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Sample-Path Solution of Stochastic Variational Inequalities

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Sample-path optimization is a simulation-based method for solving optimization problems that arise in the study of complex stochastic systems. In this paper we broaden its applicability to include the solution of stochastic variational inequalities. This formulation can model equilibrium phenomena in physics, economics, and operations research. We describe the method, provide general conditions for convergence, and present numerical results of an application of the method to a stochastic economic equilibrium model of the European natural gas market. We also point out some current limitations of the method and indicate areas in which research might help to remove those limitations.

Key words: Variational inequality, sample-path optimization, simulation, coherent orientation, strong regularity

1 Introduction

In this paper we present a method for solving a variational inequality defined by a polyhedral convex set C and a function f_∞ that is an almost-sure

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limit of a computable sequence of random functions f_n. Such a situation commonly arises in simulation optimization and stochastic economic equilibrium problems involving expectations or steady-state functions. The method is a modification of *sample-path optimization* proposed by Plambeck *et al.* [23,24] and analyzed in [29]. That method, in turn, is closely related to M-estimation [15,16,38,39] and to the stochastic counterpart method [32]; for additional references see [29]. As in the original version, the functions f_n may be outputs of a simulation run of length n: from observing the f_n and making appropriate computations with them, we hope to find an approximate solution of the variational inequality defined by C and f_∞. The approach we present here was outlined without proofs in the proceedings paper [9].

In our view, the present work makes the following contributions:

- To propose a modeling structure for stochastic variational inequalities in which the expectation or limit functions must be estimated by simulation,
- To present conditions under which the approximating problems can be shown to have solutions with probability 1, provided the simulation run length is sufficiently long,
- To provide bounds for the closeness of those solutions to solutions of the limit problem, in terms of the goodness of approximation,
- To present numerical experiments on problems of moderate size to show that the technique can be implemented and to illustrate its performance.

In the process of establishing these points, we hope to show also that by using simulation – together with gradient estimation techniques – one may provide an effective alternative to discrete scenario representations of uncertainty with their associated data management problems (see [20] for example).

The deterministic variational inequality problem has been much used since its origins in the middle 1960s, for which see e.g. [11,36]. Its application to the modeling of economic equilibrium in the Project Independence study during the 1970s generated great interest; see Hogan [14]. Harker and Pang [10] present an excellent survey of the developments in the subject up to 1990, whereas Ferris and Pang [6] examine a large number of applications of complementarity problems in engineering and economics.

Josephy [17] introduced a variant of the Newton method for numerical solution of generalized equations, which are reformulations of variational inequalities. Since then, much progress has been made in globalizing the Newton idea and making it more robust. One of the best current production software packages embodying this idea is the PATH solver of Dirkse and Ferris [3], a stabilized Newton method based on a generalization of the line search idea.

As far as we are aware, the idea of using stochastic variational inequalities to model uncertainty in equilibrium phenomena has not been extensively used.

Exceptions are the works of Haurie *et al.* [12] and De Wolf and Smeers [2]. Haurie *et al.* used a dynamic stochastic model to find the equilibrium price and quantity of the natural gas to be traded in the European market. However, their model contained an affine price/demand relation that resulted in an *integrable* model: that is, one which reduces to an optimization problem. In Section 4, we extend their model to incorporate a different price/demand relationship, resulting in a non-integrable model, and introduce a different stochastic structure. In particular, while the model of [12] employed a scenario representation of uncertainty with a scenario tree of only four branches, we propose using simulation to observe a large number of instances of the random parameters, averaging their effects, and solving the resulting problem.

Of course, these ideas are not new in the case of optimization: see [23,24,29,34]. Shapiro considers similar questions in [33], where the asymptotic behavior of solutions of simulation optimization problems is studied. The conditions developed there differ from those we provide here in two main respects; first, our conditions apply to general variational inequalities, not just those arising from optimization problems; second, we prove the existence and convergence of solutions of the approximating problems, whereas Shapiro assumes them. In order to obtain these stronger results we make stronger assumptions than does Shapiro: in particular, we use a *coherent orientation* condition that is equivalent to the well-known condition of strong regularity, for which see e.g. [5,22,25].

The remainder of this paper is organized in four sections. In Section 2, we formulate the problem and describe the method that we propose. We also provide the necessary background on variational inequalities including definitions and discussions of the normal map, critical cone, normal manifold, coherent orientation, and continuous convergence. In Section 3, we analyze the method using the concepts explained in Section 2. In Section 4, we present the results of numerical experiments applying the method to an energy planning problem in the European natural gas market. We solve two versions of this problem, modeling uncertainty with discrete and with continuous probability distributions. Finally, in Section 5 we summarize the work we have presented and suggest some directions for future research.

2 Background and Formulation

In this section we define the variational inequality formulation that we shall use, explain some general aspects of the methodology, and present definitions of certain concepts from analysis that will be needed in what follows. These include a generalization of nonsingularity that is of particular importance to our method.

For a closed convex set C (in general, a subset of a Hilbert space, but here a subset of $\mathbb{R}^k$) and a function f from an open set $\Theta \subset \mathbb{R}^k$ to $\mathbb{R}^k$, the *variational inequality* problem is to find a point $x_0 \in C \cap \Theta$, if any exists, satisfying

$$\text{for each } x \in C, \quad \langle x - x_0, f(x_0) \rangle \geq 0, \tag{1}$$

where $\langle y, z \rangle$ denotes the inner product of y and z. Geometrically, (1) means that $f(x_0)$ is an inward normal to C at x_0. Many problems from applications involve polyhedral sets C, and in this paper we restrict our attention to such sets. Two immediate examples are a system of k nonlinear equations in k unknowns, and the first-order necessary optimality conditions for a nonlinear-programming problem with continuously differentiable objective and constraint functions.

Naturally, not all variational inequality problems arise from optimization. In some economic equilibrium models the lack of certain symmetry properties results in a model that is said to be *non-integrable.* In such a model, it is not possible to find the equilibrium prices and quantities by substituting an associated optimization problem for the variational inequality. For discussion of an actual model of this type that was heavily used in policy analysis, see [14]. The theory that we shall develop here does not require any symmetry properties, so it applies to non-integrable models. In fact, the application to the European gas market given in Section 4 required the solution of a non-integrable stochastic economic equilibrium model.

Our method works with a vector-valued stochastic process $\{f_n(\omega, x) \mid n = 1, 2, \dots\}$ and a vector-valued function $f_\infty(x)$ with $x \in \mathbb{R}^k$. For all $n \geq 1$ and $x \in \mathbb{R}^k$, the random variables $f_n(\omega, x)$ are defined on a common probability space $(\Omega, \mathcal{F}, P)$, and for almost all ω the $f_n(\omega, \cdot)$ converge pointwise to $f_\infty(\cdot)$. The method seeks a point x_0 solving (1) with $f = f_\infty$; it works by fixing a large n and a sample point ω, solving the deterministic variational inequality with $f(\cdot) = f_n(\omega, \cdot)$, and taking the solution $x_n(\omega)$ as an estimate of x_0. In Section 3 we give conditions ensuring that with probability 1, when n is sufficiently large the $x_n(\omega)$ exist and are close to x_0.

One way of envisioning this setup is to regard $f_n(\omega, x)$ as an estimate of the function value $f_\infty(x)$ obtained from a simulation run of length n, where x is the decision variable and ω represents the random element, i.e., the random number streams used in the simulation. One can use the method of common random numbers to evaluate $f_n(\omega, x)$ for different values of x. Furthermore, exact values for the derivatives or directional derivatives of the f_n can be obtained using well-established methods of gradient estimation such as infinitesimal perturbation analysis; see [8,13,37]. The method proposed here thus inherits most of the advantages of sample-path optimization, such as the ability to use deterministic techniques to solve the variational inequality.

In the particular case of an unconstrained optimization problem the method takes a special form. If we write the first order optimality conditions for the problem, we obtain k nonlinear equations in k unknowns. Then solving the associated variational inequality would amount to finding a zero of the gradient of the objective function. In the stochastic context the approximate solution of this problem will be an estimate of a critical point of this objective function. Such a point may not be an optimizer unless certain second-order conditions are satisfied. However, when the objective function is known to be locally convex (or locally concave, depending on the sense of optimization), around the critical point, the solution point will be an optimizer. See [9] for an actual implementation in that case.

At this point it may be helpful to correct a possible misconception about this formulation. One might think, because we use an expectation or limit function in the variational inequality that we try to solve, that this method is some variant of the so-called "expected value method" in stochastic optimization, in which a random variable is replaced by its expected value before solution (thereby producing an incorrect formulation). This is not so, as one can see by considering the simple example of minimizing (in x) the function $\Phi(x) := \mathrm{E}_\xi \phi(x, \xi)$, where

$$\phi(x,\xi) = \begin{cases} 3\alpha(\xi - x)^2 & \text{if } \xi \le x, \\ 3\beta(\xi - x)^2 & \text{if } \xi \ge x. \end{cases}$$

Here we suppose that $0 < \beta \le \alpha$, and that ξ is uniformly distributed on [0,1]. This model formulates the problem of selecting x to estimate an uncertain quantity ξ, with a quadratic loss whose magnitude depends on whether one has overestimated or underestimated ξ.

A computation shows that

$$\Phi(x) = \begin{cases} \beta x^3 + \beta(1-x)^3, & \text{if } x \le 0, \\ \alpha x^3 + \beta(1-x)^3, & \text{if } x \in [0,1], \\ \alpha x^3 + \alpha(1-x)^3, & \text{if } x \ge 1, \end{cases}$$

so that Φ is convex and C^2; its minimizer is

$$\hat{x} := (\alpha - \beta)^{-1}[(\alpha\beta)^{1/2} - \beta].$$

To illustrate our approach on this simple problem, we note that ϕ satisfies the conditions for interchange of derivative and expectation, so that we can express the necessary and sufficient optimality condition

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\Phi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{E}_\xi \phi(x,\xi)$$

as

$$0 = \mathrm{E}_\xi \frac{\mathrm{d}}{\mathrm{d}x}\phi(x,\xi).$$

Our method would then proceed to approximate $\mathrm{E}_\xi \frac{\mathrm{d}}{\mathrm{d}x}\phi(x,\xi)$ by simulation and to find a zero of the resulting function. This procedure would approximate the correct minimizer. By contrast, if we replaced the random variable ξ by its expectation of $1/2$, we would obtain the function

$$\tilde{\Phi}(x) = \phi(x, \mathrm{E}\xi) = \begin{cases} 3\alpha(.5 - x)^2 & \text{if } x \geq .5, \\ 3\beta(.5 - x)^2 & \text{if } x \leq .5, \end{cases}$$

whose minimizer is evidently $x = .5$, a point that does not minimize $\Phi(x)$ unless $\alpha = \beta$.

Returning to the description of our method, we note that in order to guarantee the closeness of the estimate $x_n(\omega)$ to the true solution x_0 we need to impose certain functional convergence properties on the sequence $\{f_n\}$. The specific property we require is called *continuous convergence*; it is equivalent to uniform convergence to a continuous limit on compact sets [18], and is defined as follows:

Definition 1 *A sequence f_n of extended-real-valued functions defined on $\mathbb{R}^k$* converges continuously *to an extended-real-valued function f_∞ defined on $\mathbb{R}^k$ (written $f_n \xrightarrow{C} f_\infty$) if for any $x \in \mathbb{R}^k$ and any sequence $\{x_n\}$ converging to x, one has $f_n(x_n) \to f_\infty(x)$. A sequence of functions from $\mathbb{R}^k$ into $\mathbb{R}^m$ converges continuously if each of the m component functions does so.*

To understand the rationale for requiring continuous convergence, consider a sequence of functions f_n and of points x_n such that for each n, x_n solves the variational inequality defined by f_n and C, and $x_n \to x$ as $n \to \infty$. Now if $f_n \xrightarrow{C} f_\infty$ then the limit point x is a solution of the limit variational inequality defined by f_∞ and C. Therefore we might reasonably use solutions of the former as estimates of the limit problem. However, although this result is useful, it unfortunately guarantees neither the existence of the solutions x_n nor their convergence.

To guarantee such existence and convergence we need to impose a certain generalized nonsingularity condition. We begin with several definitions required to explain this condition. The first is that of the *normal map* associated with (1). This map is used to convert a variational inequality defined by f and C to a single-valued equation; it is defined by $f_C(z) = f(\Pi_C(z)) + z - \Pi_C(z)$, where Π_C is the Euclidean projector on C. If x_0 solves (1) then $z_0 = x_0 - f(x_0)$ satisfies $f_C(z_0) = 0$. Further, if z_0 is a zero of f_C then $x_0 = \Pi_C(z_0)$ solves (1); see [28] for example.

Next, we define the *normal cone* $N_C(x)$ of C at x to be the set

$$\{y^* \mid \text{for each } c \in C, \quad \langle y^*, c - x\rangle \leq 0\}$$

provided that $x \in C$, and to be empty otherwise. An equivalent way of expressing (1) is then the *generalized equation*

$$0 \in f(x_0) + N_C(x_0). \tag{2}$$

If $x \in C$ then the *tangent cone* of C at x, written $T_C(x)$, is the polar of $N_C(x)$: that is, the set of all y such that $\langle y^*, y\rangle \leq 0$ for each $y^* \in N_C(x)$. The *critical cone* defined by C and a given point $z \in \mathbb{R}^k$ (not necessarily in C) is then defined by

$$K(z) = T_C(\Pi_C(z)) \cap \{y^* \in \mathbb{R}^k \mid \langle y^*, z - \Pi_C(z)\rangle = 0\}.$$

Now fix any z and write $K = K(z)$. As K is polyhedral it has only finitely many faces; for each nonempty face F the normal cone of K takes a constant value, say N_F, on the relative interior of F. Then the set $\sigma_F = F + N_F$ is a nonempty polyhedral convex set of dimension k in $\mathbb{R}^k$. The collection $\mathcal{N}_K$ of all these σ_F for nonempty faces F of K is called the *normal manifold* of K; see [27]. In each of these σ_F the projector Π_K reduces to an affine map (generally different for different σ_F). If A is a linear transformation from $\mathbb{R}^k$ to $\mathbb{R}^k$, then we say that the normal map A_K is *coherently oriented* if in each σ_F the determinant of the affine map obtained by restricting A_K to σ_F has the same nonzero sign. As a simple illustration of this property, we can consider the case in which K happens to be a subspace (the "strict complementary slackness" situation in the optimization literature). Then the coherent orientation requirement reduces to nonsingularity of the *section* of A in K: that is, the linear map from K to K given by $\Pi_K \circ (A|K)$. In particular, if $C = \mathbb{R}^k$ (the case of nonlinear equations), then $K(z) = \mathbb{R}^k$ for each $z \in \mathbb{R}^k$, and then $\mathcal{N}_{K(z)}$ has only one cell, namely $\mathbb{R}^k$ itself. Then A_K is coherently oriented exactly when A is nonsingular. In general, the coherent orientation condition is a way of generalizing nonsingularity to the case of a nontrivial set C.

3 Solution of Stochastic Variational Inequalities

In this section we discuss the solution of stochastic variational inequalities via the sample-path method. The main result, Theorem 5, shows that under mild regularity conditions on the limit function f and on a solution x_0 of the original problem (1), for any continuous sample function f_n that is sufficiently close to f the variational inequality

$$\text{find } x \in C \text{ such that } \langle f_n(x), c - x \rangle \geq 0 \text{ for all } c \in C,$$

has a solution close to x_0. We first prove a few auxiliary technical results, then proceed with the main convergence result, and finally comment on its implications.

Theorem 2 *Let f be a continuous function from an open set Θ in $\mathbb{R}^k$ to $\mathbb{R}^k$ having an inverse that is Lipschitzian on $f(\Theta)$ with modulus $\lambda > 0$. Let x_* be a point of Θ such that $f(x_*) = 0$. Then there is a positive number α_0 with $B(x_*, \alpha_0) \subset \Theta$ such that for any $\alpha \in (0, \alpha_0]$ and any continuous function $g : B(x_*, \alpha) \to \mathbb{R}^k$ satisfying*

$$\gamma := \sup_{x \in B(x_*, \alpha)} \|f(x) - g(x)\| < \lambda^{-1}\alpha, \tag{3}$$

the function g has at least one zero, and each such zero lies in $B(x_*, \lambda\gamma)$.

PROOF. $f(\Theta)$ is open by invariance of domain ([4], Prop. 7.4); hence it is a neighborhood of the origin in $\mathbb{R}^k$. Continuity of f then implies that there is a positive α_0 such that $B(x_*, \alpha_0) \subset \Theta$ and

$$f(B(x_*, \alpha_0)) + \lambda^{-1}\alpha_0 B \subset f(\Theta), \tag{4}$$

where B is the unit ball. Fix $\alpha \in (0, \alpha_0]$ and observe that (4) still holds if we substitute α for α_0. For brevity write $B_* = B(x_*, \alpha)$. Let g be a function from B_* to $\mathbb{R}^k$ with the properties listed in the statement of the theorem. Then for each $x \in B_*$ and each $t \in [0, 1]$ we have

$$\begin{aligned}(1-t)g(x) + tf(x) &= f(x) + (1-t)[g(x) - f(x)] \\ &\subset f(B_*) + \lambda^{-1}\alpha B \subset f(\Theta).\end{aligned}$$

Therefore $H(x, t) = f^{-1}[(1-t)g(x) + tf(x)]$ is a well defined and continuous function from $B_* \times [0, 1]$ into $\mathbb{R}^k$.

Observe that for each x on the boundary of B_* and any $t \in [0,1]$, $H(x,t) \neq x_*$. Otherwise, for some such x and t we would have

$$f(x_*) - f(x) = (1-t)[g(x) - f(x)].$$

But then

$$\lambda^{-1}\alpha = \lambda^{-1}\|x_* - x\| \leq \|f(x_*) - f(x)\| \leq \gamma < \lambda^{-1}\alpha,$$

and we reach a contradiction.

Now let i be the identity of B_*. Applying the homotopy invariance principle ([21], Th. 6.2.2), we find that

$$\deg(f^{-1} \circ g, \operatorname{int} B_*, x_*) = \deg(i, \operatorname{int} B_*, x_*) = 1.$$

Accordingly, there must be at least one point $\bar{x}$ in B_* such that $f^{-1}(g(\bar{x})) = x_*$: that is, such that $g(\bar{x}) = f(x_*) = 0$.

Next, let x belong to B_* but with $\|x - x_*\| > \lambda\gamma$. Since

$$\begin{aligned}\|f^{-1}[(1-t)g(x) + tf(x)] - f^{-1}[f(x)]\| &\leq \lambda\|(1-t)[g(x) - f(x)]\| \\ &\leq (1-t)\lambda\gamma,\end{aligned}$$

we must have

$$\begin{aligned}\|H(x,t) - x_*\| &\geq \|x - x_*\| - \|f^{-1}[(1-t)g(x) + tf(x)] - f^{-1}[f(x)]\| \\ &\geq \|x - x_*\| - (1-t)\lambda\gamma. \end{aligned} \tag{5}$$

Taking $t = 0$ we find that $f^{-1}[g(x)] \neq x_*$ and therefore $g(x) \neq 0$. Hence each zero of g lies in $B(x_*, \lambda\gamma)$. □

The next result is a simple corollary of Theorem 2 that investigates the behavior of approximate solutions of the function g in that theorem.

Corollary 3 *Let f, Θ, λ, x_*, and α_0 be as in Theorem 2, and let $\alpha \in (0, \alpha_0]$. Let h be a continuous function from $B(x_*, \alpha)$ to $\mathbb{R}^k$ and y a point of $\mathbb{R}^k$ such that*

$$\eta := \sup_{x \in B(x_*, \alpha)} \|f(x) - h(x)\|, \qquad \eta + \|y\| < \lambda^{-1}\alpha.$$

Then the function $h - y$ has at least one zero, and each such zero lies in $B(x_*, \lambda(\eta + \|y\|))$.

PROOF. In Theorem 2 take $g = h - y$, noting that then $\gamma \le \eta + \|y\|$. □

In order to exhibit a class of functions to which the foregoing analysis applies, we put together some known results to obtain a simple inverse-function theorem for normal maps.

Theorem 4 *Let C be a polyhedral convex set in $\mathbb{R}^k$, z_0 be a point of $\mathbb{R}^k$ and x_0 be the Euclidean projection of z_0 on C. Let f be a function from an open set Θ containing x_0 to $\mathbb{R}^k$. Suppose that $f_C(z_0) = 0$, and let K be the critical cone defined by C and z_0, i.e.,*

$$K = T_C(x_0) \cap \{v \in \mathbb{R}^k \mid \langle f(x_0), v\rangle = 0\}.$$

Assume that f has a strong Fréchet derivative $\mathrm{d}f(x_0)$ at x_0, and that $\mathrm{d}f(x_0)_K$ is coherently oriented. Then there is an open set U containing z_0 such that f_C is well defined on U, $f_C(U)$ is open, and $f_C|U$ has a Lipschitzian inverse on $f_C(U)$.

PROOF. The normal map f_C is well defined on the open set $\Pi_C^{-1}(\Theta)$, which contains z_0. Define a function L from $\mathbb{R}^k$ to $\mathbb{R}^k$ by $L(x) = f(x_0) + \mathrm{d}f(x_0)(x - x_0)$, and note that $L_C(z_0) = f_C(0) = 0$. By applying Proposition 4.1 of [26] to the function $F(x,y) = f(x) - y$ we find that for each $y \in \mathbb{R}^k$ L_C strongly approximates $F(\,\cdot\,, y)_C(z)$ at (z_0, y) with respect to the variable z. Further, by Theorem 5.2 of [27], L_C is a local Lipschitzian homeomorphism at z_0. It follows that L_C carries a neighborhood N of z_0 onto a neighborhood of the origin in $\mathbb{R}^k$, and on the latter neighborhood L_C^{-1} is Lipschitzian with some modulus λ_0. Now apply Theorem 3.2 of [26] to conclude that for each $\lambda > \lambda_0$ there are neighborhoods U_0 of z_0 and V_0 of the origin, and a map $z : V_0 \to U_0$ that is Lipschitzian with modulus λ, such that for each $y \in V_0$, $z(y)$ is the unique solution in U_0 of $F(\,\cdot\,, y)_C(z) = 0$ (*i.e.*, of $f_C(z) = y$). As z is continuous there is an open subset V of V_0 such that $0 \in V$ and $z(V) =: U \subset \operatorname{int} U_0$. It is then straightforward to show that U is open, that $f_C(U) = V$, and that $z|V$ is the inverse of $f_C|U$. □

The proof of Theorem 4 shows how the Lipschitz modulus for f_C^{-1} can be estimated, via that of L_C^{-1}, if it is desired to do so.

Note again that if K happens to be a subspace (the "strict complementary slackness" situation), then the coherent orientation requirement reduces to nonsingularity of the section of $\mathrm{d}f(x_0)$ in K. In particular, if $C = \mathbb{R}^k$ (the case of nonlinear equations), then $K = \mathbb{R}^k$ and the requirement is that $\mathrm{d}f(x_0)$

be nonsingular, as one would expect. In that case λ can be taken to be any number greater than $\|df(x_0)^{-1}\|$.

We next prove the main result of the paper. Roughly speaking, it says that if the variational inequality defined by the limit function has a solution x_0 satisfying a generalized nonsingularity condition, then for sufficiently good approximations of the limit function the approximating problems must have solutions close to x_0.

Theorem 5 *Let Θ be an open subset of $\mathbb{R}^k$ and let C be a polyhedral convex set in $\mathbb{R}^k$. Let x_0 be a point of Θ, and suppose f is a function from Θ to $\mathbb{R}^k$. Let $\{f_n \mid n = 1, 2, \dots\}$ be random functions from Θ to $\mathbb{R}^k$ such that for all $x \in \Theta$ and all finite n the random variables $f_n(x)$ are defined on a common probability space $(\Omega, \mathcal{F}, P)$. Let $z_0 = x_0 - f(x_0)$ and assume the following:*

(a) With probability one, each f_n for $n = 1, 2, \dots$ is continuous and $f_n \xrightarrow{C} f$.
(b) x_0 solves the variational inequality defined by f and C.
(c) f has a strong Fréchet derivative $df(x_0)$ at x_0, and $df(x_0)_K$ is coherently oriented, where K is as defined in Theorem 4.

Then the restriction of f_C to a neighborhood of z_0 has an inverse that is Lipschitzian with some modulus λ. Further, there exist a compact subset $C_0 \subset C \cap \Theta$ containing x_0, a neighborhood $\Theta_1 \subset \Theta$ of x_0, a scalar $\beta > 0$, and a set $\Delta \subset \Omega$ of measure zero, with the following properties: for $n = 1, 2, \dots$, $\omega \in \Omega$, and $y \in \mathbb{R}^k$ with $\|y\| \le \beta$, let

$$\xi_n(\omega) = \sup_{x \in C_0} \|f_n(\omega, x) - f(x)\|,$$

and

$$X_n(\omega, y) := \{x \in C \cap \Theta_1 \mid \text{for each } c \in C,\ \langle f_n(\omega, x) - y, c - x \rangle \ge 0\}.$$

For each $\omega \notin \Delta$ there is then a finite integer N_ω such that for each $n \ge N_\omega$ the set $X_n(\omega, y)$ is a nonempty, compact subset of $B(x_0, \lambda(\xi_n(\omega) + \|y\|))$.

PROOF. Determine Δ having measure zero so that off Δ the properties listed in hypothesis (a) hold for all n. Let $\omega \notin \Delta$. We suppress ω from here on. Note that continuous convergence is equivalent to uniform convergence on compacts to a continuous limit [18], so f is continuous. We next show that $(f_n)_C \xrightarrow{C} f_C$. Suppose $x_n \to x_0$. Then

$$\begin{aligned}(f_n)_C(x_n) - f_C(x_0) = {} & [f_n(\Pi_C(x_n)) - f(\Pi_C(x_0))] \\ & + [(x_n - \Pi_C(x_n)) - (x_0 - \Pi_C(x_0))].\end{aligned}$$

The first term in brackets approaches zero by the continuous convergence of f_n (hypothesis (a)) and the continuity of Π_C; the second approaches zero because of the continuity of Π_C. Therefore $(f_n)_C \xrightarrow{C} f_C$. Next, use hypotheses (b) and (c) together with Theorem 4 to conclude that the restriction of the function f_C to some open neighborhood $U \subset \Pi_C^{-1}(\Theta)$ of z_0 has an inverse that is Lipschitzian on $f_C(U)$ with some modulus λ. Apply Theorem 2 to f_C on U to produce an α_0 with the properties listed in that theorem.

The Minty map $M(z) = (\Pi_C(z), z - \Pi_C(z))$ is a Lipschitzian homeomorphism from $\mathbb{R}^k$ onto N_C, and we know that $M(z_0) = (x_0, -f(x_0)) =: w_0$. Choose positive numbers ξ, η, and ϵ so that the balls $\Theta_1 = B(x_0, \xi)$ and $Y_1 = B(-f(x_0), \eta)$ satisfy the following conditions: (1) $B_0 := B(z_0, \alpha_0) \supset M^{-1}(W)$, where W is the neighborhood $N_C \cap (\Theta_1 \times Y_1)$ of w_0 in N_C; (2) $-f(\Theta_1) \subset Y_1$; (3) $\xi + \eta + \epsilon \le \alpha_0$. Now choose a positive α so that $\alpha < \min\{\alpha_0, \lambda\epsilon\}$ and $M^{-1}(W) \supset B(z_0, \alpha) =: B_1$. Let $C_0 = \Pi_C(B_0)$. First note that for any n, $(f_n)_C(z) - f_C(z) = f_n(\Pi_C(z)) - f(\Pi_C(z))$, so that

$$\xi_n := \sup_{x \in C_0} \|f_n(x) - f(x)\| = \sup_{z \in B_0} \|(f_n)_C(z) - f_C(z)\|.$$

Choose $\beta < \lambda^{-1}\alpha$, y with $\|y\| \le \beta$, and N large enough so that if $n \ge N$ then

$$\xi_n + \|y\| < \lambda^{-1}\alpha.$$

For $n \ge N$, Corollary 3 tells us that $(f_n - y)_C$ has a zero in B_1. But the projection onto C of any point of B_1 lies in $C \cap \Theta_1$, so the projection of this zero onto C lies in $C \cap \Theta_1$ and solves the variational inequality for $f_n - y$ and C. Therefore X_n is nonempty; it is compact because f_n is continuous and $X_n \subset \Theta_1$. Now suppose that some point $x_n' \in \Theta_1$ solves the variational inequality for $f_n - y$ and C. Then $x_n' \in C$. Let $z_n' = x_n' - f_n(x_n') + y$. Then $(f_n - y)_C(z_n') = 0$. Further,

$$\begin{aligned}\|z_n' - z_0\| &\le \|x_n' - x_0\| + \|f(x_n') - f(x_0)\| + \|f_n(x_n') - f(x_n')\| + \|y\| \\ &\le \xi + \eta + \xi_n + \|y\|.\end{aligned}$$

But $\xi_n + \|y\| < \lambda^{-1}\alpha < \epsilon$, so z_n' belongs to B_0. But then Corollary 3 tells us that in fact

$$\|z_n' - z_0\| \le \lambda(\sup_{z \in B_0} \|(f_n)_C(z) - f_C(z)\| + \|y\|) = \lambda(\xi_n + \|y\|),$$

and nonexpansivity of the projection implies $\|x_n' - x_0\| \le \|z_n' - z_0\|$. Therefore $X_n \subset B(x_0, \lambda(\xi_n + \|y\|))$. □

In Theorem 5 we could have considered a random limit function f instead of a deterministic one, in which case λ, C_0, Θ_1, and x_0 would depend on the sample point ω. However, in most problems for which we think the method might be useful, f is a deterministic function, (for example, the gradient of a steady-state performance measure in a stochastic system). The form of the theorem is very general in that it allows us to work not only with the f_n but also with small perturbations of the f_n. This is important when using a numerical method having finite precision to solve the variational inequality defined by f_n and C.

Also note that the conditions derived in Theorem 5 are directly applicable to a special case of stochastic variational inequalities: namely, mixed complementarity problems involving expectations or steady-state performance functions. When the original problem to be solved is a constrained stochastic optimization problem, the coherent orientation condition reduces to strong regularity of the generalized equation expressing the first order necessary optimality conditions. In the unconstrained case, this condition is equivalent to the non-singularity of the Hessian at the solution point. For applications to simulation optimization, see [22].

4 Numerical Example

In this section we illustrate an application of the foregoing methodology to a model of the European natural gas market. We describe the problem, explain briefly the procedure used, and finally present numerical results. Theoretically, the method we propose is capable of handling both discrete and continuous probability distributions; by considering both types of uncertainties in the gas market problem we illustrate its practical utility as well.

The problem is to find the market price and quantity of natural gas to be produced and shipped to various markets from various producers. We adopt the model and partially the notation of Haurie *et al.* [12], modifying several parameters and functional relations. In particular, we use a different price/demand relation that results in a non-integrable model. As was done in [12], we consider a decomposition of the European gas market into the producers and the consumers of natural gas. This is a simple preliminary model in the sense that it does not take into account the possibility of the transmission companies' and/or distributors' acting as players as well.

The model has m players (producers and exporters of natural gas), each player controlling a set of production units, n markets, and T time periods. Let U_i be the set of production units controlled by the ith producer, for $i = 1, \ldots, m$. For each $\ell \in U_i$, $j = 1, \ldots, n$, and $t = 1, \ldots, T$, the variables are defined as follows:

R_ℓ^t = remaining reserves at beginning of period t (R_ℓ^1 is given).

K_ℓ^t = available capacity at beginning of period t (K_ℓ^1 is given).

$q_{\ell j}^t$ = annual production that is shipped to market j during period t.

D_j^t = annual domestic production of market j during period t (given).

Q_j^t = total annual quantity available on market j during period t: that is,

$$Q_j^t = D_j^t + \sum_{i=1}^{m} \sum_{\ell \in U_i} q_{\ell j}^t. \tag{6}$$

I_ℓ^t = physical capacity invested in during period t for use in period $t+1$. Investment is assumed to take place in one lump sum at the end of period t.

$c_{\ell j}^t$ = constant marginal transportation cost for shipping from production unit ℓ to market j during period t.

y_t = number of years in period t. We used $y_s = 5$ for $s = 1, 2, 3$ and $y_4 = 20$.

r = the (yearly) interest rate in effect; we used $r = 0.10$.

During each period, each producer incurs a (constant) annual cash flow for every year in that period. To compute the present value of these cash flows we use a factor f_t expressing the time value of money: for period t,

$$f_t = [(1+r)^{y_t} - 1]/[r(1+r)^{\sum_{s=1}^{t} y_s}].$$

The investment costs are linear with marginal cost $\Gamma_\ell \geq 0$, whereas marginal production costs follow a curve that has a reverse L shape in order to ensure that the unit production cost is approximately constant when away from the production capacity but goes to infinity at the capacity limit; see [19]. Therefore the production cost function is taken to be $G_\ell(x) = a_\ell x - b_\ell \ln(X_\ell - x)$, where a_ℓ and b_ℓ are parameters, x represents the quantity produced, and X_ℓ is the capacity limit. Let $P_j^t(Q)$ be the price of natural gas in market j during period t when the amount available in that market is Q.

In contrast to the affine demand law used in [12], we use the following:

$$P_j^t(Q) = p0_j^t (Q/q0_j^t)^{1/e_j^t}, \tag{7}$$

where $p0_j^t$ and $q0_j^t$ are base prices and base demands respectively and e_j^t is the price elasticity of demand for natural gas in market j during period t. Unlike the affine demand law, (7) is generally non-integrable, so that the equilibrium condition cannot be expressed as an optimization problem.

We now develop the model itself, beginning with calculation of the net present value (NPV) of a producer's profit. In each year of period t, the revenue from shipments to market j from unit ℓ is $q_{\ell j}^t P_j^t(Q_j^t)$, and the corresponding shipping cost is $c_{\ell j}^t q_{\ell j}^t$, while the yearly production cost for unit ℓ is $G_\ell(\sum_{j=1}^n q_{\ell j}^t)$. The investment cost paid at the end of period t for unit ℓ is $\Gamma_\ell I_\ell^t$. Therefore the NPV of profit for producer i is

$$F_i(q_{\ell j}^t, I_\ell^t) = \sum_{t=1}^{T} \sum_{\ell \in U_i} \Big[f_t \Big\{ \sum_{j=1}^{n} [P_j^t(Q_j^t) - c_{\ell j}^t] q_{\ell j}^t - G_\ell(\sum_{j=1}^{n} q_{\ell j}^t) \Big\} - \Gamma_\ell I_\ell^t (1+r)^{-\sum_{s=1}^{t} y_s} \Big]. \tag{8}$$

For each i and each $\ell \in U_i$, reserves satisfy

$$R_\ell^{t+1} = R_\ell^t - y_t \sum_{j=1}^{n} q_{\ell j}^t, \qquad t = 1, \ldots, T, \tag{9}$$

and we require

$$R_\ell^{t+1} \geq 0, \qquad t = 1, \ldots, T; \tag{10}$$

recall that R_ℓ^1 is given. Similarly, the capacities satisfy

$$K_\ell^{t+1} = K_\ell^t + I_\ell^t, \qquad t = 1, \ldots, T, \tag{11}$$

and

$$K_\ell^t - \sum_{j=1}^{n} y_t q_{\ell j}^t \geq 0, \qquad t = 1, \ldots, T. \tag{12}$$

In addition to these, we have the quantity balance equation (6), and we require that the decision variables $q_{\ell j}^t$ and I_ℓ^t be nonnegative.

The producers are the USSR (the former Soviet Union), the Netherlands, Norway, and Algeria. Among those Norway has two producing units, whereas all the others have one each. There are six markets: UK, the Netherlands, FRGer (the former West Germany), BelLux (Belgium/Luxembourg), Italy, and France.

In each period, we consider a shutdown possibility in Algeria which results in an interruption of production/transportation of natural gas in the current period and in all future periods. Define ψ_t to be 0 if Algeria interrupts its production in period t and 1 otherwise; then this induces a change in the reserve constraints for Algeria in the following fashion (Algeria has one production unit; hence $\ell = 1$):

$$R_1^{t+1} = R_1^t \psi_t - \sum_{j=1}^{6} y_t q_{1j}^t. \tag{13}$$

This should not be regarded as a physical change in the reserves; it is imposed only to guarantee that no natural gas is traded between Algeria and the rest of Europe.

Our next step is to formulate the first-order necessary conditions for an equilibrium as a variational inequality in the variables $q_{\ell j}^t$ and I_ℓ^t. We begin by discarding the auxiliary variables Q_j^t, R_ℓ^t, and K_ℓ^t and rewriting the reserve depletion constraints by replacing (9) and (10) by

$$R_\ell^1 - \sum_{s=1}^{t} y_s \sum_{j=1}^{n} q_{\ell j}^s \geq 0, \quad \ell \in U_i, \quad i = 1, \dots 4, \quad t = 1, \dots, 4, \tag{14}$$

and replacing (11) and (12) by

$$K_\ell^1 + \sum_{s=1}^{t-1} I_\ell^s - \sum_{j=1}^{n} y_t q_{\ell j}^t \geq 0, \quad \ell \in U_i, \quad i = 1, \dots 4, \quad t = 1, \dots, 4, \tag{15}$$

with the standard convention that $\sum_1^0$ is vacuous.

The ith producer wishes to maximize $F_i(q_{\ell j}^t, I_\ell^t)$, given by (8) with Q_j^t replaced by the right-hand side of (6), subject to the constraints (14) and (15). These individual maximization problems are coupled by the presence in each problem of the production quantity decisions of all the other producers. For an equilibrium we wish to find values of the decision variables $q_{\ell j}^t$ and I_ℓ^t so that these maximizations take place simultaneously. We therefore write the first-order necessary optimality conditions for each producer's problem of maximizing (8) subject to the constraints (14) and (15) as well as nonnegativity of the decision variables $q_{\ell j}^t$ and I_ℓ^t, and try to solve these simultaneously. Note that with the demand law (7), the objective function is not generally convex, so the first-order necessary conditions may not always be sufficient.

To write these conditions we regard the left-hand sides of (14) and (15) as vectors of dimension γ_i, where $\gamma_i = 4$ for $i = 1, 2, 4$ and $\gamma_3 = 8$ (since Norway has two producing units, whereas all the others have one each), and we denote these vectors by $-g_i^1$ and $-g_i^2$ respectively. We then associate dual variables u_1^i and u_2^i, also of dimension γ_i, with the constraints $-g_i^1 \geq 0$ and $-g_i^2 \geq 0$ respectively. We write the quantities $q_{\ell j}^t$ for producer i (that is, for $\ell \in U_i$ and for all t and j) as a vector x_i of dimension α_i, where $\alpha_i = 4 \times 6 = 24$ for $i = 1, 2, 4$ and $\alpha_3 = 2 \times 4 \times 6 = 48$. Similarly, we write the physical capacity investments I_ℓ^t for $\ell \in U_i$ as a vector z_i of dimension β_i, where $\beta_i = 3$ for $i = 1, 2, 4$ and $\beta_3 = 6$.

As the constraints are affine, no constraint qualification is needed, and simultaneous satisfaction of the first-order conditions for all producers can then be expressed by the following generalized equation: for $i = 1, \ldots, 4$ find nonnegative vectors x_i, z_i, u_i^1, and u_i^2 such that

$$\begin{array}{llll} 0 \in -\frac{\partial F_i}{\partial x_i} & +u_i^1 \frac{\partial g_i^1}{\partial x_i} & +u_i^2 \frac{\partial g_i^2}{\partial x_i} & +N_{\mathbb{R}_+^{\alpha_i}}(x_i) \\ 0 \in -\frac{\partial F_i}{\partial z_i} & & +u_i^2 \frac{\partial g_i^2}{\partial z_i} & +N_{\mathbb{R}_+^{\beta_i}}(z_i) \\ 0 \in \ -g_1^i & & & +N_{\mathbb{R}_+^{\gamma_i}}(u_1^i) \\ 0 \in \ -g_2^i & & & +N_{\mathbb{R}_+^{\gamma_i}}(u_2^i) \end{array} \tag{16}$$

Note again that the instances of (16) for different i are coupled by the quantity decisions, so that (16) is actually a single variational inequality containing 175 variables: 120 quantities $q_{\ell j}^t$, 15 investment decisions I_ℓ^t, and a total of 40 dual variables.

The parameters used in the experiments are in Tables 1–3; subscripts ℓ are omitted. Prices are in 1983 dollars and quantities are in *mtoe* (million tons of oil equivalent).

Table 1
Cost coefficients and parameters for production units

Producer	a	b	X	K^1	R^1	Γ	c^1	c^2	c^3	c^4
USSR	1.606	51	80	3750	3750	0.0	0.58	0.56	0.55	0.55
Netherlands	1.212	67	80	1900	1900	0.0	0.14	0.13	0.13	0.12
Norway1	1.507	85	80	300	300	0.0	0.35	0.34	0.34	0.33
Norway2	1.507	85	80	0	1936	0.5	0.35	0.34	0.34	0.33
Algeria	2.102	96	80	3087	3087	0.0	0.70	0.69	0.64	0.62

For each period t, we assumed ψ_t to be 0 with probability 0.40. After sampling from the uniform distribution and generating sequences ψ_t^n, we computed the sample functions f_n by averaging the constraints (13). To compute the limiting distribution will generally not be possible for more complicated distributions; however in this case we can solve the variational inequality with f_∞ and compare the solution with the results of the simulations. Table 4 shows the results for different simulation lengths, $n = 20$, $n = 200$, $n = 1000$, $n = 15000$, and $n = 100000$, as well as for the limit function, $n = \infty$. To model the problem we used GAMS [1], and in the solution we used the PATH solver [3] that is implemented in GAMS.

Table 2
Base prices and base demands in markets, $p0$ and $q0$

Market	Period 1		Period 2		Period 3		Period 4	
BelLux	5.12	7.8	2.56	9.4	3.41	9.4	5.12	9.5
FRGer	5.27	40.7	2.64	46.2	3.52	46.5	5.27	44.6
France	5.25	23.6	2.62	28.3	3.50	29.8	5.25	28.5
Italy	5.15	25.3	2.57	34.9	3.43	37.5	5.15	37.2
Netherlands	5.16	28.9	2.58	29.9	3.44	32.2	5.16	29.7
UK	4.54	43.8	2.27	50.3	3.03	56.4	4.54	53.7

Table 3
Price elasticities and domestic production in markets, e and D

Market	Period 1		Period 2		Period 3		Period 4	
BelLux	-1.07	0.00	-1.26	0.00	-1.34	0.00	-1.42	0.00
FRGer	-1.46	13.70	-1.58	13.80	-1.68	13.80	-1.79	13.80
France	-0.81	4.80	-1.19	2.90	-1.57	3.00	-2.01	3.00
Italy	-1.15	10.40	-1.36	10.00	-1.45	10.00	-1.54	10.40
Netherlands	-0.94	22.93	-1.13	20.96	-1.29	24.11	-1.45	23.90
UK	-0.61	33.70	-0.87	35.00	-1.10	37.00	-1.30	38.00

Table 4
Prices and quantities for Period 4 (when shutdown is uncertain)

Market	$n = 20$		$n = 200$		$n = 1000$		$n = 15000$		$n = 100000$		$n = \infty$	
BelLux	5.19	9.31	5.26	9.15	5.29	9.07	5.27	9.11	5.27	9.13	5.27	9.12
FRGer	4.77	53.28	4.83	52.17	4.86	51.64	4.84	51.94	4.84	52.02	4.84	52.01
France	4.82	33.77	4.88	32.95	4.91	32.56	4.90	32.78	4.89	32.84	4.89	32.83
Italy	4.87	40.50	4.93	39.77	4.96	39.42	4.94	39.62	4.94	39.67	4.94	39.66
Netherlands	5.99	23.90	5.99	23.90	5.99	23.90	5.99	23.90	5.99	23.90	5.99	23.90
UK	4.53	53.79	4.58	53.06	4.61	52.71	4.59	52.91	4.59	52.96	4.59	52.95

The solution is a very detailed report containing prices of natural gas in every market and the amount each producer ships to each market. Since the values of these quantities in different periods exhibited similar convergence behavior, we report in Table 4 only the price and quantity information for the last period. For each n, the number of iterations and the number of function evaluations required by PATH were 12 and 16 respectively.

Table 4, incidentally, shows that we get quite close to the true solution with a simulation run length as small as 200 random numbers. That table also shows that the prices in the Netherlands are the same for each value of n. This is due to the fact that the Netherlands does not import natural gas and hence its market price is not affected by the shutdown in Algeria.

We also considered a different version of this problem in which there is no

Table 5
Parameters for model with uncertain oil price

t	η_t	or_t	π_t^L	π_t^U
1	-0.10	30	16	34
2	-0.12	15	12	18
3	-0.24	30	24	36
4	-0.36	35	28	42

shutdown in Algeria but in which the prices and demands for natural gas depend on the uncertain price of oil, and therefore are themselves uncertain. In this model the demand law (7) is replaced by

$$P_j^t(Q) = \hat{p}_j^t(Q/\hat{q}_j^t)^{1/e_j^t}, \tag{17}$$

where

$$\hat{p}_j^t = p0_j^t(op_t/or_t), \qquad \hat{q}_j^t = q0_j^t(op_t/or_t)^{\eta_t}.$$

Here op_t is a random oil price taken to be uniformly distributed on $[\pi_t^L, \pi_t^U]$, or_t is a fixed reference price for oil, and η_t is an elasticity relating the relative demand for natural gas to the relative price of oil. The values of η_t, or_t, π_t^L, and π_t^U are given in Table 5.

For approximate solution of the limit variational inequality, we sampled from the uniform distribution at each realization, then averaged the resulting functions to obtain the f_n for the approximating variational inequality. Table 6 presents the results for Period 4 for simulation lengths $n = 20$, $n = 200$, $n = 1000$, and $n = 20000$.

For comparison purposes, we then computed the expectation of the quantity $\hat{p}_j^t/(\hat{q}_j^t)^{1/e_j^t}$ and solved the variational inequality with the limit function; the solution is given in the column corresponding to $n = \infty$ in Table 6. The PATH solver required the same numbers of iterations and function evaluations as in the previous model. In both versions, we observed that the amount of time required to solve the variational inequality did not vary much with the length of the simulation runs.

For illustrative purposes we formulated these two example problems so that we could compute the limit functions exactly and could thereby verify the accuracy of the computed results. In practice, of course, this would not be possible (otherwise one would not need to use simulation), and it is in no way necessary for application of the method we have described. Two application areas in which this method has been successfully used and in which the limit function is not computable are option pricing [9] and network design [22].

Table 6
Prices and quantities for Period 4 (with demand law (17) in effect)

Market	$n = 20$		$n = 200$		$n = 1000$		$n = 20000$		$n = \infty$	
BelLux	5.17	9.28	5.18	9.28	5.19	9.31	5.19	9.31	5.19	9.30
FRGer	4.75	53.03	4.76	53.07	4.77	53.27	4.77	53.24	4.77	53.23
France	4.80	33.58	4.81	33.62	4.82	33.78	4.82	33.75	4.82	33.74
Italy	4.85	40.34	4.86	40.36	4.87	40.48	4.87	40.46	4.87	40.46
Netherlands	5.95	23.90	5.97	23.90	5.99	23.90	5.99	23.90	5.99	23.90
UK	4.51	53.65	4.52	53.66	4.53	53.77	4.53	53.75	4.53	53.75

5 Conclusion

In this paper we have shown how to extend a simulation-based method, sample-path optimization, to solve stochastic variational inequalities. We have justified the method by exhibiting sufficient conditions for convergence, and we have presented the results of a small numerical experiment on a non-integrable economic equilibrium model of the European natural gas market. These results indicate that the method is implementable. However, we have not presented any analysis of its speed of convergence, nor have we shown how to obtain confidence regions for the computed results. These questions are the focus of current research.

One important practical aspect of the method concerns the approximating functions f_n. For the theory these need only be continuous and converge continuously to a relatively nice limit f_∞. However, for computational purposes more is needed. For example, in the European gas market problem of Section 4 we used the GAMS modeling language for computational solution. To be able to use GAMS we need a closed form expression for the f_n; the automatic differentiation capability of GAMS then is used to evaluate gradients of these functions in order to apply an efficient solution algorithm. This means that for this example we needed the f_n to be expressible in closed form and differentiable; in fact, the underlying theory for the PATH solver requires some additional conditions on the derivatives [3].

The requirement for a closed-form expression arose because we wanted to use the automatic differentiation capability of GAMS, so this is not really a basic requirement. However, most efficient methods for solving variational inequalities will require numerical evaluation of both the f_n and their derivatives. To avoid this, one might resort to techniques that do not require derivative values, such as the resolvent method (see [30,31] for example), but the performance of such methods is usually considerably worse than that of methods that utilize gradient information. In addition, except in special cases it may be difficult or impossible to compute the resolvent. An alternate approach when evaluating exact gradients of the f_n is not possible might be to use approximations of these gradients in the solution of the variational inequality. Josephy [17] provides conditions for local convergence of such solutions. However, we are not aware of any systematic use of this idea for practical computation, and therefore we cannot assess its numerical effectiveness.

The model presented in Section 4 is static; uncertainty enters via one or more parameters appearing in the model. In such situations it is often realistic to expect to be able to obtain closed-form expressions for the f_n. There are other situations in which this can be considerably more difficult. One example of such a problem is the option-pricing application of [9]. This was a problem of unconstrained optimization, so f_∞ was the gradient of a function ϕ_∞ and the problem was to find a zero of $d\phi_\infty$. Unfortunately, it was not possible to find well-behaved approximating functions ϕ_n whose gradients could be used as the f_n. Instead, the f_n were obtained by applying methods of perturbation analysis from [7] and [35] to obtain estimators that could then be averaged to construct the f_n.

A difficult case in general appears to be that in which one is modeling a dynamic system and the f_n relate to performance of the system as observed over some time interval, say one of length n. If the underlying problem is one of optimization and we are solving a variational inequality derived from the first order optimality conditions, then the f_n will contain gradients of the objective function, presumably a function related to system performance, and possibly also gradients of nonlinear constraints (constraints that are entirely linear can be subsumed in the definition of C). Although it is often possible to obtain numerical values for such gradients — and hence for the f_n — by using general methods of perturbation analysis [8,13], at present we know of no generally applicable method for obtaining derivatives of the f_n in this case. However, methods are available for some special cases; for example, see [40] for results for the system times of customers as a function of the parameters of the service time distributions in a GI/GI/1 queue.

In summary, if the f_n are differentiable and have closed-form expressions, then one can conveniently apply modeling languages, such as GAMS, having automatic differentiation capabilities. When the underlying system is static, as in the option pricing problem of [9], one may be able to use methods of perturbation analysis to find closed-form expressions for suitable f_n and their gradients. When the underlying system is dynamic, the f_n may not have closed forms, but again one may use perturbation analysis to evaluate them. How to evaluate *their* gradients in this case is, however, still largely an open question; existing methods are tailor-made to specific systems. This seems to be an important area for further research.

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