Safety Factor and Inverse Reliability Measures

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Abstract:

Probability performance measure and probability sufficiency factor are two inverse reliability measures that have gained importance as alternate measures of safety. Inverse measures have several advantages, including improving accuracy in response surface approximations, computational efficiency, and allowing easy estimates of resources needed for achieving the target safety levels. This paper establishes the relationship between the two inverse measures, and describes their advantages compared to the direct measures of probability and reliability index. Methods to compute the inverse measures are also described. Reliability based design optimization with inverse measure is demonstrated with a beam design example.

1
Introduction

Traditionally structural safety was defined in terms of safety factor to compensate for uncertainties in loading and material properties and for inaccuracies in geometry and theory. Safety factors permit design using inexpensive deterministic optimization. In addition, it is relatively easy to estimate the change in structural weight needed to satisfy a target safety factor requirement.

Probabilistic approaches provide more accurate measures of uncertainty incorporating available uncertainty data. Structural safety is measured in terms of probability of failure to satisfy a performance criterion. The probability of failure is often expressed in terms of reliability index, which is the ratio of the mean to the standard deviation of the safety margin distribution.

Optimization for safety using probabilistic approaches called Reliability Based Design Optimization (RBDO) is computationally expensive. In addition, the difference between the probability of failure or the reliability index and their target values does not provide the designer with easy estimates of the necessary resources to achieve these target values. Finally, when the probability of failure is low, probabilities calculated through Monte Carlo Simulation (MCS) are computed as zero. This zero probability of failure does not provide useful gradient information for optimization.

One safety measure that combines the advantages of safety factors and probability of failure was proposed by Birger (1970). It is an inverse measure that quantifies the level of safety in terms of the change in structural response needed to meet the target probability of failure. More recently, several researchers developed inverse reliability methods (Tu et al., 1999, Lee et al., 2002, Qu and Haftka., 2003) that are closely related to the Birger measure.

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Tu et al. (1999) developed a method called Probabilistic Measure Approach (PMA). This method involves computing an inverse measure they call the probability performance measure (PPM). They showed that this method allows faster RBDO compared to the traditional use of the Reliability Index Approach (RIA). Qu and Haftka (2003) developed a similar inverse measure, which they called the probability sufficiency factor (PSF) for use with response surface approximation (RSA) and Monte Carlo Simulation (MCS). They showed that PSF leads to more accurate RSA, more effective RBDO and permits estimating the resources needed to meet the target reliability.

The objectives of this work are to establish the relationship between the PSF and PPM, to discuss methods available for calculating these inverse measures, and to explore the advantages of using inverse measures. Section 2 describes inverse reliability measures. Calculation of inverse measures by MCS is discussed in section 3. Section 4 describes calculation of inverse measures using moment-based techniques, followed by discussion of using inverse measures in RBDO in section 5. Section 6 demonstrates the concepts with the help of a beam design example, and section 7 provides concluding remarks.

2
Inverse Reliability Measures

The safety factor, Q, is defined as the ratio of the capacity of the system g_c (e.g., strength allowable) to the response g_r. To account for uncertainties, the design safety factor is greater than one (e.g., 1.5 in aeronautical applications). To address the probabilistic interpretation of safety factor, Birger (1970) proposed to consider the probability distribution function F_Q of the safety factor:

$$F_Q(q) = Prob(\frac{g_c}{g_r} \pounds q) \tag{1}$$

Note that unlike the deterministic safety factor, which is normally calculated for the mean value of the random variables, $\frac{g_c}{g_r}$ in (1), is a random function. Given a target probability, $P_{ft\,arget}$, Birger suggested calculating a safety factor q^* (which we call here the Birger safety factor) by solving

$$F_Q(q) = Prob(\frac{g_c}{g_r} \pounds q^*) = Prob(Q \pounds q^*) = P_{ftarget} \tag{2}$$

That is, the Birger safety factor is found by setting the cumulative density function (CDF) of the safety factor equal to the target probability and solving for the safety factor. This is illustrated in Figure 1.

Qu and Haftka (2003) developed a similar measure to the Birger safety factor, calling it first the probabilistic safety factor and then the probabilistic sufficiency factor (PSF). They found the reference to Birger's work in Elishakoff (2001) excellent review of safety factors and their relations to probabilities. It is desirable to avoid the term safety factor for this entity because the common use of the term safety factor is mostly deterministic and independent of the target safety level. Therefore, while

noting the identity of the Birger safety factor and the probabilistic sufficiency factor, we will use the latter term in the following.

Failure happens when the actual safety factor Q is less than 1.The basic design condition that the probability of failure should be smaller than the target probability for a safe design may then be written as:

$$P_f = Prob(Q \pounds 1) = F_Q(1) \pounds P_{ft\,arget} \qquad (3)$$

Using inverse transformation (3) can be expressed as:

$$1 \pounds F_Q^{-1}(P_{ftarget}) = q^* \qquad (4)$$

The use of the inverse transformation accounts for calling the probability sufficiency factor an inverse measure.

The PSF concept is illustrated in Figure 1. $P_{ftarget}$, the design requirement is known and the corresponding area in the CDF of the safety factor is the shaded region in Figure 1. The upper bound of the abscissa q^* is the value of PSF. The region to the
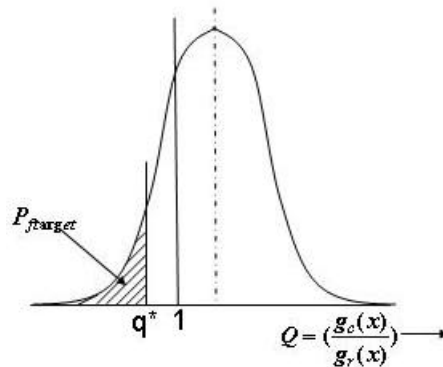

Figure 1: Schematic distribution of safety factor Q. PSF is the value of the safety factor corresponding to the target probability of failure.

left of the vertical line $Q = 1$ represent failure. To satisfy the basic design condition, q^* should be larger than or equal to 1. In order to achieve this, it is possible to either increase g_c or decrease g_r. The PSF, q^* represents the factor that has to multiply the response g_r or divide the capacity g_c so that the safety factor be raised to 1.

For example, a PSF of 0.8 means that g_r has to be multiplied by 0.8 or g_c be divided by 0.8 so that the safety factor ratio increases to one. In other words, it means that g_r has to be decreased by 20 %(1-0.8) or g_c has to be increased by 25% ((1/0.8)-1) in order to achieve the target failure probability.

PSF is useful in estimating the resources to achieve the required target probability of failure. For example, in a stress dominated design, if the target probability of failure is 10^{-5} and a current design yields a probability of failure of 10^{-3}, one cannot easily estimate the change in the weight required to achieve the target failure probability. Instead, if the failure probability corresponds to a PSF of 0.8, this

supplies the designer with an estimate that the weight of the overstressed component has to be increased by about 20% to meet the target.

In probabilistic approaches, instead of safety factor it is customary to use a performance function or a limit state function to define failure (or success) of a system. For example, the limit state function can be expressed as:

$$G(x) = g_c(x) - g_r(x) \tag{5a}$$

In terms of safety factor, limit state function is:

$$G(x) = Q - 1 \tag{5b}$$

Failure happens when $G(x)$ is less than zero, so the probability of failure P_f is:

$$P_f = Prob(G(x) \le 0) \tag{6}$$

Using (6), (3) can be rewritten as:

$$P_f = Prob(G(x) \le 0) = F_G(0) \le P_{ftarget} \tag{7}$$

Inverse transformation allows us to write (7) as,

$$0 \le F_G^{-1}(P_{ftarget}) = g* \tag{8}$$

Here, $g*$ is the Probabilistic Performance Measure (PPM, Tu et al, 1999). PPM can be defined as the solution to (7). Figure 2 illustrates the concept of PPM.
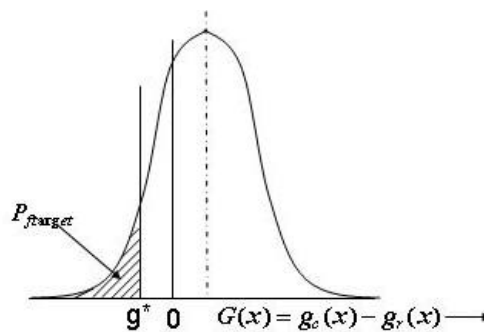

Figure 2: Schematic distribution of limit state function. PPM is the value of the limit state function corresponding to the target probability of failure.

The shaded area corresponds to target failure probability. The area to the left of the line $G = 0$ indicates failure. $g*$ is the factor that has to be subtracted from (5a) in order to make the vertical line at $g*$ move to or further to the right of $G = 0$ line and hence facilitating a safe design.

For example, a PPM of -0.8 means that the design is not safe enough, and -0.8 has to be subtracted from $G(x)$ in order to achieve the target probability of failure. A PPM value of 0.3 means that it is an over qualified design and we have a safety margin of 0.3 to reduce the cost function while meeting the target failure probability.

Considering $g*$ as the solution for (7), it can be rewritten in terms of safety factor as:

$$Prob(G(x) = Q - 1 \pounds g^*) = P_{ftarget} \qquad (9)$$

Comparing (4), (8) and (9), we can observe a relationship between q^* and g^*. PSF (q^*) is related to PPM (g^*) as:

$$q^* = g^* + 1 \qquad (10)$$

This simple relationship between PPM and PSF shows that they are closely related. Both are inverse measures, and the difference is only in the way the limit state function is written. If the limit state function is expressed as the difference between capacity and response as in (5a), failure probability formulation allows us to define PPM. Alternately, if the limit state function is expressed in terms of safety factor as in (5b), corresponding failure probability formulation allows us to define PSF.

3
Inverse Measure Calculation by MCS

Conceptually, the simplest approach to evaluate PSF or PPM is by Monte Carlo simulation (MCS), which involves generation of random sample points according to the statistical distribution of the variables. The sample points which violate the performance criteria are considered failed. Figure 3 illustrates the concept
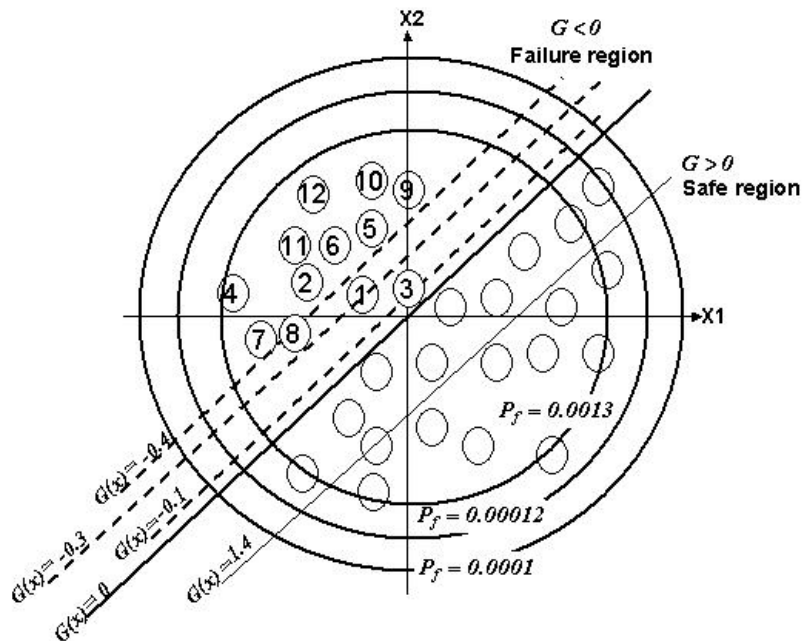

Figure 3: Illustration of the calculation of PSF with Monte Carlo Simulation for linear performance function.

of MCS. A two variable problem with linear limit state function is considered. The straight lines are the contour lines of the limit state function and sample points generated by MCS are represented by small circles, with the numbered circles representing failed samples. The limit value of the limit state function divides the distribution space to safe region and failure region. The dashed lines represent failed conditions and the continuous lines represent safe conditions.

The failure probability is estimated as the ratio of number of samples failed to the total number of samples

$$P_f \approx \frac{num(G(\hat{x}) \leq 0)}{N} \qquad (11)$$

where, $\hat{x}$ is the randomly chosen sample point, $num(G(\hat{x}) \leq 0)$ denotes the number of trials for which $(G(\hat{x}) \leq 0)$ and N is the total number of trials. For example, in Fig. 3, the number of sample points that fall in the failure region above the $G = 0$ curve is 12. Considering that the total number of samples is 100,000 the failure probability is estimated at $1.2*10^{-4}$. For a fixed number of samples, the accuracy of MCS deteriorates with the decrease in failure probability. For example, with only 12 failure points out of the 100,000 samples, the accuracy of the estimate of the probability is not high. In fact, the standard deviation of the estimate is 0.35×10^{-5}, more than a quarter of the estimate. When the probability of failure is smaller than one over the number of sample points, its calculated value by MCS is likely to be zero.

PSF is estimated by MCS as the nth smallest safety factor among the N sampled safety factors. For example, considering the example illustrated in Figure 3, if the target failure probability is 10^{-4}, to satisfy the target probability of failure, no more than 10 samples out of the 100,000 should fail. With the given $P_{ftarget}$, 10 samples are allowed to fail, so the focus is on the two extra samples that failed. PSF is equal to the value of the highest safety factor among the n (in this case, it is 10) lowest safety factors. The numbered small circles are the sample points that failed. Of these, three sample points with the highest safety factor are marked with their corresponding limit state curves. The tenth smallest safety factor corresponds to the sample numbered 8 and has a limit state value of -0.4, which is the value of PPM. Mathematically expressed as:

$$Q_n = n^{th} \min_{i=1}^{N}(Q(x_i)) \qquad (12)$$

where, n^{th} min is the nth smallest safety factor. So, the calculation of PSF requires only sorting the lowest safety factors in the MCS sample.

It is observed from Figure 1 and Figure 3 that the probability of failure changes by several orders of magnitude but the PSF varies by less than one order of magnitude. The accuracy of PSF is maintained in regions where the target failure probability is very low.

For complex problems, response surface approximations can be used to reduce the computational cost. The noisy response produced by MCS can be filtered out by fitting design response surface approximations to probability of failure. Accurate values of probability of failure requires higher order design response surface. It is difficult to construct highly accurate design response surface because of the huge variation of failure probability. To overcome these difficulties, Qu and Haftka (2003) discusses the usage of PSF to improve the accuracy of design response surface. It is shown that design response surface based on PSF has higher accuracy and accelerates the convergence of RBDO. Since PSF exhibits very less variation when compared to P_f and reliability index, it is easier to approximate PSF rather than reliability index or P_f.

4
Inverse Measure Calculation by Moment Based Methods

Moment based methods provide for less expensive calculation of the probability of failure compared to MCS, though they are limited to single failure mode. These methods require a search for the most probable point (MPP) on the failure surface in the standard normal space. First Order Reliability Method (FORM) is the most widely used moment based technique. FORM is based on the idea of linear approximation of the limit state function and is accurate as long as the curvature of the limit state function is not too high. Second Order Reliability Method (SORM) approximates the effect of the curvature of the limit state function. All the random variables are to be transformed to the standard normal variables with zero mean and unit variance.

When the limit state has a significant curvature, linear approximation for the limit state becomes less accurate. Methods that deal with non-linearity of the limit state function are termed as 'second order' methods. The second order methods approximate the limit state function by second order Taylor series expansion.

Moment based methods are employed to calculate reliability index, which is denoted by $\boldsymbol{b}$ and related to probability of failure as:

$$P_f = \boldsymbol{F}(-\boldsymbol{b}) \tag{13}$$

where $\boldsymbol{F}$ is the standard normal cumulative distribution function. Respective target values of $\boldsymbol{b}$ and failure probability are also related in the same manner. $\boldsymbol{b}$ can be calculated using standard reliability analysis. In the standard normal space, the point on the first order limit state function at which the distance from the origin is minimum is the MPP and its distance from the origin is the reliability index.

Figure 4 illustrates the concept of reliability index and MPP search for a two variable case in the standard normal space. In reliability analysis, concerns are first
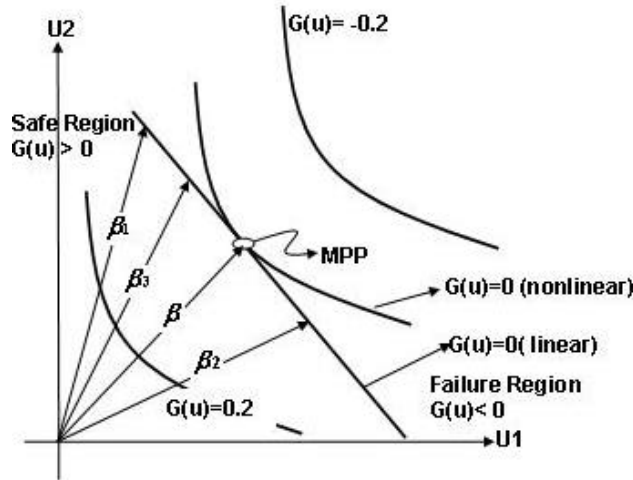

Figure 4: Reliability analysis and MPP

focused on the $G(u) = 0$ curve. Next, among the various $\boldsymbol{b}$ values possible (denoted by $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$) the minimum $\boldsymbol{b}$ is sought. The corresponding point is the MPP. This process can be mathematically expressed as:

$$\text{Minimize} \quad \boldsymbol{b} = \sqrt{u^T u}$$
$$\text{Subject to:} \quad \boldsymbol{G(u) = 0} \qquad\qquad (14)$$

Inverse reliability measures can also be computed through moment based methods. Figure 5 illustrates the concept of inverse reliability analysis and MPP search. The circles represent the $\boldsymbol{b}$ curves with the target $\boldsymbol{b}$ curve represented by dashed circle.
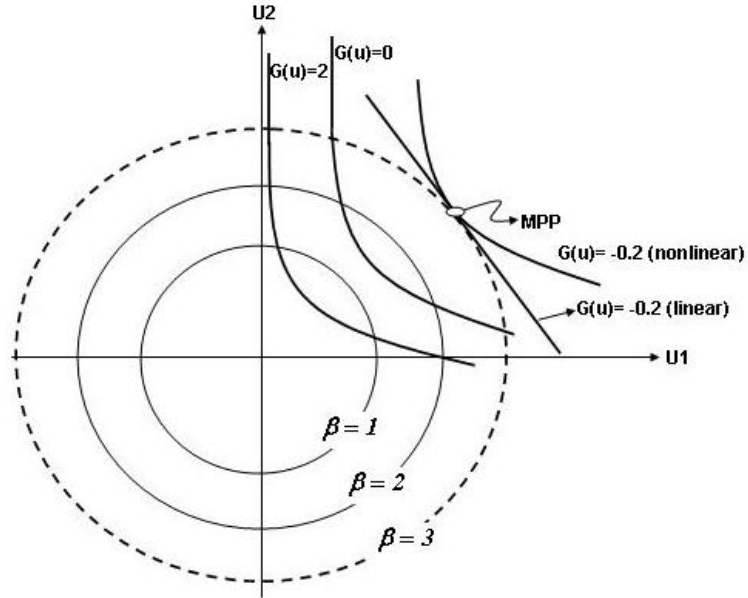

Figure 5: Inverse reliability analysis and MPP for target probability of failure of 0.00135 ($\boldsymbol{b}$ =3).

Here, among the different values of limit state functions that pass through the $\boldsymbol{b}_{t\,\mathrm{arg}et}$ curve, the one with minimum value is sought. The value of this minimal limit state function is the PPM. The point on the target circle with the minimal limit state function is searched. In this case, the value of the minimal limit state function or the PPM is -0.2. This process can be expressed as:

$$\text{Minimize} \quad \boldsymbol{G(u)}$$
$$\text{Subjected to:} \quad |u| = \sqrt{u^T u} = \boldsymbol{b}_{t\,\mathrm{arg}et} \qquad (15)$$

Reliability analysis and inverse reliability analysis search for different MPP. In reliability analysis the MPP is on the $\boldsymbol{G(u) = 0}$ failure surface. In inverse reliability analysis, the MPP search is the $\boldsymbol{b}_{t\,\mathrm{arg}et}$ curve.

5
RBDO with Inverse Measures:

Generally, RBDO problems are formulated as:

$$\text{Minimize: Cost function (design variables)}$$
$$\text{Subject to: probabilistic constraint} \qquad (16)$$

The probabilistic constraint can be prescribed by several methods like the Reliability Index Approach (RIA), the Performance Measure Approach (PMA), the Probability Sufficiency Factor approach (PSF), see. Table1.

Table1: Different approaches to prescribe probabilistic constraint

Method	RIA	PMA	PSF
Probabilistic Constraint	$\boldsymbol{b} \geq \boldsymbol{b}_{t\arg et}$	$g* \geq 0$	$q* \geq 1$
Quantity to be computed	Reliability Index ($\boldsymbol{b}$)	PPM ($g*$)	PSF ($q*$)

To date, most researchers have used RIA to prescribe the probabilistic constraint. The inverse reliability measures led to the usage of PMA (Tu et al. 1999) and PSF (Qu and Haftka, 2003). RIA is efficient in solving violated probabilistic constraints but yields singularity in some cases. Whereas, PMA is very efficient for inactive constraints and the convergence rate is higher compared to RIA, hence making it a computationally attractive method.

In RIA, $\boldsymbol{b}$ can be computed as the product of reliability analysis as discussed in the previous section. In PMA, PPM can be computed through inverse reliability analysis. Using (11) PSF can be computed from PPM, for PSF approach.

6
Beam Design Example:

The cantilever beam shown in Figure 6 is taken from Wu et al. (2001). The objective of the design is to minimize the weight or equivalently the cross sectional area: $\boldsymbol{A = wt}$ subject to two reliability constraints, which require the safety indices for strength and deflection constraints to be larger than three. The first two failure modes are expressed as:

Yielding:
$$g_s = R - \boldsymbol{s} = R - \left(\frac{600}{wt^2} Y + \frac{600}{w^2 t} X \right) \tag{17}$$

Tip Displacement:
$$g_d = D_O - D = D_O - \frac{4L^3}{Ewt} \sqrt{\left(\left(\frac{Y}{t^2} \right)^2 + \left(\frac{X}{w^2} \right)^2 \right)} \tag{18}$$

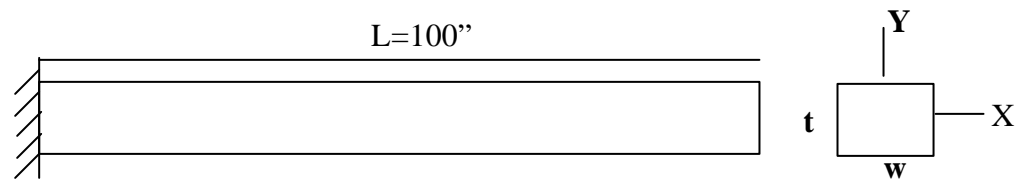where R is the yield strength, X and Y are the horizontal and vertical loads and w and

Figure 6: Cantilever beam subjected to horizontal and vertical random loads

t are the design parameters. L is the length and E is the elastic modulus. R, X, Y and E are random in nature and are defined in table 2.

Table 2. Random variables

Random Variables	X	Y	R	E
Distribution	Normal (500,100)lb	Normal (1000,100)lb	Normal (40000,2000) psi	Normal (29E6,1.45E6) psi

The relation between the stresses, displacement and weight for this problem is presented to demonstrate the utility of PSF in estimating the required resources to achieve a safe design. Considering a design with a PSF of $q*$ which is lesser than one with the dimensions of $A_0 = w_0 t_0$, the structure can be made safer by scaling both w and t by a factor c. This will change the stress and displacement expressed in (17) and (18) by a factor of c^3 and the area by a factor of c^2. Since PSF is inversely proportional to the most critical stress and displacement, the relation between PSF and area can be expressed as:

$$PSF = q* \left(\frac{A}{A_0} \right)^{1.5} \tag{19}$$

(19) indicates that a one percent increase in area will increase the PSF by 1.5 percent. Thus, for example, considering a design with a PSF of 0.97, the safety factor deficiency is 3% and the structure can be made safer with a weight increase of less than two percent.

The design with strength reliability constraint is solved first followed by design for system reliability constraint. The results for the strength constraint are presented in Table 3. The yield strength case has a linear limit state function and FORM gives reasonably accurate results for this case. The MCS is performed with 100,000 samples. The standard deviation in the failure probability calculated by MCS can be estimated as:

$$\sigma_p = \sqrt{\frac{P_f(1 - P_f)}{M}} \tag{20}$$

In this case, the failure probability of 0.0013 calculated from 100,000 samples has a standard deviation of 1.1394E-4.

Table 3. Comparison of optimum design for strength constraint

Minimize objective function A such that $\beta >= 3$

Method		Reliability Analysis FORM	Inverse Reliability Analysis		Exact Optimum (Wu et al, 2001)
			FORM	MCS (Qu and Haftka.,2003)	
Optima	w	2.4460	2.4460	2.4526	2.4484
	t	3.8922	3.8920	3.8840	3.8884
Objective Function		9.5202	9.5202	9.5367	9.5204
Safety Index		3.00	3.00	3.0162	3.00
Failure Probability		0.00135	0.00135	0.0013	0.00135

In an attempt to verify the relation in (10) numerically, inverse reliability analysis is conducted by adapting FORM for the optimal values of design variables obtained from inverse reliability analysis using MCS. The results are presented in table 4.

Table 4. Comparison of inverse measures

Inverse Reliability Analysis with $w = 2.4526$ and $t = 3.8840$

Method	FORM	MCS(10E7 samples)
Failure Probability	0.001238	0.001241
Inverse Measure	PPM: 0.00258	PSF: 1.002619

It is observed from table 4 that the relationship between the two inverse measures expressed in (10) is numerically verified. The results of inverse reliability analysis for reliability design with system reliability constraint by MCS with 100,000 samples are presented in table 5. The allowable deflection is chosen to be 2.5" in order to have competing constraints (Wu et al., 2001).

Table 5. Design for System Reliability by MCS

Minimize objective function A such that $b >= 3$

Optima	Objective Function	Failure Probability	Safety Index	PSF
$w = 2.6881$, $t = 3.500$	9.4084	0.00314	2.7328	0.9733

In this case, the safety factor deficiency is 2.67% and the structure can be made safe by scaling the area by a factor of 1.0182 according to (19) and the design variables w and t by 1.0091 (i.e., $1.0182^{0.5}$). Hence the final scaled design has dimensions of $w = 2.7123$ and $t = 3.5315$. The failure probability of the scaled design is 0.001302 with a PSF of 1.0011 evaluated by MCS with 1,000,000 samples.

7
Concluding Remarks

The relationship between two inverse safety measures, PPM and PSF is established. The computation of inverse measure by Monte Carlo simulation and moment based techniques were discussed. Usage of inverse measures in reliability based design optimization (RBDO) accelerates convergence. They can be employed to increase the accuracy of design response surface. The accuracy of inverse measures are maintained even when the failure probability is very low. Moreover, inverse measures can be employed to estimate the additional cost change to achieve the target reliability. These features of inverse measure make it a valuable resource in RBDO. A simple beam example was used to demonstrate some of the concepts. The proposed paper will have additional comparisons, including the use of second order reliability method (SORM).

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