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Sample path deviations of the Wiener and the Ornstein–Uhlenbeck process from its bridges

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Abstract. We study sample path deviations of the Wiener process from three different representations of its bridge: anticipative version, integral representation and space–time transform. Although these representations of the Wiener bridge are equal in law, their sample path behavior is quite different. Our results nicely demonstrate this fact. We calculate and compare the expected absolute, quadratic and conditional quadratic path deviations of the different representations of the Wiener bridge from the original Wiener process. It is further shown that the presented qualitative behavior of sample path deviations is not restricted only to the Wiener process and its bridges. Sample path deviations of the Ornstein–Uhlenbeck process from its bridge versions are also considered and we give some quantitative answers also in this case.

1 Introduction

Let $(W_t)_{t\geq 0}$ be a standard one-dimensional Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where the filtration $(\mathcal{F}_t)_{t\geq 0}$ is the usual augmentation of the natural filtration of the Wiener process W [see, e.g., Karatzas and Shreve (1991)]. We consider the following versions of the Wiener bridge from a to b over the time-interval $[0, T]$, where $a, b \in \mathbb{R}$ [see, e.g., Karatzas and Shreve (1991, Section 5.6.B)]:

1. *Anticipative version*

$$W_t^{\mathrm{av}} = a + (b-a)\frac{t}{T} + \left(W_t - \frac{t}{T}W_T\right), \qquad 0 \leq t \leq T.$$

2. *Integral representation*

$$W_t^{\mathrm{ir}} = \begin{cases} a + (b-a)\dfrac{t}{T} + \displaystyle\int_0^t \frac{T-t}{T-s}\,\mathrm{d}W_s & \text{if } 0 \leq t < T, \\ b & \text{if } t = T. \end{cases}$$

3. *Space–time transform*

$$W_t^{\mathrm{st}} = \begin{cases} a + (b-a)\dfrac{t}{T} + \dfrac{T-t}{T} W_{(tT)/(T-t)} & \text{if } 0 \leq t < T, \\ b & \text{if } t = T. \end{cases}$$

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The attribute *anticipative* indicates that for the definition of W_t^{av} we use the random variable W_T, where the time point T follows the time point t. In the sequel we will use the notation $(W_t^{\mathrm{br}})_{t\in[0,T]}$ if the version of the bridge is not specified. All the bridge versions above are Gauss processes with the same finite-dimensional distributions. This can be easily calculated, since the versions all have mean function $\mathbb{E}(W_t^{\mathrm{br}}) = a + (b-a)\frac{t}{T}$, $0 \le t \le T$, and covariance function

$$\mathrm{Cov}(W_s^{\mathrm{br}}, W_t^{\mathrm{br}}) = s\frac{T-t}{T}, \qquad 0 \le s \le t < T. \tag{1.1}$$

We note that the finite-dimensional distributions of the above Wiener bridge versions coincide with the conditional finite-dimensional distributions of the Wiener process $(a + W_t)_{t\in[0,T]}$ starting in a and conditioned on $\{a + W_T = b\}$; see, for example, Problem 5.6.13 in Karatzas and Shreve (1991) or Chapter IV.4 in Borodin and Salminen (2002). Bridges of Gaussian processes have been generally defined by Gasbarra et al. (2007), while from the Markovian point of view the reader may consult Fitzsimmons et al. (1992), Barczy and Pap (2005), Chaumont and Uribe Bravo (2011), and the more recent Bryc and Wesołowski (2009) which deals with the inhomogeneous case.

It follows from the definitions that all bridge versions have almost sure continuous sample paths. The (left) continuity of the trajectories at $t = T$ is not obvious in the case of the integral representation and space–time transform. But, since the three bridge representations above have the same finite-dimensional distributions and are continuous on $[0, T)$, (left) continuity at $t = T$ of the integral representation and the space–time transform follows from the obvious continuity at $t = T$ of the anticipative representation, similarly to the proof of Proposition 1.10(iv) in Revuz and Yor (2001, p. 21). [The desired continuity also follows by Karatzas and Shreve (1991, Corollary 5.6.10 and Problem 2.9.3).]

Hence, the anticipative version W^{av}, the integral representation W^{ir} and the space–time transform W^{st} induce the same probability measure on $(C[0,T], \mathcal{B}(C[0,T]))$, where $C[0,T]$ is the space of continuous functions from $[0,T]$ into $\mathbb{R}$ and $\mathcal{B}(C[0,T])$ denotes the Borel σ-algebra on $C[0,T]$. All these underline and explain the commonly used definition of a Wiener bridge from a to b over the time-interval $[0,T]$ [see, e.g., Karatzas and Shreve (1991, Definition 5.6.12)], namely, it is any almost surely continuous Gauss process having mean function $a + (b-a)\frac{t}{T}$, $t \in [0,T]$, and covariance function given in (1.1).

Furthermore, according to Section 5.6.B in Karatzas and Shreve (1991) or Example 8.5 in Chapter IV in Ikeda and Watanabe (1981), the above versions of the Wiener bridge are solutions to the linear stochastic differential equation (SDE)

$$\mathrm{d}W_t^{\mathrm{br}} = \frac{b - W_t^{\mathrm{br}}}{T-t}\,\mathrm{d}t + \mathrm{d}W_t, \qquad 0 \le t < T, \text{ with } W_0^{\mathrm{br}} = a. \tag{1.2}$$

By Theorem 5.2.1 in Øksendal (2003), strong uniqueness holds for the SDE (1.2), and $(W_t^{\mathrm{ir}})_{t\in[0,T)}$ is the unique strong solution of this SDE being adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T)}$, whereas $(W_t^{\mathrm{av}})_{t\in[0,T)}$ and $(W_t^{\mathrm{st}})_{t\in[0,T)}$ are only weak solutions to the SDE (1.2). The anticipative representation $(W_t^{\mathrm{av}})_{t\in[0,T)}$ cannot be a strong solution, since its definition formally requires information about W_T, although W_t^{av} and W_T are independent for every $t \in [0, T]$, indeed, $\operatorname{Cov}(W_t^{\mathrm{av}}, W_T) = \operatorname{Cov}(W_t, W_T) - \frac{t}{T}\operatorname{Cov}(W_T, W_T) = 0$ for all $t \in [0, T]$. The space–time transform representation $(W_t^{\mathrm{st}})_{t\in[0,T)}$ is only a weak solution, since it is adapted only to the filtration $(\mathcal{F}_{tT/(T-t)})_{t\in[0,T)}$ and $\mathcal{F}_{tT/(T-t)} \supsetneqq \mathcal{F}_t$, $t \in (0, T)$. We also note that, even though the three bridge versions have the same law on $(C[0, T], \mathcal{B}(C[0, T]))$, their joint laws together with the Wiener process through which they are constructed are different (see Propositions 2.1 and 2.4). Our aim is to elucidate the sample path deviations compared to the original Wiener process $(a + W_t)_{t\in[0,T]}$ starting in a. A motivation for our study is given at the end of this section.

By visual inspection inferred from a moderate number of simulated sample paths, for a typical sample path of the Wiener process the deviations from its anticipative bridge version and its space–time transform are larger than from its integral representation of the bridge; see Figure 1. Note that in general the deviation from the space–time transform bridge version is hard to compare with the other deviations, since $(W_t^{\mathrm{st}})_{t\in[T/2,T)}$ depends on the future part $(W_t)_{t\in[T,\infty)}$ of the Wiener process, which is not visible in the pictures of Figure 1. Our aim is to give quantitative answers to this qualitative behavior observed from simulations and thus to

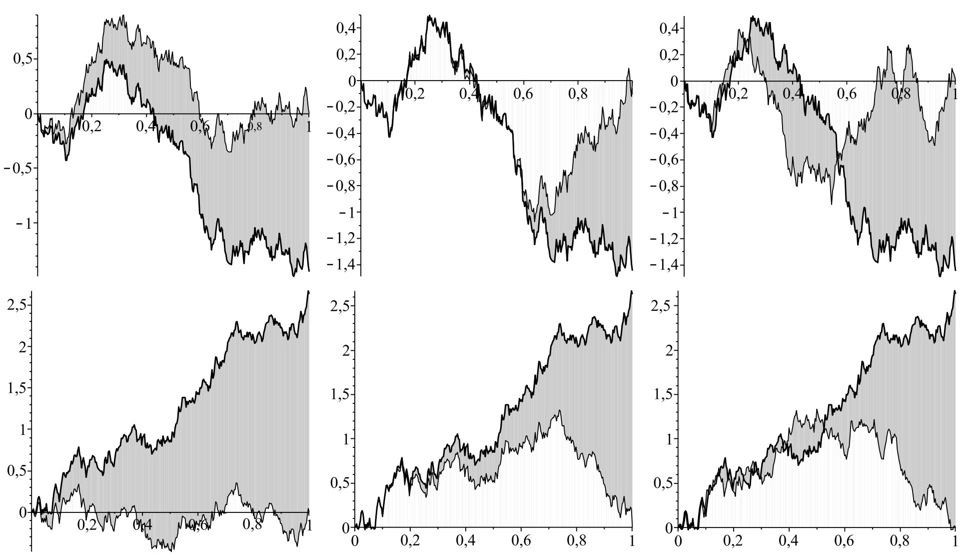

Figure 1 *Two typical sample paths of the Wiener process (rows, thick lines) and its deviations from the anticipative version (left column), the integral representation (middle column) and the space–time transform (right column) of the Wiener bridge from* 0 *to* 0 *over the time-interval* [0, 1].

study the path deviations on $[0, T)$:

$$\begin{aligned} a + W_t - W_t^{\mathrm{av}} &= (a-b)\frac{t}{T} + \frac{t}{T} W_T, \\ a + W_t - W_t^{\mathrm{ir}} &= (a-b)\frac{t}{T} + \int_0^t \frac{t-s}{T-s}\,\mathrm{d}W_s, \\ a + W_t - W_t^{\mathrm{st}} &= (a-b)\frac{t}{T} + \left(W_t - \frac{T-t}{T} W_{(tT)/(T-t)} \right). \end{aligned} \tag{1.3}$$

Note that the dependence of the path deviations in (1.3) upon the starting and endpoint of the bridge (a and b) is only via their difference $a - b$. Hence, without loss of generality we can and will assume $a = 0$ in the sequel.

Simulation studies also show that the above typical behavior is reversed in case the endpoint W_T of the Wiener sample path is close to the prescribed endpoint b of its bridge, namely, for such a sample path of the Wiener process the deviation from its anticipative bridge version is smaller than from its integral representation of the bridge or from its space–time bridge version; see Figure 2. We aim to give quantitative answers to this effect and, thus, in Section 2 we will particularly compare the so-called expected pth order sample path deviations

$$\mathbb{E}\left(\int_0^T |W_t - W_t^{\mathrm{br}}|^p \,\mathrm{d}t \right) = \int_0^T \mathbb{E}(|W_t - W_t^{\mathrm{br}}|^p)\,\mathrm{d}t$$

for $p = 1, 2$ and in case of $p = 2$ we will explicitly calculate the conditional analogue

$$\mathbb{E}\left(\int_0^T (W_t - W_t^{\mathrm{br}})^2 \,\mathrm{d}t \,\Big|\, W_T = d \right) = \int_0^T \mathbb{E}\big((W_t - W_t^{\mathrm{br}})^2 | W_T = d\big)\,\mathrm{d}t$$

for prescribed endpoints $W_T = d$, $d \in \mathbb{R}$ of the original Wiener process. The reason for not considering a general natural number p is that we just want to demonstrate the phenomenon that the bridge versions have different sample path behavior. In the above formulas, integration over the time-interval $[0, T]$ and taking expectations can be interchanged. Indeed, since we have continuous sample paths, we

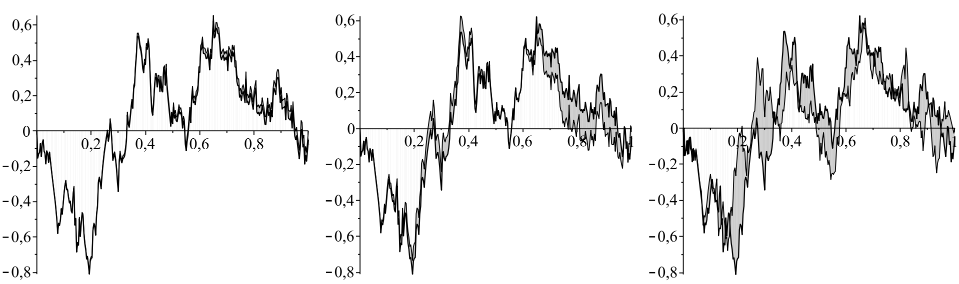

Figure 2 *A sample path of the Wiener process with* $W_1 \approx 0$ (*thick line*) *and its deviations from the anticipative version* (*left*), *the integral representation* (*middle*), *and the space–time transform* (*right*) *of the Wiener bridge from* 0 *to* 0 *over the time-interval* $[0, 1]$.

can consider monotone approximations of the integrals by Riemannian sums with non-negative summands and then apply the monotone convergence theorem for (conditional) expectations. In what follows expected first and second order sample path deviations will be called expected absolute and quadratic path deviations, respectively.

We will further show in Section 3 that the above mentioned qualitative behavior of sample path deviations is not restricted only to the Wiener process and its bridge versions: sample path deviations of the Ornstein–Uhlenbeck process from its bridge versions are also considered. Here we give some quantitative answers, too; see Theorem 3.6.

Our results are to be seen as paradigmatic examples that give rise for future work concerning more broad questions of how certain pathwise constructions of Gaussian or Markovian bridges can differ, although they obey the same law. The reason for concentrating on the Wiener and on the Ornstein–Uhlenbeck process here is the possibility of giving explicit expressions for some quantities (such as second moment) related to the path deviations of different bridge versions to the original process through which they are constructed. In particular, the case of an Ornstein–Uhlenbeck process shows that explicit expressions for path deviations can soon become unwieldily. As a future task, one may also address the question of existence of a bridge version that minimizes the distance to the unconditioned stochastic process in a certain sense.

To further motivate our study, we point out that similar problems were considered by DasGupta (1996), Bharath and Dey (2011), and Balabdaoui and Pitman (2011). Namely, DasGupta (1996, Theorem 1) gave an infinite series representation of the expectations

$$\mathbb{E}\bigg(\int_0^\delta |W_t^{\mathrm{br}} - \mu t - W_t| \, \mathrm{d}t\bigg), \qquad \delta \in (0, 1], \mu \in \mathbb{R},$$

where $(W_t)_{t\in[0,1]}$ and $(W_t^{\mathrm{br}})_{t\in[0,1]}$ denote respectively a standard Wiener process and an independent Wiener bridge with $a = b = 0$ and $T = 1$. For some special values of δ and μ the exact values were also calculated. The motivation of DasGupta for calculating the expectations above is to understand whether distinguishing between a Wiener bridge and an independent Wiener process with possible drift on the basis of observations at discrete times is intrinsically difficult. It turned out that distinguishing one from the other is not an easy task. DasGupta studied the likelihood ratio test for testing the null-hypothesis $H_0 : X_t = W_t^{\mathrm{br}}, t \in [0, 1]$, against the alternative hypothesis $H_1 : X_t = W_t + \mu t, t \in [0, 1]$, for some $\mu \in \mathbb{R}$, based on discrete observations from a process $(X_t)_{t\in[0,1]}$. Recently, the question of distinguishing a Wiener process from a Wiener bridge was also considered by Bharath and Dey (2011). Note that in our setup $(W_t)_{t\in[0,1]}$ and $(W_t^{\mathrm{br}})_{t\in[0,1]}$ are not independent. Hence, our results may be useful to answer the question of distinction in case the Wiener bridge is constructed by the help of the original Wiener process and not an independent copy. One can address the same question for Ornstein–

Uhlenbeck bridges or for more general process bridges. Our calculations in the Ornstein–Uhlenbeck case can be considered as a first step towards the corresponding calculations of Section 2 in DasGupta (1996). Balabdaoui and Pitman (2011) gave a representation of the maximal difference between a Wiener bridge and its (least) concave majorant on the unit interval. As an application, expressions for the distribution, density function and moments of this difference were derived.

The presented results might also be applied to the study of animal movements. Horne et al. (2007) use a two-dimensional Wiener bridge to model the unknown movement of an animal between two consecutively observed positions of the animal. The model is used to investigate questions on the mean occupation frequency $\mathbb{E}(\frac{1}{T}\int_0^T 1_A(X_{1,t}^{\mathrm{br}}, X_{2,t}^{\mathrm{br}})\,\mathrm{d}t)$ in a region $A \in \mathcal{B}(\mathbb{R}^2)$, where $(X_{1,t}^{\mathrm{br}})_{t\in[0,T]}$ and $(X_{2,t}^{\mathrm{br}})_{t\in[0,T]}$ are independent Wiener bridges such that $(X_{1,0}^{\mathrm{br}}, X_{2,0}^{\mathrm{br}})$ and $(X_{1,T}^{\mathrm{br}}, X_{2,T}^{\mathrm{br}})$ are the starting and ending positions of the animal at time 0 and T, respectively. If the region A depends on the original (independent) Wiener processes $(X_{1,t})_{t\in[0,T]}$, $(X_{2,t})_{t\in[0,T]}$, for example, for questions concerning the closeness of the animal's path to the path of a Wiener process, our results show that the expected occupation frequency heavily depends on the chosen version of the bridge.

Finally, we remark that the present paper has an arXiv version [Barczy and Kern (2010b)] containing more technical details. Passages in the present paper, for which the arXiv version contains significantly more information, are specially indicated in the sequel.

2 Path deviation of the Wiener process from its bridges

The full information about the considered path deviations is hidden in the joint distribution of $(W_t^{\mathrm{br}}, W_t)_{t\in[0,T]}$ which is a two-dimensional centered Gauss process. A first indicator for different sample path behavior of the bridge versions is the correlation function $\varrho(W_t^{\mathrm{br}}, W_t)$ of these bridge versions and the original Wiener process.

Proposition 2.1. *For all $t \in (0, T)$, we have*

$$\varrho(W_t^{\mathrm{av}}, W_t) = \varrho(W_t^{\mathrm{st}}, W_t) = \sqrt{\frac{T-t}{T}}$$

and

$$\varrho(W_t^{\mathrm{ir}}, W_t) = \frac{\sqrt{T(T-t)}}{t}\log\frac{T}{T-t}.$$

Proof. By (1.1), we get for every $0 \le t \le T$

$$\mathrm{Var}(W_t^{\mathrm{br}}) = \mathrm{Cov}(W_t^{\mathrm{br}}, W_t^{\mathrm{br}}) = t\frac{T-t}{T}.$$

We easily calculate for every $0 \le t < T$

$$\operatorname{Cov}(W_t^{\mathrm{av}}, W_t) = \operatorname{Cov}(W_t, W_t) - \frac{t}{T}\operatorname{Cov}(W_T, W_t) = t - \frac{t^2}{T} = t\frac{T-t}{T},$$

$$\operatorname{Cov}(W_t^{\mathrm{ir}}, W_t) = \operatorname{Cov}\left(\int_0^t \frac{T-t}{T-s}\,\mathrm{d}W_s, \int_0^t 1\,\mathrm{d}W_s\right) = \int_0^t \frac{T-t}{T-s}\,\mathrm{d}s$$
$$= (T-t)\log\frac{T}{T-t},$$

and

$$\operatorname{Cov}(W_t^{\mathrm{st}}, W_t) = \operatorname{Cov}\left(\frac{T-t}{T} W_{(tT)/(T-t)}, W_t\right) = t\frac{T-t}{T}.$$

Thus, we get for every $0 < t < T$,

$$\varrho(W_t^{\mathrm{av}}, W_t) = \frac{t(T-t)/T}{\sqrt{(t/T)(T-t)\cdot t}} = \sqrt{\frac{T-t}{T}} = \varrho(W_t^{\mathrm{st}}, W_t) \tag{2.1}$$

and

$$\varrho(W_t^{\mathrm{ir}}, W_t) = \frac{(T-t)\log(T/(T-t))}{\sqrt{(t/T)(T-t)\cdot t}} = \frac{\sqrt{T(T-t)}}{t}\log\frac{T}{T-t},$$

concluding the proof. □

Remark 2.2. For all $T \in (0, \infty)$, the function $(0, T) \ni t \mapsto \varrho(W_t^{\mathrm{br}}, W_t)$ is strictly decreasing. For the anticipative version and space–time transform, it is an immediate consequence of (2.1). For the integral representation one can show that the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\varrho(W_t^{\mathrm{ir}}, W_t)$ is negative for $t \in (0, T)$ (see also our arXiv preprint [Barczy and Kern (2010b, Remark 2.2)]). Note also that $\varrho(W_t^{\mathrm{br}}, W_t) \to 1$ as $t \downarrow 0$, and $\varrho(W_t^{\mathrm{br}}, W_t) \to 0$ as $t \uparrow T$. Hence, W_t^{br} and W_t, $t \in (0, T)$, are positively correlated for all bridge versions. Moreover,

$$\frac{\sqrt{T(T-t)}}{t}\log\frac{T}{T-t} > \sqrt{\frac{T-t}{T}}, \qquad t \in (0, T). \tag{2.2}$$

Indeed, (2.2) is equivalent to $-\frac{t}{T} > \log(1 - \frac{t}{T})$ for all $t \in (0, T)$, which follows by $\log(1-x) \le -x$ for all $0 \le x < 1$. Hence, the integral representation is more positively correlated to the original process than the anticipative version and the space–time transform. □

2.1 Gauss and conditional Gauss distribution of path deviations

Proposition 2.3. *For all $t \in [0, T)$ and $a = 0$, $b \in \mathbb{R}$, the path deviation $W_t - W_t^{\mathrm{br}}$ is normally distributed with mean $\mathbb{E}(W_t - W_t^{\mathrm{br}}) = -b\frac{t}{T}$ and with variance*

$$\operatorname{Var}(W_t - W_t^{\mathrm{av}}) = \operatorname{Var}(W_t - W_t^{\mathrm{st}}) = \frac{t^2}{T},$$

$$\operatorname{Var}(W_t - W_t^{\mathrm{ir}}) = t\left(1 + \frac{T-t}{T}\right) + 2(T-t)\log\frac{T-t}{T} =: \sigma^2(t).$$

Proof. With $a = 0$, by (1.3), for every $0 \le t < T$ the path deviation $W_t - W_t^{\mathrm{br}}$ is normally distributed with mean $\mathbb{E}(W_t - W_t^{\mathrm{br}}) = -b\frac{t}{T}$ and with variance

$$\operatorname{Var}(W_t - W_t^{\mathrm{av}}) = \operatorname{Var}\left(\frac{t}{T}W_T\right) = \frac{t^2}{T},$$

$$\begin{aligned}\operatorname{Var}(W_t - W_t^{\mathrm{ir}}) &= \operatorname{Var}\left(\int_0^t \frac{t-s}{T-s}\,\mathrm{d}W_s\right) = \int_0^t \left(1 - \frac{T-t}{T-s}\right)^2 \mathrm{d}s \\ &= t + 2(T-t)\log\frac{T-t}{T} + (T-t)^2\left(\frac{1}{T-t} - \frac{1}{T}\right) = \sigma^2(t),\end{aligned}$$

and

$$\begin{aligned}\operatorname{Var}(W_t - W_t^{\mathrm{st}}) &= \operatorname{Var}\left(W_t - \frac{T-t}{T}W_{(tT)/(T-t)}\right) \\ &= \operatorname{Var}\left(-\frac{T-t}{T}(W_{(tT)/(T-t)} - W_t) + \frac{t}{T}W_t\right) \\ &= \left(\frac{T-t}{T}\right)^2\left(\frac{tT}{T-t} - t\right) + \frac{t^3}{T^2} = \frac{(T-t)t^2 + t^3}{T^2} = \frac{t^2}{T},\end{aligned}$$

concluding the proof. □

By Proposition 2.3, for every $0 < t < T$, the variance of the path deviation of the integral representation from the original Wiener process is smaller than those of the anticipative version or the space–time transform, since we have $\sigma^2(t) = 2t - \frac{t^2}{T} + 2(T-t)\log(1 - \frac{t}{T})$ and, thus,

$$\sigma^2(t) < \frac{t^2}{T}, \qquad t \in (0, T). \tag{2.3}$$

Indeed, (2.3) is equivalent to $-\frac{t}{T} > \log(1 - \frac{t}{T})$ for all $t \in (0, T)$, which holds, since $\log(1-x) \le -x$ for all $0 \le x < 1$.

Next we examine the conditional distribution of the path deviation $W_t - W_t^{\mathrm{br}}$ given the endpoint W_T.

Proposition 2.4. *For all $t \in [0, T)$, $a = 0$, $b \in \mathbb{R}$ and $d \in \mathbb{R}$, the conditional distribution of the path deviation $W_t - W_t^{\mathrm{br}}$ given $W_T = d$ is normal with mean*

$$\mathbb{E}(W_t - W_t^{\mathrm{av}} | W_T = d) = (d-b)\frac{t}{T}, \tag{2.4}$$

$$\mathbb{E}(W_t - W_t^{\mathrm{ir}} | W_T = d) = (d-b)\frac{t}{T} + d\frac{T-t}{T}\log\frac{T-t}{T}, \tag{2.5}$$

$$\mathbb{E}(W_t - W_t^{\mathrm{st}} | W_T = d) = -b\frac{t}{T} + \frac{d}{T}(2t - T)\cdot 1_{[T/2,T)}(t), \tag{2.6}$$

and with variance

$$\operatorname{Var}(W_t - W_t^{\mathrm{av}} | W_T = d) = 0, \tag{2.7}$$

$$\begin{aligned} \operatorname{Var}(W_t - W_t^{\mathrm{ir}} | W_T = d) = 2t\frac{T-t}{T} + 2\frac{(T-t)^2}{T}\log\frac{T-t}{T} \\ - \frac{(T-t)^2}{T}\left(\log\frac{T-t}{T}\right)^2, \end{aligned} \tag{2.8}$$

$$\operatorname{Var}(W_t - W_t^{\mathrm{st}} | W_T = d) = \frac{t^2}{T} - \frac{(2t-T)^2}{T} \cdot 1_{[T/2,T)}(t). \tag{2.9}$$

Proof. For all $0 \le t < T$, the joint distribution $(W_t - W_t^{\mathrm{br}}, W_T)$ of the path deviation and the endpoint is a two-dimensional normal distribution and, by Theorem 2 and Problem 5 in Chapter II, Section 13 of Shiryaev (1996), it is known that the conditional distribution of $W_t - W_t^{\mathrm{br}}$ given $W_T = d$ is normal with mean

$$\mathbb{E}(W_t - W_t^{\mathrm{br}}) + \frac{d - \mathbb{E}(W_T)}{\operatorname{Var}(W_T)} \operatorname{Cov}(W_t - W_t^{\mathrm{br}}, W_T) \tag{2.10}$$

and with variance

$$\operatorname{Var}(W_t - W_t^{\mathrm{br}}) - \frac{(\operatorname{Cov}(W_t - W_t^{\mathrm{br}}, W_T))^2}{\operatorname{Var}(W_T)}. \tag{2.11}$$

Here we have

$$\operatorname{Cov}(W_t - W_t^{\mathrm{av}}, W_T) = \operatorname{Cov}\left(\frac{t}{T}W_T, W_T\right) = t, \qquad t \in [0, T),$$

and, thus, (2.10), (2.11) and Proposition 2.3 yield that

$$\begin{aligned} \mathbb{E}(W_t - W_t^{\mathrm{av}} | W_T = d) &= -b\frac{t}{T} + \frac{d}{T}t = (d-b)\frac{t}{T}, \\ \operatorname{Var}(W_t - W_t^{\mathrm{av}} | W_T = d) &= \frac{t^2}{T} - \frac{t^2}{T} = 0. \end{aligned}$$

We note that the above formulae follow immediately, since in case of $W_T = d$, we have $W_t - W_t^{\mathrm{av}} = (d-b)\frac{t}{T}$. Further, we have

$$\begin{aligned} \operatorname{Cov}(W_t - W_t^{\mathrm{ir}}, W_T) &= \operatorname{Cov}\left(\int_0^t \frac{t-s}{T-s}\,\mathrm{d}W_s, \int_0^T 1\,\mathrm{d}W_s\right) = \int_0^t \frac{t-s}{T-s}\,\mathrm{d}s \\ &= \int_0^t \left(1 - \frac{T-t}{T-s}\right)\mathrm{d}s = t + (T-t)\log\frac{T-t}{T}, \end{aligned}$$

and, thus, (2.10), (2.11) and Proposition 2.3 yield that

$$\mathbb{E}(W_t - W_t^{\mathrm{ir}} | W_T = d) = -b\frac{t}{T} + \frac{d}{T}\left(t + (T-t)\log\frac{T-t}{T}\right),$$

$$\begin{aligned}
\operatorname{Var}(W_t - W_t^{\mathrm{ir}} | W_T = d) &= \sigma^2(t) - \frac{(t + (T-t)\log((T-t)/T))^2}{T} \\
&= 2t - \frac{t^2}{T} + 2(T-t)\log\frac{T-t}{T} - \frac{t^2}{T} \\
&\quad - 2(T-t)\frac{t}{T}\log\frac{T-t}{T} - \frac{(T-t)^2}{T}\left(\log\frac{T-t}{T}\right)^2.
\end{aligned}$$

This implies (2.5) and (2.8). Finally, we have

$$\begin{aligned}
&\operatorname{Cov}(W_t - W_t^{\mathrm{st}}, W_T) \\
&\quad = \operatorname{Cov}(W_t, W_T) - \frac{T-t}{T}\operatorname{Cov}\big(W_{(tT)/(T-t)}, W_T\big) \\
&\quad = t - \frac{T-t}{T}\min\left(\frac{tT}{T-t}, T\right) \\
&\quad = \begin{cases} t - \dfrac{T-t}{T}\dfrac{tT}{T-t} = 0 & \text{if } 0 \le t \le \dfrac{T}{2}, \\ t - \dfrac{T-t}{T}T = 2t - T & \text{if } \dfrac{T}{2} \le t < T, \end{cases}
\end{aligned}$$

and, thus, (2.10), (2.11) and Proposition 2.3 yield (2.6) and (2.9). □

2.2 Expected absolute path deviations

Lemma 2.5. *For all $t \in (0, T)$ and $a = 0$, $b \in \mathbb{R}$, we have*

$$\mathbb{E}(|W_t - W_t^{\mathrm{av}}|) = \mathbb{E}(|W_t - W_t^{\mathrm{st}}|) > \mathbb{E}(|W_t - W_t^{\mathrm{ir}}|).$$

Proof. For a normally distributed random variable Y_{μ,σ^2} with mean $\mu \in \mathbb{R}$ and with variance $\sigma^2 > 0$ first note that $\mathbb{E}(|Y_{\mu,\sigma^2}|)$ is a strictly increasing function in $\sigma > 0$. Indeed, by standard calculations,

$$\mathbb{E}(|Y_{\mu,\sigma^2}|) = 2\sigma\Phi'\left(\frac{\mu}{\sigma}\right) + \mu\left(2\Phi\left(\frac{\mu}{\sigma}\right) - 1\right), \tag{2.12}$$

where Φ denotes the distribution function of a standard normally distributed random variable and, by differentiating with respect to $\sigma > 0$, we have

$$\frac{\partial}{\partial\sigma}\mathbb{E}(|Y_{\mu,\sigma^2}|) = 2\Phi'\left(\frac{\mu}{\sigma}\right) > 0.$$

For more detailed calculations, see our arXiv preprint [Barczy and Kern (2010b, Lemma 2.5)]. By Section 2.1 together with (2.3), we get for all $0 < t < T$,

$$\begin{aligned}
\mathbb{E}(|W_t - W_t^{\mathrm{av}}|) &= \mathbb{E}(|W_t - W_t^{\mathrm{st}}|) \\
&= \frac{2t}{\sqrt{T}}\Phi'\left(\frac{b}{\sqrt{T}}\right) + b\frac{t}{T}\left(2\Phi\left(\frac{b}{\sqrt{T}}\right) - 1\right)
\end{aligned}$$

$$> 2\sigma(t)\Phi'\Big(\frac{bt}{T\sigma(t)}\Big) + b\frac{t}{T}\Big(2\Phi\Big(\frac{bt}{T\sigma(t)}\Big) - 1\Big)$$
$$= \mathbb{E}(|W_t - W_t^{\mathrm{ir}}|),$$

concluding the proof.

We note that one can give another argument for the monotonicity of $\mathbb{E}(|Y_{\mu,\sigma^2}|)$ in $\sigma > 0$. Namely, Tanaka's formula [see, e.g., Revuz and Yor (2001, Chapter VI, Theorem 1.2)] yields that $\mathbb{E}(|\mu + W_t|) = |\mu| + \mathbb{E}(L_t^\mu)$, $t \geq 0$, where L^μ denotes the local time of the Wiener process at level μ. Now, since the local time L_t^μ is increasing in $t > 0$, the same holds true for $\mathbb{E}(|\mu + W_t|) = \mathbb{E}(|Y_{\mu,t}|)$. □

Next we compare expected absolute path deviations $\mathbb{E}(\int_0^T |W_t - W_t^{\mathrm{br}}|\,\mathrm{d}t)$. Using that integration over the time-interval $[0, T]$ and taking expectation can be interchanged (as explained in the Introduction), by Lemma 2.5, we also get

$$\mathbb{E}\Big(\int_0^T |W_t - W_t^{\mathrm{av}}|\,\mathrm{d}t\Big) = \mathbb{E}\Big(\int_0^T |W_t - W_t^{\mathrm{st}}|\,\mathrm{d}t\Big) > \mathbb{E}\Big(\int_0^T |W_t - W_t^{\mathrm{ir}}|\,\mathrm{d}t\Big).$$

Using (2.12) and Proposition 2.4, it might also be possible to calculate and to compare expected conditional absolute path deviations given $W_T = d$. This task is more complicated, since now the mean is different for different versions of the bridge; see Proposition 2.4. Instead we will now consider expected (conditional) quadratic path deviations which have much nicer forms.

2.3 Expected quadratic path deviations

Theorem 2.6. *For all $t \in [0, T)$ and $a = 0$, $b \in \mathbb{R}$, we have*

$$\mathbb{E}\big((W_t - W_t^{\mathrm{av}})^2\big) = \mathbb{E}\big((W_t - W_t^{\mathrm{st}})^2\big) = \frac{t^2}{T} + b^2\frac{t^2}{T^2}, \tag{2.13}$$

$$\mathbb{E}\big((W_t - W_t^{\mathrm{ir}})^2\big) = \sigma^2(t) + b^2\frac{t^2}{T^2}, \tag{2.14}$$

where $\sigma^2(t)$ is defined in Proposition 2.3.

Moreover, the expected quadratic path deviations take the following forms:

$$\mathbb{E}\Big(\int_0^T (W_t - W_t^{\mathrm{av}})^2\,\mathrm{d}t\Big) = \mathbb{E}\Big(\int_0^T (W_t - W_t^{\mathrm{st}})^2\,\mathrm{d}t\Big) = \frac{T}{3}(T + b^2),$$
$$\mathbb{E}\Big(\int_0^T (W_t - W_t^{\mathrm{ir}})^2\,\mathrm{d}t\Big) = \frac{T}{3}\Big(\frac{T}{2} + b^2\Big).$$

Proof. For a normally distributed random variable Y_{μ,σ^2} with mean μ and with variance $\sigma^2 \geq 0$ we clearly have $\mathbb{E}(Y^2_{\mu,\sigma^2}) = \sigma^2 + \mu^2$. Hence, by Proposition 2.3

we get (2.13) and (2.14). Then we have

$$\mathbb{E}\Big(\int_0^T (W_t - W_t^{\mathrm{av}})^2\,\mathrm{d}t\Big) = \mathbb{E}\Big(\int_0^T (W_t - W_t^{\mathrm{st}})^2\,\mathrm{d}t\Big) = \int_0^T \frac{t^2}{T^2}(T+b^2)\,\mathrm{d}t = \frac{T}{3}(T+b^2),$$

and by change of variables $s = (T-t)/T$ and partial integration we get

$$\begin{aligned}\mathbb{E}\Big(\int_0^T (W_t - W_t^{\mathrm{ir}})^2\,\mathrm{d}t\Big) &= \int_0^T \Big(\sigma^2(t) + b^2\frac{t^2}{T^2}\Big)\,\mathrm{d}t\\ &= \int_0^T \Big[t\Big(2-\frac{t}{T}\Big) + 2(T-t)\log\frac{T-t}{T} + b^2\frac{t^2}{T^2}\Big]\,\mathrm{d}t\\ &= T^2 - \frac{1}{3}T^2 + 2T^2\int_0^1 s\log s\,\mathrm{d}s + \frac{1}{3}b^2T\\ &= \frac{1}{6}T^2 + \frac{1}{3}b^2T = \frac{T}{3}\Big(\frac{T}{2}+b^2\Big),\end{aligned}$$

concluding the proof. □

Note that by Theorem 2.6 and (2.3) for all $t \in (0,T)$

$$\mathbb{E}\big((W_t - W_t^{\mathrm{av}})^2\big) = \mathbb{E}\big((W_t - W_t^{\mathrm{st}})^2\big) > \sigma^2(t) + b^2\frac{t^2}{T^2} = \mathbb{E}\big((W_t - W_t^{\mathrm{ir}})^2\big).$$

Integrating over $[0,T]$, the case $b=0$ of Theorem 2.6 shows that the expected quadratic path deviation of the integral representation is half of those of the anticipative version or the space–time transform of the bridge. This is in accordance with the typical observations from simulation studies as in Figure 1.

2.4 Expected conditional quadratic path deviations

Theorem 2.7. *For all $t \in [0,T)$, $a=0$, $b \in \mathbb{R}$ and $d \in \mathbb{R}$, we have*

$$\mathbb{E}\big((W_t - W_t^{\mathrm{av}})^2 | W_T = d\big) = (d-b)^2\frac{t^2}{T^2}, \tag{2.15}$$

$$\begin{aligned}\mathbb{E}\big((W_t - W_t^{\mathrm{ir}})^2 | W_T = d\big) &= 2t\frac{T-t}{T} + 2\frac{(T-t)^2}{T}\log\frac{T-t}{T}\\ &\quad - \frac{(T-t)^2}{T}\Big(\log\frac{T-t}{T}\Big)^2 \qquad (2.16)\\ &\quad + \Big((d-b)\frac{t}{T} + d\frac{T-t}{T}\log\frac{T-t}{T}\Big)^2,\end{aligned}$$

$$\mathbb{E}\big((W_t - W_t^{\mathrm{st}})^2 | W_T = d\big) = \frac{t^2}{T} - \frac{(2t-T)^2}{T} 1_{[T/2,T)}(t) + \left(b\frac{t}{T} - d\frac{(2t-T)}{T} 1_{[T/2,T)}(t)\right)^2. \tag{2.17}$$

Moreover, the expected conditional quadratic path deviations take the following forms:

$$\mathbb{E}\left(\int_0^T (W_t - W_t^{\mathrm{av}})^2 \,\mathrm{d}t \,\Big|\, W_T = d\right) = \frac{1}{3}(d-b)^2 T, \tag{2.18}$$

$$\mathbb{E}\left(\int_0^T (W_t - W_t^{\mathrm{ir}})^2 \,\mathrm{d}t \,\Big|\, W_T = d\right) = \frac{7}{54}(b-d)^2 T + \frac{11}{54} b^2 T - \frac{7}{54} dbT + \frac{1}{27} T^2, \tag{2.19}$$

$$\mathbb{E}\left(\int_0^T (W_t - W_t^{\mathrm{st}})^2 \,\mathrm{d}t \,\Big|\, W_T = d\right) = \frac{1}{6}(d-b)^2 T + \frac{1}{6} b^2 T - \frac{1}{12} dbT + \frac{1}{6} T^2. \tag{2.20}$$

Proof. By (2.4) and (2.7) for $0 < t < T$ we get (2.15). Using that integration over the time-interval $[0, T]$ and taking conditional expectation can be interchanged (as explained in the Introduction), we get (2.15) yields (2.18). By (2.5) and (2.8) we have (2.16), hence, by change of variables $s = (T-t)/T$ and partial integration

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T (W_t - W_t^{\mathrm{ir}})^2 \,\mathrm{d}t \,\Big|\, W_T = d\right) \\
&= \int_0^T \bigg[2t\frac{T-t}{T} + (d-b)^2 \frac{t^2}{T^2} + 2d(d-b)\frac{t}{T}\frac{T-t}{T}\log\frac{T-t}{T} \\
&\qquad + 2T\frac{(T-t)^2}{T^2}\log\frac{T-t}{T} + (d^2 - T)\frac{(T-t)^2}{T^2}\left(\log\frac{T-t}{T}\right)^2\bigg] \,\mathrm{d}t \\
&= \frac{1}{27}T^2 + \frac{7}{54}d^2 T - \frac{7}{18} dbT + \frac{1}{3} b^2 T,
\end{aligned}$$

which yields (2.19). Finally, by (2.6) and (2.9) we have (2.17), hence, by change of variables $s = 2t - T$ we get

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T (W_t - W_t^{\mathrm{st}})^2 \,\mathrm{d}t \,\Big|\, W_T = d\right) \\
&= \frac{1}{3}T^2 + \frac{1}{3}b^2 T - \int_{T/2}^T \left[\frac{(2t-T)^2}{T} + \frac{2dbt(2t-T)}{T^2} - \frac{d^2}{T^2}(2t-T)^2\right] \mathrm{d}t \\
&= \frac{1}{6}T^2 + \frac{1}{3}b^2 T - \frac{5}{12} dbT + \frac{1}{6} d^2 T,
\end{aligned}$$

which yields (2.20). □

In what follows we give a complete comparison of the quantities (2.18), (2.19) and (2.20). Let $\tilde{b} = b/\sqrt{T}$ and $\tilde{d} = d/\sqrt{T}$. Using the notation

$$e_{\mathrm{br}} := \mathbb{E}\left(\int_0^T (W_t - W_t^{\mathrm{br}})^2 \,\mathrm{d}t \,\Big|\, W_T = d\right),$$

by Theorem 2.7, we have

$$\begin{aligned}
e_{\mathrm{av}} &= \frac{1}{3}(\tilde{b} - \tilde{d})^2 T^2, \\
e_{\mathrm{ir}} &= \left(\frac{7}{54}(\tilde{b} - \tilde{d})^2 + \frac{11}{54}\tilde{b}^2 - \frac{7}{54}\tilde{b}\tilde{d} + \frac{1}{27}\right)T^2, \\
e_{\mathrm{st}} &= \left(\frac{1}{6}(\tilde{b} - \tilde{d})^2 + \frac{1}{6}\tilde{b}^2 - \frac{1}{12}\tilde{b}\tilde{d} + \frac{1}{6}\right)T^2.
\end{aligned}$$

Hence, we easily calculate

$$\begin{aligned}
e_{\mathrm{av}} > e_{\mathrm{ir}} \quad &\iff \quad \frac{11}{54}(\tilde{b} - \tilde{d})^2 > \frac{11}{54}\tilde{b}^2 - \frac{7}{54}\tilde{b}\tilde{d} + \frac{1}{27} \\
&\iff \quad \left|\tilde{d} - \frac{15}{22}\tilde{b}\right| > \sqrt{\frac{2}{11} + \left(\frac{15}{22}\right)^2 (\tilde{b})^2}, \\
e_{\mathrm{av}} > e_{\mathrm{st}} \quad &\iff \quad \frac{1}{6}(\tilde{b} - \tilde{d})^2 > \frac{1}{6}\tilde{b}^2 - \frac{1}{12}\tilde{b}\tilde{d} + \frac{1}{6} \\
&\iff \quad \left|\tilde{d} - \frac{3}{4}\tilde{b}\right| > \sqrt{1 + \frac{9}{16}(\tilde{b})^2}
\end{aligned}$$

and

$$\begin{aligned}
e_{\mathrm{st}} < e_{\mathrm{ir}} \quad &\iff \quad \frac{1}{27}(\tilde{b} - \tilde{d})^2 < \frac{1}{27}\tilde{b}^2 - \frac{5}{108}\tilde{b}\tilde{d} - \frac{7}{54} \\
&\iff \quad \left|\tilde{d} - \frac{3}{8}\tilde{b}\right| < \sqrt{\frac{9}{64}(\tilde{b})^2 - \frac{7}{2}} \quad \text{and} \quad (\tilde{b})^2 \geq \frac{224}{9}.
\end{aligned}$$

The corresponding regions are graphically illustrated in Figure 3.

Finally, we remark that Theorem 2.7 justifies our simulation results in case the endpoint W_T of the Wiener sample path is close to the prescribed endpoint b of its bridge. Indeed, in case of $d = b$ by Theorem 2.7 it can be easily seen that the expected conditional quadratic path deviation of the Wiener process from the anticipative version of its bridge is 0 being smaller than from the integral representation of the bridge or from the space–time bridge version.

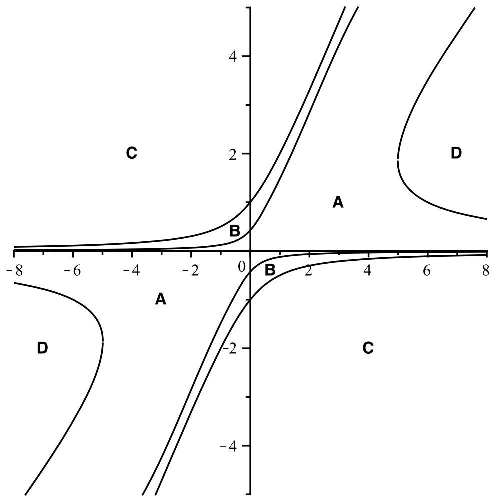

Figure 3 *Regions in the* $(\tilde{b}, \tilde{d})$*-plain for which* A: $e_{\text{av}} < e_{\text{ir}} < e_{\text{st}}$, B: $e_{\text{ir}} < e_{\text{av}} < e_{\text{st}}$, C: $e_{\text{ir}} < e_{\text{st}} < e_{\text{av}}$, *and* D: $e_{\text{av}} < e_{\text{st}} < e_{\text{ir}}$.

3 Path deviation of the Ornstein–Uhlenbeck process from its bridges

Let $(U_t^a)_{t\geq 0}$ be a one-dimensional Ornstein–Uhlenbeck process starting in $a \in \mathbb{R}$, that is, it is the unique strong solution of the SDE

$$\mathrm{d}U_t^a = qU_t^a \,\mathrm{d}t + \sigma \,\mathrm{d}W_t \quad \text{with initial condition } U_0^a = a$$

for some $q \neq 0$ and $\sigma > 0$, where $(W_t)_{t\geq 0}$ is a standard Wiener process. It is well known that the Ornstein–Uhlenbeck process has the integral representation

$$U_t^a = \mathrm{e}^{qt}\left(a + \sigma \int_0^t \mathrm{e}^{-qs} \,\mathrm{d}W_s\right), \qquad t \geq 0,$$

which is a Gauss process with mean function $\mathbb{E}(U_t^a) = a\mathrm{e}^{qt}$ and covariance function $\operatorname{Cov}(U_s^a, U_t^a) = \sigma^2 \frac{\mathrm{e}^{qt}}{q} \sinh(qs)$ for $0 \leq s \leq t$. We also have $U_t^a = a\mathrm{e}^{qt} + U_t^0$, $t \geq 0$, where $(U_t^0)_{t\geq 0}$ is a one-dimensional Ornstein–Uhlenbeck process starting in 0.

We consider the following versions of the Ornstein–Uhlenbeck bridge from a to b over the time-interval $[0, T]$, where $a, b \in \mathbb{R}$:

1. *Anticipative version*

$$U_t^{\text{av}} = a\frac{\sinh(q(T-t))}{\sinh(qT)} + b\frac{\sinh(qt)}{\sinh(qT)} + \left(U_t^0 - \frac{\sinh(qt)}{\sinh(qT)} U_T^0\right), \qquad 0 \leq t \leq T.$$

Up to our knowledge this anticipative version of the Ornstein–Uhlenbeck bridge first appears on page 378 of Donati-Martin (1990) for $a = b = 0$ and in Lemma 1 of Papież and Sandison (1990) for special values of q and σ. It is also an easy consequence of Theorem 2 in Delyon and Hu (2006) and of Proposition 4 in Gasbarra et al. (2007).

2. *Integral representation*

$$U_t^{\mathrm{ir}} = a\frac{\sinh(q(T-t))}{\sinh(qT)} + b\frac{\sinh(qt)}{\sinh(qT)} + \sigma\int_0^t \frac{\sinh(q(T-t))}{\sinh(q(T-s))}\,\mathrm{d}W_s$$

for $0 \le t < T$ and $U_T^{\mathrm{ir}} = b$. This integral representation of the Ornstein–Uhlenbeck bridge is the unique strong solution of the below given SDE (3.2); see, for example, Barczy and Kern (2010a, Remark 3.9).

3. *Space–time transform*

$$U_t^{\mathrm{st}} = a\frac{\sinh(q(T-t))}{\sinh(qT)} + b\frac{\sinh(qt)}{\sinh(qT)} + \sigma\mathrm{e}^{qt}\frac{\kappa(T)-\kappa(t)}{\kappa(T)}W_{\kappa(t)\kappa(T)/(\kappa(T)-\kappa(t))}$$

for $0 \le t < T$ and $U_T^{\mathrm{st}} = b$, with the strictly increasing time-change

$$\mathbb{R} \ni t \mapsto \kappa(t) = \frac{\mathrm{e}^{-qt}\sinh(qt)}{q} = \frac{1-\mathrm{e}^{-2qt}}{2q}.$$

This space–time transform of the Ornstein–Uhlenbeck bridge goes back to the proof of Lemma 1 in Papież and Sandison (1990) and is, roughly speaking, a time-transformation by κ and a rescaling with the coefficient e^{qt} of the space–time transform representation $(W_t^{\mathrm{st}})_{t\in[0,T]}$ of the Wiener bridge from a to b over the time-interval $[0, T]$.

Remark 3.1. We note that the previous versions of an Ornstein–Uhlenbeck bridge are in accordance with the corresponding versions of a usual standard Wiener bridge introduced in the Introduction. By this we mean that for all $T > 0$, $t \in [0, T]$ and $\sigma = 1$, U_t^{br} converges to W_t^{br} in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ as $q \to 0$ (see our arXiv preprint [Barczy and Kern (2010b, Remark 3.1)]).

In all that follows we will use the notation $(U_t^{\mathrm{br}})_{t\in[0,T]}$ if the version of the bridge is not specified.

First we present a lemma about a time-transformation which will be useful for calculating $\mathrm{Var}(U_t^a - U_t^{\mathrm{st}})$ and $\mathrm{Cov}(U_s^{\mathrm{st}}, U_t^{\mathrm{st}})$, $0 \le s, t < T$.

Lemma 3.2. *For the time-transformation* $\kappa_T^*(t) := \frac{\kappa(t)\kappa(T)}{\kappa(T)-\kappa(t)}$, $t \in [0, T)$, *with* $\kappa(t) := \frac{1-\mathrm{e}^{-2qt}}{2q}$, $t \in \mathbb{R}$, *we get* κ_T^* *is strictly increasing and* $\kappa_T^*(t) \ge t$ *for all* $t \in [0, T)$.

Proof. Since the function $[0, T) \ni t \mapsto \frac{tT}{T-t}$ is strictly increasing and $\mathbb{R} \ni t \mapsto \kappa(t) = \frac{1-\mathrm{e}^{-2qt}}{2q}$ is strictly increasing for every $q \ne 0$, we get that $[0, T) \ni t \mapsto \frac{\kappa(t)\kappa(T)}{\kappa(T)-\kappa(t)} =: \kappa_T^*(t)$ is strictly increasing. Further, easy calculations show that

1. $\kappa_T^*(0) = 0$ and $\lim_{t\uparrow T} \kappa_T^*(t) = \infty$.
2. κ_T^* is differentiable on $[0, T)$, namely,

$$(\kappa_T^*)'(t) = \frac{\kappa'(t)\kappa(T)(\kappa(T) - \kappa(t)) + \kappa(t)\kappa(T)\kappa'(t)}{(\kappa(T) - \kappa(t))^2} = \frac{\kappa'(t)\kappa^2(T)}{(\kappa(T) - \kappa(t))^2}$$

for $t \in [0, T)$ with $\kappa'(t) = \mathrm{e}^{-2qt}$ and, hence, $(\kappa_T^*)'(0) = 1$.
3. For the second derivative we get

$$\begin{aligned}(\kappa_T^*)''(t) &= \frac{\kappa''(t)\kappa^2(T)(\kappa(T) - \kappa(t))^2 + 2(\kappa(T) - \kappa(t))(\kappa')^2(t)\kappa^2(T)}{(\kappa(T) - \kappa(t))^4} \\ &= \frac{\kappa^2(T)(\kappa''(t)(\kappa(T) - \kappa(t)) + 2(\kappa')^2(t))}{(\kappa(T) - \kappa(t))^3}, \qquad t \in [0, T).\end{aligned}$$

Since $\kappa''(t)(\kappa(T) - \kappa(t)) + 2(\kappa')^2(t) = \mathrm{e}^{-4qt} + \mathrm{e}^{-2q(T+t)} > 0$, we have $(\kappa_T^*)'$ is strictly increasing.

Altogether this shows that $(\kappa_T^*)'(t) \geq (\kappa_T^*)'(0) = 1$, $t \in [0, T)$ and, hence, $\kappa_T^*(t) \geq t$ for all $t \in [0, T)$. □

Proposition 3.3. *Let $(U_t^{\mathrm{br}})_{t\in[0,T]}$ be an Ornstein–Uhlenbeck bridge from a to b over the time-interval $[0, T]$, where $a, b \in \mathbb{R}$. Then $(U_t^{\mathrm{br}})_{t\in[0,T]}$ is a Gauss process with mean function*

$$\mathbb{E}(U_t^{\mathrm{br}}) = a\frac{\sinh(q(T - t))}{\sinh(qT)} + b\frac{\sinh(qt)}{\sinh(qT)}, \qquad 0 \leq t < T,$$

and with covariance function

$$\operatorname{Cov}(U_s^{\mathrm{br}}, U_t^{\mathrm{br}}) = \frac{\sigma^2}{q}\frac{\sinh(qs)\sinh(q(T - t))}{\sinh(qT)}, \qquad 0 \leq s \leq t < T. \tag{3.1}$$

Hence, all the bridge versions above have the same finite-dimensional distributions.

Proof. For $0 \leq s \leq t < T$, we have the covariance function

$$\begin{aligned}\operatorname{Cov}(U_s^{\mathrm{av}}, U_t^{\mathrm{av}}) &= \operatorname{Cov}\left(U_s^0 - \frac{\sinh(qs)}{\sinh(qT)}U_T^0, U_t^0 - \frac{\sinh(qt)}{\sinh(qT)}U_T^0\right) \\ &= \operatorname{Cov}(U_s^0, U_t^0) - \frac{\sinh(qt)}{\sinh(qT)}\operatorname{Cov}(U_s^0, U_T^0) \\ &\quad - \frac{\sinh(qs)}{\sinh(qT)}\operatorname{Cov}(U_t^0, U_T^0) + \frac{\sinh(qs)\sinh(qt)}{\sinh^2(qT)}\operatorname{Cov}(U_T^0, U_T^0) \\ &= \sigma^2\frac{\mathrm{e}^{qt}}{q}\sinh(qs) - \frac{\sinh(qt)}{\sinh(qT)}\sigma^2\frac{\mathrm{e}^{qT}}{q}\sinh(qs) \\ &= \frac{\sigma^2}{q}\frac{\sinh(qs)\sinh(q(T - t))}{\sinh(qT)}\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Cov}(U_s^{\mathrm{ir}}, U_t^{\mathrm{ir}})
&= \sigma^2 \operatorname{Cov}\biggl(\int_0^s \frac{\sinh(q(T-s))}{\sinh(q(T-r))}\,\mathrm{d}W_r, \int_0^t \frac{\sinh(q(T-t))}{\sinh(q(T-r))}\,\mathrm{d}W_r\biggr)\\
&= \sigma^2 \int_0^s \frac{\sinh(q(T-s))\sinh(q(T-t))}{\sinh^2(q(T-r))}\,\mathrm{d}r\\
&= \frac{\sigma^2}{q}\sinh\bigl(q(T-s)\bigr)\sinh\bigl(q(T-t)\bigr)\int_{q(T-s)}^{qT} \frac{1}{\sinh^2 v}\,\mathrm{d}v\\
&= \frac{\sigma^2}{q}\sinh\bigl(q(T-s)\bigr)\sinh\bigl(q(T-t)\bigr)\biggl(\frac{\cosh(q(T-s))}{\sinh(q(T-s))} - \frac{\cosh(qT)}{\sinh(qT)}\biggr)\\
&= \frac{\sigma^2}{q}\sinh\bigl(q(T-s)\bigr)\sinh\bigl(q(T-t)\bigr)\frac{\sinh(qT-q(T-s))}{\sinh(q(T-s))\sinh(qT)}\\
&= \frac{\sigma^2}{q}\frac{\sinh(qs)\sinh(q(T-t))}{\sinh(qT)}.
\end{aligned}$$

By Lemma 3.2, for $0 \le s \le t < T$ we get

$$\begin{aligned}
\operatorname{Cov}(U_s^{\mathrm{st}}, U_t^{\mathrm{st}}) &= \sigma^2 \mathrm{e}^{q(s+t)} \frac{\kappa(T)-\kappa(s)}{\kappa(T)}\frac{\kappa(T)-\kappa(t)}{\kappa(T)}\\
&\quad \times \operatorname{Cov}\bigl(W_{\kappa(s)\kappa(T)/(\kappa(T)-\kappa(s))}, W_{\kappa(t)\kappa(T)/(\kappa(T)-\kappa(t))}\bigr)\\
&= \sigma^2 \mathrm{e}^{q(s+t)} \frac{\kappa(T)-\kappa(s)}{\kappa(T)}\frac{\kappa(T)-\kappa(t)}{\kappa(T)}\frac{\kappa(s)\kappa(T)}{\kappa(T)-\kappa(s)}\\
&= \sigma^2 \mathrm{e}^{qt} \frac{\sinh(qs)}{\mathrm{e}^{-qT}\sinh(qT)}\frac{\mathrm{e}^{-2qt}-\mathrm{e}^{-2qT}}{2q}\\
&= \frac{\sigma^2}{q}\frac{\sinh(qs)\sinh(q(T-t))}{\sinh(qT)},
\end{aligned}$$

concluding the proof. □

It follows from the definitions that all bridge versions have almost sure continuous sample paths on $[0, T)$. The (left) continuity of the trajectories at $t = T$ is also obvious in case of the anticipative version. As explained in the Introduction, this induces (left) continuity at $t = T$ of the integral representation and of the space–time transform (one can also refer to Karatzas and Shreve (1991, Problem 2.9.3) and Barczy and Kern (2010a, Lemma 4.5)). Hence, the anticipative version U^{av}, the integral representation U^{ir} and the space–time transform U^{st} induce the same probability measure on $(C[0,T], \mathcal{B}(C[0,T]))$. This underlines and explains the definition of an Ornstein–Uhlenbeck bridge from a to b over the time-interval

$[0, T]$, by which we mean any almost surely continuous Gauss process having mean function and covariance function given in Proposition 3.3.

We also note that the finite-dimensional distributions of the Ornstein–Uhlenbeck bridge versions coincide with the conditional finite-dimensional distributions of the Ornstein–Uhlenbeck process $(U_t^a)_{t\in[0,T]}$ (starting in a) and conditioned on $\{U_T^a = b\}$; see, for example, Delyon and Hu (2006, Theorem 2), Gasbarra et al. (2007, Proposition 4) or Barczy and Kern (2010a, Proposition 3.5).

3.1 Different sample path behavior of Ornstein–Uhlenbeck bridge versions

Let us consider the linear SDE

$$\mathrm{d}U_t^{\mathrm{br}} = q\left(-\coth\bigl(q(T-t)\bigr)U_t^{\mathrm{br}} + \frac{b}{\sinh(q(T-t))}\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t \tag{3.2}$$

for $0 \le t < T$ with initial condition $U_0^{\mathrm{br}} = a$. Then the integral representation of the Ornstein–Uhlenbeck bridge is the unique strong solution of this SDE (see, e.g., Delyon and Hu (2006, Proposition 3) or Barczy and Kern (2010a, Remark 3.10)), while the anticipative version and the space–time transform are only weak solutions. We also note that one could present other indicators for different sample path behavior of the Ornstein–Uhlenbeck bridge versions, for example, by calculating the covariances $\mathrm{Cov}(U_t^{\mathrm{br}}, U_t^a)$ or the correlations $\rho(U_t^{\mathrm{br}}, U_t^a)$ of the coordinates of the two-dimensional Gauss process $(U_t^{\mathrm{br}}, U_t^a)_{t\in[0,T]}$. Namely, standard calculations yield for $0 < t < T$

$$\mathrm{Cov}(U_t^{\mathrm{av}}, U_t^a) = \frac{\sigma^2}{q}\frac{\sinh(q(T-t))\sinh(qt)}{\sinh(qT)},$$

$$\mathrm{Cov}(U_t^{\mathrm{ir}}, U_t^a) = \frac{\sigma^2}{q}\mathrm{e}^{-q(T-t)}\sinh\bigl(q(T-t)\bigr)\left(qt + \log\left(\frac{\sinh(qT)}{\sinh(q(T-t))}\right)\right),$$

$$\mathrm{Cov}(U_t^{\mathrm{st}}, U_t^a) = \frac{\sigma^2}{q}(\mathrm{e}^{qt}-1)\frac{\sinh(q(T-t))}{\sinh(qT)};$$

see also our arXiv preprint [Barczy and Kern (2010b, Proposition 3.4)] for details. This also shows that, even though the three bridge versions have the same law on $(C[0,T], \mathcal{B}(C[0,T]))$, their joint laws together with the Ornstein–Uhlenbeck process U^a are different. Since the above formulas for the covariances are hard to compare in general, in the sequel we concentrate on a comparison of expected quadratic path deviations. Our aim is to analyze the sample path deviations of the Ornstein–Uhlenbeck bridge versions to the original Ornstein–Uhlenbeck process $(U_t^a)_{t\in[0,T)}$ (starting in a) by calculating and comparing expected quadratic path deviations $\mathbb{E}(\int_0^T (U_t^a - U_t^{\mathrm{br}})^2\,\mathrm{d}t)$.

Simulation studies show the same qualitative behavior of typical sample path deviations of the anticipative version, the integral representation and the space–time transform of the Ornstein–Uhlenbeck bridge as we have for the Wiener bridge; see the upper row of Figure 4.

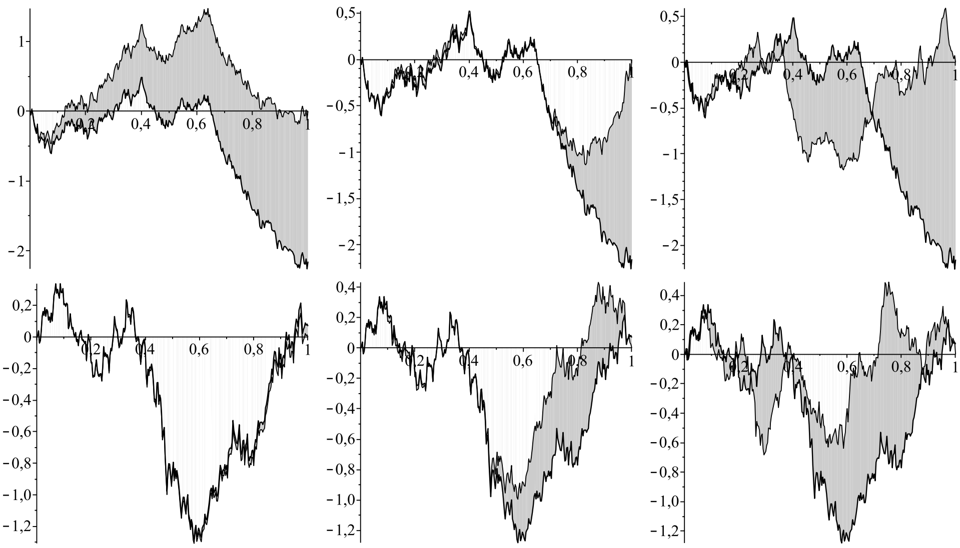

Figure 4 *Two sample paths of the Ornstein–Uhlenbeck process with* $\sigma = 1$ *and* $q = -1$ (*upper row, thick line*), *respectively* $q = 2$ (*lower row, thick line*) *and their deviations from the anticipative version* (*left column*), *the integral representation* (*middle column*), *and the space–time transform* (*right column*) *of the Ornstein–Uhlenbeck bridge from* 0 *to* 0 *over the time-interval* [0, 1].

Note that in general the deviation from the space–time transform bridge version is hard to compare with the other deviations, since $(U_t^{\mathrm{st}})_{t\in[t^*,T)}$ depends on the future part $(U_t^a)_{t\in[T,\infty)}$ of the Ornstein–Uhlenbeck process, which is not visible in the pictures of Figure 4. Here, $t^* \in (0,T)$ is defined as follows. Due to the strict monotonicity of κ_T^* and $\lim_{t\uparrow T}\kappa_T^*(t) = \infty$ there is a unique $t^* \in (0,T)$ such that $\kappa_T^*(t^*) = T$; see the analysis of the time-transform κ_T^* in Lemma 3.2.

From simulation studies we also get that the above typical behavior is again reversed in case the endpoint U_T^a of the Ornstein–Uhlenbeck sample path is close to the prescribed endpoint b of its bridge, namely, for such a sample path of the Ornstein–Uhlenbeck process the deviation from its anticipative bridge version is smaller than from its integral representation of the bridge; see the lower row of Figure 4.

Our aim is again to give quantitative answers to this qualitative behavior observed from simulation studies by studying the path deviations on $[0,T)$:

$$
\begin{aligned}
U_t^a - U_t^{\mathrm{av}} &= (a\mathrm{e}^{qT} - b)\frac{\sinh(qt)}{\sinh(qT)} + \frac{\sinh(qt)}{\sinh(qT)}U_T^0,\\
U_t^a - U_t^{\mathrm{ir}} &= (a\mathrm{e}^{qT} - b)\frac{\sinh(qt)}{\sinh(qT)}\\
&\quad + \sigma\int_0^t\left(\mathrm{e}^{q(t-s)} - \frac{\sinh(q(T-t))}{\sinh(q(T-s))}\right)\mathrm{d}W_s, \qquad (3.3)
\end{aligned}
$$

$$U_t^a - U_t^{\rm st} = (a\mathrm{e}^{qT} - b)\frac{\sinh(qt)}{\sinh(qT)} + \left(U_t^0 - \sigma \mathrm{e}^{qt}\frac{\kappa(T)-\kappa(t)}{\kappa(T)} W_{\kappa(t)\kappa(T)/(\kappa(T)-\kappa(t))}\right).$$

Note that all path deviations depend only on the transformed difference $(a\mathrm{e}^{qT} - b)$ of starting and endpoint of the bridge. Hence, without loss of generality we can and will assume that $a = 0$ in the sequel. For simplicity we will concentrate on calculating the Gauss distributions of path deviations and to compare the expected quadratic path deviations only.

3.2 Gauss distribution of path deviations

Proposition 3.4. *For all $t \in [0, T)$ and $a = 0$, $b \in \mathbb{R}$, the path deviation $U_t^0 - U_t^{\rm br}$ is normally distributed with mean*

$$\mathbb{E}(U_t^0 - U_t^{\rm br}) = -b\frac{\sinh(qt)}{\sinh(qT)},$$

and with variance

$$\mathrm{Var}(U_t^0 - U_t^{\rm av}) = \sigma^2 \frac{\mathrm{e}^{qT}}{q}\frac{\sinh^2(qt)}{\sinh(qT)}, \tag{3.4}$$

$$\mathrm{Var}(U_t^0 - U_t^{\rm ir}) = \frac{\sigma^2}{q}\left(\sinh(qt)\left(\mathrm{e}^{qt} + \frac{\sinh(q(T-t))}{\sinh(qT)}\right) - 2\mathrm{e}^{-q(T-t)}\sinh\big(q(T-t)\big) \times \left(qt + \log\frac{\sinh(qT)}{\sinh(q(T-t))}\right)\right), \tag{3.5}$$

$$\mathrm{Var}(U_t^0 - U_t^{\rm st}) = \frac{\sigma^2}{q}\left(\sinh(qt)\left(\mathrm{e}^{qt} + \frac{\sinh(q(T-t))}{\sinh(qT)}\right) + 2(1-\mathrm{e}^{qt})\frac{\sinh(q(T-t))}{\sinh(qT)}\right). \tag{3.6}$$

Proof. With $a = 0$, by (3.3), for every $0 \le t < T$ the path deviation $U_t^0 - U_t^{\rm br}$ is normally distributed with mean $\mathbb{E}(U_t^0 - U_t^{\rm br}) = -b\frac{\sinh(qt)}{\sinh(qT)}$ and with variance

$$\mathrm{Var}(U_t^0 - U_t^{\rm av}) = \frac{\sinh^2(qt)}{\sinh^2(qT)}\mathrm{Var}(U_T^0) = \sigma^2\frac{\mathrm{e}^{qT}}{q}\frac{\sinh^2(qt)}{\sinh(qT)},$$

and

$$
\begin{aligned}
&\operatorname{Var}(U_t^0 - U_t^{\mathrm{ir}}) \\
&\quad = \sigma^2 \int_0^t \biggl(\mathrm{e}^{q(t-s)} - \frac{\sinh(q(T-t))}{\sinh(q(T-s))} \biggr)^2 \, \mathrm{d}s \\
&\quad = \sigma^2 \int_0^t \biggl[\mathrm{e}^{2q(t-s)} - 2\mathrm{e}^{q(t-s)} \frac{\sinh(q(T-t))}{\sinh(q(T-s))} + \frac{\sinh^2(q(T-t))}{\sinh^2(q(T-s))} \biggr] \, \mathrm{d}s \\
&\quad = \sigma^2 \biggl(\frac{1}{2q}(\mathrm{e}^{2qt}-1) - 2\int_0^t \frac{\mathrm{e}^{q(T-s)} - \mathrm{e}^{-q(T-2t+s)}}{\mathrm{e}^{q(T-s)} - \mathrm{e}^{-q(T-s)}} \, \mathrm{d}s \\
&\qquad\qquad + \sinh^2\bigl(q(T-t)\bigr) \int_0^t \frac{1}{\sinh^2(q(T-s))} \, \mathrm{d}s \biggr) \\
&\quad = \sigma^2 \biggl(\frac{1}{2q}(\mathrm{e}^{2qt}-1) - \frac{2(1-\mathrm{e}^{-2q(T-t)})}{q} \int_{\mathrm{e}^{q(T-t)}}^{\mathrm{e}^{qT}} \frac{v}{v^2-1} \, \mathrm{d}v \\
&\qquad\qquad + \sinh^2\bigl(q(T-t)\bigr) \frac{1}{q} \biggl(\frac{\cosh(q(T-t))}{\sinh(q(T-t))} - \frac{\cosh(qT)}{\sinh(qT)} \biggr) \biggr) \\
&\quad = \sigma^2 \biggl(\frac{1}{2q}(\mathrm{e}^{2qt}-1) - \frac{(1-\mathrm{e}^{-2q(T-t)})}{q} \log \frac{\mathrm{e}^{2qT}-1}{\mathrm{e}^{2q(T-t)}-1} \\
&\qquad\qquad + \sinh^2\bigl(q(T-t)\bigr) \frac{1}{q} \frac{\sinh(qt)}{\sinh(q(T-t))\sinh(qT)} \biggr),
\end{aligned}
$$

which yields (3.5). Using Lemma 3.2, we get

$$
\begin{aligned}
\operatorname{Var}(U_t^0 - U_t^{\mathrm{st}}) &= \operatorname{Var}\biggl(U_t^0 - \sigma \mathrm{e}^{qt} \frac{\kappa(T)-\kappa(t)}{\kappa(T)} W_{\kappa(t)\kappa(T)/(\kappa(T)-\kappa(t))} \biggr) \\
&= \operatorname{Var}(U_t^0) + \sigma^2 \mathrm{e}^{2qt} \biggl(\frac{\kappa(T)-\kappa(t)}{\kappa(T)} \biggr)^2 \operatorname{Var}\bigl(W_{\kappa(t)\kappa(T)/(\kappa(T)-\kappa(t))}\bigr) \\
&\quad - 2\sigma \mathrm{e}^{qt} \frac{\kappa(T)-\kappa(t)}{\kappa(T)} \operatorname{Cov}\bigl(U_t^0, W_{\kappa(t)\kappa(T)/(\kappa(T)-\kappa(t))}\bigr) \\
&= \sigma^2 \frac{\mathrm{e}^{qt}}{q} \sinh(qt) + \sigma^2 \mathrm{e}^{2qt} \kappa(t) \frac{\kappa(T)-\kappa(t)}{\kappa(T)} \\
&\quad - 2\sigma^2 \mathrm{e}^{qt} \frac{\kappa(T)-\kappa(t)}{\kappa(T)} \int_0^{\min\{t,\kappa_T^*(t)\}} \mathrm{e}^{q(t-s)} \, \mathrm{d}s \\
&= \sigma^2 \frac{\mathrm{e}^{qt}}{q} \biggl(\sinh(qt) + \mathrm{e}^{qt} \frac{1-\mathrm{e}^{-2qt}}{2} \frac{\mathrm{e}^{-2qt} - \mathrm{e}^{-2qT}}{1-\mathrm{e}^{-2qT}} \\
&\qquad - 2 \frac{\mathrm{e}^{-2qt} - \mathrm{e}^{-2qT}}{1-\mathrm{e}^{-2qT}} (\mathrm{e}^{qt}-1) \biggr),
\end{aligned}
$$

which yields (3.6). □

Note that in the proof of Proposition 3.5 below we will give different representations of the variances $\operatorname{Var}(U_t^0 - U_t^{\mathrm{br}})$ calculated in Proposition 3.4.

3.3 Expected quadratic path deviations

Proposition 3.5. *For all $t \in (0, T)$ and $a = 0$, $b \in \mathbb{R}$, we have*

$$\mathbb{E}\big((U_t^0 - U_t^{\mathrm{ir}})^2\big) < \begin{cases} \mathbb{E}\big((U_t^0 - U_t^{\mathrm{st}})^2\big) < \mathbb{E}\big((U_t^0 - U_t^{\mathrm{av}})^2\big), & \text{if } q > 0, \\ \mathbb{E}\big((U_t^0 - U_t^{\mathrm{av}})^2\big) < \mathbb{E}\big((U_t^0 - U_t^{\mathrm{st}})^2\big), & \text{if } q < 0. \end{cases} \tag{3.7}$$

Proof. First note that $\mathbb{E}((U_t^0 - U_t^{\mathrm{br}})^2) = \operatorname{Var}(U_t^0 - U_t^{\mathrm{br}})$, $t \in [0, T)$, since the mean function of path deviations is the same for all bridge versions (see Proposition 3.4). Next we give different representations of $\operatorname{Var}(U_t^0 - U_t^{\mathrm{ir}})$ and $\operatorname{Var}(U_t^0 - U_t^{\mathrm{st}})$ calculated in Proposition 3.4 that are more suitable for comparison. By Proposition 3.4, we have

$$\begin{aligned}
&\operatorname{Var}(U_t^0 - U_t^{\mathrm{ir}}) \\
&\quad = \frac{\sigma^2}{q}\bigg[2\sinh\big(q(T-t)\big)\bigg(\frac{\sinh(qt)}{\sinh(qT)} - \mathrm{e}^{-q(T-t)}\bigg(qt + \log\frac{\sinh(qT)}{\sinh(q(T-t))}\bigg)\bigg) \\
&\qquad\qquad + \sinh(qt)\bigg(\mathrm{e}^{qt} - \frac{\sinh(q(T-t))}{\sinh(qT)}\bigg)\bigg] \\
&\quad = 2\frac{\sigma^2}{q}\sinh\big(q(T-t)\big)\bigg(\frac{\sinh(qt)}{\sinh(qT)} - \mathrm{e}^{-q(T-t)}\bigg(qt + \log\frac{\sinh(qT)}{\sinh(q(T-t))}\bigg)\bigg) \\
&\qquad + \mathrm{e}^{qT}\frac{\sigma^2}{q}\frac{\sinh^2(qt)}{\sinh(qT)},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Var}(U_t^0 - U_t^{\mathrm{st}}) &= \frac{\sigma^2}{q}\bigg(\sinh(qt)\bigg(\mathrm{e}^{qt} - \frac{\sinh(q(T-t))}{\sinh(qT)}\bigg) \\
&\qquad\qquad + 2\frac{\sinh(q(T-t))}{\sinh(qT)}\big(\sinh(qt) + 1 - \mathrm{e}^{qt}\big)\bigg) \\
&= \mathrm{e}^{qT}\frac{\sigma^2}{q}\frac{\sinh^2(qt)}{\sinh(qT)} + 2\frac{\sigma^2}{q}\frac{\sinh(q(T-t))(1-\cosh(qt))}{\sinh(qT)}.
\end{aligned}$$

The advantage of this new representation is that now the variances include the term $\mathrm{e}^{qT}\frac{\sigma^2}{q}\frac{\sinh^2(qt)}{\sinh(qT)}$ for all versions of path deviations.

For the comparison $\mathbb{E}(U_t^0 - U_t^{\mathrm{st}})^2$ with $\mathbb{E}(U_t^0 - U_t^{\mathrm{av}})^2$ we consider the continuous function h_q on $[0, T]$ defined by

$$h_q(t) := 2\frac{\sigma^2}{q}\frac{\sinh(q(T-t))(1-\cosh(qt))}{\sinh(qT)}, \qquad t \in [0, T].$$

Clearly, $h_q(t) = 0$ if and only if $t \in \{0, T\}$ and, further, for all $0 < t < T$ we have $h_q(t) < 0$ if $q > 0$ and $h_q(t) > 0$ if $q < 0$. Thus, we get

$$\mathbb{E}\big((U_t^0 - U_t^{\mathrm{st}})^2\big) \begin{cases} < \mathbb{E}\big((U_t^0 - U_t^{\mathrm{av}})^2\big), & \text{if } q > 0, \\ > \mathbb{E}\big((U_t^0 - U_t^{\mathrm{av}})^2\big), & \text{if } q < 0. \end{cases} \tag{3.8}$$

For the other comparisons, we show that

$$\frac{\sinh(qt)}{\sinh(qT)} - \mathrm{e}^{-q(T-t)}\left(qt + \log\frac{\sinh(qT)}{\sinh(q(T-t))}\right) < \begin{cases} 0 & \text{if } q < 0, \\ \dfrac{1-\cosh(qt)}{\sinh(qT)} & \text{if } q > 0. \end{cases}$$

Using that

$$\left|\frac{\sinh(q(T-t))}{\sinh(qT)} - 1\right| < 1, \qquad t \in (0, T),$$

by $\log(1+x) \le x$, $|x| < 1$, we have for all $0 < t < T$,

$$\begin{aligned} &\frac{\sinh(qt)}{\sinh(qT)} - \mathrm{e}^{-q(T-t)}\left(qt + \log\frac{\sinh(qT)}{\sinh(q(T-t))}\right) \\ &\quad = \frac{\sinh(qt)}{\sinh(qT)} + \mathrm{e}^{-q(T-t)}\left(\log\frac{\sinh(q(T-t))}{\sinh(qT)} - qt\right) \\ &\quad \le \frac{\sinh(qt)}{\sinh(qT)} + \mathrm{e}^{-q(T-t)}\frac{\sinh(q(T-t)) - (1+qt)\sinh(qT)}{\sinh(qT)} \\ &\quad = \frac{1}{2\sinh(qT)}\big(\mathrm{e}^{qt} - \mathrm{e}^{-qt} + 1 - \mathrm{e}^{-2q(T-t)} - (1+qt)(\mathrm{e}^{qt} - \mathrm{e}^{-2qT+qt})\big) \\ &\quad =: \frac{1}{2\sinh(qT)} g_q(t). \end{aligned}$$

For $q < 0$ it is enough to show that $g_q(t) > 0$ for all $0 < t < T$. Now

$$\begin{aligned} g_q(t) &= \big(\mathrm{e}^{-2qT+qt} - \mathrm{e}^{-2q(T-t)}\big) + (1 - \mathrm{e}^{-qt}) - qt(\mathrm{e}^{qt} - \mathrm{e}^{-2qT+qt}) \\ &= \mathrm{e}^{-2q(T-t)}(\mathrm{e}^{-qt} - 1) + (1 - \mathrm{e}^{-qt}) - qt\mathrm{e}^{qt}(1 - \mathrm{e}^{-2qT}) \\ &= \big(\mathrm{e}^{-2q(T-t)} - 1\big)(\mathrm{e}^{-qt} - 1) + qt\mathrm{e}^{qt}(\mathrm{e}^{-2qT} - 1) \\ &\le (\mathrm{e}^{-2qT} - 1)(\mathrm{e}^{-qt} - 1) + qt\mathrm{e}^{qt}(\mathrm{e}^{-2qT} - 1) \\ &= (\mathrm{e}^{-2qT} - 1)(\mathrm{e}^{-qt} - 1 + qt) + qt(\mathrm{e}^{qt} - 1)(\mathrm{e}^{-2qT} - 1), \end{aligned}$$

which is obviously positive for $q < 0$ and $0 < t < T$. For $q > 0$ we have to show that $g_q(t) < 2 - \mathrm{e}^{qt} - \mathrm{e}^{-qt}$ for all $0 < t < T$. Now

$$\begin{aligned} &2 - \mathrm{e}^{qt} - \mathrm{e}^{-qt} - g_q(t) \\ &\quad = 1 - \mathrm{e}^{qt} + \mathrm{e}^{-2q(T-t)} - \mathrm{e}^{-2qT+qt} + qt\mathrm{e}^{qt}(1 - \mathrm{e}^{-2qT}) \\ &\quad =: \tilde{g}_q(t) \end{aligned}$$

for which $\tilde{g}_q(0) = 0$ holds and we have

$$\begin{aligned} \tilde{g}_q'(t) &= -q\mathrm{e}^{qt} + 2q\mathrm{e}^{-2q(T-t)} - q\mathrm{e}^{-2qT+qt} \\ &\quad + q\mathrm{e}^{qt}(1 - \mathrm{e}^{-2qT}) + q^2 t\mathrm{e}^{qt}(1 - \mathrm{e}^{-2qT}) \\ &= 2q\mathrm{e}^{-2qT+qt}(\mathrm{e}^{qt} - 1) + q^2 t\mathrm{e}^{qt}(1 - \mathrm{e}^{-2qT}) > 0 \end{aligned}$$

for $q > 0$ and $0 < t < T$, which completes the proof. Hence, by (3.4) and (3.8), we get (3.7). □

Moreover, by (3.7), the expected quadratic path deviations satisfy the following inequalities: if $q > 0$, then

$$\int_0^T \mathbb{E}\big((U_t^0 - U_t^{\mathrm{ir}})^2\big)\,\mathrm{d}t < \int_0^T \mathbb{E}\big((U_t^0 - U_t^{\mathrm{st}})^2\big)\,\mathrm{d}t < \int_0^T \mathbb{E}\big((U_t^0 - U_t^{\mathrm{av}})^2\big)\,\mathrm{d}t,$$

and if $q < 0$, then

$$\int_0^T \mathbb{E}\big((U_t^0 - U_t^{\mathrm{ir}})^2\big)\,\mathrm{d}t < \int_0^T \mathbb{E}\big((U_t^0 - U_t^{\mathrm{av}})^2\big)\,\mathrm{d}t < \int_0^T \mathbb{E}\big((U_t^0 - U_t^{\mathrm{st}})^2\big)\,\mathrm{d}t.$$

In the next theorem we get more explicit representations of the expected quadratic path deviations.

Theorem 3.6. *For $a = 0$ and $b \in \mathbb{R}$, we have*

$$\mathbb{E}\bigg(\int_0^T (U_t^0 - U_t^{\mathrm{av}})^2\,\mathrm{d}t\bigg) = \frac{b^2}{4q}\cdot\frac{\sinh(2qT) - 2qT}{\sinh^2(qT)} + \frac{\sigma^2\mathrm{e}^{qT}}{4q^2}\cdot\frac{\sinh(2qT) - 2qT}{\sinh(qT)},$$

$$\begin{aligned} \mathbb{E}\bigg(\int_0^T (U_t^0 - U_t^{\mathrm{ir}})^2\,\mathrm{d}t\bigg) &= \frac{b^2}{4q}\cdot\frac{\sinh(2qT) - 2qT}{\sinh^2(qT)} + \frac{\sigma^2\mathrm{e}^{qT}}{4q^2}\cdot\frac{\sinh(2qT) - 2qT}{\sinh(qT)} \\ &\quad - \frac{\sigma^2}{q^2} + \frac{T\sigma^2\cosh(qT)}{q\sinh(qT)} - \frac{\sigma^2 T^2}{2} + \frac{\sigma^2 T}{2q} - \frac{\sigma^2}{4q^2} \\ &\quad + \frac{\sigma^2}{4q^2}\mathrm{e}^{-2qT} - \frac{\sigma^2}{q^2}\int_0^{qT}(1 - \mathrm{e}^{-2x})\log\frac{\sinh(qT)}{\sinh(x)}\,\mathrm{d}x, \end{aligned}$$

$$\mathbb{E}\bigg(\int_0^T (U_t^0 - U_t^{\mathrm{st}})^2\,\mathrm{d}t\bigg) = \frac{\sigma^2\mathrm{e}^{qT}}{4q^2}\cdot\frac{\sinh(2qT) - 2qT}{\sinh(qT)} + \frac{2\sigma^2}{q^2}\cdot\frac{\cosh(qT) - 1}{\sinh(qT)}$$

$$
- \frac{\sigma^2}{q}\Big(\frac{\sinh(2qT)}{2q} + T\Big) + \frac{\sigma^2 \cosh(qT)}{2q^2 \sinh(qT)}\big(\cosh(2qT) - 1\big).
$$

Proof. For the anticipative version, by Proposition 3.4, we get

$$
\begin{aligned}
&\mathbb{E}\Big(\int_0^T (U_t^0 - U_t^{\mathrm{av}})^2 \,\mathrm{d}t\Big) \\
&\quad = \int_0^T \mathbb{E}(U_t^0 - U_t^{\mathrm{av}})^2 \mathrm{d}t \\
&\quad = \int_0^T \mathrm{e}^{qT} \frac{\sigma^2}{q} \frac{\sinh^2(qt)}{\sinh(qT)} \,\mathrm{d}t + \int_0^T b^2 \frac{\sinh^2(qt)}{\sinh^2(qT)} \,\mathrm{d}t \\
&\quad = \frac{\mathrm{e}^{qT}\sigma^2}{q \sinh(qT)} \int_0^T \sinh^2(qt) \,\mathrm{d}t + \frac{b^2}{\sinh^2(qT)} \int_0^T \sinh^2(qt) \,\mathrm{d}t \\
&\quad = \frac{\mathrm{e}^{qT}\sigma^2}{2q \sinh(qT)}\Big(\frac{\sinh(2qT)}{2q} - T\Big) + \frac{b^2}{2\sinh^2(qT)}\Big(\frac{\sinh(2qT)}{2q} - T\Big).
\end{aligned}
$$

For the integral representation, by Proposition 3.4 and the previous calculations for the anticipative version, we get

$$
\begin{aligned}
&\mathbb{E}\Big(\int_0^T (U_t^0 - U_t^{\mathrm{ir}})^2 \,\mathrm{d}t\Big) \\
&\quad = \frac{b^2}{4q} \frac{\sinh(2qT) - 2qT}{\sinh^2(qT)} + \frac{\sigma^2 \mathrm{e}^{qT}}{4q^2} \frac{\sinh(2qT) - 2qT}{\sinh(qT)} \\
&\qquad - \frac{2\sigma^2}{q} \int_0^T \sinh\big(q(T-t)\big)\mathrm{e}^{-q(T-t)}\Big(qt + \log \frac{\sinh(qT)}{\sinh(q(T-t))}\Big) \,\mathrm{d}t \\
&\qquad + \frac{2\sigma^2}{q \sinh(qT)} \int_0^T \sinh\big(q(T-t)\big) \sinh(qt) \,\mathrm{d}t.
\end{aligned}
$$

Here

$$
\begin{aligned}
&\int \sinh\big(q(T-t)\big) \sinh(qt) \,\mathrm{d}t \\
&\quad = \int \big(\sinh(qT)\cosh(qt) - \cosh(qT)\sinh(qt)\big) \sinh(qt) \,\mathrm{d}t \\
&\quad = \sinh(qT) \int \cosh(qt) \sinh(qt) \,\mathrm{d}t - \cosh(qT) \int \sinh^2(qt) \,\mathrm{d}t \\
&\quad = \frac{\sinh(qT)}{2} \int \sinh(2qt) \,\mathrm{d}t - \frac{\cosh(qT)}{2}\Big(\frac{\sinh(2qt)}{2q} - t\Big)
\end{aligned}
$$

$$= \frac{\sinh(qT)\cosh(2qt)}{4q} - \frac{\cosh(qT)\sinh(2qt)}{4q} + \frac{t\cosh(qT)}{2}$$
$$= \frac{\sinh(q(T-2t))}{4q} + \frac{t\cosh(qT)}{2},$$

and, hence,

$$\frac{2\sigma^2}{q\sinh(qT)}\int_0^T \sinh\bigl(q(T-t)\bigr)\sinh(qt)\,\mathrm{d}t$$
$$= \frac{2\sigma^2}{q\sinh(qT)}\biggl(-\frac{\sinh(qT)}{4q} + \frac{T\cosh(qT)}{2} - \frac{\sinh(qT)}{4q}\biggr)$$
$$= -\frac{\sigma^2}{q^2} + \frac{T\sigma^2\cosh(qT)}{q\sinh(qT)}.$$

We also have, by partial integration,

$$\int \sinh\bigl(q(T-t)\bigr)\mathrm{e}^{-q(T-t)}qt\,\mathrm{d}t$$
$$= \int \frac{1-\mathrm{e}^{-2q(T-t)}}{2}qt\,\mathrm{d}t$$
$$= \frac{q}{2}\biggl(\frac{t^2}{2} - \int t\mathrm{e}^{-2q(T-t)}\,\mathrm{d}t\biggr)$$
$$= \frac{qt^2}{4} - \frac{q}{2}\biggl(\frac{t\mathrm{e}^{-2q(T-t)}}{2q} - \int \frac{\mathrm{e}^{-2q(T-t)}}{2q}\,\mathrm{d}t\biggr)$$
$$= \frac{qt^2}{4} - \frac{t\mathrm{e}^{-2q(T-t)}}{4} + \frac{\mathrm{e}^{-2q(T-t)}}{8q},$$

and, hence,

$$-\frac{2\sigma^2}{q}\int_0^T \sinh\bigl(q(T-t)\bigr)\mathrm{e}^{-q(T-t)}qt\,\mathrm{d}t = -\frac{2\sigma^2}{q}\biggl(\frac{qT^2}{4} - \frac{T}{4} + \frac{1}{8q} - \frac{1}{8q}\mathrm{e}^{-2qT}\biggr).$$

Moreover, by the change of variables $q(T-t) = x$, we get

$$-\frac{2\sigma^2}{q}\int_0^T \sinh\bigl(q(T-t)\bigr)\mathrm{e}^{-q(T-t)}\log\frac{\sinh(qT)}{\sinh(q(T-t))}\,\mathrm{d}t$$
$$= -\frac{2\sigma^2}{q}\int_0^{qT}\frac{1-\mathrm{e}^{-2x}}{2}\log\frac{\sinh(qT)}{\sinh(x)}\,\mathrm{d}x,$$

and then we get the formula for $\mathbb{E}(\int_0^T (U_t^0 - U_t^{\mathrm{ir}})^2 \,\mathrm{d}t)$. We note that we are unable to solve the integral $\int_0^{qT} (1 - \mathrm{e}^{-2x}) \log \frac{\sinh(qT)}{\sinh(x)} \,\mathrm{d}x$.

Finally, for the space–time transform, by Proposition 3.4 and the previous calculations for the anticipative version, we get

$$\begin{aligned}
\mathbb{E}\biggl(\int_0^T (U_t^0 - U_t^{\mathrm{st}})^2 \,\mathrm{d}t\biggr) &= \int_0^T \mathbb{E}(U_t^0 - U_t^{\mathrm{st}})^2 \,\mathrm{d}t \\
&= \frac{\sigma^2 \mathrm{e}^{qT}}{4q^2} \cdot \frac{\sinh(2qT) - 2qT}{\sinh(qT)} \\
&\quad + \frac{2\sigma^2}{q \sinh(qT)} \int_0^T \sinh\bigl(q(T-t)\bigr)\bigl(1 - \cosh(qt)\bigr) \,\mathrm{d}t \\
&= \frac{2\sigma^2}{q \sinh(qT)} \biggl(-\frac{1}{q}\bigl(1 - \cosh(qT)\bigr) \\
&\qquad - \int_0^T \sinh\bigl(q(T-t)\bigr) \cosh(qt) \,\mathrm{d}t\biggr) \\
&\quad + \frac{\sigma^2 \mathrm{e}^{qT}}{4q^2} \cdot \frac{\sinh(2qT) - 2qT}{\sinh(qT)}.
\end{aligned}$$

Here

$$\begin{aligned}
&\int_0^T \sinh\bigl(q(T-t)\bigr) \cosh(qt) \,\mathrm{d}t \\
&\quad = \sinh(qT) \int_0^T \cosh^2(qt) \,\mathrm{d}t - \cosh(qT) \int_0^T \sinh(qt) \cosh(qt) \,\mathrm{d}t \\
&\quad = \frac{\sinh(qT)}{2} \biggl(\frac{\sinh(2qT)}{2q} + T\biggr) - \frac{\cosh(qT)}{4q}\bigl(\cosh(2qT) - 1\bigr),
\end{aligned}$$

and, hence, we get the formula for $\mathbb{E}(\int_0^T (U_t^0 - U_t^{\mathrm{st}})^2 \,\mathrm{d}t)$. □

We note that the formulas $\mathbb{E}(\int_0^T (U_t^0 - U_t^{\mathrm{br}})^2 \,\mathrm{d}t)$ are harder to compare than the variances in Proposition 3.4 are. It might also be possible to calculate the Gauss conditional distribution of path deviations given $U_T^0 = d$ using Theorem 2 and Problem 5 in Chapter II, Section 13 of Shiryaev (1996), and to calculate corresponding formulas for conditional quadratic path deviations. But even if these formulas are present, in general it will not be easy to compare the conditional quadratic path deviations, since they will depend on the four parameters q, b, d, T and possibly also on σ. We renounce to give these explicit and likewise very long calculations.

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