0$ are parameters. It is easy to verify that (H_1) is satisfied.

We first consider the case when $m = 0$. Since $\frac{k}{f(k)} = \frac{k}{e^{\frac{ck}{c+k}} - 1} \longrightarrow \infty$ as $k \longrightarrow \infty$, there exists $k_0 > 0$ (independent of α) such that for $k > k_0$

$$\frac{k}{f(k)} > \max\{A_1, 1\}. \max\{\|v\|_\infty, \|w\|_\infty\}. \tag{4.2}$$

Let $k > k_0$. Next, for $\alpha > k$, since $\frac{\alpha}{f(\alpha)} = \frac{\alpha}{[e^{\frac{\alpha}{2}} - e^{\frac{\alpha k}{\alpha+k}}] + [e^{\frac{ck}{c+k}} - 1]} \longrightarrow 0$ as $\alpha \longrightarrow \infty$, there exists $\alpha_0(k)(> k)$ such that for $\alpha > \alpha_0(k)$

$$A_1 > \frac{\alpha}{f(\alpha)} \cdot \frac{2NC_N}{R^2}. \tag{4.3}$$

Thus, choosing $a_1 = a_2 = k$, $b_1 = b_2 = \alpha$, by (4.2), (4.3), it is easy to see that both (H_2) and (H_3) are also satisfied when $k > k_0$ and $\alpha > \alpha_0(k)$. Hence Theorems 1.1 - 1.2 and Corollary 1.3 hold for this example when $k > k_0$ and $\alpha > \alpha_0(k)$.

By continuity, it follows that Theorems 1.1-1.2 and Corollary 1.3 also hold for this example when $k > k_0$, $\alpha > \alpha_0(k)$ and $m \approx 0$.

5 Approximation to the exact bifurcation diagrams for (4.1) when $\Omega = (0, 1)$

In this case, we note that the solutions of (4.1) can be completely analyzed by the quadrature method discussed in [3]. Here, (4.1) reduces to

$$\begin{cases} -u'' = \lambda f(u); \ (0, 1) \\ -u'(0) + \sqrt{\lambda} u(0) = 0 \\ u'(1) + \sqrt{\lambda} u(1) = 0, \end{cases} \tag{5.1}$$

and the positive solutions to (5.1) are symmetric about $x = \frac{1}{2}$. Namely, the solutions take the shape as in Figure 9.

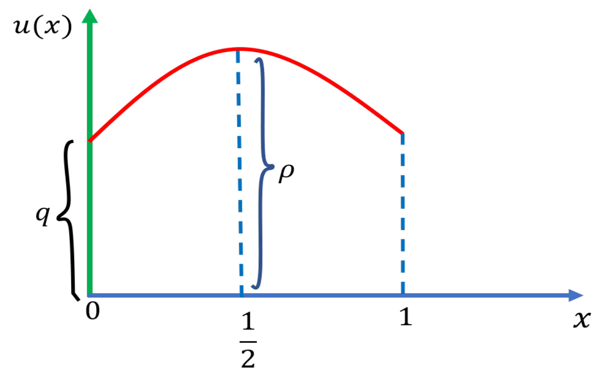


Fig. 9: The shape of the solutions of (5.1).

Further, the exact bifurcation diagrams for positive solutions to (5.1) are described by the equations:

$$\lambda = 2\Big(\int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}}\Big)^2 \tag{5.2}$$

and

$$2[F(\rho) - F(q)] = q^2 \tag{5.3}$$

where, $\rho = u(\frac{1}{2})$, $q = u(0) = u(1)$, and $F(s) = \int_0^s f(t)dt$.

Below we provide some bifurcation diagrams for the example discussed in the previous section via Mathematica computation of (5.2)-(5.3). In fact, we obtain exact Σ–shaped bifurcation curves for certain parameter values.

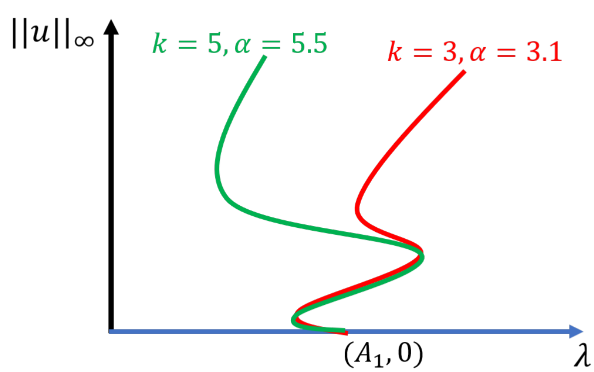


Fig. 10: The local view of the bifurcation diagrams near the bifurcation point $(A_1, 0)$ when m = 0 and $c = 2.5$.

Appendix

Proof of Remark 1.1: First, we show the non-existence of positive solutions for $\lambda \approx 0$. Let u be a positive solution of (1.1). Then by the Green's second identity we obtain:

$$0 = \int_\Omega [\theta_{\lambda,m+1}\Delta u - u\Delta\theta_{\lambda,m+1}]dx$$

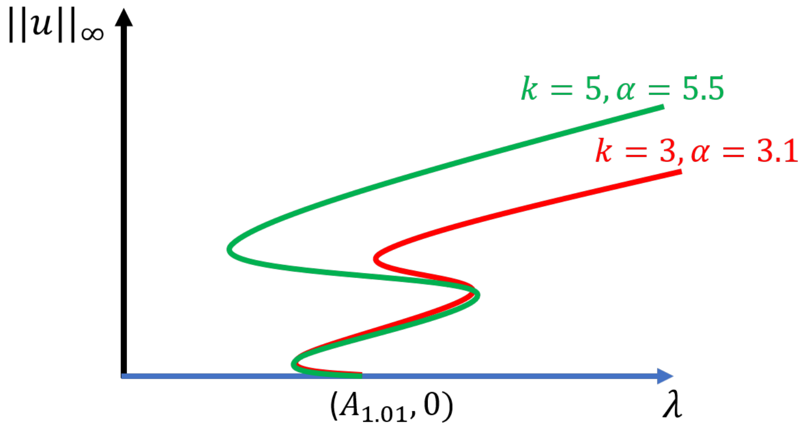


Fig. 11: The local view of the bifurcation diagrams near the bifurcation point $(A_{1.01}, 0)$ when m = 0.01 and $c = 2.5$.

$$\begin{aligned} &= \int_\Omega [-\lambda f(u) + u(\sigma_{\lambda,m+1} + \lambda)(m+1)]\theta_{\lambda,m+1} dx \\ &\geq \int_\Omega [-\lambda M u + u(\sigma_{\lambda,m+1} + \lambda)(m+1)]\theta_{\lambda,m+1} dx \\ &= \int_\Omega \lambda \left\{ \frac{(m+1)\sigma_{\lambda,m+1}}{\lambda} - [M - (m+1)] \right\} u\theta_{\lambda,m+1} dx \end{aligned} \tag{.4}$$

where $M > (m+1)$ is such that $f(s) \leq Ms$ for all $s \in [0, \infty)$. Now for $\lambda < A_{m+1}$, $\sigma_{\lambda,m+1} > 0$, and $\lim_{\lambda \to 0} \frac{\sigma_{\lambda,m+1}}{\lambda} = \infty$ (see [11]). This contradicts (.4) for $\lambda \approx 0$ and hence (1.1) has no positive solution for $\lambda \approx 0$.
Next, when $m > 0$, if u is a positive solution of (1.1), then again by the Green's second identity we obtain:

$$\begin{aligned} 0 &= \int_\Omega [\theta_{\lambda,m} \Delta u - u \Delta \theta_{\lambda,m}] dx \\ &= \int_\Omega [-\lambda f(u) + u(\sigma_{\lambda,m} + \lambda)m]\theta_{\lambda,m} dx \\ &\leq \int_\Omega [-\lambda m u + u(\sigma_{\lambda,m} + \lambda)m]\theta_{\lambda,m} dx \\ &= \int_\Omega m\sigma_{\lambda,m} u\theta_{\lambda,m} dx \end{aligned} \tag{.5}$$

since $f(s) \geq ms$ on $[0, \infty)$. Now if $\lambda > A_m$ then $\sigma_{\lambda,m} < 0$ which contradicts (.5). Hence (1.1) has no positive solution for $\lambda > A_m$.

Conflict of interest: The authors state no conflict of interest.

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