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SAMPLE QUANTILES OF ADDITIVE RENEWAL REWARD PROCESSES

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Abstract

The distribution of the sample quantiles of random processes is important for the pricing of some of the so-called financial 'look-back' options. In this paper a representation of the distribution of the α-quantile of an additive renewal reward process is obtained as the sum of the supremum and the infimum of two rescaled independent copies of the process. This representation has already been proved for processes with stationary and independent increments. As an example, the distribution of the α-quantile of a randomly observed Brownian motion is obtained.

SAMPLE QUANTILES; LOOK-BACK FINANCIAL OPTIONS; MARKED RENEWAL PROCESSES

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1. Introduction

Look-back options are becoming increasingly important in mathematical finance. They are also called path-dependent options. An important example is α-quantile (also called α-percentile) options. In order to define them, let $(Y(t), t \geqq 0)$ denote the price of the underlying asset. For $0<\alpha<1$, define the α-quantile of $Y(t)$, $M_Y(\alpha, t)$ as

$$M_Y(\alpha, t)=\inf\left\{x : \int_0^t \mathbf{1}(Y(s) \leqq x)ds > \alpha t\right\}.$$

The α-quantile is the level at which the process spends a proportion of size at least α of its time below that level and a proportion of size at least $1-\alpha$ above. Pricing such options involves calculating $\boldsymbol{E}^*(h(M_Y(\alpha, t_2)) \mid \mathscr{F}_{t_1})$, where the expectation is calculated under a changed measure, h is a known function, $0 \leqq t_1 < t_2$ are fixed times and $\mathscr{F}_t$ is the filtration generated by $Y(t)$. The problem is usually studied for a general measurable function h. Two examples that are interesting in practice are $h(x)=(x-b)^+$ (associated with the so-called *call* option) and $h(x)=(b-x)^+$ (associated with the so-called *put* option). This in general involves finding the distribution of $M_Y(\alpha, t)$.

This option was first introduced by Miura [12]. One of the advantages of quantile options over other path-dependent options is that the problem of finding the distribution of the α-quantile of a stochastic process is equivalent to finding the distribution of the

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quantile of any monotone function of the process. Usually, such a function is the natural logarithm. The problem of pricing an α-quantile option, when $X(t)=\ln(Y(t))$ is a Brownian motion with a drift, has been solved by Akahori [1] and Dassios [3]. Also, for the driftless case, see [15]. In [3] the following representation for the distribution of the quantiles of a Brownian motion with a drift was obtained.

Proposition 1. *Let $X(t)=\sigma B(t)+\mu t$, where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$ and $(B(t),\, t \geqq 0)$ is a standard Brownian motion with $B(0)=0$. Furthermore, let $X^{(1)}(t)$ and $X^{(2)}(t)$ be independent copies of $X(t)$. Then,*

$$M_X(\alpha, t) \overset{\text{(law)}}{=} \sup_{0 \leqq s \leqq \alpha t} X^{(1)}(s) + \inf_{0 \leqq s \leqq (1-\alpha)t} X^{(2)}(s). \tag{1.1}$$

This decomposition provides us with a closed form expression for the distribution of $M(\alpha, t)$, since the distributions of the maximum and the minimum of a Brownian motion with drift are easy to obtain using well established results on hitting times. Proposition 1 was proved with the help of the Feynman–Kac formula. Embrechts *et al.* [8] gave two further proofs of the result.

An interesting problem is finding the distribution of an α-quantile of other stochastic processes. Embrechts and Samorodnitsky [9] studied the tail behaviour of the quantiles of a class of heavy tailed stochastic processes. In [4], the following generalisation of Proposition 1 can be found.

Proposition 2. *Let $X(t)$ be a process with stationary and independent increments and paths in $D([0, \infty))$ (the paths of $X(t)$ are right continuous with left limits; see p.* 307 *of* [2]). *Furthermore, let $X^{(1)}(t)$ and $X^{(2)}(t)$ be independent copies of $X(t)$. Then,*

$$M_X(\alpha, t) \overset{\text{(law)}}{=} \sup_{0 \leqq s \leqq \alpha t} X^{(1)}(s) + \inf_{0 \leqq s \leqq (1-\alpha)t} X^{(2)}(s).$$

In fact, Embrechts *et al.* [8] and Dassios [3] have obtained generalised versions of Propositions 1 and 2 respectively that include a similar representation for the joint distribution of $M_X(\alpha, t)$ and $X(t)$. Furthermore, Wendel [14] derived a discrete analogue of this representation in discrete time for sums of exchangeable random variables. We will obtain the version for sums of i.i.d. variables as a corollary of our results.

We will now define the process whose quantiles we will examine in this paper. Let $((T_i, Y_i),\, i=1,2,\cdots)$ be a sequence of independent and identically distributed pairs of random variables on a probability space $(\Omega, \mathscr{F}, \Pr)$ taking values in $\mathbb{R}^+ \times \mathbb{R}$ and having joint distribution function $G(u, y)$. Let

$$S_0=0, \quad S_n=\sum_{i=1}^{n} T_i, \qquad n=1,2,\cdots$$

and define the renewal process $(N(t),\, t \geqq 0)$ by

$$N(t)=\sup_{n=0,1,2,\cdots} \{n : S_n \leqq t\}.$$

We define $(X(t),\, t \geqq 0)$ by

$$X(t)=\begin{cases}\sum_{i=1}^{N(t)} Y_i, & N(t)=1,2,\cdots.\\ 0, & N(t)=0.\end{cases} \tag{1.2}$$

It should be noted that $X(t)$ is semi-Markov, but not a Markov process. However, the pair $(X(t), U(t))$, where $U(t)=t-S_{N(t)}$ is the time elapsed since the last jump in $X(t)$, is a Markov process. It can also be formulated as a piecewise deterministic Markov process as defined by Davis [6], [7]. For details on how to formulate $(X(t), U(t))$ as a piecewise deterministic Markov process see [5], p. 199. We call our process a renewal reward process, as in [13], p. 77–83.

$X(t)$ can be used as an alternative model for the price of securities in mathematical finance. In some respects it is more realistic than a Brownian motion as the price only changes at specific points in time where a transaction occurs or some information becomes available instead of changing continuously. It is also closely connected to the insurance model described by Dassios and Embrechts [5], p. 198.

We will organise this paper as follows. In Section 2 we will state and prove an important analytical result on the solution of an integral equation. As a corollary, we will derive a decomposition for the α-quantile of the sums of i.i.d. random variables. In Section 3 we will prove our main result, which is that Proposition 2 is true for $X(t)$ as defined above. Finally, in Section 4 we will apply our results to an example. We will obtain the distribution of the α-quantile of randomly observed Brownian motion.

2. An integral equation

We start by obtaining equations for appropriate functionals of the maximum and the minimum of sums of independent and identically distributed random variables. It is easy to check that the following result is true.

Lemma 1. *Let $Y_1, Y_2,\cdots$ be a sequence of independent and identically distributed random variables with distribution function F. Let $X_0=0$ and $X_n=\Sigma_{i=1}^n Y_i$ for $n=1,2,\cdots$. Furthermore let $0\leqq\psi<1$ and define*

$$H_1(x;\psi)=(1-\psi)\sum_{n=0}^{\infty}\psi^n \Pr\left(\max_{0\leqq i\leqq n}(X_i)\leqq x\right) \tag{2.1}$$

and

$$H_2(x;\psi)=(1-\psi)\sum_{n=0}^{\infty}\psi^n \Pr\left(\min_{0\leqq i\leqq n}(X_i)\leqq x\right). \tag{2.2}$$

Then $H_1(x;\psi)$ satisfies the equation

$$H_1(x;\psi)=\mathbf{1}(x\geqq 0)\left((1-\psi)+\psi\int_{-\infty}^{\infty}H_1(x-y;\psi)dF(y)\right) \tag{2.3}$$

and $H_2(x;\psi)$ satisfies the equation

(2.4) $$H_2(x;\psi)=\mathbf{1}(x\geqq 0)+\psi\mathbf{1}(x<0)\int_{-\infty}^{\infty}H_2(x-y;\psi)dF(y).$$

Remarks.

1. Equations (2.3) and (2.4) are Wiener–Hopf-type equations like the ones described in [10], Section XII.3, with a defective probability measure.

2. From their definition we can see that $H_1(x;\psi)$ and $H_2(x;\psi)$ are distribution functions. $H_1(x;\psi)$ is the distribution function of $\max_{0\leqq i\leqq N}(X_i)$ and is the distribution function of $\min_{0\leqq i\leqq N}(X_i)$, where N is a random variable independent of $Y_1, Y_2,\cdots$ with a geometric distribution; i.e. $\Pr(N=n)=(1-\psi)\psi^n$, $n=0, 1, 2,\cdots$.

We will now prove the main result of this section.

Lemma 2. *Let $\bar{D}(\mathbb{R})$ be the space of bounded real valued functions of $\mathbb{R}$ that are right continuous with left limits existing for all points and let $G_1(x)$, $G_2(x)$ be distribution functions. Then, for all $0\leqq\psi<1$ and $0\leqq\phi<1$, the equation*

(2.5) $$H(x)=(1-\psi)\mathbf{1}(x\geqq 0)+\psi\mathbf{1}(x\geqq 0)\int_{-\infty}^{\infty}H(x-y)dG_1(y)$$
$$+\phi\mathbf{1}(x<0)\int_{-\infty}^{\infty}H(x-y)dG_2(y)$$

has a unique solution $H(x;\psi,\phi)$ in $\bar{D}(\mathbb{R})$. Furthermore

(2.6) $$H(x;\psi,\phi)=\int_{-\infty}^{\infty}H_1(x-y;\psi)dH_2(y;\phi),$$

where $H_1(x;\psi)$ is the unique solution of

(2.7) $$H_1(x)=(1-\psi)\mathbf{1}(x\geqq 0)+\psi\mathbf{1}(x\geqq 0)\int_{-\infty}^{\infty}H_1(x-y)dG_1(y)$$

in $\bar{D}(\mathbb{R})$ and $H_2(x;\phi)$ is the unique solution of

(2.8) $$H(x)=\mathbf{1}(x\geqq 0)+\phi\mathbf{1}(x<0)\int_{-\infty}^{\infty}H_2(x-y;\psi)dG_2(y)$$

in $\bar{D}(\mathbb{R})$.

Proof. Define the metric $d(H,K)$ in $\bar{D}(\mathbb{R})$, by

(2.9) $$d(H,K)=\sup_{x\in\mathbb{R}}|H(x)-K(x)|.$$

Also, define $\boldsymbol{T}:\bar{D}(\mathbb{R})\to\bar{D}(\mathbb{R})$, by

$$TH(x) = (1-\psi)\mathbf{1}(x \geqq 0) + \psi \mathbf{1}(x \geqq 0) \int_{-\infty}^{\infty} H(x-y)dG_1(y)$$
$$+\phi \mathbf{1}(x<0) \int_{-\infty}^{\infty} H(x-y)dG_2(y).$$

Observe that for $0 \leqq \psi < 1$ and $0 \leqq \phi < 1$, $\boldsymbol{T}$ is a contraction mapping on $\bar{D}(\mathbb{R})$, using the metric defined by (2.9). Then by the fixed point theorem for contraction mappings, (2.5) has a unique solution. Moreover, (2.7) and (2.8) have unique solutions, since they are special cases of (2.5) for $\varphi = 0$ and $\psi = 0$ respectively. By Lemma 1 and the second remark following its proof, these solutions, $H_1(x; \psi)$ and $H_2(x; \phi)$, are distribution functions and let U and V be random variables on a suitable probability space with distribution functions $H_1(x; \psi)$ and $H_2(x; \phi)$ respectively. Let $H(x; \psi, \phi)$ be the convolution of $H_1(x; \psi)$ and $H_2(x; \phi)$, as defined by (2.6). We will prove that $H(x; \psi, \phi)$ satisfies (2.5) and therefore is its unique solution in $\bar{D}(\mathbb{R})$.

Remark 1. Observe that $H(x; \psi, \phi) = \Pr(U+V \leqq x)$. Now, suppose $x \geqq 0$, and condition on $V = v$, then $\Pr(U+V \leqq x \mid V=v) = H_1(x-v; \psi)$. Note that $\Pr(V \leqq 0) = 1$ and so we only need to consider $c \leqq 0$, in which case $x - v \geqq 0$ and from (2.7) we then get that

$$H_1(x-v; \psi) = (1-\psi) + \psi \int_{-\infty}^{\infty} H_1(x-v-y; \psi)dG_1(y). \tag{2.10}$$

Averaging over all non-positive v, we get that for $x \geqq 0$,

$$H(x; \psi, \phi) = (1-\psi) + \psi \int_{-\infty}^{\infty} H(x-y; \psi, \phi)dG_1(y). \tag{2.11}$$

Similarly for $x < 0$, condition on $U = u$; then $\Pr(U+V \leqq x \mid U=u) = H_2(x-u; \psi)$. Note that $\Pr(U \geqq 0) = 1$ and so we only need to consider $u \geqq 0$, in which case $x - u < 0$ and from (2.8) we then get that

$$H_2(x-u; \phi) = \phi \int_{-\infty}^{\infty} H_2(x-u-y; \psi)dG_2(y). \tag{2.12}$$

Averaging over all non-negative u, we get that for $x < 0$,

$$H(x; \psi, \phi) = \phi \int_{-\infty}^{\infty} H(x-y; \psi, \phi)dG_2(y). \tag{2.13}$$

Combining (2.11) and (2.13) we see that $H(x; \psi, \phi)$ satisfies (2.5) and therefore is its unique solution in $D(\mathbb{R})$.

A corollary of this lemma is the following result on the quantiles of sums of i.i.d. random variables.

Corollary 3. *Let* $Y_1, Y_2, \cdots$ *be a sequence of independent and identically distributed random variables with distribution function F. Let* $X_0=0$ *and* $X_n=\Sigma_{i=1}^n Y_i$ *for* $n=1,2,\cdots$. *Furthermore let*

$$M_{j,n}=\inf\left\{z:\sum_{i=0}^{n}\mathbf{1}(X_i\leqq z)>j\right\}. \tag{2.14}$$

(*So* $M_{0,n}$ *denotes the smallest of* $X_0, X_1,\cdots, X_n$, $M_{1,n}$ *the second smallest and so on, with* $M_{n,n}$ *denoting the largest*). *Let also* $X_0^{(1)}, X_1^{(1)},\cdots$ *and* $X_0^{(2)}, X_1^{(2)},\cdots$ *be two independent copies of the sequence* $X_0, X_1,\cdots$. *Then*

$$M_{j,n}\overset{\text{(law)}}{=}\max_{0\leqq i\leqq j}(X_i^{(1)})+\min_{0\leqq i\leqq n-j}(X_i^{(2)}). \tag{2.15}$$

Proof. Consider the occupation time $L_n(x)=\Sigma_{i=0}^n \mathbf{1}(X_i\leqq x)$ and let

$$h(x)=\sum_{n=0}^{\infty}\phi^n \boldsymbol{E}(\eta^{L_n(x)}),$$

where $0\leqq\phi<1$ and $0\leqq\eta<1/\phi$. Using the fact that $Y_1, Y_2,\cdots$ are independent and therefore exchangeable, condition on $Y_1=y$ and observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\phi^n\boldsymbol{E}(\eta^{L_n(x)}\mid Y_1=y)&=\eta^{\mathbf{1}(x\geqq 0)}+\phi\eta^{\mathbf{1}(x\geqq 0)}\sum_{n=1}^{\infty}\phi^{n-1}\boldsymbol{E}(\eta^{\Sigma_{i=1}^n\mathbf{1}(\Sigma_{r=1}^i Y_r\leqq x)}\mid Y_1=y)\\&=\eta^{\mathbf{1}(x\geqq 0)}+\phi\eta^{\mathbf{1}(x\geqq 0)}\sum_{n=1}^{\infty}\phi^{n-1}\boldsymbol{E}(\eta^{\mathbf{1}(0\leqq x-y)+\Sigma_{i=2}^n\mathbf{1}(\Sigma_{r=2}^i Y_r\leqq x-y)})\\&=\eta^{\mathbf{1}(x\geqq 0)}+\phi\eta^{\mathbf{1}(x\geqq 0)}\sum_{n=1}^{\infty}\phi^{n-1}\boldsymbol{E}(\eta^{\mathbf{1}(0\leqq x-y)+\Sigma_{i=1}^{n-1}\mathbf{1}(\Sigma_{r=1}^i Y_r\leqq x-y)})\\&=\eta^{\mathbf{1}(x\geqq 0)}+\phi\eta^{\mathbf{1}(x\geqq 0)}h(x-y).\end{aligned}$$

Averaging over all values of y we therefore get

$$h(x)=\eta^{\mathbf{1}(x\geqq 0)}+\phi\eta^{\mathbf{1}(x\geqq 0)}\int_{-\infty}^{\infty}h(x-y)dF(y). \tag{2.16}$$

Now, note that

$$\boldsymbol{E}(\eta^{L_n(x)})=1-(1-\eta)\sum_{j=0}^{n}\eta^j\Pr(L_n(x)>j),$$

and therefore

$$\begin{aligned}h(x)&=\frac{1}{1-\phi}-(1-\eta)\sum_{n=0}^{\infty}\phi^n\sum_{j=0}^{n}\eta^j\Pr(L_n(x)>j)\\&=\frac{1}{1-\phi}-(1-\eta)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\phi^k(\eta\phi)^j\Pr(L_{j+k}(x)>j).\end{aligned} \tag{2.17}$$

Setting $\psi = \eta\phi$ and observing that the events $\{L_{j+k}(x) > j\}$ and $\{M_{j,j+k} \leqq x\}$ are identical, we can rewrite (2.17) as

$$h(x) = \frac{1}{1-\phi} - \frac{\phi-\psi}{\phi} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \phi^k \psi^j \Pr(M_{j,j+k} \leqq x) \tag{2.18}$$

$$= \frac{1}{1-\phi} - \frac{\phi-\psi}{\phi(1-\phi)(1-\psi)} H(x),$$

where

$$H(x) = (1-\phi)(1-\psi) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \phi^k \psi^j \Pr(M_{j,j+k} \leqq x). \tag{2.19}$$

From (2.18) and (2.16), we get

$$H(x) = (1-\psi)\mathbf{1}(x \geqq 0) + \psi \mathbf{1}(x \geqq 0) \int_{-\infty}^{\infty} H(x-y)dF(y) \tag{2.20}$$

$$+ \phi \mathbf{1}(x < 0) \int_{-\infty}^{\infty} H(x-y)dF(y).$$

From Lemma 2 we have that

$$H(x) = \int_{-\infty}^{\infty} H_1(x-y)dH_2(y), \tag{2.21}$$

where $H_1(x)$ is the unique solution in $\bar{D}(\mathbb{R})$ of

$$H_1(x) = (1-\psi)\mathbf{1}(x \geqq 0) + \psi \mathbf{1}(x \geqq 0) \int_{-\infty}^{\infty} H_1(x-y)dF(y),$$

and $H_2(x)$ is the unique solution in $\bar{D}(\mathbb{R})$ of

$$H_2(x) = \mathbf{1}(x \geqq 0) + \phi \mathbf{1}(x < 0) \int_{-\infty}^{\infty} H_2(x-y)dF(y).$$

From Lemma 1 we see that

$$H_1(x) = (1-\psi) \sum_{j=0}^{\infty} \psi^j \Pr\left(\max_{0 \leqq i \leqq j} (X_i) \leqq x \right) \tag{2.22}$$

and

(2.23) $$H_2(x)=(1-\phi)\sum_{k=0}^{\infty}\phi^k \Pr\left(\min_{0\leqq i\leqq k}(X_i)\leqq x\right).$$

From (2.19), (2.21), (2.22), (2.23) and the uniqueness of the relevant expansion we conclude that

$$M_{j,j+k}\overset{\text{(law)}}{=}\max_{0\leqq i\leqq j}(X_i^{(1)})+\min_{0\leqq i\leqq k}(X_i^{(2)}),$$

for all $j\geqq 0$ and $k\geqq 0$. This concludes the proof of the corollary.

Remarks.

1. This result has already been proved somewhat differently by Wendel [14].
2. A similar decomposition to (2.15) can be obtained for the joint distribution of $M_{j,n}$ and X_n (see [4]).
3. Note that for the proof of the corollary $G_1=G_2=F$. The more general situation of Lemma 2 will be applicable to the result of the next section.

3. Sample quantiles of the semi-Markov process

In this section we will prove the main result of the paper. $X(t)$ is defined as in the introduction, $X(t)=\Sigma_{i=1}^{N(t)} Y_i$ for $N(t)>0$ and $X(t)=0$ for $N(t)=0$. Recall that the pairs (T_i, Y_i), $i=1,2,\cdots$ are i.i.d. random variables on a probability space $(\Omega, \mathscr{F}, \Pr)$ and have joint distribution function $G(u,y)$, where $T_1, T_2,\cdots$ are the renewal intervals of the renewal process $N(t)$.

Theorem 1. *Let $X(t)$ be defined by* (1.2). *Furthermore, define*

(3.1) $$M_X(\alpha,t)=\inf\left\{x:\int_0^t \mathbf{1}(X(s)\leqq x)ds>\alpha t\right\}.$$

Let $X^{(1)}(t)$, $X^{(2)}(t)$ be independent copies of $X(t)$; then

(3.2) $$M_X(\alpha,t)\overset{\text{(law)}}{=}\sup_{0\leqq s\leqq \alpha t}X^{(1)}(s)+\inf_{0\leqq s\leqq (1-\alpha)t}X^{(2)}(s).$$

Proof. Define the occupation time

$$L(x,t)=\int_0^t \mathbf{1}(X(x)\leqq x)ds$$

and

(3.3) $$h(x)=\int_0^\infty e^{-\beta t}\boldsymbol{E}(e^{-\gamma L(x,t)})dt,$$

where $\beta>0$ and $\gamma>-\beta$. Conditioning on $T_1=u$ and $Y_1=y$, we get

$$\int_0^\infty e^{-\beta t} \boldsymbol{E}(e^{-\gamma L(x,t)} \mid T_1 = u,\ Y_1 = y) dt$$

$$= \int_0^u e^{-\beta t} e^{-\gamma \mathbf{1}(x \geqq 0) t} dt + e^{-\beta u} e^{-\gamma \mathbf{1}(x \geqq 0) u} \int_0^\infty e^{-\beta t} \boldsymbol{E}(e^{-\gamma L(x-y,t)}) dt.$$

Averaging over all u and y we get

$$h(x) = \frac{1 - \hat{G}(\beta+\gamma)}{\beta+\gamma} \mathbf{1}(x \geqq 0) + \frac{1 - \hat{G}(\beta)}{\beta} \mathbf{1}(x < 0)$$

$$+ \mathbf{1}(x \geqq 0) \hat{G}(\beta+\gamma) \int_{-\infty}^{\infty} h(x-y) d\tilde{G}(y; \beta+\gamma) \tag{3.4}$$

$$+ \mathbf{1}(x < 0) \hat{G}(\beta) \int_{-\infty}^{\infty} h(x-y) d\tilde{G}(y; \beta)$$

where, for $\rho > 0$,

$$\hat{G}(\rho) = \iint_{\mathbb{R}^+ \times \mathbb{R}} e^{-\rho u} dG(u, z)$$

and

$$\tilde{G}(y; \rho) = \frac{1}{\hat{G}(\rho)} \iint_{\mathbb{R}^+ \times (-\infty, y)} e^{-\rho u} dG(u, z). \tag{3.5}$$

Clearly $\tilde{G}(y; \rho)$ is a distribution function in y for fixed ρ. Now, note that

$$\boldsymbol{E}(e^{-\gamma L(x,t)}) = 1 - \gamma \int_0^t e^{-\gamma v} \Pr(L(x, t) > v) dv$$

and therefore

$$h(x) = \frac{1}{\beta} - \gamma \int_0^\infty e^{-\beta t} \int_0^t e^{-\gamma v} \Pr(L(x, t) > v) dv dt.$$

Observe that there is a set $\Omega_1 \subset \Omega$, with $\Pr(\Omega_1) = 1$ such that for all $\omega \in \Omega_1$, $L(x, t)$ is piecewise constant with right continuous paths, so the infimum in (3.1) is attained, the events $\{L(x, t) > v\} \cap \Omega_1$ and $\{M(v/t, t) \leqq x\} \cap \Omega_1$ are identical and therefore

$$\Pr(L(x, t) > v) = \Pr\left(M\left(\frac{v}{t}, t\right) \leqq x\right). \tag{3.6}$$

Setting $\theta = \beta + \gamma$ we get

$$h(x)=\frac{1}{\beta}-\frac{\theta-\beta}{\beta\theta}H(x)$$

where

$$H(x)=\beta\theta\int_0^\infty\int_0^\infty e^{-\theta v}e^{-\beta s}\Pr\left(M_X\left(\frac{v}{v+s},v+s\right)\leqq x\right)dvds. \tag{3.7}$$

From (3.7) and (3.4) we see that $H(x)$ satisfies

$$\begin{aligned} H(x)&=(1-\hat{G}(\theta))\mathbf{1}(x\geqq 0)+\mathbf{1}(x\geqq 0)\hat{G}(\theta)\int_{-\infty}^\infty H(x-y)d\tilde{G}(y;\theta)\\ &\quad+\mathbf{1}(x<0)\hat{G}(\beta)\int_{-\infty}^\infty H(x-y)d\tilde{G}(y;\beta). \end{aligned} \tag{3.8}$$

We can now find an equation for a similar functional of $\sup_{0\leqq s\leqq t}X(s)$. Let

$$H_1(x)=\theta\int_0^\infty e^{-\theta t}\Pr\left(\sup_{0\leqq s\leqq t}X(s)\leqq x\right)dt. \tag{3.9}$$

Conditioning on $T_1=u$ and $Y_1=y$, we get

$$\begin{aligned} &\theta\int_0^\infty e^{-\theta t}\Pr\left(\sup_{0\leqq s\leqq t}X(s)\leqq x\mid T_1=u,\ Y_1=y\right)dt\\ &\quad=\mathbf{1}(x\geqq 0)\left((1-e^{-\theta u})+\theta e^{-\theta u}\int_0^\infty e^{-\theta t}\Pr\left(\sup_{0\leqq s\leqq t}X(s)\leqq x-y\right)dt\right). \end{aligned}$$

Averaging over all values of u and y, we have that

$$H_1(x)=(1-\hat{G}(\theta))\mathbf{1}(x\geqq 0)+\hat{G}(\theta)\mathbf{1}(x\geqq 0)\int_{-\infty}^\infty H_1(x-y)d\tilde{G}(y;\theta). \tag{3.10}$$

Similarly, define

$$H_2(x)=\beta\int_0^\infty e^{-\beta t}\Pr\left(\inf_{0\leqq s\leqq t}X(s)\leqq x\right)dt. \tag{3.11}$$

Conditioning on $T_1=u$ and $Y_1=y$, we get

$$\beta \int_0^\infty e^{-\beta t} \Pr\left(\inf_{0 \leqq s \leqq t} X(s) \leqq x \mid T_1 = u,\ Y_1 = y \right) dt$$

$$= \mathbf{1}(x \geqq 0) + \mathbf{1}(x < 0) \beta e^{-\beta u} \int_0^\infty e^{-\beta t} \Pr\left(\inf_{0 \leqq s \leqq t} X(s) \leqq x - y \right) dt.$$

Averaging over all values of u and y, we have

$$H_2(x) = \mathbf{1}(x \geqq 0) + \hat{G}(\beta) \mathbf{1}(x < 0) \int_{-\infty}^{\infty} H_2(x-y) d\tilde{G}(y;\beta). \tag{3.12}$$

From (3.8), (3.10) and (3.12), we see that applying Lemma 2 with $G_1(y) = \tilde{G}(y;\theta)$ and $G_2(y) = \tilde{G}(y;\beta)$ yields

$$H(x) = \int_{-\infty}^{\infty} H_1(x-y) dH_2(y). \tag{3.13}$$

From (3.7), (3.9), (3.11), (3.13) and the uniqueness of Laplace transforms we conclude that

$$M_X\left(\frac{v}{v+s}, v+s \right) \overset{\text{(law)}}{=} \sup_{0 \leqq r \leqq v} X^{(1)}(r) + \inf_{0 \leqq r \leqq s} X^{(2)}(r) \tag{3.14}$$

for all $v > 0$ and $s > 0$. Setting $v = \alpha t$ and $s = (1-\alpha)t$ completes the proof of the theorem.

Remarks.

1. The results in [4] might have suggested that the property described by (3.2) characterises processes with stationary and independent increments. The theorem just proved shows that this is not true.

2. An interesting process to consider, with a view towards insurance applications, is $X^*(t) = ct + X(t)$. One could try to investigate whether (3.2) holds for $X^*(t)$. Unfortunately, Lemma 2 is not directly applicable and arguments such as (3.6) are not valid either. However, for the special case where $N(t)$ is a Poisson process and the sequence $Y_1, Y_2, \cdots$ is independent of $N(t)$, $X^*(t)$ has stationary and independent increments and from the results in [4] we can see that (3.2) is true.

4. An example

Let $(W(t), t \geqq 0)$ be a standard Brownian motion. Let $T_1, T_2, \cdots$ be a sequence of i.i.d. random variables that are exponentially distributed with parameter λ. Define

$$X(t) = \begin{cases} 0, & \text{for } 0 \leqq t < T_1, \\ W(T_i), & \text{for } T_i \leqq t < T_{i+1}, \end{cases} \qquad i = 1, 2, \cdots. \tag{4.1}$$

We can reformulate $X(t)$ be defining $N(t)$ as the Poisson process with renewal times $T_1, T_2, \cdots$, where $((T_i, Y_i), i = 1, 2, \cdots)$ is a sequence of independent and identically distributed pairs of random variables such that for all i, T_i is exponentially distributed with parameter λ and conditionally on $T_i=u$, Y_i is normally distributed with mean 0 and variance u. Then, we set

$$X(t)=\begin{cases}\sum_{i=1}^{N(t)} Y_i, & N(t)=1,2,\cdots.\\ 0, & N(t)=0.\end{cases} \tag{4.2}$$

By the symmetry of the distribution of Y_i we see that $\sup_{0\leqq s\leqq t} X^{(1)}(s)$ and $-\inf_{0\leqq s\leqq t} X^{(2)}(s)$ have the same distribution. Theorem 1 suggests that in order to find the distribution of $M_X(\alpha, t)$ for any α it suffices to find the distribution of $\sup_{0\leqq s\leqq t} X^{(1)}(s)$.

Consider a random variable $\tilde{T}$ which is exponentially distributed with parameter $\lambda+\theta$ and is independent of $W(t)$. Then, by (3.5) we have that $\tilde{G}(y;\theta)=\Pr(W(\tilde{T})\leqq y)$. Furthermore, by symmetry if T_y denotes the first hitting time of y by $W(t)$, for $y<0$ this is

$$\tfrac{1}{2}\Pr(T_{|y|}<\tilde{T})=\tfrac{1}{2}E(\exp[-(\lambda+\theta)T_{|y|}])=\tfrac{1}{2}\exp[-|y|\sqrt{2(\lambda+\theta)}]$$

and for $y\geqq 0$,

$$\tfrac{1}{2}\Pr(T_y<\tilde{T})+\Pr(\tilde{T}<T_y)=1-\tfrac{1}{2}\Pr(T_y<\tilde{T})=1-\tfrac{1}{2}\exp[-y\sqrt{2(\lambda+\theta)}].$$

Defining

$$H_1(x)=\theta\int_0^\infty e^{-\theta t}\Pr\left(\sup_{0\leqq s\leqq t} X(s)\leqq x\right)dt, \tag{4.3}$$

we see that $H_1(x)$ satisfies (3.10); then from the above, this can be written as

$$H_1(x)=\begin{cases}\dfrac{\theta}{\lambda+\theta}+\dfrac{\lambda}{\sqrt{2(\lambda+\theta)}}\displaystyle\int_0^x H_1(x-y)\exp(-y\sqrt{2(\lambda+\theta)})dy\\ \quad+\dfrac{\lambda}{\sqrt{2(\lambda+\theta)}}\displaystyle\int_0^\infty H_1(x+y)\exp(-y\sqrt{2(\lambda+\theta)})dy, & \text{for } x\geqq 0\\ 0, & \text{for } x<0.\end{cases} \tag{4.4}$$

To solve (4.4), set for $x\geqq 0$, $H_1(x)=[\theta/(\lambda+\theta)]+A(x)+B(x)$, where $A(x)$ and $B(x)$ are the second and third terms respectively of the first leg of (4.4). Then,

$$\frac{dA(x)}{dx}=-\sqrt{2(\lambda+\theta)}A(x)+\frac{\lambda}{\sqrt{2(\lambda+\theta)}}H_1(x)$$

$$\frac{dB(x)}{dx}=\sqrt{2(\lambda+\theta)}B(x)-\frac{\lambda}{\sqrt{2(\lambda+\theta)}}H_1(x)$$

and so

$$\frac{d^2A(x)}{dx^2} = 2(\lambda+\theta)A(x) - \lambda H_1(x) + \frac{\lambda}{\sqrt{2(\lambda+\theta)}}\frac{dH_1(x)}{dx}$$

$$\frac{d^2B(x)}{dx^2} = 2(\lambda+\theta)B(x) - \lambda H_1(x) - \frac{\lambda}{\sqrt{2(\lambda+\theta)}}\frac{dH_1(x)}{dx}$$

and therefore

$$\frac{d^2H_1(x)}{dx^2} = 2\theta H_1(x) - 2\theta.$$

Thus, the solution of (4.4) is given by

$$H_1(x) = \begin{cases} 1 - \dfrac{\sqrt{\lambda+\theta}-\sqrt{\theta}}{\sqrt{\lambda+\theta}}\,\mathrm{e}^{-x\sqrt{2\theta}}, & \text{for } x \geqq 0. \\ 0, & \text{for } x < 0, \end{cases} \tag{4.5}$$

Inverting the Laplace transform given by (4.3) and (4.5) we get that, for $x \geqq 0$, $\Pr(\sup_{0\leqq s\leqq t} X(s) \leqq x) = K(x, t)$, where

$$K(x, t) = \Psi\left(\frac{x}{\sqrt{t}}\right) + \int_0^t \frac{\exp(-x^2/2u)\exp[-\lambda(t-u)]}{\pi\sqrt{u(t-u)}}\,du. \tag{4.6}$$

and

$$\Psi(x) = \int_0^x \sqrt{\frac{2}{\pi}}\exp\left(-\frac{y^2}{2}\right)dy.$$

From (4.6) we also have

$$\Pr\left(\sup_{0\leqq s\leqq t} X(s) = 0\right) = K(0, t) = \int_0^t \frac{\mathrm{e}^{-\lambda(t-u)}}{\pi\sqrt{u(t-u)}}\,du = \mathrm{e}^{-\lambda t/2} I_0\left(\frac{\lambda t}{2}\right), \tag{4.7}$$

where

$$I_r(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+r}}{n!\,(n+r)!}, \qquad r = 0, 1, 2, \cdots$$

is the modified Bessel function of the first kind. We can also write

$$\Pr\left(\sup_{0\leqq s\leqq t} X(s) \leqq x\right) = K(0, t) + \int_0^x k(y, t)dy$$

where $K(0, t)$ is given by (4.7) and $k(x, t)$ by

$$k(x, t) = \sqrt{\frac{2}{\pi t}}\exp\left(-\frac{x^2}{2t}\right) - \int_0^t \frac{x\exp(-x^2/2u)\exp[-\lambda(t-u)]}{\pi\sqrt{u^3(t-u)}}\,du. \tag{4.8}$$

Using Theorem 1 and the fact that $\sup_{0\leqq s\leqq t} X^{(1)}(s)$ and $-\inf_{0\leqq s\leqq t} X^{(2)}(s)$ have the same distribution, we can obtain the distribution function of $M_X(\alpha, t)$ for $0<\alpha<1$. This is

$$\Pr(M_X(\alpha,t)\leqq x)=\begin{cases} K(x,\alpha t)K(0,(1-\alpha)t) \\ \quad +\displaystyle\int_0^\infty K(x+y,\alpha t)k(y,(1-\alpha)t)dy, & \text{for } x\geqq 0 \\ 1-K(-x,(1-\alpha)t)K(0,\alpha t) \\ \quad -\displaystyle\int_0^\infty K(y-x,(1-\alpha)t)k(y,\alpha t)dy, & \text{for } x<0. \end{cases} \tag{4.9}$$

Observe that

$$\Pr(M_X(\alpha,t)=0)=\mathrm{e}^{-\lambda t/2}I_0\left(\frac{\alpha\lambda t}{2}\right)I_0\left(\frac{(1-\alpha)\lambda t}{2}\right). \tag{4.10}$$

The moments of $M_X(\alpha, t)$ can be found by first finding the moments of $\sup_{0\leqq s\leqq t}X(s)$. Note that from (4.5) we can get

$$\int_0^\infty \mathrm{e}^{-\theta t}\boldsymbol{E}\left(\sup_{0\leqq s\leqq t}X(s)\right)dt=\frac{\sqrt{\lambda+\theta}-\sqrt{\theta}}{\theta\sqrt{2\theta}\sqrt{\lambda+\theta}}. \tag{4.11}$$

Inverting the Laplace transform in (4.11), we have

$$\boldsymbol{E}\left(\sup_{0\leqq s\leqq t}X(s)\right)=\sqrt{\frac{2t}{\pi}}-\sqrt{\frac{1}{\lambda}}\,\Phi(\sqrt{2\lambda t})$$

and therefore

$$\boldsymbol{E}(M_X(\alpha,t))=\sqrt{\frac{2\alpha t}{\pi}}-\sqrt{\frac{2(1-\alpha)t}{\pi}}-\sqrt{\frac{1}{\lambda}}\,(\Psi(\sqrt{2\alpha\lambda t})-\Psi(\sqrt{2(1-\alpha)\lambda t})). \tag{4.12}$$

Furthermore, from (4.5) we see that

$$\int_0^\infty \mathrm{e}^{-\theta t}\boldsymbol{E}\left(\left(\sup_{0\leqq s\leqq t}X(s)\right)^2\right)dt=\frac{\sqrt{\lambda+\theta}-\sqrt{\theta}}{\theta^2\sqrt{\lambda+\theta}}. \tag{4.13}$$

Inverting the Laplace transform in (4.13), we have

$$\boldsymbol{E}\left(\left(\sup_{0\leqq s\leqq t}X(s)\right)^2\right)=t-t\mathrm{e}^{-\lambda t/2}\left(I_0\left(\frac{\lambda t}{2}\right)+I_1\left(\frac{\lambda t}{2}\right)\right)$$

and therefore

$$\begin{aligned} \operatorname{Var}(M_X(\alpha,t)) &= t-\alpha t\mathrm{e}^{-\lambda\alpha t/2}\left(I_0\left(\frac{\alpha\lambda t}{2}\right)+I_1\left(\frac{\alpha\lambda t}{2}\right)\right) \\ &\quad -(1-\alpha)t\mathrm{e}^{-\lambda(1-\alpha)t/2}\left(I_0\left(\frac{(1-\alpha)\lambda t}{2}\right)+I_1\left(\frac{(1-\alpha)\lambda t}{2}\right)\right) \\ &\quad -\left(\sqrt{\frac{2\alpha t}{\pi}}-\sqrt{\frac{1}{\lambda}}\,\Psi(\sqrt{2\alpha\lambda t})\right)^2-\left(\sqrt{\frac{2(1-\alpha)t}{\pi}}-\sqrt{\frac{1}{\lambda}}\,\Psi(\sqrt{2(1-\alpha)\lambda t})\right)^2. \end{aligned} \tag{4.14}$$

Remarks.

1. If we let $\lambda \to \infty$, we then see that (4.9) converges to $\int_{-\infty}^{x} c(y)dy$, where $c(x)$ is given by

$$c(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\, e^{-x^2/2}\left(1-\Psi\left(x\sqrt{\frac{1-\alpha}{\alpha}}\right)\right), & \text{for } x \geqq 0, \\ \sqrt{\frac{2}{\pi}}\, e^{-x^2/2}\left(1-\Psi\left(-x\sqrt{\frac{\alpha}{1-\alpha}}\right)\right), & \text{for } x \leqq 0, \end{cases}$$

which is the distribution function of the α-quantile of the standard Brownian motion $W(t)$, as first obtained by Yor [15].

2. The results of this section can be generalised for $W(t)$ being a Brownian motion with drift μ, in which case, conditionally on $T_i = u$, Y_i is normally distributed with mean μu and variance u. The analogue of (4.4) admits a solution of the same form as (4.5).

3. Note that the unconditional distribution of Y_i will then be double exponential. The use of the double exponential distribution for asset returns is not new; see, for example, [11].

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