

*S<sup>1</sup>-Actions on Cohomology Complex Projective Spaces  
with Three Components of the Fixed Point Sets*

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§0. Introduction

A closed oriented smooth  $2n$ -manifold  $M$  is called a *cohomology complex projective  $n$ -space* (cohomology  $CP^n$ ), if the integral cohomology ring of  $M$  is isomorphic to that of the complex projective  $n$ -space  $CP^n$ , i.e., if there exists an element  $\alpha \in H^2(M; \mathbb{Z})$  such that

$$(0.1) \quad H^*(M; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1}), \quad \langle \alpha^n, [M] \rangle = 1.$$

Then the conjecture of T. Petrie [4; Intro.] for homotopy complex projective spaces follows immediately from the following statement:

(0.2) *Assume that  $M$  is a cohomology  $CP^n$  with  $\alpha \in H^2(M; \mathbb{Z})$  satisfying (0.1). If  $M$  admits a non-trivial (smooth)  $S^1$ -action, then the total Pontrjagin class of  $M$  is given by*

$$p(M) = (1 + \alpha^2)^{n+1}.$$

K. Wang [6] and T. Yoshida [7] have proved independently the conjecture of T. Petrie for semi-free actions by the following

**THEOREM** ([6; Prop. 2.2-3, Cor. 2.5]). (0.2) is valid, if  $M$  admits an  $S^1$ -action whose fixed point set has two (connected) components.

The purpose of this note is to prove the following

**THEOREM 1.** (0.2) and the conjecture of T. Petrie are valid, if  $M$  admits an  $S^1$ -action whose fixed point set has three components.

We shall prove this theorem by the following

**THEOREM 2.** Let  $M$  be a cohomology  $CP^n$ . Then any effective  $S^1$ -action on  $M$  with three components of the fixed point set is of the linear type.

Here, an  $S^1$ -action on  $M$  is defined to be of the *linear type*, if its normal representations of the fixed point set are of the same type as those of a linear  $S^1$ -action on  $CP^n$  (cf. Definition 1.6).

Since a non-trivial  $S^1$ -action on  $M$  induces an effective  $S^1$ -action on  $M$ , Theorem 1 follows immediately from Theorem 2 and the following theorem; recently A. Hattori [3] has proved it, and our original proof of it in the special case is thus excluded.

**THEOREM** ([3; Prop. 4.15]). (0.2) is valid, if  $M$  admits a non-trivial  $S^1$ -action of the linear type.

We note that there exists an exotic  $S^1$ -action on cohomology  $CP^3$  which is not of the linear type ([4; II, §4]), but (0.2) is valid for  $n=3$  by the results of J. Dejter [2] and T. Yoshida [8].

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### §1. Preliminaries

In this section, we recall some results due to G. Bredon [1], T. Petrie [4] and J. C. Su [5].

Let  $M$  be a cohomology  $CP^n$ , and suppose that  $M$  admits a non-trivial (smooth)  $S^1$ -action and its fixed point set  $F(S^1, M)$  has  $l$  (connected) components  $F_0, F_1, \dots, F_{l-1}$ . Then

(1.1) (cf. [1; VII, Th. 5.1]) each  $S^1$ -submanifold  $F_i$  of  $M$  is a cohomology  $CP^{n_i}$  and

$$n + 1 = \sum_{i=0}^{l-1} (n_i + 1).$$

Let  $\eta$  be the Hopf bundle over  $M$ , i.e., the complex line bundle over  $M$  whose first Chern class is equal to  $\alpha \in H^2(M; \mathbb{Z})$  in (0.1). Then

(1.2) ([4; II, Def. 1.2, Cor. 1.15]) there are distinct integers  $a_0, a_1, \dots, a_{l-1}$ , which are well-defined up to translation, with the following property: An  $S^1$ -action on  $\eta$ , which is a lifting of the given  $S^1$ -action on  $M$ , acts on the fibre  $\eta|_x$  ( $x \in F_i$ ) via the complex representation  $t^{a_i} \in R(S^1)$ .

(1.3) (cf. [1; VII, §5]) For each  $i=0, 1, \dots, l-1$ , let  $v_i$  be the normal bundle of  $F_i$  in  $M$ . Then there are positive integers  $m_{i,u}$  ( $u=1, \dots, n-n_i$ ) with the following property: An  $S^1$ -action on the complex  $(n-n_i)$ -plane bundle  $v_i$ , which is induced by the given  $S^1$ -action on  $M$ , acts on the fibre  $v_i|_x$  ( $x \in F_i$ ) via the complex representation

$$\sum_{u=1}^{n-n_i} t^{m_{i,u}} \in R(S^1).$$

This representation is called the normal representation of  $F_i$  in  $M$ .

For any subgroup  $H$  of  $S^1$ , let  $F(H, M)$  denote the fixed point set of the  $H$ -action on  $M$  which is the restriction of the given  $S^1$ -action on  $M$ .

LEMMA 1.4. *Let  $H$  be a subgroup of  $S^1$  of order  $h$ , and  $Y$  be the component of  $F(H, M)$  containing  $F_i$ . Then*

$$\dim Y = \dim F_i + 2\#\{u: h|m_{i,u}\},$$

where  $\#A$  is the number of elements of a finite set  $A$ .

PROOF. By noticing that  $Y$  is an  $S^1$ -submanifold of  $M$  and by studying the complex dimension of the normal bundle of  $F_i$  in  $Y$ , we see immediately the lemma by the definition of the normal representation. q. e. d.

(1.5) (cf. [1; VII, Th. 5.5]) *The integers in (1.2) and (1.3) satisfy*

$$\prod_{u=1}^{n-n_i} m_{i,u} = \prod_{0 \leq j \leq l-1, j \neq i} |a_j - a_i|^{n_j+1}$$

for  $i=0, \dots, l-1$ .

DEFINITION 1.6. The given  $S^1$ -action on  $M$  is said to be of the *linear type*, if  $|a_j - a_i|$  occurs  $(n_j + 1)$ -times in the integers  $m_{i,u}$  ( $u=1, \dots, n-n_i$ ), i.e., if the normal representation of  $F_i$  in  $M$  is given by

$$\sum_{0 \leq j \leq l-1, j \neq i} (n_j + 1)t^{|a_j - a_i|}$$

for  $i=0, \dots, l-1$ .

(1.7) ([5; Th. 5.1]) *For any prime  $p$  and any positive integer  $r$ , each component  $F_k(p^r)$  ( $k=0, 1, \dots, l(p^r)-1$ ) of  $F(Z_{p^r}, M)$  is  $F_k(p^r) \widetilde{\simeq} CP^{n_k(p^r)}$  and*

$$n + 1 = \sum_{k=0}^{l(p^r)-1} (n_k(p^r) + 1).$$

Here, the notation  $N \widetilde{\simeq} CP^m$  means that  $N$  is a closed oriented (smooth)  $2m$ -manifold and the cohomology ring  $H^*(N; Z_p)$  is given by

$$H^*(N; Z_p) = Z_p[\alpha]/(\alpha^{m+1}), \quad \alpha \in H^2(N; Z_p).$$

COROLLARY 1.8. *In (1.1) and (1.7),*

$$n_k(p^r) + 1 = \sum_{i \in L_k} (n_i + 1),$$

where  $L_k = \{i: i=0, 1, \dots, l-1, F_i \subset F_k(p^r)\}$ .

PROOF. Since  $F_k(p^r)$  is an  $S^1$ -submanifold of  $M$ ,

$$\chi(F_k(p^r)) = \chi(F(S^1, F_k(p^r)))$$

( $\chi$  is the Euler characteristic) by [1; III, Th. 10.9]. Further  $F(S^1, F_k(p^r)) = \bigcup_{i \in L_k} F_i$ . Thus we see the desired equality. q. e. d.

(1.9) ([5; Th. 5.4]) *Two components  $F_i$  and  $F_j$  of  $F(S^1, M)$  are contained in a component of  $F(Z_{p^r}, M)$  if and only if  $|a_j - a_i|$  is a multiple of  $p^r$ .*

(1.10) ([5; Prop. 6.3]) *Let  $H$  be a subgroup of  $S^1$  and  $Y$  be a component of  $F(H, M)$ . If  $Y \cap F(S^1, M) \neq \emptyset$ , then*

$$\dim Y \leq 2(\chi(Y) - 1).$$

**COROLLARY 1.11.** *In (1.10), if  $Y$  contains exactly one component  $F_i$  of  $F(S^1, M)$ , then  $Y = F_i$  and  $\dim Y = 2(\chi(Y) - 1)$ .*

*If the order of  $H$  is  $h$  in addition, then none of the integer  $m_{i,u}$  in (1.3) is a multiple of  $h$ .*

**PROOF.** By the assumption,  $F(S^1, Y) = F_i$  and  $\chi(Y) = \chi(F(S^1, Y))$ . Thus

$$\dim Y \geq \dim F_i = 2(\chi(F_i) - 1) = 2(\chi(Y) - 1) \geq \dim Y$$

by (1.1) and (1.10). Thus  $\dim Y = \dim F_i$  and  $Y = F_i$ .

The last half follows immediately from Lemma 1.4. q. e. d.

## §2. Proof of Theorem 2

In the rest of this note, we assume that the given  $S^1$ -action on  $M$  is effective and that the fixed point set  $F(S^1, M)$  has three components  $F_0, F_1$  and  $F_2$ .

**LEMMA 2.1.** *For the integers  $a_0, a_1$  and  $a_2$  in (1.2), any two of  $|a_0 - a_1|$ ,  $|a_0 - a_2|$  and  $|a_1 - a_2|$  are relatively prime.*

**PROOF.** Assume that  $p^r$  ( $p$ : prime,  $r \geq 1$ ) divides  $|a_0 - a_1|$  and  $|a_0 - a_2|$ . Then some component  $F_0(p^r)$  of  $F(Z_{p^r}, M)$  contains  $F_0, F_1$  and  $F_2$  by (1.9). Hence  $\dim F_0(p^r) = 2n_0(p^r) = 2n = \dim M$  by (1.7), Corollary 1.8 and (1.1). Thus  $F_0(p^r) = M$ , which contradicts the effectivity of the  $S^1$ -action on  $M$ . q. e. d.

**LEMMA 2.2.** *If  $|a_i - a_j|$  ( $i \neq j$ ) is a multiple of  $p^r$  ( $p$ : prime,  $r \geq 1$ ), then the fixed point set  $F(Z_{p^r}, M)$  has two components  $F_0(p^r)$  and  $F_k$ , where*

$$F_0(p^r) \supset F_i \cup F_j, \quad \{i, j, k\} = \{0, 1, 2\}.$$

**PROOF.** By (1.9) and the above proof,  $F(Z_{p^r}, M)$  has components  $F_0(p^r)$  and  $F_1(p^r)$  such that  $F_0(p^r) \supset F_i \cup F_j$  and  $F_1(p^r) \supset F_k$ . Then  $n_0(p^r) = n_i + n_j + 1$  and  $n_1(p^r) = n_k$  by Corollary 1.8, and these equalities imply that

$$F(Z_{p^r}, M) = F_0(p^r) \cup F_1(p^r) \quad \text{and} \quad F_1(p^r) = F_k$$

by (1.1) and (1.7).

q. e. d.

LEMMA 2.3. *Let*

$$\sum_{u=1}^{n-n_0} t^{m_{0,u}} \in R(S^1) \quad (n - n_0 = n_1 + n_2 + 2)$$

be the normal representation of  $F_0$  in  $M$  (cf. (1.3)).

(i) *If  $p^r | a_i - a_0 |$  ( $p$ : prime,  $r \geq 1$ ), then  $\{u: p^r | m_{0,u}\}$  consists of exactly  $n_i + 1$  elements, where  $i = 1$  or  $2$ .*

(ii) *If  $p^r | a_1 - a_0 |$  and  $q^s | a_2 - a_0 |$  ( $p, q$ : prime;  $r, s \geq 1$ ), then  $\{u: p^r | m_{0,u}\} \cap \{u: q^s | m_{0,u}\} = \emptyset$ .*

PROOF. (i) By Lemma 2.2,  $F(Z_{p^r}, M)$  has a component  $F_0(p^r)$  such that  $F_0(p^r) \supset F_0 \cup F_i$  and  $F_0(p^r) \cap F_j = \emptyset$  ( $\{i, j\} = \{1, 2\}$ ). Then we see (i) by Lemma 1.4 and Corollary 1.8.

(ii) By Lemma 2.2, there are components  $F_0(p^r)$  and  $F_0(q^s)$  of  $F(Z_{p^r}, M)$  and  $F(Z_{q^s}, M)$ , respectively, such that

$$F_0(p^r) \supset F_0 \cup F_1, F_0(p^r) \cap F_2 = \emptyset; F_0(q^s) \supset F_0 \cup F_2, F_0(q^s) \cap F_1 = \emptyset.$$

Then,  $F(Z_{p^r q^s}, M) = F(Z_{p^r}, M) \cap F(Z_{q^s}, M)$  has a component  $Y$  with  $Y \supset F_0$ ,  $Y \cap (F_1 \cup F_2) = \emptyset$ . Thus, since  $(p, q) = 1$  by Lemma 2.1, (ii) follows immediately from the latter half by Corollary 1.11. q. e. d.

Now, we can prove Theorem 2 in §0 by Definition 1.6 and the following

LEMMA 2.4. *The normal representation in the above lemma is given by*

$$(n_1 + 1)t^{|a_1 - a_0|} + (n_2 + 1)t^{|a_2 - a_0|}.$$

PROOF. It is sufficient to prove the lemma by assuming  $|a_1 - a_0| \geq |a_2 - a_0|$ .

Case I:  $|a_1 - a_0| = 1$  or  $2$ , and  $|a_2 - a_0| = 1$ .

For this case, the lemma follows immediately from (1.5) and (i) of the above lemma.

Case II:  $|a_1 - a_0| \geq |a_2 - a_0| \geq 2$ .

By the above lemma, we see easily that

$$\{u: p_1^{r_1} | m_{0,u}\} = \{u: p_2^{r_2} | m_{0,u}\}$$

if  $p_k^{r_k} | a_i - a_0 |$  ( $p_k$ : prime,  $r_k \geq 1$ ) for  $k = 1, 2$ , where  $i = 1$  or  $2$ . Thus the lemma follows from (1.5).

Case III:  $|a_1 - a_0| \geq 3$  and  $|a_2 - a_0| = 1$ .

For this case,  $|a_2 - a_1| \geq 2$  and we see by Case II that

(2.5) the normal representation of  $F_1$  in  $M$  is given by

$$(n_0 + 1)t^{|a_0 - a_1|} + (n_2 + 1)t^{|a_2 - a_1|}.$$

Now, suppose  $p^r q^s | |a_0 - a_1|$ , where  $p, q$  are distinct primes and  $r, s \geq 1$ , and let  $Y$  be the component of  $F(Z_{p^r q^s}, M)$  containing  $F_1$ . Then Lemmas 1.4, 2.1 and (2.5) imply that

$$(2.6) \quad \dim Y = \dim F_1 + 2(n_0 + 1) = 2(n_0 + n_1 + 1).$$

Furthermore,

$$(2.7) \quad Y \cap F_2 = \emptyset \quad \text{and} \quad Y \supset F_0 \cup F_1.$$

In fact, since  $F(Z_{p^r q^s}, M) = F(Z_{p^r}, M) \cap F(Z_{q^s}, M)$ , Lemma 2.2 implies  $Y \cap F_2 = \emptyset$ . By (2.6),  $\dim Y > \dim F_1$  and  $Y \cong F_1$ . Thus we see  $Y \supset F_0$  by the first half of Corollary 1.11.

Therefore, by (2.6–7) and Lemma 1.4, we see that

$$2(n_0 + n_1 + 1) = \dim Y = \dim F_0 + 2\#\{u: p^r q^s | m_{0,u}\},$$

which implies  $\#\{u: p^r q^s | m_{0,u}\} = n_1 + 1$ . This and Lemma 2.3 (i) imply that

$$(2.8) \quad \{u: p^r | m_{0,u}\} = \{u: q^s | m_{0,u}\} \quad \text{if} \quad p^r q^s | |a_0 - a_1|.$$

Thus the lemma follows from (2.8) and (1.5).

q. e. d.

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